Magnetic Hardy inequalities in $L^p$ setting

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The $p$-Laplacian $-\Delta_p$

- **The case** $p = 2$: $-\Delta_p = -\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$.

$$-\Delta_p u := -\text{div}(|\nabla u|^{p-2} \nabla u), \quad p > 1.$$ 

The associated $L^2(\mathbb{R}^d)$ quadratic form $h_p$ of $-\Delta_p$ is given by

$$h_p[u] = \int_{\mathbb{R}^d} |\nabla u|^p \, dx, \quad \forall u \in \mathcal{D}(h_p) := W^{1,p}(\mathbb{R}^d). \quad (1)$$

and the sesquilinear form: for $u \in \mathcal{D}(h_p), v \in \mathcal{D}(h_{p'})$

$$h_p(u, v) := (-\Delta_p u, v)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (-\Delta_p u) v \, dx$$

$$= \int_{\mathbb{R}^d} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx.$$
Dirichlet/Lagrange principle

A solution to the problem $-\Delta_p u = f$ + Dirichlet BC:

$$u \in W_0^{1,p} : h_p(u,v) = \langle f, v \rangle, \quad \forall v \in W_0^{1,p}. \quad (2)$$

Define $E : W_0^{1,p} \rightarrow \mathbb{R}$,

$$E(\varphi) := \frac{1}{p} h_p[\varphi] - \langle f, \varphi \rangle.$$

- $u$ solution for (2) $\iff E(u) = \min_{\varphi \in W_0^{1,p}} E(v)$.
- $E$ has a minimum if $h_p[\cdot]$ is ”positive” enough (coercive, etc).
Some definitions

- We say that $-\Delta_p$ is a **non-negative operator** if

$$-\Delta_p \geq 0 :\iff h_p[u] \geq 0, \quad \forall u \in \mathcal{D}(h_p);$$

- $-\Delta_p$ is a **subcritical operator** if $-\Delta_p$ satisfies a Hardy-type inequality, i.e. there exists $V \in L^1_{\text{loc}}(\mathbb{R}^d)$, $V \neq 0$, such that

$$-\Delta_p \cdot \geq V|\cdot|^{p-2}\cdot,$$

in the sense of $L^2$ quadratic forms:

$$h_p[u] \geq \int_{\mathbb{R}^d} V|u|^p \, dx, \quad \forall u \in W^{1,p}(\mathbb{R}^d).$$

- Otherwise, $-\Delta_p$ is a **critical operator** (i.e. there is NO Hardy inequality for $-\Delta_p$).
Let $d \geq 2$ and $1 \leq p < d$. If $u \in W^{1,p}(\mathbb{R}^d)$ then $u/|x| \in L^p(\mathbb{R}^d)$ and it satisfies

$$\int_{\mathbb{R}^d} |\nabla u|^p \, dx \geq \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx,$$

$$\mu_{p,d} := \left( \frac{d - p}{p} \right)^p. \quad (3)$$

Moreover, the constant $\mu_{p,d}$ is optimal in the sense that (3) does not hold with any bigger constant.
Criticality versus sub-criticality of $-\Delta_p$

- $p < d \Rightarrow -\Delta_p$ is sub-critical (by Hardy Inequality): with $V(x) := \mu_{p,d}/|x|^p$, i.e.

$$- \Delta_p \cdot \geq \mu_{p,d} \frac{|x|^{p-2}}{|x|^p} \quad (4)$$

- $p \geq d \Rightarrow -\Delta_p$ is critical:

**Proposition**

*Let $p \geq d$. If $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ is a non-negative potential such that

$$\int_{\mathbb{R}^d} |\nabla u|^p \, dx \geq \int_{\mathbb{R}^d} V|u|^p \, dx, \quad \forall u \in C_c^\infty(\mathbb{R}^d), \quad (5)$$

then $V = 0$ a.e. in $\mathbb{R}^d$.***

- $p < d \Rightarrow H := -\Delta_p - \mu_{p,d} \frac{|x|^{p-2}}{|x|^p}$? (Obviously $H \geq 0$).
$H := -\Delta_p - \mu_{p,d} \frac{|p-2|}{|x|^p}$ is critical for $p < d$:

**Proposition**

Let $1 \leq p < d$. If $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ is a non-negative potential such that

$$
\int_{\mathbb{R}^d} |\nabla u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \geq \int_{\mathbb{R}^d} V|u|^p \, dx, \quad \forall u \in C_c^\infty(\mathbb{R}^d),
$$

(6)

then $V = 0$ a.e. in $\mathbb{R}^d$. 
Consider a smooth magnetic potential $A : \mathbb{R}^d \to \mathbb{R}^d$, The magnetic $p$-Laplacian is formally defined on $C^\infty_c(\mathbb{R}^d)$ by

$$\Delta_{A,p} u := \text{div}_A(|\nabla_A u|^{p-2}\nabla_A u),$$

(7)

where the magnetic gradient and magnetic divergence are given by

$$\nabla_A u := \nabla u + iA(x)u; \quad \text{div}_A F := \text{div}F + iA \cdot F,$$

(8)

for any smooth vector field $F : \mathbb{R}^d \to \mathbb{C}^d$.

- Of course, if $A = 0$ then $\Delta_{A,p} = \Delta_p$. 
The associated form $h_{A,p}$ of the magnetic $p$-Laplacian $\Delta_{A,p}$

For all $u \in \mathcal{D}(h_{A,p}) := C_c^\infty(\mathbb{R}^d)$

$$h_{A,p}[u] := \int_{\mathbb{R}^d} |\nabla_A u|^p \, dx = \int_{\mathbb{R}^d} |\nabla u + iA(x)u|^p \, dx,$$

where the norm $\| \cdot \|$ with respect to which the closure is taken is given by

$$\|u\| := \sqrt[p]{h_{A,p}[u] + \|u\|_{L^p(\mathbb{R}^d)}^p}.$$

- We extend the notions of subcriticality/criticality also to $-\Delta_{A,p}$. 
• The magnetic field (2 diff. form):
\[ B : \mathbb{R}^d \to \mathbb{R}^{d \times d} \text{ smooth, } dB = 0 \]
, i.e. \( \exists A \) with \( dA = B \), \( (B_{ij} = A_{j,x_i} - A_{i,x_j}) \)

• The choice of \( A \) does not matter to much...

If \( A, \tilde{A} : \mathbb{R}^d \to \mathbb{R}^d \) s.t. \( dA = d\tilde{A} = B \) then there exists a scalar field \( \phi : \mathbb{R}^d \to \mathbb{R} \) such that \( A - \tilde{A} = d\phi \). It is easy to see that

\[ \mathcal{D}(h_{A,p}) = \mathcal{D}(h_{\tilde{A},p}) \quad \text{and} \quad h_{A,p}[\psi] = h_{\tilde{A},p}[\psi e^{i\phi}], \quad \forall \psi \in C_c^\infty(\mathbb{R}^d). \]

(9)
The diamagnetic inequality/Kato’s inequality

- Diamagnetic inequality:

\[ |\nabla_A u(x)| \geq |\nabla |u|(x)| \quad \text{a.e. } x \in \mathbb{R}^d, \forall u \in \mathcal{D}(h_{A,p}). \quad (10) \]

Then

\[ \int_{\mathbb{R}^d} |\nabla_A u|^p \, dx \geq \int_{\mathbb{R}^d} |\nabla |u|^p \, dx \]

- So, all the inequalities valid for the standard $p$-Laplacian transfer to the magnetic $p$-Laplacian.

- BUT can we improve them?
Let \( p \geq d \) and \( B \) be a smooth and closed magnetic field with \( B \neq 0 \). Then there exists a constant \( C_{B,p,d} > 0 \) such that for any magnetic potential \( A \) with \( dA = B \) we have

\[
\int_{\mathbb{R}^d} |\nabla A u|^p \, dx \geq C_{B,p,d} \int_{\mathbb{R}^d} \rho(x) |u|^p \, dx, \quad \forall u \in \mathcal{D}(h_{A,p}), \quad (11)
\]

where

\[
\rho(x) := \frac{1}{|x|^d (|\log |x||^p + |x|^{p-d})}.
\]

- \( p \geq d \Rightarrow -\Delta_{A,p} \) is sub-critical \((-\Delta_p \) is critical)!
Previously known results (the case $p = d = 2$)

- $B \neq 0$, with $\rho(x) = \frac{1}{1+|x|^2|\log|x||^2}$ in [C.-Krejcirik 2016]
- $B \neq 0$, under the additional condition $\frac{1}{2\pi} \int_{\mathbb{R}^2} *B \ dx \notin \mathbb{Z}$ where $*B := B_{12}$ it was proved with $\rho(x) = \frac{1}{1+|x|^2}$ in [Laptev-Weidl, 1998].
- $B \neq 0 +$ compactly supported + unbounded $\rho$, done in [Cassano-Franceschi-Krejcirik-Prandi, 2023]
- For Aharonov-Bohm type $A(x) = \psi \left( \frac{x}{|x|} \right) \frac{(-x_2,x_1)}{|x|^2}$ it was shown with $\rho(x) = 1/|x|^2$ also in [Laptev-Weidl, 1998].
Main results

Sketch of proof \( \int_{\mathbb{R}^d} |\nabla_A u|^p \, dx \geq C \int_{\mathbb{R}^d} \rho(x) |u|^p \, dx \),
\[
\rho(x) := \frac{1}{|x|^d (|\log |x||^p + |x|^{p-d})}.
\]

Step 1 If \( p \geq d \) then for all \( u \in C_c^\infty (B_{\tilde R}(0)) \)
\[
\int_{B_{\tilde R}(0)} |\nabla u|^p \, dx \geq \left( \frac{p-1}{p} \right)^p \frac{1}{\tilde R^{p-d}} \int_{B_{\tilde R}(0)} \frac{|u|^p}{|x|^d (\log \frac{\tilde R}{|x|})^p} \, dx.
\]

Step 2 If \( p \neq d \) then
\[
\int_{B_c^\varepsilon_{\tilde R}(0)} |\nabla u|^p \, dx \geq \left| \frac{d-p}{p} \right|^p \int_{B_c^\varepsilon_{\tilde R}(0)} \frac{|u|^p}{|x|^p} \, dx, \quad \forall u \in C_c^\infty (B_c^\varepsilon_{\tilde R}(0)).
\]

Step 3 If \( p = d \) then \( \forall u \in C_c^\infty (B_c^\varepsilon_{\tilde R}(0)) \)
\[
\int_{B_c^\varepsilon_{\tilde R}(0)} |\nabla u|^d \, dx \geq \left( \frac{d-1}{d} \right)^d \int_{B_c^\varepsilon_{\tilde R}(0)} \frac{|u|^d}{|x|^d (\log \frac{\tilde R}{|x|})^d} \, dx
\]
Sketch of proof of \( \int_{\mathbb{R}^d} |\nabla_A u|^p \, dx \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p \, dx \),

\[ \rho(x) := \frac{1}{|x|^d(|\log|x|)|p+|x|^{p-d})}. \]

**Lemma**

Let \( d \geq 2 \) and \( 1 < p < \infty \). Assume also that \( B \neq 0 \) and let \( A \) be such that \( B = dA \). Let \( R > 1 \) be fixed and consider the annular domain \( \Omega_R := B_R(0) \setminus B_{\frac{1}{R}}(0) \). Then we define

\[ \mu_B(R) := \inf_{u \in W^{1,p}(\Omega_R), u \neq 0} \frac{\int_{\Omega_R} |(\nabla + iA)u|^p \, dx}{\int_{\Omega_R} |u|^p \, dx}. \]  \hspace{1cm} (12)

Then \( \mu_B \neq 0 \) on \((1, \infty)\).

**PROOF** = Steps 1-3 + Lemma + diamagnetic inequality + a localization argument.
What about the sub-criticality of $H_A := -\Delta_{A,p} - \mu_{p,d} \frac{|p-2|}{|x|^p}$ when $p < d$?

**Theorem (C-Krejcirik-Lam-Laptev, NON 2024)**

Let $2 \leq p < d$ and $B$ be a smooth and closed magnetic field with $B \neq 0$. Then there exists a constant $c(p) > 0$ such that for any vector field $A$ with $dA = B$ we have

$$\int_{\mathbb{R}^d} |\nabla_A u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \geq c(p) \int_{\mathbb{R}^d} \left| \nabla_A (u |x|^{d-p}) \right|^p |x|^{p-d} \, dx,$$

(13)

The constant $c(p)$ in (13) is explicitly given by

$$c(p) := \inf_{(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{\left[ t^2 + s^2 + 2s + 1 \right]^{p/2}}{[t^2 + s^2]^{p/2}} - 1 - ps \in (0, 1) \tag{14}$$

- The optimal value of the constant $c(p)$ is an interesting open problem.
- The case $1 < p < 2$ remains open.
Proof of Theorem (ineq (13))

(Step 1) First we prove the free-magnetic case:

**Corollary**

For any $2 \leq p < d$ there exists a positive constant $c(p)$ such that

\[
\int_{\mathbb{R}^d} |\nabla u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \geq c(p) \int_{\mathbb{R}^d} \left| \nabla (u|x|^{\frac{d-p}{p}}) \right|^p |x|^{p-d} \, dx, \quad \forall u
\]

(15)

- For $p = 2$ (15) becomes an identity with $c(2) = 1$ ([Brezis-Vazquez 1997]):

\[
\int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{(d - 2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx = \int_{\mathbb{R}^d} \left| \nabla (u|x|^{\frac{d-2}{2}}) \right|^2 |x|^{2-d} \, dx
\]

which gives rise to many applications...

- With the help of Corollary... $H := -\Delta_p - \mu_{p,d} \frac{|.|^{p-2}}{|x|^p}$ is subcritical in bounded domains, etc.
Proof of Theorem (ineq (13))

(15) is a consequence of the identity

\[
\int_{\mathbb{R}^d} |\nabla u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx
= \int_{\mathbb{R}^d} C_p \left( \nabla u, |x|^{d-p} \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right) \, dx,
\]

where

\[
C_p (\alpha, \beta) = |\alpha|^p - |\alpha - \beta|^p - p |\alpha - \beta|^{p-2} \text{Re} (\alpha - \beta) \cdot \bar{\beta}
\]

Then easily show that

\[
C_p (\alpha, \beta) \geq c_p |\beta|^p , \alpha, \beta \in \mathbb{C}^d,
\]

where \( c(p) := \inf_{(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{[t^2 + s^2 + 2s + 1]^{\frac{p}{2}} - 1 - ps}{[t^2 + s^2]^{\frac{p}{2}}} \in (0, 1) \)
Proof of Theorem (ineq. (13))

(Step 2)

\[
|\nabla_A u|^p - C_p \left( \nabla_A u, |x|^{-\frac{d-p}{p}} \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right) \\
= |\nabla u|^p - C_p \left( \nabla u, |x|^{-\frac{d-p}{p}} \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right) .
\]

where \( C_p (\alpha, \beta) = |\alpha|^p - |\alpha - \beta|^p - p |\alpha - \beta|^{p-2} \text{Re} (\alpha - \beta) \cdot \overline{\beta} \)

(Step 3) In view of (Step 2) + (16):

\[
\int_{\mathbb{R}^d} |\nabla_A u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx
\]

\[
= \int_{\mathbb{R}^d} C_p \left( \nabla_A u, |x|^{-\frac{d-p}{p}} \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right) \, dx, \quad (17)
\]

(Step 4) \( C_p (\alpha, \beta) \geq c_p |\beta|^p + (17) = \text{END OF PROOF.} \)
Theorem (C.-Krejcirik-Lam-Laptev, NON 2024)

Let $2 \leq p < d$ and $B$ be a smooth and closed magnetic field with $B \neq 0$. Then there exists a constant $C_{B,p,d} > 0$ such that for any vector field $A$ with $dA = B$ we have

$$\int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \geq C_{B,p,d} \int_{\mathbb{R}^d} \rho(x)|u|^p \, dx, \quad \forall u \in \mathcal{D}(h_{A,p}),$$

(18)

where

$$\rho(x) := \frac{1}{|x|^p (1 + |\log|x||^p)}.$$  

- $2 \leq p < d \Rightarrow H_A = -\Delta_{A,p} - \mu_{p,d} \frac{|\cdot|^{p-2}}{|x|^p}$ is sub-critical

  $(-\Delta_p - \mu_{p,d} \frac{|\cdot|^{p-2}}{|x|^p}$ is critical !)

- This improves our previous result in [C.-Krejcirik, 2016, Thm. 1.1] from $L^2$ to the $L^p$ setting by obtaining also an unbounded weight $\rho$.  

\[2 \leq p < d \Rightarrow H_A = -\Delta_{A,p} - \mu_{p,d} \frac{|\cdot|^{p-2}}{|x|^p}\text{ is sub-critical}
\]

\[(-\Delta_p - \mu_{p,d} \frac{|\cdot|^{p-2}}{|x|^p}\text{ is critical !})
\]

\[\text{This improves our previous result in [C.-Krejcirik, 2016, Thm. 1.1] from } L^2 \text{ to the } L^p \text{ setting by obtaining also an unbounded weight } \rho.
\]
Main results

Sketch of proof

\[
\int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \, dx \geq C \int_{\mathbb{R}^d} \rho(x) |u|^p \, dx,
\]

\[
\rho(x) := \frac{1}{|x|^d (|\log |x||^{p+1})}.
\]

Step 2 If \( p < d \) then

\[
\int_{B_{\tilde{R}}^c(0)} \left| \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right|^p |x|^{p-d} \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{B_{\tilde{R}}(0)} \frac{|u|^p}{|x|^p \left( \log \frac{\tilde{R}}{|x|} \right)^p} \, dx
\]

Step 3 If \( p \neq d \) then

\[
\int_{B_{\tilde{R}}^c(0)} \left| \nabla \left( u |x|^{\frac{d-p}{p}} \right) \right|^p |x|^{p-d} \, dx \geq \left( \frac{p-1}{p} \right)^p \int_{B_{\tilde{R}}^c(0)} \frac{|u|^p}{|x|^p \left( \log \frac{\tilde{R}}{|x|} \right)^p} \, dx
\]
Sketch of proof of

\[ \int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, dx \, dx \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p \, dx, \]

\[ \rho(x) := \frac{1}{|x|^p(|\log|x||)^{p+1}}. \]

Lemma

Let \( d \geq 2 \) and \( 1 < p < \infty \). Assume also that \( B \neq 0 \) and let \( A \) be such that \( B = dA \). Let \( R > 1 \) be fixed and consider the annular domain \( \Omega_R := B_R(0) \setminus B_{\frac{1}{R}}(0) \). Then we define

\[ \mu_B(R) := \inf_{u \in W^{1,p}(\Omega_R), u \neq 0} \frac{\int_{\Omega_R} |(\nabla + iA)u|^p \, dx}{\int_{\Omega_R} |u|^p \, dx}. \]  

(19)

Then \( \mu_B \neq 0 \) on \((1, \infty)\).

PROOF = Steps 1-3 + Lemma + diamagnetic inequality + a localization argument
Finally, let us discuss the Aharonov–Bohm potential

\[ A_\beta(x) = \beta \frac{(x_2, -x_1)}{|x|^2}, \quad \beta \in \mathbb{R}, \]

in the case of dimension \( d = 2 \).
Main results

Aharonov-Bohm potentials

**Theorem (C.-Krejcirik-Lam-Laptev, NON 2024)**

Let $d = 2$, $1 \leq p < 2$ and let $A_{\beta}$ be given by (20). If $\beta \notin \mathbb{Z}$, then there exists a constant

$$
\lambda_{\beta}(p) > \left( \frac{2 - p}{p} \right)^{p}
$$

such that

$$
\int_{\mathbb{R}^2} |\nabla A_{\beta} u|^p \, dx \geq \lambda_{\beta}(p) \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \, dx, \quad \forall u \in C^\infty_c(\mathbb{R}^2).
$$

(21)

- This improves the sharp constant $\lambda_{\beta}(p)$ with respect to the non-magnetic case.
- The case $p = 2$: $\lambda(2) = \text{dist}(\beta, \mathbb{Z})^2$ (for test functions $u \in C^\infty_c(\mathbb{R}^2 \setminus \{0\})$ due to [Laptev-Weidl, 1998]
• **Open problem:** \( \lambda_\beta(p) =? \)

**Sketch of proof of (21).**

\[
\lambda(\beta, p) := \inf_{u \in W^{1,p}(0,2\pi), u(0) = u(2\pi)} \frac{\int_0^{2\pi} |\partial_\varphi u + i\beta u|^p \, d\varphi}{\int_0^{2\pi} |u|^p \, d\varphi}
\]  

(22)

Then we have \( \lambda(\beta, p) > 0 \) if \( \beta \notin \mathbb{Z} \).

• **Open problem:** \( \lambda(\beta, p) =? \)
\[
\left( \int_{\mathbb{R}^2} |\nabla_A u|^p \, dx \right)^{\frac{2}{p}} = \left( \int_0^\infty \int_0^{2\pi} \left[ |\partial_r u|^2 + \frac{|\partial_\varphi u + i\beta u|^2}{r^2} \right] \frac{p}{2} \, d\varphi \, dr \right)^{\frac{2}{p}}
\]

\[
= \left\| \frac{|\partial_r u|^2}{p} + \frac{|\partial_\varphi u + i\beta u|^2}{r^2} \right\|_{\frac{p}{2}}
\]

\[
\frac{p}{2} < 1 : \quad \geq \left\| \frac{|\partial_r u|^2}{p} \right\|_{\frac{p}{2}} + \left\| \frac{|\partial_\varphi u + i\beta u|^2}{r^2} \right\|_{\frac{p}{2}}
\]

\[
= \left\| \frac{|\partial_r u|^p}{p^1} \right\|_{\frac{2}{p}} + \left\| \frac{|\partial_\varphi u + i\beta u|^p}{r^p} \right\|_{1}
\]

\[
(Hardy+ (22)) \geq \left[ \left( \frac{2-p}{p} \right)^2 + \lambda (\beta, p) \right] \left( \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \, dx \right)^{\frac{2}{p}}
\]
A mean value magnetic $L^p$ inequality:

**Theorem (cf. Thm. 2.1.1, Aermark, PhD Thesis Stockholm 2014)**

Let $d = 2$, $1 < p < 2$ and let $A_\beta$ be given by (20). Then

$$
\left( \frac{\| \nabla A_\beta u \|_{L^p(\mathbb{R}^2)} + \| \nabla A_\beta \bar{u} \|_{L^p(\mathbb{R}^2)} }{2} \right)^p \geq \left( \frac{\sqrt{(2 - p)^2 + \beta^2 p^2}}{p} \right)^p \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \ dx
$$

(23)

for any $u \in C_\infty^c(\mathbb{R}^2)$.

- Notice that $|\nabla A_\beta \bar{u}| = |\nabla - A_\beta u|$, but not $|\nabla A_\beta \bar{u}| = |\nabla A_\beta u|$ in general, unless $u$ is real valued test function. In this latter case inequality (23) reduces to (21) with

$$
\lambda(p) = \left( \frac{\sqrt{(2 - p)^2 + \beta^2 p^2}}{p} \right) > \left( \frac{2 - p}{p} \right)^p , \text{ provided } \beta \neq 0.
$$

- Although this answers partially to our question the general case still remains open.
Some other recent developments on magnetic inequalities: [Fanelli-Krejcirik-Laptev-Vega 2020], [Lam-Lu, 2023], [Lu-Yang, 2024], [Fanelli-Kovarik], etc..

Thank you for your attention!