Noncompact Sobolev embeddings, quantitative aspects

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- When $\Omega \subseteq \mathbb{R}^n$ is a "nice enough" bounded open set, $n \geq 2$, and u vanishes at ∂G, then:

$$|\nabla u| \in L^p(\Omega) \Rightarrow u \in \begin{cases} L^q(\Omega) \text{ for every } q \in \left[1, \frac{np}{n-p}\right] & \text{if } p \in [1, n), \\ L^q(\Omega) \text{ for every } q \in [1, \infty) & \text{if } p = n, \\ \mathcal{C}^{0,1-\frac{n}{p}}(\overline{\Omega}) & \text{if } p > n. \end{cases}$$

- $V_0^1 L^p(\Omega) \ldots |u| \in L^p(\Omega)$ and $\|u\|_{V_0^1 L^p(\Omega)} = \||\nabla u|\|_{L^p(\Omega)}$

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- $V_0^1 L^p(\Omega) \ldots |u| \in L^p(\Omega)$ and $\|u\|_{V_0^1 L^p(\Omega)} = \||\nabla u|\|_{L^p(\Omega)}$
- $E \colon V_0^1 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$
- $V_0^1 L^p(\Omega) = V_0^1 X(\Omega)$, $L^{\frac{np}{n-p}}(\Omega) = Y(\Omega)$

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$$\|\nabla u_\kappa\|_{L^p(\Omega)} = \kappa^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\Omega)}$$
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  But $-\frac{n}{q} = 1 - \frac{n}{p}$ for $q = \frac{np}{n-p}$.
- Quantitative questions: What is the quality of compactness of E? How severe is the lack of compactness of E?

Measure of noncompactness

- Given a positively homogeneous operator T between (quasi)Banach space X and Y, its measure of noncompactness is defined as

$$\beta(T) = \lim_{m \to \infty} \inf\{r > 0 : T(B_X) \text{ can be covered in } Y \text{ by } 2^{m-1} \text{ balls with radius at most } r\}.$$

- $0 \leq \beta(T) \leq \|T\|$, and T is compact if and only if $\beta(T) = 0$.

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- $0 \leq \beta(T) \leq \|T\|$, and T is compact if and only if $\beta(T) = 0$.
- $\beta(E\colon V_0^1 L^p(\Omega) \to L^q(\Omega)) \approx_{n,p,q} \lim_{m\to\infty} m^{-\frac{1}{n}}$ for $q \in [1, \frac{np}{n-p})$.
  - D. E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators*. Cambridge University Press, 1996.
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Question

We know that $\beta(E\colon V_0^1 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)) > 0$. Is it any better than $\beta(E\colon V_0^1 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)) = \|E\|$?

Maximal noncompactness

Theorem (S. Hencl, 2003)

For $p \in [1, n)$, the Sobolev embedding $E\colon V_0^1 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$ is maximally noncompact in the sense that $\beta(E\colon V_0^1 L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)) = \|E\|$.

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- In fact:

Theorem (O. Bouchala, 2020)

For $p \in [1, n)$ and $1 \leq q \leq r < \infty$, the Sobolev-Lorentz embedding $E\colon V_0^1 L^{p,q}(\Omega) \to L^{\frac{np}{n-p},r}(\Omega)$ is maximally noncompact in the sense that $\beta(E\colon V_0^1 L^{p,q}(\Omega) \to L^{\frac{np}{n-p},r}(\Omega)) = \|E\|$.

If $p = 1$, then also $q = 1$.

Here $V_0^1 L^{p,q}(\Omega)$ and $L^{\frac{np}{n-p},r}(\Omega)$ are defined in such a way that they are Banach spaces.

- Key ingredients in their proofs are:
  - The target function space is disjointly superadditive with a suitable power, and its (quasi)norm is monotone.
  - It is* a Banach space.
  - The explicit description of its norm can be exploited.
  - There exist extremals with vanishing supports.
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- The disjoint superadditivity may not be at our disposal.
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- While some of these ingredients are natural, some are rather restrictive and in fact not necessary.
- The disjoint superadditivity may not be at our disposal.
- Important function spaces are often merely equivalent to Banach spaces, but problems of geometrical nature may forbid us from simply "passing to an equivalent norm".
- In particular, their method fails for noncompact limiting Sobolev embeddings of $V_0^1 L^n$.

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- $\|f\|_{L^{\infty,s,\alpha}(\Omega)} = \|t^{-\frac{1}{s}} \log(\frac{2|\Omega|}{t})^{\alpha} f^*(t)\|_{L^s(0,|\Omega|)}$

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- $L^\infty(\Omega) \subsetneq L^{\infty,n,-1}(\Omega)$
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- $L^{\infty,\infty,-1+\frac{1}{n}}(\Omega) = \exp L^{\frac{n}{n-1}}(\Omega)$
- $\beta(E\colon V_0^1L^n(\Omega) \to L^{\infty,s,-1+\frac{1}{n}-\frac{1}{s}}(\Omega)) = ?$, $s \in [n,\infty]$
- Spaces $L^{\infty,s,\alpha}(\Omega)$ are not disjointly superadditive with any power, and $\|\cdot\|_{L^{\infty,s,\alpha}(\Omega)}$ is[†] merely a quasinorm.

[†]with our definition

General theorem for obtaining lower bounds

Theorem (J. Lang, Z. M., L. Pick, 2024)

Let T be a linear operator from a quasi-Banach space X to a quasi-Köthe function space $Y \subseteq M_0(R,\mu)$. Assume that Y has absolutely continuous and uniformly separating quasinorm.
Let $\lambda > 0$. If there is a sequence $\{x_j\}_{j=1}^{\infty} \subseteq B_X$ such that

$$\operatorname{supp} Tx_j \to \emptyset \quad \text{as } j \to \infty$$

and

$$\|Tx_j\|_Y \geq \lambda \quad \text{for every } j \in \mathbb{N},$$

then $\beta(T\colon X \to Y) \geq \lambda$.

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quasi-Köthe function space = monotone + $\chi_E \in Y$ and $f\chi_E \in L^1(R,\mu)$ for every $f \in Y$ and $E \subseteq R$, $\mu(E) < \infty$

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- $\beta(E\colon V_0^1 L^n(\Omega) \to L^{\infty, s, -1+\frac{1}{n}-\frac{1}{s}}(\Omega)) = \|E\|$ for every $s \in [n, \infty)$
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- $\|h_j - Tx_j\|_Y \leq r$
- On the one hand, $\|h_j\|_Y = \|Tx_j + h_j - Tx_j\|_Y \geq \varepsilon_{r,\lambda} > 0$
- On the other hand,

$$\operatorname{supp} h_j \subseteq \operatorname{supp} Tx_j \to \emptyset \quad \text{as } j \to \infty$$

and

$$|h_j| \leq \sum_{k=1}^{m} |g_k| \in Y \quad \mu\text{-a.e. in } R \text{ for every } j \in \mathbb{N},$$

and so

$$\lim_{j\to\infty} \|h_j\|_Y = 0.$$

- The case $r = \infty$ in $\beta(E\colon V_0^1 L^{p,q}(\Omega) \to L^{\frac{np}{n-p},r}(\Omega))$ was solved by J. Lang, V. Musil, M. Olšák, and L. Pick in 2021.
- The case $s = \infty$ was omitted in $\beta(E\colon V_0^1 L^{n}(\Omega) \to L^{\infty,s,-1+\frac{1}{n}-\frac{1}{s}}(\Omega))$ because the general theorem cannot be used.

Marcinkiewicz spaces

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- $\|f\|_{m_\varphi(\Omega)} = \sup_{t\in(0,|\Omega|)} f^*(t)\varphi(t)$
    - $\varphi(t) = t^{\frac{1}{r}} \ldots L^{r,\infty}$
    - $\varphi(t) = \log(\frac{2|\Omega|}{t})^{-1+\frac{1}{n}} \ldots L^{\infty,\infty,-1+\frac{1}{n}}(\Omega)$

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    - $\varphi(t) = \log(\frac{2|\Omega|}{t})^{-1+\frac{1}{n}} \ldots L^{\infty,\infty,-1+\frac{1}{n}}(\Omega)$
- The method used by LMOP requires

$$\lim_{a\to\infty} \inf_{t\in(0,|\Omega|/a)} \frac{\varphi(at)}{\varphi(t)} = \infty,$$

which fails for $\varphi(t) = \log(\frac{2|\Omega|}{t})^{-1+\frac{1}{n}}$.

Theorem (J. Malý, Z. M., V. Musil, L. Pick, 2024)

Let X be a quasinormed space of measurable functions on (R, μ). Let $\varphi\colon (0, \mu(R)) \to (0, \infty)$ be almost quasiconcave. Let $T\colon X \to m_\varphi(R, \mu)$ be a bounded positively homogeneous operator. Let $r \in (0, \|T\|)$.

Assume that there are $\tau \in (0, 1]$ and a set $S \subseteq R$ of finite positive measure such that for each $\sigma > 0$ and $\varepsilon \in (0, 1)$ there are $m \in \mathbb{N}$, functions $f_i \in X$ and pairwise disjoint sets $E_i \subseteq S$, $i = 1, \ldots, m$, each of positive measure, such that the following properties hold:

- $\|f_i\|_X = 1$ *for every* $i \in \{1, \ldots, m\}$,
- $\sum_{i=1}^{m} \mu(E_i) \geq \tau \mu(S)$,
- $s_i \geq \sigma$ *for every* $i \in \{1, \ldots, m\}$,
- $s_i \varphi(\mu(E_i)) \geq (1 - \varepsilon) r$ *for every* $i \in \{1, \ldots, m\}$,

where

$$s_i = \operatorname*{essinf}_{E_i} |T f_i|$$

Then $\beta(T\colon X \to m_\varphi) \geq r$.

Another point of view, s-numbers

- Various delicate properties of operators (such as compactness of Sobolev embeddings) can be studied quantitatively by means of various s-numbers.
    - An axiomatic approach introduced by A. Pietsch in 1974.

Definition (Approximation numbers)

Let $T \in \mathcal{B}(X,Y)$. Its m-th, $m \in \mathbb{N}$, approximation number $a_m(T)$ is defined as

$$a_m(T) = \inf\{\|T - F\| : F \in \mathcal{B}(X,Y) \text{ with } \operatorname{rank} F < m\}.$$

- $a_m(E\colon V_0^1 L^p(\Omega) \to L^q(\Omega)) \approx m^{-(\frac{1}{n} - (\frac{1}{p} - \frac{1}{q})_+)}$ for every $1 \le q < \frac{np}{n-p}$.
- The approximation numbers are the largest s-numbers.

Noncompact Sobolev embeddings

Definition (Bernstein numbers)

Let $T \in \mathcal{B}(X, Y)$. Its m-th, $m \in \mathbb{N}$, Bernstein number $b_m(T)$ is defined as

$$b_m(T) = \sup_{\substack{X_m \subset X \\ \dim X_m = m}} \inf_{\substack{x \in X_m \\ \|x\|_X = 1}} \|Tx\|_X.$$

- In approximation theory, Bernstein numbers are useful for providing lower bounds on various (not necessarily linear) approximation quantities (including some other s-numbers).
  - R. J. Kunsch. *Bernstein numbers and lower bounds for the Monte Carlo error*. Springer Proc. Math. Stat., 163. 2016

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  - R. J. Kunsch. *Bernstein numbers and lower bounds for the Monte Carlo error*. Springer Proc. Math. Stat., 163. 2016
- In operator/spectral theory, they are connected with (finitely) strictly singular operators.
  - $\mathcal{K}(X,Y) \subseteq \mathcal{FSS}(X,Y) \subseteq \mathcal{SS}(X,Y) \subseteq \mathcal{B}(X,Y)$
  - P. Lefèvre and L. Rodríguez-Piazza. Finitely strictly singular operators in harmonic analysis and function theory. Adv. Math. 255(2014), 119–152.

Question

Is one of the embeddings $E\colon V_0^1L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$ and $E\colon V_0^1L^p(\Omega) \to L^{\frac{np}{n-p},p}(\Omega)$ in some sense more noncompact than the other?

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Theorem

For every $p \in [1, n)$ and $m \in \mathbb{N}$.

1. *We have $b_m(E\colon V^1_0L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)) \approx m^{-\frac{1}{n}}$, and the embedding $E\colon V^1_0L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega)$ is finitely strictly singular.*
2. *We have $b_m(E\colon V^1_0L^p(\Omega) \to L^{\frac{np}{n-p},p}(\Omega)) = \|E\|$, and the embedding $E\colon V^1_0L^p(\Omega) \to L^{\frac{np}{n-p},p}(\Omega)$ is not strictly singular.*

The first-order Lebesgue case with $p = 1$ is due to J. Bourgain and M. Gromov, 1989. The rest (and higher-order variants) appeared in a paper by J. Lang, Z. M., 2023

Thank you!


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