Kato-Ponce Inequality With $A_{\vec{P}}$ Weights

Sean Douglas
University of Missouri

Analysis Seminar

Ohio State University

7 December, 2023

Fourier Transform

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform and inverse Fourier transform are respectively defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot \xi} dy$$

Fourier Transform

For $f \in L^1(\mathbb{R}^n)$ the Fourier transform and inverse Fourier transform are respectively defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(y) e^{-2\pi i y \cdot \xi} dy$$

$$\mathcal{F}^{-1}(f)(\xi) = \int_{\mathbb{R}^n} f(y) e^{2\pi i y \cdot \xi} dy.$$

Fractional Differentiation

For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{F}^{-1}(\underbrace{2\pi i(\cdot)\widehat{\varphi}}_{2\pi i\xi\widehat{\varphi}(\xi)}) = \varphi'.$$

Fractional Differentiation

For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{F}^{-1}(\underbrace{2\pi i(\cdot)\widehat{\varphi}}_{2\pi i\xi\widehat{\varphi}(\xi)}) = \varphi'.$$

Thus multiplication by ξ in frequency is essentially the same as taking the derivative in space.

Fractional Differentiation

For $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$\mathcal{F}^{-1}(\underbrace{2\pi i(\cdot)\widehat{\varphi}}_{2\pi i\xi\widehat{\varphi}(\xi)}) = \varphi'.$$

Thus multiplication by ξ in frequency is essentially the same as taking the derivative in space.
More generally,

$$\mathcal{F}^{-1}(\underbrace{(2\pi i\cdot)^{m}\widehat{\varphi}}_{(2\pi i\xi)^{m}\widehat{\varphi}(\xi)}) = \frac{d^{m}}{dx^{m}}\varphi.$$

What about when m is not an integer?

Fractional Differentiation

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$ we define the homogeneous differential operator

$$D^s\varphi := \mathcal{F}^{-1}\big(\underbrace{|\cdot|^s\widehat{\varphi}}_{|\xi|^s\widehat{\varphi}(\xi)}\big).$$

Fractional Differentiation

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $s > 0$ we define the homogeneous differential operator

$$D^s\varphi := \mathcal{F}^{-1}\big(\underbrace{|\cdot|^s\widehat{\varphi}}_{|\xi|^s\widehat{\varphi}(\xi)}\big).$$

Similarly, the inhomogenous differential operator, where $\langle\cdot\rangle := (1+|\cdot|^2)^{\frac{1}{2}}$, is given by

$$J^s\varphi := \mathcal{F}^{-1}\big(\langle\cdot\rangle^s\widehat{\varphi}\big).$$

Fractional Differentiation

Notice if s is an even integer, $s = 2k$, then

$$|\xi|^{2k}\widehat{\varphi}(\xi) = (\xi_1^2 + \cdots + \xi_n^2)^k \widehat{\varphi}(\xi)$$
$$= \sum_{t_1+\cdots+t_n=k} \binom{k}{t_1, \cdots, t_n} \prod_{j=1}^{n} \xi_j^{2t_j} \widehat{\varphi}(\xi),$$

which will give the derivative in the classical sense. For this reason some authors use $(-\Delta)^{\frac{s}{2}}$ is used in place of D^s, and $(I-\Delta)^{\frac{s}{2}}$ in place of J^s (modulo a $2\pi i$).

Leibniz Rule

Standard Leibniz rule is given by

$$\partial^\alpha(fg) = \sum_{\beta:\beta\leq\alpha} \binom{\alpha}{\beta} f^{(\beta)} g^{(\alpha-\beta)}.$$

Leibniz Rule

Standard Leibniz rule is given by

$$\partial^\alpha(fg) = \sum_{\beta:\beta\leq\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} f^{(\beta)} g^{(\alpha-\beta)}.$$

We are interested in controlling the derivative of a product by *only* the higher order derivative terms. This may not be possible pointwise, but it is in norm. For example,

$$\begin{aligned}\left\| \frac{d^2}{dx^2}(fg) \right\|_{L^r} &= \|f''g + g''f + 2f'g'\|_{L^p} \\ &\leq \|f''g\|_{L^p} + \|g''f\|_{L^p} + \|2f'g'\|_{L^p} \\ &\lesssim \|f''\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|g''\|_{L^{p_1}} \|f\|_{L^{p_2}}.\end{aligned}$$

Fractional Leibniz Rule

For the fractional derivative we study analogous estimates called-

Kato-Ponce Inequality

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Fractional Leibniz Rule

For the fractional derivative we study analogous estimates called-

Kato-Ponce Inequality

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|J^s g\|_{L^{p_2}}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. More generally we are interested in inequalities of the form,

Weighted Multifactor Kato-Ponce Inequality

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}$$

where $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Kato, *Ponce* (1988) used a normed fractional Leibniz type rule, with $s > 0, 1 < p < \infty$, $1 < p_1 < \infty$, $p_2 = \infty$ in the study of Euler and Naiver Stokes equations.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Kato, *Ponce* (1988) used a normed fractional Leibniz type rule, with $s > 0, 1 < p < \infty$, $1 < p_1 < \infty$, $p_2 = \infty$ in the study of Euler and Naiver Stokes equations.

Christ, *Weinstein* (1991) proved the homogeneous KP inequality for $0 < s < 1$ and $1 < p < \infty$, $1 < p_1, p_2 \leq \infty$ in the analysis of Korteweg-de Vries equations.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Kato, Ponce (1988) used a normed fractional Leibniz type rule, with $s > 0, 1 < p < \infty$, $1 < p_1 < \infty$, $p_2 = \infty$ in the study of Euler and Naiver Stokes equations.

Christ, Weinstein (1991) proved the homogeneous KP inequality for $0 < s < 1$ and $1 < p < \infty$, $1 < p_1, p_2 \leq \infty$ in the analysis of Korteweg-de Vries equations.

Gulisashvili, Kon (1996) obtained the homogeneous and inhomogeneous KP inequality for $0 < s$ and $1 < p < \infty$, $1 < p_1, p_2 \leq \infty$ and used it in the analysis of Schrödinger semigroups.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Bernicot, Maldonado, Moen, Naibo (2014) obtained the KP inequality for $\frac{1}{2} < p < \infty, 1 < p_1, p_2 \leq \infty$ and $s > n$.

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Bernicot, Maldonado, Moen, Naibo (2014) obtained the KP inequality for $\frac{1}{2} < p < \infty, 1 < p_1, p_2 \leq \infty$ and $s > n$.

Grafakos, Oh (2014) obtained the KP inequality for $\frac{1}{2} < p < 1, 1 < p_1, p_2 \leq \infty$ and $s > \max(n(\frac{1}{p} - 1), 0)$ or $s \in 2\mathbb{N}$.
Furthermore, they gave counterexamples for s out of that range.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Bernicot, Maldonado, Moen, Naibo (2014) obtained the KP inequality for $\frac{1}{2} < p < \infty, 1 < p_1, p_2 \leq \infty$ and $s > n$.

Grafakos, Oh (2014) obtained the KP inequality for $\frac{1}{2} < p < 1, 1 < p_1, p_2 \leq \infty$ and $s > \max(n(\frac{1}{p} - 1), 0)$ or $s \in 2\mathbb{N}$. Furthermore, they gave counterexamples for s out of that range.

Grafakos, Maldonado, Naibo (2014) obtained the KP inequality when the target space is BMO; and posed the question when the target space is L^∞.

Background: Classical KP

$$\|J^s(fg)\|_{L^p} \lesssim \|J^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}}$$

Bernicot, Maldonado, Moen, Naibo (2014) obtained the KP inequality for $\frac{1}{2} < p < \infty, 1 < p_1, p_2 \leq \infty$ and $s > n$.

Grafakos, Oh (2014) obtained the KP inequality for $\frac{1}{2} < p < 1, 1 < p_1, p_2 \leq \infty$ and $s > \max(n(\frac{1}{p} - 1), 0)$ or $s \in 2\mathbb{N}$. Furthermore, they gave counterexamples for s out of that range.

Grafakos, Maldonado, Naibo (2014) obtained the KP inequality when the target space is BMO; and posed the question when the target space is L^∞.

Cruz-Uribe, Naibo (2022) obtained the KP inequality for variable Lebesgue spaces.

Background: KP Endpoints

- The classical KP inequality uses Calderón-Zygmund theory.
- Calderón-Zygmund theory fails at the endpoints i.e. $p = \infty$ or when either of p_1, p_2 are equal to 1.
- CZ techniques give weaker results at the endpoints namely $L^1 \times L^{p_2} \to L^{p,\infty}$ and $L^\infty \times L^\infty \to BMO$.
- But the endpoint Kato-Ponce cases are true in the strong sense. This distinguishes KP inequalities from other bilinear estimates.

Background: KP endpoint cases

Bourgain, Li (2004) used a new technique for the L^∞ endpoint (i.e. $p = \infty$) case where s is in a optimal range.

Background: KP endpoint cases

Bourgain, *Li* (2004) used a new technique for the L^∞ endpoint (i.e. $p = \infty$) case where s is in a optimal range.

Oh, *Wu* (2020) expanded on this averaging technique to get the L^1 endpoint case (i.e. $p_1 = 1$ or $p_2 = 1$ or both are 1), where s is in a optimal range.

Background: KP endpoint cases

Bourgain, *Li* (2004) used a new technique for the L^∞ endpoint (i.e. $p = \infty$) case where s is in a optimal range.

Oh, *Wu* (2020) expanded on this averaging technique to get the L^1 endpoint case (i.e. $p_1 = 1$ or $p_2 = 1$ or both are 1), where s is in a optimal range.

We now have the KP inequality in the full range of indices i.e.

Theorem

Let $f, g \in \mathcal{S}(\mathbb{R}^n)$ $\frac{1}{2} \leq p \leq \infty$, $1 \leq p_1, p_2 \leq \infty$ *be related by* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. *Let* $s > \max(n(\frac{1}{p} - 1), 0)$ *or* $s \in 2\mathbb{N}$, *then*

$$\|J^s(fg)\|_{L^p} \leq C_{n,s,p_1,p_2}\Big(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|f\|_{L^{p_1}}\|J^s g\|_{L^{p_2}}\Big).$$

Background: Weighted KP Inequalities

Naibo, *Thomson* (2019) obtained the KP inequality in function spaces with Muckenhoupt weights for $\frac{1}{2} < p < \infty$, $1 < p_1, p_2 \leq \infty$, where s is in a optimal range depending on p and the weights.

Background: Weighted KP Inequalities

Naibo, Thomson (2019) obtained the KP inequality in function spaces with Muckenhoupt weights for $\frac{1}{2} < p < \infty$, $1 < p_1, p_2 \leq \infty$, where s is in a optimal range depending on p and the weights.

Oh, Wu (2021) obtained the KP inequality for polynomial weights [i.e. weights of the form $(1 + |\cdot|^2)^{\frac{a}{2}}$ for $a \geq 0$] for $\frac{1}{2} \leq p \leq \infty$, $1 \leq p_1, p_2 \leq \infty$, where s is in a optimal range.

Background: Weighted KP Inequalities

Naibo, Thomson (2019) obtained the KP inequality in function spaces with Muckenhoupt weights for $\frac{1}{2} < p < \infty$, $1 < p_1, p_2 \leq \infty$, where s is in a optimal range depending on p and the weights.

Oh, Wu (2021) obtained the KP inequality for polynomial weights [i.e. weights of the form $(1 + |\cdot|^2)^{\frac{a}{2}}$ for $a \geq 0$] for $\frac{1}{2} \leq p \leq \infty$, $1 \leq p_1, p_2 \leq \infty$, where s is in a optimal range.

- For Muckenhoupt weights s is *dependent* on the weights.
- For polynomial weights s is *independent* of the weights.
- Oh and Wu's result just requires that the power on the polynomial is positive; hence the polynomial weights need not be Muckenhoupt weights.

Outline

Outline

Notation

A weight, w, is a nonnegative, measurable function such that $0 < w < \infty$ a.e.

Notation

A weight, w, is a nonnegative, measurable function such that $0 < w < \infty$ a.e.

For $0 < p < \infty$ we define,

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

Notation

A weight, w, is a nonnegative, measurable function such that $0 < w < \infty$ a.e.

For $0 < p < \infty$ we define,

$$\|f\|_{L^p(w)} := \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \Big)^{\frac{1}{p}}$$

For $p = \infty$ and w a weight we define,

$$\|f\|_{L^\infty(w)} := \inf\{C \geq 0 : |f(x)| \leq C \text{ for a.e. x}\} = \|f\|_{L^\infty}$$

A weight, w, is a nonnegative, measurable function such that $0 < w < \infty$ a.e.

For $0 < p < \infty$ we define,

$$\|f\|_{L^p(w)} := \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\Big)^{\frac{1}{p}}$$

For $p = \infty$ and w a weight we define,

$$\|f\|_{L^\infty(w)} := \inf\{C \geq 0 : |f(x)| \leq C \text{ for a.e. } x\} = \|f\|_{L^\infty}$$

Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator:

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|\, dy =: \sup_{r>0} \fint_{Q(x,r)} |f(y)|\, dy.$$

Definition (Muckenhoupt Weight)

Let w be a locally integrable weight, and $1 < p < \infty$. Then w is a Muckenhoupt weight if it satisfies

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

Moreover, if $[w]_{A_p} < \infty$, we say $w \in A_p$.

A_p Weights

Definition (Muckenhoupt Weight)

Let w be a locally integrable weight, and $1 < p < \infty$. Then w is a Muckenhoupt weight if it satisfies

$$[w]_{A_p} := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty.$$

Moreover, if $[w]_{A_p} < \infty$, we say $w \in A_p$.

$$\tau_w := \inf\{p \geq 1 : w \in A_p\}$$

Muckenhoupt's theorem

Theorem (Muckenhoupt 1972)

For $p > 1$

$$\|\mathcal{M}f\|_{L^p(w)} \;\leq\; C\|f\|_{L^p(w)}$$

$$\Longleftrightarrow$$

$$w \in A_p.$$

- The theorem is also true with the Hilbert or Riesz transform in place of $\mathcal{M}$.
- $A_q \subset A_p$ for $q \leq p$.
- A_p weights are doubling (i.e. $w(\lambda Q) \lesssim \lambda^{np}[w]_{A_p} w(Q)$).
- A_p weights satisfy the reverse Hölder property.

$A_{\vec{P}}$ Weights

- A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González (2009)

$A_{\vec{P}}$ Weights

- A. K. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González (2009)

Definition (Multiple Weights)

Let $\vec{P} = (p_1, \dots, p_m)$ with $1 < p_1, \dots, p_m < \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Given $\vec{w} = (w_1, \dots, w_m)$, where w_j are weights, set

$$w = \prod_{j=1}^{m} w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition (or $\vec{w} \in A_{\vec{P}}$) if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w \right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{1/p_i'} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes.

Multi(sub)linear Maximal Function

Definition

Given $\vec{f} = (f_1, \ldots, f_m)$ where each entry is measurable, we define the maximal operator $\mathscr{M}$ by

$$\mathscr{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| \, dy_j,$$

where the supremum is taken over all cubes Q containing x.

Multi(sub)linear Maximal Function

Definition

Given $\vec{f} = (f_1, \ldots, f_m)$ where each entry is measurable, we define the maximal operator $\mathscr{M}$ by

$$\mathscr{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|\, dy_j,$$

where the supremum is taken over all cubes Q containing x.

Theorem (L-O-P-T-G 2009)

Let $1 < p_j < \infty$, $j = 1, \ldots, m$, and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m}$. Then the inequality

$$\|\mathscr{M}(\vec{f})\|_{L^p(w)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

holds for every measurable $\vec{f}$ if and only if $\vec{w} \in A_{\vec{P}}$.

$A_{\vec{P}}$ VS. A_p

There are some key similarities and differences between these two weight classes, and the corresponding maximal operators.

- Trivially $\mathscr{M}(\vec{f})(x) \leq \prod_{j=1}^{m} \mathcal{M}(f_j)(x)$.

$A_{\vec{P}}$ VS. A_p

There are some key similarities and differences between these two weight classes, and the corresponding maximal operators.

- Trivially $\mathscr{M}(\vec{f})(x) \leq \prod_{j=1}^m \mathcal{M}(f_j)(x)$.

- If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ then $w = \prod_{j=1}^m w_j^{p/p_j} \in A_{mp}$.

$A_{\vec{P}}$ VS. A_p

There are some key similarities and differences between these two weight classes, and the corresponding maximal operators.

- Trivially $\mathscr{M}(\vec{f})(x) \leq \prod_{j=1}^m \mathcal{M}(f_j)(x)$.

- If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ then $w = \prod_{j=1}^m w_j^{p/p_j} \in A_{mp}$.

- However, w_j may not even be locally integrable!

$A_{\vec{P}}$ VS. A_p

There are some key similarities and differences between these two weight classes, and the corresponding maximal operators.

- Trivially $\mathscr{M}(\vec{f})(x) \leq \prod_{j=1}^{m} \mathcal{M}(f_j)(x)$.
- If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ then $w = \prod_{j=1}^{m} w_j^{p/p_j} \in A_{mp}$.
- However, w_j may not even be locally integrable!
- $A_{p_1} \times \cdots \times A_{p_m} \subset A_{\vec{P}}$ is a proper subset.

$A_{\vec{P}}$ VS. A_p

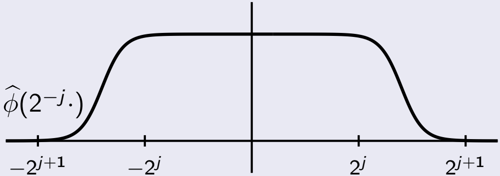
There are some key similarities and differences between these two weight classes, and the corresponding maximal operators.

- Trivially $\mathscr{M}(\vec{f})(x) \leq \prod_{j=1}^{m} \mathcal{M}(f_j)(x)$.
- If $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{P}}$ then $w = \prod_{j=1}^{m} w_j^{p/p_j} \in A_{mp}$.
- However, w_j may not even be locally integrable!
- $A_{p_1} \times \cdots \times A_{p_m} \subset A_{\vec{P}}$ is a proper subset.

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q w \Big)^{1/p} \prod_{j=1}^{m} \Big(\frac{1}{|Q|} \int_Q w_i^{1-p_i'} \Big)^{1/p_i'} < \infty,$$

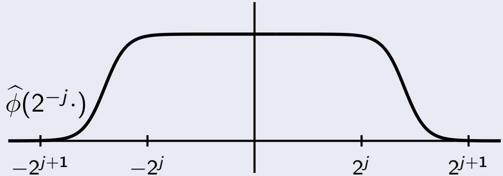
Notation: Littlewood-Paley operators

$$S_j f := \mathcal{F}^{-1}(\widehat{\phi}(2^{-j}\cdot)\,\widehat{f}\,)$$

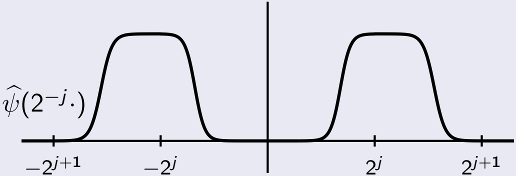


Notation: Littlewood-Paley operators

$$S_j f := \mathcal{F}^{-1}(\widehat{\phi}(2^{-j}\cdot)\,\widehat{f}\,)$$



$$\widehat{\psi}(\xi) := \widehat{\phi}(\xi) - \widehat{\phi}(2\xi) \qquad \Delta_j f := \mathcal{F}^{-1}(\widehat{\psi}(2^{-j}\cdot)\,\widehat{f}\,)$$



Notice that $\widehat{\psi}$ gives rise to a partition of unity

$$\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1 \text{ or } \sum_{j\in\mathbb{Z}} \Delta_j = I.$$

As well as the useful identity

$$\sum_{j\leq j_0} \widehat{\psi}(2^{-j}\xi) = \widehat{\phi}(2^{-j_0}\xi) \text{ or } \sum_{j\leq j_0} \Delta_j = S_{j_0}.$$

Outline

The 2-factor $\not\Longrightarrow$ 3-factor in full range of indices

Let

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

If $p < 1$, we will show that the 2-factor KP inequality does not inductively imply the 3-factor KP inequality.

The 2-factor $\not\Longrightarrow$ 3-factor in full range of indices

Let

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

If $p < 1$, we will show that the 2-factor KP inequality does not inductively imply the 3-factor KP inequality. For example let

$$\mathrm{p}_1 = p_2 = \tfrac{3}{2}, \text{ and } p_3 = 2.$$

Let q_1 and q_2 be such that

$$\frac{2}{3} + \frac{2}{3} + \frac{1}{2} = \frac{1}{q_1} + \frac{1}{2} = \frac{2}{3} + \frac{1}{q_2}.$$

So we have

$$q_1 = \frac{3}{4} \text{ and } q_2 = \frac{6}{7}.$$

It follows we can not directly apply the 2-factor KP inequality.

Outline

Kato-Ponce For Multiple Weights (Main result)

Theorem (Douglas 2023)

Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < p < \infty$, $1 < p_1, \dots, p_m < \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Let* $\vec{w} \in A_{\vec{P}}$, *and let* $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$. *If* $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, *then there exists a constant* $C = C(n, m, w, s, p_1, \dots, p_m) < \infty$ *such that for all* $f_t \in \mathcal{S}(\mathbb{R}^n)$ *with* $t \in \{1, \dots, m\}$ *we have*

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots$$
$$\cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

Furthermore, the same estimate holds with D^s *in place of* J^s.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- Extends the KP inequality from a product of 2 functions to a product of m functions.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1\cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- Extends the KP inequality from a product of 2 functions to a product of m functions.
- This implies the KP inequality for Muckenhoupt weights.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1\cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)} + \cdots \\ \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- Extends the KP inequality from a product of 2 functions to a product of m functions.
- This implies the KP inequality for Muckenhoupt weights.
- The weights w_t may not even be locally integrable.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1\cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- Extends the KP inequality from a product of 2 functions to a product of m functions.
- This implies the KP inequality for Muckenhoupt weights.
- The weights w_t may not even be locally integrable.
- The inhomogeneous version implies the homogeneous version via a dilation argument.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- The range of the smoothness index is given by $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, which implies s depends on the choice of weights.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1\cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- The range of the smoothness index is given by $s > n(\frac{1}{\min(p/\tau_w,1)} - 1)$, which implies s depends on the choice of weights.
- The range of s is sharp; that is the inequality can fail for s outside of that range.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- The range of the smoothness index is given by $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, which implies s depends on the choice of weights.
- The range of s is sharp; that is the inequality can fail for s outside of that range.
- The integrability index does NOT include the endpoints i.e. $1 < p_1, \ldots, p_m < \infty$.

Kato-Ponce For Multiple Weights (Main result)

$$\|J^s(f_1\cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|f_m\|_{L^{p_m}(w_m)} + \cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)}\cdots\|J^s f_m\|_{L^{p_m}(w_m)}.$$

Keypoints

- The range of the smoothness index is given by $s > n(\frac{1}{\min(p/\tau_w,1)} - 1)$, which implies s depends on the choice of weights.
- The range of s is sharp; that is the inequality can fail for s outside of that range.
- The integrability index does NOT include the endpoints i.e. $1 < p_1, \ldots, p_m < \infty$.
- What can be said about the weighted endpoint case?

L^1 endpoint with A_p weights (Different result)

Theorem (Douglas 2022)

Let $m \in \mathbb{Z}^+$, $\frac{1}{m} \le p \le \infty$, $1 \le p_1, \dots, p_m \le \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Let $w_t \in A_{p_t}$ for $t \in \{1, \dots, m\}$, and let $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$. If $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, then there exists a constant $C = C(n, m, w, s, p_1, \dots, p_m) < \infty$ such that for all $f_t \in \mathcal{S}(\mathbb{R}^n)$ with $t \in \{1, \dots, m\}$ we have

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots \\ \cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

Furthermore, the same estimate holds with D^s in place of J^s.

Outline

Bernstein Type Expressions

We need a Bernstein's type inequality i.e.

$$\|J^s \Delta_j^\psi f\|_{L^p(w)} \sim 2^{js} \|\Delta_j^\psi f\|_{L^p(w)},$$

but without the norm.

Proposition

Let $s \in \mathbb{R}$, and let $\widehat{\psi}$ be a $\mathcal{C}^\infty(\mathbb{R}^n)$ function supported in the annulus $\frac{1}{2} \leq |\xi| \leq 2$. Define $\Delta_j^\psi f$ to be convolution with $2^{jn}\psi(2^j \cdot)$, and $\Delta_{j,\mu}^\psi$ to be convolution with $2^{jn}\psi(2^j \cdot + \mu)$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and let $j \in \mathbb{Z}$. Then one has

$$J^s \Delta_j^\psi f(x) = 2^{js} \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} \Delta_{j,\mu}^\psi f(x) \text{ and } 2^{js} \Delta_j^\psi f(x) = \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} \Delta_{j,\mu}^\psi J^s f(x)$$

where $|c_{j,\mu}| \lesssim (1 + |\mu|)^{-N}$ for any $N \in \mathbb{N}$, when $j \geq 0$, the implicit constant is independent of j.

Proof of Proposition

Let

$$\sigma_j(\xi) \equiv (2^{-2j} + |\xi|^2)^{\frac{s}{2}} \widehat{\psi_\star}(\xi) = \chi_{[-4,4]^n}(\xi) \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} e^{2\pi i \xi \cdot \frac{\mu}{8}}$$

where the coefficients decay rapidly independently of j.

Observe for $j \geq 0$,

$$\begin{aligned} J^s \Delta_j^\psi f(x) &= \int (1 + |\xi|^2)^{\frac{s}{2}} \widehat{\psi_\star}(2^{-j}\xi) \widehat{\Delta_j^\psi f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= \int 2^{js} (2^{-2j} + |2^{-j}\xi|^2)^{\frac{s}{2}} \widehat{\psi_\star}(2^{-j}\xi) \widehat{\Delta_j^\psi f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= 2^{js} \int \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} e^{2\pi i \xi \cdot 2^{-j-3}\mu} \widehat{\Delta_j^\psi f}(\xi) e^{2\pi i \xi \cdot x} d\xi \\ &= 2^{js} \sum_{\mu \in \mathbb{Z}^n} c_{j,\mu} \Delta_{j,\mu}^\psi f(x). \end{aligned}$$

Averaging lemma

Lemma (Oh, Wu 2020)

If $a_k \lesssim \min(2^{ka}A, 2^{-kb}B)$ *for some* $a, b, A, B > 0$ *and every* $k \in \mathbb{Z}$, *then for any* $u > 0$, *we have* $\{a_k\}_{k\in\mathbb{Z}} \in \ell^u(\mathbb{Z})$ *and*

$$\|\{a_k\}_{k\in\mathbb{Z}}\|_{\ell^u} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}.$$

In particular, if $\|f_k\|_{L^r(w)} \lesssim \min(2^{ka}A, 2^{-kb}B)$ *for* $0 < r \leq \infty$, *every* $k \in \mathbb{Z}$, *and a weight* w *then*

$$\Big\|\sum_{k\in\mathbb{Z}} f_k\Big\|_{L^r(w)} \lesssim A^{\frac{b}{a+b}} B^{\frac{a}{a+b}}.$$

Bourgain and Li were the first to use this technique to obtain the L^∞ endpoint. Oh and Wu later refined it and found a creative way to apply it to the L^1 endpoint.

Controlled By The Multilinear Maximal Function

Analogous to how the Hardy-Littlewood maximal function pointwise controls the convolution of a function with the L^1 dilate of a Schwartz function we have

Proposition

Let $\vec{f} = (f_1, \ldots, f_m)$ *where* $f_j \in L^1_{loc}(\mathbb{R}^n)$ *and* $\varphi^j \in \mathscr{S}(\mathbb{R}^n)$ *for* $j \in \{1, \ldots, m\}$*. For* $t \in \mathbb{R}_{>0}$ *define the operator* Υ^j_t *to be convolution with* $t^{-n}\varphi^j(t^{-1}\cdot)$*, then there is a finite constant independent of* t *such that*

$$|(\Upsilon^1_t f_1) \cdots (\Upsilon^m_t f_m)| \leq C_{n,m,\varphi^1,\ldots,\varphi^m} \mathscr{M}(\vec{f}).$$

Controlled By The Multilinear Maximal Function

Analogous to how the Hardy-Littlewood maximal function pointwise controls the convolution of a function with the L^1 dilate of a Schwartz function we have

Proposition

Let $\vec{f} = (f_1, \ldots, f_m)$ where $f_j \in L^1_{loc}(\mathbb{R}^n)$ and $\varphi^j \in \mathscr{S}(\mathbb{R}^n)$ for $j \in \{1, \ldots, m\}$. For $t \in \mathbb{R}_{>0}$ define the operator Υ^j_t to be convolution with $t^{-n}\varphi^j(t^{-1}\cdot)$, then there is a finite constant independent of t such that

$$|(\Upsilon^1_t f_1) \cdots (\Upsilon^m_t f_m)| \le C_{n,m,\varphi^1,\ldots,\varphi^m} \mathscr{M}(\vec{f}).$$

Suppose the Υ^j_t were replaced by the shifted operators $\Upsilon^j_{t,\mu}$ defined by convolution with $t^{-n}\varphi^j(t^{-1} \cdot + \mu)$ for $\mu \in \mathbb{R}^n$. Then the final constant grows polynomially in $|\mu|$, i.e.

$$|(\Upsilon^1_{t,\mu} f_1) \cdots (\Upsilon^m_{t,\mu} f_m)| \le (1 + |\mu|)^{n+\gamma} C_{n,m,\varphi^1,\ldots,\varphi^m} \mathscr{M}(\vec{f}).$$

Outline

Strategy of proof: Decomposition

We start by rewriting the fractional derivative of the product,

$$J^s(f_1 f_2 \cdots f_m)(x) =$$

$$\int_{\mathbb{R}^{mn}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}$$

$$= \sum_{\vec{j}\in\mathbb{Z}^m} \int_{\mathbb{R}^{mn}} (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{\psi}(2^{-j_1}\xi_1)\cdots\widehat{\psi}(2^{-j_m}\xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}$$

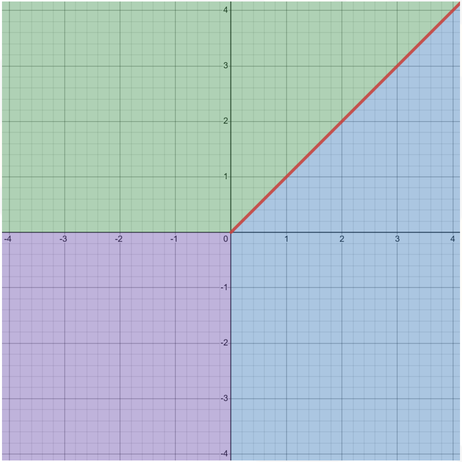
$$= \sum_{\vec{\eta}\in\{0,1\}^m} \int_{\mathbb{R}^{mn}} \sum_{\vec{j}\in\mathscr{B}_{\vec{\eta}}} \Lambda_{\vec{j}}(\vec{\xi})(1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}} \widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m) e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x} d\vec{\xi}$$

where

$$\mathbb{Z}^m = \bigsqcup_{\vec{\eta}\in\{0,1\}^m} \mathscr{B}_{\vec{\eta}}.$$

Strategy of proof: Decomposition

For example the decomposition of $\mathbb{Z}^2$ is



$$(j, k) \in \mathbb{Z}^2$$

- $(1{,}1) \sim j = k > 0$
- $(1{,}0) \sim 0 < k < j$
- $(0{,}1) \sim 0 < j < k$
- $(0{,}0) \sim j \leq 0$ and $k \leq 0$

Strategy of proof: Decomposition

$$\mathbb{Z}^m = \bigsqcup_{\vec{\eta}\in\{0,1\}^m} \mathscr{B}_{\vec{\eta}}.$$

We define

$$\mathscr{B}_{\vec{\eta}} := \{(j_1,\dots,j_m)\in\mathbb{Z}^m : \text{if } \eta_t = 1 \text{ for some } 1\le t\le m \text{ then, } \max(j_1,\dots,j_m) = j_t \text{ and } j_t > 0. \text{ If } \eta_t = 0 \text{ then } \max(j_1,\dots,j_m) > j_t\}.$$

$\mathscr{B}_{\vec{\eta}}$ is the elements of $\mathbb{Z}^m$ where the coordinates containing a 1 are the same, positive and strictly bigger then the remaining entries.

Strategy of proof: Decomposition

To get a sense of how this decomposition looks in higher dimensions and to see that it produces a paraproduct decomposition lets consider (1,0,0).

$$(1,0,0) \approx \mathscr{B}_{(1,0,0)} = \{(j_1, j_2, j_3) \in \mathbb{Z}^3 : j_1 > j_2 \text{ and } j_1 > j_3 \text{ and } j_1 > 0\}$$

Then

$$\begin{aligned}
&\sum_{\vec{j} \in \mathscr{B}_{(\mathbf{1},\mathbf{0},\mathbf{0})}} \widehat{\psi}(2^{-j_1}\xi_1)\widehat{\psi}(2^{-j_2}\xi_2)\widehat{\psi}(2^{-j_3}\xi_3) \\
&= \sum_{j_1>0} \sum_{j_2<j_1} \sum_{j_3<j_1} \widehat{\psi}(2^{-j_1}\xi_1)\widehat{\psi}(2^{-j_2}\xi_2)\widehat{\psi}(2^{-j_3}\xi_3) \\
&= \sum_{j>0} \widehat{\psi}(2^{-j}\xi_1)\widehat{\phi}(2^{-(j-1)}\xi_2)\widehat{\phi}(2^{-(j-1)}\xi_3) \\
&\approx \sum_{j>0} (\Delta_j f_1)(S_{j-1} f_2)(S_{j-1} f_3)
\end{aligned}$$

Strategy of proof: Decomposition

The fractional derivative

$$J^{s}(f_1 \cdots f_m)$$

is broken into paraproducts of two types:

Strategy of proof: Decomposition

The fractional derivative

$$J^s(f_1 \cdots f_m)$$

is broken into paraproducts of two types:

The Diagonal Paraproduct ($b > 1$)

$$\sum_{j>0} J^s\Big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1} f_{b+1})\cdots(S_{j-1} f_m)\Big)$$

and

Strategy of proof: Decomposition

The fractional derivative

$$J^s(f_1 \cdots f_m)$$

is broken into paraproducts of two types:

The Diagonal Paraproduct $(b > 1)$

$$\sum_{j>0} J^s\Big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1} f_{b+1})\cdots(S_{j-1} f_m)\Big)$$

and

The Off-Diagonal Paraproduct $(b = 1)$

$$\sum_{j>0} J^s\Big((\Delta_j f_1)(S_{j-1} f_2)\cdots(S_{j-1} f_m)\Big)$$

Strategy of proof: Diagonal Paraproduct-Decay

Expanding high frequency term we have

$$\sum_{j>0} J^{s}\Big((\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1}f_{b+1})\cdots(S_{j-1}f_m)\Big)(x)$$

$$= \sum_{j>0}\int 2^{js}(2^{-2j}+|2^{-j}\xi_1+\cdots+2^{-j}\xi_m|^2)^{\frac{s}{2}}\widehat{\phi}(2^{-j-m}(\xi_1+\cdots+\xi_m))$$

$$\times\, \widehat{\Delta_j f_1}(\xi_1)\cdots\widehat{\Delta_j f_b}(\xi_b)\widehat{S_{j-1}f_{b+1}}(\xi_{b+1})\cdots\widehat{S_{j-1}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\vec{\xi}$$

- In the unweighted case expanding the part in blue in Fourier series is not an issue i.e.

$$(2^{-2j}+|\xi|^2)^{\frac{s}{2}}\widehat{\phi}(2^{-m}\xi) = \chi_{[-4,4]}(2^{-m}\xi)\sum_{\mu\in\mathbb{Z}^n} c_{j,\mu}e^{2\pi i\xi\cdot 2^{-m-3}\mu}.$$

- The decay from the coefficients is *just* enough to overcome the effects of modulation.

Strategy of proof: Diagonal Paraproduct-Decay

In the unweighted case the decay of the Fourier coefficients is bounded by $(1+|\mu|)^{-n-s}$ and the effects of modulation are logarithmic.

Lemma (Grafakos, Oh 2014)

Let $\mu \in \mathbb{Z}^n$ let $\Delta_{j,\mu}$ be convolution with $2^{jn}\psi(2^{-j}\cdot+\mu)$. Then for all $1 < q < \infty$

$$\left\| \sqrt{\sum_{j\in\mathbb{Z}} |\Delta_{j,\mu} f|^2} \right\|_{L^q} \leq C_n \max(q,(q-1)^{-1}) \ln(2+|\mu|) \|f\|_{L^q}.$$

Strategy of proof: Diagonal Paraproduct-Decay

In the unweighted case the decay of the Fourier coefficients is bounded by $(1+|\mu|)^{-n-s}$ and the effects of modulation are logarithmic.

Lemma (Grafakos, Oh 2014)

Let $\mu \in \mathbb{Z}^n$ let $\Delta_{j,\mu}$ be convolution with $2^{jn}\psi(2^{-j}\cdot+\mu)$. Then for all $1 < q < \infty$

$$\left\| \sqrt{\sum_{j\in\mathbb{Z}} |\Delta_{j,\mu} f|^2} \right\|_{L^q} \leq C_n \max(q, (q-1)^{-1}) \ln(2+|\mu|) \|f\|_{L^q}.$$

- In the weighted case the smoothness estimate required for CZ theory is too rough.
- Naibo and Thomson's technique using the machinery of function spaces sidesteps this issue of decay.

Strategy of proof: Diagonal Paraproduct-Decay

Theorem (Naibo, Thomson 2019)

Let $w \in A_\infty$, and let $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{N}$. Let $0 < p < \infty$, and $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, then

$$\left\| J^s \Big(\sum_{j \in \mathbb{N}} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \Big) \right\|_{L^p(w)}$$

$$\lesssim \left\| \sum_{j \in \mathbb{N}} 2^{js} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \right\|_{L^p(w)}$$

where the implicit constant depends on m, n, s, r, w.

Strategy of proof: Diagonal Paraproduct-Decay

Theorem (Naibo, Thomson 2019)

Let $w \in A_\infty$, and let $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$ and $j \in \mathbb{N}$. Let $0 < p < \infty$, and $s > n(\frac{1}{\min(p/\tau_w, 1)} - 1)$, then

$$\left\| J^s \Big(\sum_{j \in \mathbb{N}} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \Big) \right\|_{L^p(w)}$$
$$\lesssim \left\| \sum_{j \in \mathbb{N}} 2^{js} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \right\|_{L^p(w)}$$

where the implicit constant depends on m, n, s, r, w.

A key ingredient is bounding the convolution pointwise by maximal-type operators. Specifically, when $\widehat{u}$ is compactly supported we use estimates given heuristically by

$$|\varphi * u(x)| \lesssim \Big(\mathcal{M}(|u|^t)(x) \Big)^{\frac{1}{t}}.$$

Strategy of proof: Diagonal Paraproduct-Summability

Using the previous theorem and Bernstein's inequality we can estimate a summmand of

$$\left\| \sum_{j \in \mathbb{N}} 2^{js} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \right\|_{L^p(w)}$$

above by

$$\left\| 2^{js} (\Delta_j f_1)(\Delta_j f_2) \cdots (\Delta_j f_b)(S_{j-1} f_{b+1}) \cdots (S_{j-1} f_m) \right\|_{L^p(w)}$$

which is bounded by a constant multiple of

$$2^{js} \left\| \mathscr{M}(\vec{f}) \right\|_{L^p(w)}$$

Strategy of proof: Diagonal Paraproduct-Summability

Using the averaging lemma and Bernstein's inequality we can estimate a summmand of

$$\left\| \sum_{j\in\mathbb{N}} 2^{js} (\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1} f_{b+1})\cdots(S_{j-1} f_m) \right\|_{L^p(w)}$$

above by

$$\Big\| 2^{js} 2^{-js} 2^{-js} \sum_{\mu_1\in\mathbb{Z}} \sum_{\mu_2\in\mathbb{Z}} c_{j,\mu_1} c_{j,\mu_2} (\Delta_{j,\mu_1} J^s f_1)(\Delta_{j,\mu_2} J^s f_2)\cdots(\Delta_j f_b) \times (S_{j-1} f_{b+1})\cdots(S_{j-1} f_m) \Big\|_{L^p(w)}.$$

which is bounded by a constant multiple of

$$2^{-js} \left\| \mathscr{M}\left(J^s f_1, J^s f_2, f_3, \cdots, f_m\right) \right\|_{L^p(w)}$$

Strategy of proof: Diagonal Paraproduct-Summability

Now applying the averaging lemma with estimates $a = b = s$ as well as the AMGM inequality we have

$$\begin{aligned}
&\Big\| J^s \Big(\sum_{j>0} (\Delta_j f_1)(\Delta_j f_2)\cdots(\Delta_j f_b)(S_{j-1} f_{b+1})\cdots(S_{j-1} f_m)\Big)\Big\|_{L^p(w)} \\
&\lesssim \Big(\Big\| \mathscr{M}(\vec{f})\Big\|_{L^p(w)} \Big\| \mathscr{M}(J^s f_1, J^s f_2, f_3, \cdots, f_m)\Big\|_{L^p(w)}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} \\
&\qquad \times \|J^s f_1\|_{L^{p_1}(w_1)} \|J^s f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)} \Big)^{\frac{1}{2}} \\
&\leq \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} \\
&\qquad + \|f_1\|_{L^{p_1}(w_1)} \|J^s f_2\|_{L^{p_2}(w_2)} \|f_3\|_{L^{p_3}(w_3)} \cdots \|f_m\|_{L^{p_m}(w_m)}.
\end{aligned}$$

Finishing the proof of the diagonal paraproduct.

Strategy of proof: Off-Diagonal Paraproduct

$$J^s\left(\sum_{j>0}(\Delta_j f_1)(S_{j-1}f_2)\cdots(S_{j-1}f_m)\right)$$

Fix $a \in \mathbb{N}$ to be determined later. We expand the above expression as

$$\sum_{j\in\mathbb{N}}(\Delta_j f_1)\Big(S_{j-a}f_2 + \sum_{j-a<k<j}\Delta_k f_2\Big)\cdots\Big(S_{j-a}f_m + \sum_{j-a<k<j}\Delta_k f_m\Big).$$

Multiplying out the terms we write

$$\sum_{j\in\mathbb{N}}(\Delta_j f_1)(S_{j-a}f_2)(S_{j-a}f_3)\cdots(S_{j-a}f_m)$$

plus finitely many other paraproducts with at least one Δ_k operator where $k \sim j$. These finitely many other paraproducts will behave in the same way as the case for $b > 1$.

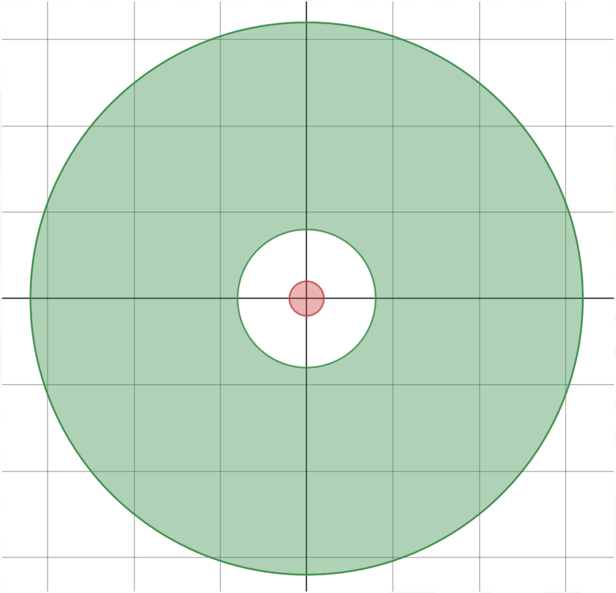
Strategy of proof: Off-Diagonal Paraproduct

Expanding the fractional derivative we have

$$\sum_{j>0} J^{s}\big((\Delta_j f_1)(S_{j-a}f_2)\cdots(S_{j-a}f_m)\big)(x)$$
$$= \sum_{j>0}\int (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\,\widehat{\Delta_j f_1}(\xi_1)$$
$$\times\, \widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,d\xi_1\cdots d\xi_m$$

- Here $a\in\mathbb{N}$ is chosen big enough so that $|\xi_1+\cdots+\xi_m|\sim|\xi_1|$.

Strategy of proof: Off-Diagonal Paraproduct



Strategy of proof: Off-Diagonal Paraproduct

Now for the high-low frequency term, expanding the fractional derivative we have

$$\sum_{j>0} J^{s}\big((\Delta_j f_1)(S_{j-a}f_2)\cdots(S_{j-a}f_m)\big)(x)$$
$$= \sum_{j>0} \int (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\,\widehat{\Delta_j f_1}(\xi_1)$$
$$\times\, \widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}\,d\xi_1\cdots d\xi_m$$

- Here $a\in\mathbb{N}$ is chosen big enough so that $|\xi_1+\cdots+\xi_m|\sim|\xi_1|$.

Strategy of proof: Off-Diagonal Paraproduct

Now for the high-low frequency term, expanding the fractional derivative we have

$$\sum_{j>0} J^s\big((\Delta_j f_1)(S_{j-a}f_2)\cdots(S_{j-a}f_m)\big)(x)$$
$$= \sum_{j>0}\int (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\widehat{\Delta_j f_1}(\xi_1)$$
$$\times\, \widehat{S_{j-a}f_2}(\xi_2)\cdots\widehat{S_{j-a}f_m}(\xi_m)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\xi_1\cdots d\xi_m$$

- Here $a\in\mathbb{N}$ is chosen big enough so that $|\xi_1+\cdots+\xi_m|\sim|\xi_1|$.
- For boundedness we will use a m-linear multiplier theorem in the setting of multiple weights.

Multiplier Theorem

- The first use of a bilinear multiplier theorem that employed a Hörmander-type smoothness condition was introduced by Tomita.
- Grafakos and Si extended this multiplier theorem to the m-linear case.
- Li and Sun proved the $A_{\vec{P}}$-weighted m-linear multiplier theorem with a Hörmander-type smoothness condition.

Multiplier Theorem

- The first use of a bilinear multiplier theorem that employed a Hörmander-type smoothness condition was introduced by Tomita.
- Grafakos and Si extended this multiplier theorem to the m-linear case.
- Li and Sun proved the $A_{\vec{P}}$-weighted m-linear multiplier theorem with a Hörmander-type smoothness condition.

Let $\sigma \in L^\infty(\mathbb{R}^{mn})$. The m-linear Fourier multiplier is defined as

$$T_\sigma(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{mn}} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)} \sigma(\xi_1, \ldots, \xi_m) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m)\, d\vec{\xi}.$$

Let Λ be a Schwartz function on $\mathbb{R}^{mn}$ satisfying

$$\operatorname{supp} \Lambda \subseteq \left\{ (\xi_1, \ldots, \xi_m) : \frac{1}{2} \leq |\xi_1| + \cdots + |\xi_m| \leq 2 \right\}$$
$$\sum_{k\in\mathbb{Z}} \Lambda(2^{-k}\xi_1, \ldots, 2^{-k}\xi_m) = 1, \forall (\xi_1, \ldots, \xi_m) \neq \vec{0}.$$

Multiplier Theorem

Theorem (Li, Sun 2012)

Let $\vec{P} = (p_1, \dots, p_m)$ *with* $1 < p_1, \dots, p_m < \infty$ *and* $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. *Suppose that* $mn/2 < t \leq mn$, *and* $\sigma \in L^\infty(\mathbb{R}^{mn})$ *with*

$$\sup_{k \in \mathbb{Z}} \left\| J^t \sigma_k \right\|_{L^2(\mathbb{R}^{mn})} < \infty,$$

where

$$\sigma_k(\xi_1, \dots, \xi_m) = \Lambda(\xi_1, \dots, \xi_m)\sigma(2^{-k}\xi_1, \dots, 2^{-k}\xi_m).$$

Let $r_0 := mn/t < p_1, \dots, p_m < \infty$ *and* $\vec{w} \in A_{\vec{P}/r_0}$. *Then*

$$\left\| T_\sigma(\vec{f}) \right\|_{L^p(w)} \lesssim \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i)}.$$

Multiplier Theorem

$$\sum_{j>0} J^{s}\big((\Delta_j f_1)(S_{j-b}f_2)\cdots(S_{j-b}f_m)\big)(x)$$

$$= \sum_{j>0}\int (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}\widehat{\Delta_j f_1}(\xi_1)$$

$$\times \widehat{S_{j-b}f_2}(\xi_1)\cdots\widehat{S_{j-b}f_2}(\xi_2)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\xi_1\cdots d\xi_m$$

$$= \sum_{j>0}\int (1+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}(1+|\xi_1|^2)^{-\frac{s}{2}}\widehat{\Delta_j J^{s} f_1}(\xi_1)$$

$$\times \widehat{S_{j-b}f_2}(\xi_1)\cdots\widehat{S_{j-b}f_2}(\xi_2)e^{2\pi i(\xi_1+\cdots+\xi_m)\cdot x}d\xi_1\cdots d\xi_m$$

Multiplier Theorem

In order to apply the multiplier theorem with $t = mn$ to the off-diagonal term we need to show the following Hörmander smoothness condition

$$\sup_{k\in\mathbb{Z}} \sum_{|\alpha|\le nm} \left\|\partial^\alpha \sigma_k\right\|_{L^2(\mathbb{R}^{nm})} < \infty$$

where

$$\sigma_k(\vec{\xi}) = \Lambda(\xi_1,\ldots,\xi_m)(1+|2^{-k}\xi_1+\cdots+2^{-k}\xi_m|^2)^{\frac{s}{2}}(1+|2^{-k}\xi_1|^2)^{-\frac{s}{2}} \times \sum_{j>-k} \widehat{\psi}(2^{-j}\xi_1)\widehat{\phi}(2^{-j+a}\xi_2)\cdots\widehat{\phi}(2^{-j+a}\xi_m).$$

Multiplier Theorem

In order to apply the multiplier theorem with $t = mn$ to the off-diagonal term we need to show the following Hörmander smoothness condition

$$\sup_{k\in\mathbb{Z}} \sum_{|\alpha|\leq nm} \left\|\partial^\alpha \sigma_k\right\|_{L^2(\mathbb{R}^{nm})} < \infty$$

where

$$\sigma_k(\vec{\xi}) = \Lambda(\xi_1,\ldots,\xi_m)(1+|2^{-k}\xi_1+\cdots+2^{-k}\xi_m|^2)^{\frac{s}{2}}(1+|2^{-k}\xi_1|^2)^{-\frac{s}{2}} \times \sum_{j>-k} \widehat{\psi}(2^{-j}\xi_1)\widehat{\phi}(2^{-j+a}\xi_2)\cdots\widehat{\phi}(2^{-j+a}\xi_m).$$

- This is advantageous since now we can use normal Leibniz rule.

Multiplier Theorem

$$\sup_{k\in\mathbb{Z}} \sum_{|\alpha|\le nm} \left\|\partial^\alpha \sigma_k\right\|_{L^2(\mathbb{R}^{nm})} < \infty$$

where

$$\sigma_k(\vec{\xi}) = \Lambda(\xi_1,\dots,\xi_m)(1+|2^{-k}\xi_1+\cdots+2^{-k}\xi_m|^2)^{\frac{s}{2}}(1+|2^{-k}\xi_1|^2)^{-\frac{s}{2}}$$
$$\times \sum_{j>-k} \widehat{\psi}(2^{-j}\xi_1)\widehat{\phi}(2^{-j+a}\xi_2)\cdots\widehat{\phi}(2^{-j+a}\xi_m)$$
$$= \Lambda(\xi_1,\dots,\xi_m)2^{-ks}(2^{2k}+|\xi_1+\cdots+\xi_m|^2)^{\frac{s}{2}}2^{ks}(2^{2k}+|\xi_1|^2)^{-\frac{s}{2}}$$
$$\times \sum_{j>-k} \widehat{\psi}(2^{-j}\xi_1)\widehat{\phi}(2^{-j+a}\xi_2)\cdots\widehat{\phi}(2^{-j+a}\xi_m).$$

Outline

Theorem (Douglas, Grafakos 2023)

Let $m \in \mathbb{Z}^+$, $\frac{1}{m} < p \le \infty$, $1 < p_1, \dots, p_m \le \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Let*

$$w_t(x) = \begin{cases} |x|^{a_t} & |x| \le 1 \\ |x|^{b_t} & |x| > 1 \end{cases}$$

with $a_t, b_t \in (-n, n(p_t - 1))$, $b_t \ge 0$ *and* $w = w_1^{\frac{p}{p_1}} \cdots w_m^{\frac{p}{p_m}}$ *with* $t \in \{1, \dots, m\}$.
If $s > \max\left(n(\frac{\tau_w}{p} - 1), 0\right)$, *then there exists a constant* $C = C(n, m, w_1, \dots, w_m, s, p_1, \dots, p_m) < \infty$ *such that for all* $f_t \in L^{p_t}_s(w_t)$ *with* $t \in \{1, \dots, m\}$ *we have*

$$\|J^s(f_1 \cdots f_m)\|_{h^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots$$
$$\cdots + \|f_1\|_{L^{p_1}(w_1)}\|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}.$$

- Note $f_t \in L^{p_t}_s(w_t)$, rather than Schwartz functions.

KP Inequality In Fractional Sobolev Spaces

- The weighted fractional Sobolev space $L^p_s(w)$ for $0 < p < \infty$, $s > 0$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $J^s u$ is a function in $L^p(w)$.

KP Inequality In Fractional Sobolev Spaces

- The weighted fractional Sobolev space $L^p_s(w)$ for $0 < p < \infty$, $s > 0$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $J^s u$ is a function in $L^p(w)$.
- The weighted local Hardy space $h^p(w)$ for $0 < p < \infty$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $\|u\|_{h^p(w)} := \| \sup_{0<t<1} |t^{-n}\phi(t^{-1}\cdot) * u| \|_{L^p(w)} < \infty$.

KP Inequality In Fractional Sobolev Spaces

- The weighted fractional Sobolev space $L^p_s(w)$ for $0 < p < \infty$, $s > 0$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $J^s u$ is a function in $L^p(w)$.
- The weighted local Hardy space $h^p(w)$ for $0 < p < \infty$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $\|u\|_{h^p(w)} := \| \sup_{0<t<1} |t^{-n}\phi(t^{-1}\cdot) * u| \|_{L^p(w)} < \infty$.
- The need for $h^p(w)$ is because $J^s(f_1 \cdots f_m)$ for $f_j \in L^{p_j}_s(w_j)$ is only (potentially) defined as a tempered distribution.

KP Inequality In Fractional Sobolev Spaces

- The weighted fractional Sobolev space $L^p_s(w)$ for $0 < p < \infty$, $s > 0$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $J^s u$ is a function in $L^p(w)$.
- The weighted local Hardy space $h^p(w)$ for $0 < p < \infty$, and $w \in A_\infty$ is defined to be the space of tempered distributions, u, such that $\|u\|_{h^p(w)} := \| \sup_{0<t<1} |t^{-n}\phi(t^{-1}\cdot) * u| \|_{L^p(w)} < \infty$.
- The need for $h^p(w)$ is because $J^s(f_1 \cdots f_m)$ for $f_j \in L^{p_j}_s(w_j)$ is only (potentially) defined as a tempered distribution.
- For $p \geq 1$ the previous theorem can be obtained via duality. For $p < 1$ the key ingredients include a weighted Sobolev embedding theorem, density of Schwartz functions, completeness of $h^p(w)$ and the fact $h^p(w)$ continuously embeds into $\mathcal{S}'$.

Thank You!

Proposition

Let $g \in L^q(w)$, $1 \leq q < \infty$ where $w \in A_q$, then g is a well defined tempered distribution.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and $\theta = w^{-\frac{q'}{q}}$ which is the dual weight of $w \in A_q$. Observe,

$$\begin{aligned} |\langle g, \varphi\rangle| &\leq \int |g||\varphi| w^{\frac{1}{q}} w^{-\frac{1}{q}} (1+|x|)^{n+1}(1+|x|)^{-(n+1)} \\ &\leq \|g\|_{L^q(w)} \|(1+|x|)^{-(n+1)}\|_{L^{q'}(\theta)} \sup_{x\in}(1+|x|)^{n+1}|\varphi(x)| \\ &\lesssim \|g\|_{L^q(w)} \|(1+|x|)^{-(n+1)}\|_{L^{q'}(\theta)} \sum_{|\alpha|\leq n+1} \sup_{x\in} |x|^\alpha |\varphi(x)|. \end{aligned}$$

Density: Well Defined Tempered Distribution

Let $Q_{\nu,m} \subset \mathbb{R}^n$ denote, for $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, the n-dimensional cube with sides parallel to the coordinate axes, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$. Furthermore, let $w(Q) = \int_Q w(x)\,dx$ for a weight w and a cube Q.

Theorem (Meyries, Veraar)

Let $s > 0$, $1 < p \leq q < \infty$, $w_0 \in A_p$, and $w_1 \in A_q$. Then $L^p_s(w_0) \hookrightarrow L^q(w_1)$ if and only if

$$\sup_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n} 2^{-\nu s} w_0(Q_{\nu,m})^{-\frac{1}{p}} w_1(Q_{\nu,m})^{\frac{1}{q}} < \infty.$$

Density: Well Defined Tempered Distribution

Let $Q_{\nu,m} \subset \mathbb{R}^n$ denote, for $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, the n-dimensional cube with sides parallel to the coordinate axes, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$. Furthermore, let $w(Q) = \int_Q w(x)\,dx$ for a weight w and a cube Q.

Theorem (Meyries, Veraar)

Let $s > 0$, $1 < p \le q < \infty$, $w_0 \in A_p$, and $w_1 \in A_q$. Then $L^p_s(w_0) \hookrightarrow L^q(w_1)$ if and only if

$$\sup_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n} 2^{-\nu s} w_0(Q_{\nu,m})^{-\frac{1}{p}} w_1(Q_{\nu,m})^{\frac{1}{q}} < \infty.$$

- This theorem can be extended to more general function spaces.
- In general two weight inequalities are challenging.
- For power weights the above theorem can be simplified.

Density: Well Defined Tempered Distribution

Let

$$w_{\beta,\alpha}(x) = \begin{cases} |x|^\beta & \text{if } |x| \leq 1, \\ |x|^\alpha & \text{if } |x| > 1. \end{cases}$$

Proposition

Let $\alpha_0, \beta_0, \alpha_1, \beta_1 > -n$, $1 < p \leq q < \infty$ and $s > 0$. Then for weights $w_0 = w_{\beta_0,\alpha_0}$, $w_1 = w_{\beta_1,\alpha_1}$ we obtain $L^p_s(w_0) \hookrightarrow L^q(w_1)$ if and only if

$$s - \frac{n+\beta_0}{p} \geq -\frac{n+\beta_1}{q}$$

$$s - \frac{n}{p} \geq -\frac{n}{q}$$

$$\frac{\alpha_0}{p} \geq \frac{\alpha_1}{q}.$$

Density: Well Defined Tempered Distribution

Let w and w_j be power weights and suppose $\frac{\tau_w}{p} > 1$. Let $\tau := \tau_w + \epsilon > 1$ such that $s > n(\tau/p - 1) > 0$. Notice this implies $\frac{\tau}{p} > 1$. We will use the previous Proposition to show if $f_j \in L^{p_j}_s(w_j)$ then $f_1 \cdots f_m \in L^\tau(w)$. Observe,

$$\|f_1 \cdots f_m\|_{L^\tau(w)} \lesssim \left(\int \left(|f_1|^{\frac{\tau}{p}} \cdots |f_m|^{\frac{\tau}{p}} w_1{}^{\frac{1}{p_1}} \cdots w_m{}^{\frac{1}{p_2}} \right)^p \right)^{\frac{1}{\tau}}$$
$$\leq \left(\int |f_1|^{\frac{\tau}{p} p_1} w_1 \right)^{\frac{1}{p_1} \frac{p}{\tau}} \cdots \left(\int |f_m|^{\frac{\tau}{p} p_m} w_m \right)^{\frac{1}{p_m} \frac{p}{\tau}}.$$

The terms on the RHS of the above inequality are finite by the Sobolev embedding theorem i.e.

$$\|f_j\|_{L^{\frac{\tau}{p} p_j}(w_j)} \lesssim \|J^s f_j\|_{L^{p_j}(w_j)}.$$

Hence $J^s(f_1 \cdots f_m)$ is well defined.

Density Argument

Let $q_j := \frac{\tau p_j}{p}$, then $\frac{1}{q_1} + \cdots \frac{1}{q_m} = \frac{1}{\tau}$.

- Pick Schwartz functions f_i^j, for $i \in \{1, \ldots, m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$.

Density Argument

Let $q_j := \frac{\tau p_j}{p}$, then $\frac{1}{q_1} + \cdots \frac{1}{q_m} = \frac{1}{\tau}$.

- Pick Schwartz functions f_i^j, for $i \in \{1, \ldots, m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$.
- It can be shown that $J^s(f_1^j, \ldots, f_m^j)$ converges to $J^s(f_1, \ldots, f_m)$ in $\mathcal{S}'$.

Density Argument

Let $q_j := \frac{\tau p_j}{p}$, then $\frac{1}{q_1} + \cdots \frac{1}{q_m} = \frac{1}{\tau}$.

- Pick Schwartz functions f_i^j, for $i \in \{1, \ldots, m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$.
- It can be shown that $J^s(f_1^j, \ldots, f_m^j)$ converges to $J^s(f_1, \ldots, f_m)$ in $\mathcal{S}'$.
- Also, by the KP inequality for Schwartz functions the sequence $J^s(f_1^j, \ldots, f_m^j)$ is Cauchy in $h^p(w)$, and thus it converges to G in $h^p(w)$, hence it converges to G in $\mathcal{S}'$.

Density Argument

Let $q_j := \frac{\tau p_j}{p}$, then $\frac{1}{q_1} + \cdots \frac{1}{q_m} = \frac{1}{\tau}$.

- Pick Schwartz functions f_i^j, for $i \in \{1, \ldots, m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$.
- It can be shown that $J^s(f_1^j, \ldots, f_m^j)$ converges to $J^s(f_1, \ldots, f_m)$ in $\mathcal{S}'$.
- Also, by the KP inequality for Schwartz functions the sequence $J^s(f_1^j, \ldots, f_m^j)$ is Cauchy in $h^p(w)$, and thus it converges to G in $h^p(w)$, hence it converges to G in $\mathcal{S}'$.
- By the uniqueness of the limit in $\mathcal{S}'$, we have that $G = J^s(f_1, \ldots, f_m)$.

Density Argument

Let $q_j := \frac{\tau p_j}{p}$, then $\frac{1}{q_1} + \cdots \frac{1}{q_m} = \frac{1}{\tau}$.

- Pick Schwartz functions f_i^j, for $i \in \{1, \ldots, m\}$ converging to f_i respectively in $L_s^{p_i}(w_i)$ as $j \to \infty$.
- It can be shown that $J^s(f_1^j, \ldots, f_m^j)$ converges to $J^s(f_1, \ldots, f_m)$ in $\mathcal{S}'$.
- Also, by the KP inequality for Schwartz functions the sequence $J^s(f_1^j, \ldots, f_m^j)$ is Cauchy in $h^p(w)$, and thus it converges to G in $h^p(w)$, hence it converges to G in $\mathcal{S}'$.
- By the uniqueness of the limit in $\mathcal{S}'$, we have that $G = J^s(f_1, \ldots, f_m)$.

$$\|J^s(f_1 \cdots f_m)\|_{L^p(w)} \lesssim \|J^s f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|f_m\|_{L^{p_m}(w_m)} + \cdots \cdots + \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)} \cdots \|J^s f_m\|_{L^{p_m}(w_m)}$$

Thank You!