Modular uniform convexity structures and applications to PDE.

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Introduction

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$$W^{1,2}(\Omega) = \left\{ u, |\nabla u| \in L^2(\Omega) \right\}, \quad \|u\|_{1,2} = \left(\int_\Omega \left(|u|^2 + |\nabla u|^2 \right) dx \right)^{\frac{1}{2}}$$

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Rigorously $W_0^{1,2}(\Omega)$ is the norm closure of $C_0^\infty(\Omega)$.

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Equivalently: minimze $2F = \|\cdot\|_{1,2}$ over $W_0^{1,2}(\Omega)$.

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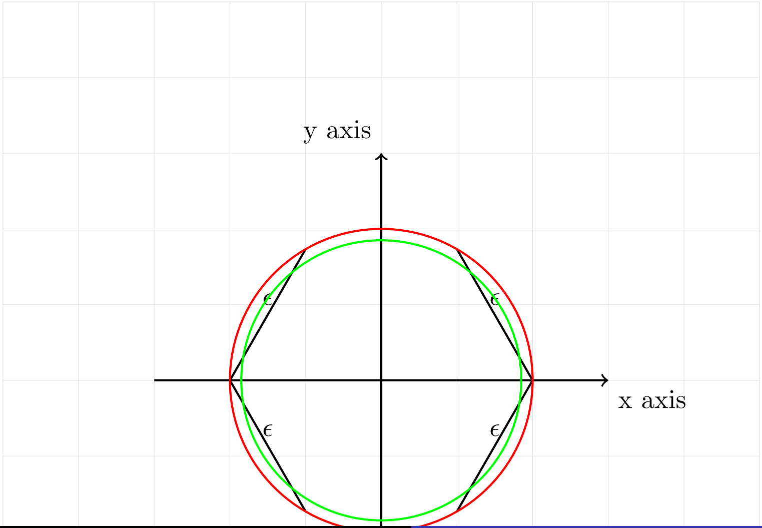

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More precisely, given any $\varphi \in X$, there exists a unique $w \in C$ such that $\|\varphi - w\| = \text{dist}(C, \varphi)$.
Make $C = W_0^{1,2}(\Omega)$, then $u = \varphi - w$ is a solution to the Dirichlet problem.

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Uniform convexity implies the sequence (w_j) is Cauchy.
pauseThen

$$W_0^{1,2}(\Omega) \ni w_j \to w.$$

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The origins of the p-Laplacian and A. Missbach, Benedikt, Girg, Kotrla, Takac, 2018.

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Then, electrorheological fluids enter the picture.

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$W^{1,p(x)}(\Omega)$ has been extensively studied as a **normed space for** $1 < \alpha \le p(x) \le \beta < \infty$. Not so for unbouded $p(x)$.

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- (*iii*) For any minimzing sequence (w_j) is defined in terms of convergence of F_p, which is weaker than convergence in norm.
- (*iv*) Even when p is bounded, the convergence of the minimizing sequence does not follow from the uniform convexity of the norm.

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This convergence defines the modular topology, τ_{ρ_p}.

Modular uniform convexity

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$$B_1^{\rho_p}(0) = \{u : \rho_p(u) \leq 1\}$$

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Corollary

Any $\emptyset \neq C$, ρ_p-closed, convex subset of $W^{1,p(x)}(\Omega)$ is proximinal.

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$V_0^{1,p(x)}(\Omega)$*, the* τ_{ρ_p} *closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(x)}(\Omega)$*, is a* τ_{ρ_p}*-closed linear subspace of* $W_0^{1,p(x)}(\Omega)$*.*

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THANK YOU FOR YOUR INTEREST.