A Hilbert space approach to state-dependent delay equations

jt Johanna Frohberg

arXiv: 2308.04730

1. Motivation & Main result
2. Proof of Main Result
3. Outlook

1

Throughout $h>0$, $T>0$.

For $x:[-h,T]\to\mathbb{R}^n$ define $x_t := \left([-h,0]\ni s\mapsto x(s+t)\in\mathbb{R}^n\right)\in(\mathbb{R}^n)^{[-h,0]}$

In this talk address w.-p. of

$$(*)\quad\begin{cases} x'(t) = g(x(t+r(x_t))) \\ x_0 = z \in C([-h,0];\mathbb{R}^n)\end{cases}$$

$g:\mathbb{R}^n\to\mathbb{R}^n$ Lipschitz

$r: \operatorname{dom}(r)\subseteq(\mathbb{R}^n)^{[-h,0]}\to[-h,0]$

Examples

Prototype $|r(\phi)-r(\psi)| \lesssim |\phi-\psi|_\infty$ where $|\phi'|_\infty, |\psi'|_\infty \leq R$

(b) $x'(t) = -x(t-\min\{|x(t)|;2\})$ $h=2$, $T>0$ small

$$x_0 = z = \begin{cases} -1 & \text{on } -2\leq t\leq -1 \\ 3(t+1)^{2/3}-1 & \text{on } -1\leq t<c \\ \text{affine} & c<t\leq 0\end{cases}$$

with $\phi(0)=1$ ϕ continuous &

not Lipschitz!

Then $x_1(t) = 1+t$, $x_2(t) = 1+t-t^3$ both solve IVP.

Main issue: Estimate $|\phi(r(\phi)) - \psi(r(\psi))| \lesssim |\phi - \psi|$

$$|\phi(r(\phi)) - \psi(r(\psi))| \leq |\phi(r(\phi)) - \psi(r(\phi))| + |\psi(r(\phi)) - \psi(r(\psi))|$$
$$\leq |\phi - \psi|_\infty + \|\psi\|_{Lip} |r(\phi) - r(\psi)|$$

Need control of Lipschitz norm of ψ and ϕ.

Known ways: Solution space C^1 (Walther '03, '04) more assumptions e.g. z.

Space of pre-histories $W^{1,\infty}$ + L_p on $(0,T]$ (Hartung, Turi '97)

Combination of both (Nishiguchi '17/'18)

As of now: No genuine Hilbert space approach! — PDE! — Kirszbraun's Theorem!

Theorem (Frohberg, W' 24)

Let $z \in H^1(-h,0)$, $g: \mathbb{R}^n \to \mathbb{R}^n$, $r: H^1(-h,0) \to [-h,0]$ all be Lipschitz continuous.

Then $\exists!\, x \in H^1(-h,T)$ satisfying (*).

(2) Problem can be reduced to the case $z = 0$.

Lemma. $x \in H^1(-h,T)$. TFAE:

x satisfies (*) $\iff$ $x(t) = \int_0^{\max\{t,0\}} g(x(s + r(x_s)))\, ds \quad (t \in (-h,T))$

Integral well-defined?

$\Theta: H^1(-h,T) \to C([0,T], H^1(-h,0))$
$x \mapsto (s \mapsto x_s)$

continuous.

+ $H^1(-h,T) \subseteq C[-h,T]$ (Sobolev-embedding)

Define

$$\Psi : x \longmapsto \left(t \mapsto \int_0^{\max\{t,0\}} g(x(s + r(x_s)))\, ds \right)$$

2 Problems: (a) Ψ Lipschitz?

(b) Ψ contractive?

(a) Consider

$$V_\beta := \{ y \in H^1(-h,0) \,;\, |y'|_\infty \leq \beta \} \subseteq H^1(-h,0)$$

closed & convex; $\pi_\beta : H^1 \longrightarrow V_\beta$ (metric) projection is contraction.

$$\Psi_\beta : x \longmapsto \left(t \mapsto \int_0^{t \vee 0} g(\pi_\beta x(s + r(\pi_\beta x_s)))\, ds \right)$$

As z Lipschitz, we find $\beta > 0$ with $|z'|_\infty < \beta$.

(b) Weighted spaces use weighted norm: $H^1_\varrho = H^1$ with norm

$$\|\varphi\|^2_{H^1_\varrho} = \int_{-h}^{T} \left(\|\varphi(s)\|^2 + \|\varphi'(s)\|^2 \right) e^{-2\varrho s}\, ds$$

The

$$\Theta_\varrho : H^1_\varrho(-h,T) \longrightarrow L_{2,\varrho}(0,T; H^1(-h,0))$$
$$x \longmapsto (s \longmapsto x_s)$$

Satisfies $\|\Theta_\varrho\| \leq \frac{1}{\sqrt{2\varrho}}$

$$\Rightarrow \|(\Psi_\beta x)' - (\Psi_\beta y)'\|^2_{L_{2,\varrho}(-h,T)} \lesssim \int_0^T \underbrace{|x_t - y_t|^2_\infty}_{\lesssim |x_t - y_t|^2_{H^1}} + \beta |x_t - y_t|^2_{H^1}\, dt \lesssim \frac{1}{2\varrho} |x - y|^2_{H^1_\varrho}$$

To conclude the proof:

Choose $\beta > \|z\|_{Lip} + \|g(z(r(z)))\|$.

Then $\exists!\, x^\beta \in H^1(-h,T)$: $x^\beta = \Phi_\beta x^\beta$.

Since $x^{\beta\prime}(t) = g(\pi_\beta x^\beta(t + r(\pi_\beta x^\beta_t)))$; thus, for small t,

$\|x^{\beta\prime}(t)\| < \beta$; so $\pi_\beta x^\beta_t = x^\beta_t$.

If $\|x^{\beta\prime}(t)\| < \beta$ for all $t \in (-h,T)$: ✓

Otherwise $\|x^{\beta\prime}(t_\beta)\| = \beta$ with t_β minimal.

Then $\exists!\, x^{2\beta} \in H^1(-h,T)$: $x^{2\beta} = \Phi_{2\beta} x^{2\beta}$,

$x^\beta\big|_{(-h,t_\beta)} = x^{2\beta}\big|_{(-h,t_\beta)}$.

$\Rightarrow$ $t_\beta < t_{2\beta} < \dots$

$(t_{k\beta})_k$ eventually constant equal T

or $\|x'(t_{k\beta})\| \longrightarrow \infty$ $(k \to \infty)$ in finite time

$(**)$ cannot happen.

Any solution of $(*)$ bdd $\Rightarrow$ $g(\dots)$ bdd

Outlook.

- SDDs on PDEs (infinite memory horizon?)
- FDEs
- Kirszbraun's Thm for non-auto PDEs