Magnetic Hardy inequalities in L^p setting

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The p-Laplacian $-\Delta_p$

- **The case $p = 2$:** $-\Delta_p = -\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$.

$$-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad p > 1.$$

The associated $L^2(\mathbb{R}^d)$ quadratic form h_p of $-\Delta_p$ is given by

$$h_p[u] = \int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{dx}, \quad \forall u \in \mathcal{D}(h_p) := W^{1,p}(\mathbb{R}^d). \tag{1}$$

and the sesquilinear form: for $u \in \mathcal{D}(h_p), v \in \mathcal{D}(h_{p'})$

$$\begin{aligned} h_p(u, v) &:= (-\Delta_p u, v)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (-\Delta_p u)\,\overline{v}\,\mathrm{dx} \\ &= \int_{\mathbb{R}^d} |\nabla u|^{p-2}\nabla u \cdot \nabla\overline{v}\,\mathrm{dx}. \end{aligned}$$

Dirichlet/Lagrange principle

A solution to the problem $-\Delta_p u = f$ +Dirichlet BC:

$$u \in W_0^{1,p} : \; h_p(u,v) =< f, v >, \quad \forall v \in W_0^{1,p}. \tag{2}$$

Define $E : W_0^{1,p} \to \mathbb{R}$,

$$E(\varphi) := \frac{1}{p} h_p[\varphi] - < f, \varphi > .$$

- u solution for (2) $\Leftrightarrow E(u) = \min_{\varphi \in W_0^{1,p}} E(v)$.
- E has a minimum if $h_p[\cdot]$ is "positive" enough (coercive, etc).

Some definitions

- We say that $-\Delta_p$ is a *non-negative* operator if

$$-\Delta_p \geq 0 \quad :\Longleftrightarrow \quad h_p[u] \geq 0, \quad \forall u \in \mathcal{D}(h_p);.$$

- $-\Delta_p$ is a subcritical operator $\quad :\Longleftrightarrow \quad -\Delta_p$ satisfies a Hardy-type inequality,
  i.e. there exists $V \in L^1_{\text{loc}}(\mathbb{R}^d)$, $V \neq 0$, such that $-\Delta_p \cdot \geq V|\cdot|^{p-2}\cdot$, in the sense of L^2 quadratic forms:

$$h_p[u] \geq \int_{\mathbb{R}^d} V|u|^p \, \mathrm{dx}, \quad \forall u \in W^{1,p}(\mathbb{R}^d).$$

- Otherwise, $-\Delta_p$ is a *critical* operator (i.e. there is NO Hardy inequality for $-\Delta_p$).

(Free) L^p-Hardy inequality

G. H. Hardy et. al. 1952, Cambridge Univ. Press.:
Let $d \geq 2$ and $1 \leq p < d$. If $u \in W^{1,p}(\mathbb{R}^d)$ then $u/|x| \in L^p(\mathbb{R}^d)$ and it satisfies

$$\int_{\mathbb{R}^d} |\nabla u|^p \, \mathrm{dx} \geq \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, \mathrm{dx}, \qquad \mu_{p,d} := \left(\frac{d-p}{p}\right)^p. \quad (3)$$

Moreover, the constant $\mu_{p,d}$ is optimal in the sense that (3) does not hold with any bigger constant.

Criticality versus sub-criticality of $-\Delta_p$

- $p < d \Rightarrow -\Delta_p$ is sub-critical (by Hardy Inequality): with $V(x) := \mu_{p,d}/|x|^p$, i.e.

$$- \Delta_p \cdot \geq \mu_{p,d} \, \frac{| \cdot |^{p-2} \cdot}{|x|^p} \tag{4}$$

- $p \geq d \Rightarrow -\Delta_p$ is critical:

Proposition

Let $p \geq d$. If $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is a non-negative potential such that

$$\int_{\mathbb{R}^d} |\nabla u|^p \, \mathrm{dx} \geq \int_{\mathbb{R}^d} V |u|^p \, \mathrm{dx}, \quad \forall u \in C_c^\infty(\mathbb{R}^d), \tag{5}$$

then $V = 0$ a.e. in $\mathbb{R}^d$.

- $p < d \Rightarrow H := -\Delta_p - \mu_{p,d} \frac{|\cdot|^{p-2}\cdot}{|x|^p}$? (Obviously $H \geq 0$).

$H := -\Delta_p - \mu_{p,d}\frac{|.|^{p-2}.}{|x|^p}$ is critical for $p < d$:

Proposition

Let $1 \leq p < d$*. If* $V \in L^1_{\text{loc}}(\mathbb{R}^d)$ *is a non-negative potential such that*

$$\int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{dx} - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \,\mathrm{dx} \geq \int_{\mathbb{R}^d} V|u|^p \,\mathrm{dx}, \quad \forall u \in C_c^\infty(\mathbb{R}^d), \tag{6}$$

then $V = 0$ *a.e. in* $\mathbb{R}^d$.

The magnetic p-Laplacian

Consider a smooth magnetic potential $A : \mathbb{R}^d \to \mathbb{R}^d$, The magnetic p-Laplacian is formally defined on $C_c^\infty(\mathbb{R}^d)$ by

$$\Delta_{A,p} u := \mathrm{div}_A(|\nabla_A u|^{p-2} \nabla_A u), \qquad (7)$$

where the magnetic gradient and magnetic divergence are given by

$$\nabla_A u := \nabla u + iA(x)u; \quad \mathrm{div}_A F := \mathrm{div} F + iA \cdot F, \qquad (8)$$

for any smooth vector field $F : \mathbb{R}^d \to \mathbb{C}^d$.

- Of course, if $A = 0$ then $\Delta_{A,p} = \Delta_p$,

The associated form $h_{A,p}$ of the magnetic p-Laplacian $\Delta_{A,p}$

For all $u \in \mathcal{D}(h_{A,p}) := \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|}$

$$h_{A,p}[u] := \int_{\mathbb{R}^d} |\nabla_A u|^p \,\mathrm{dx} = \int_{\mathbb{R}^d} |\nabla u + iA(x)u|^p \,\mathrm{dx},$$

where the norm $\|\cdot\|$ with respect to which the closure is taken is given by

$$\|u\| := \sqrt[p]{h_{A,p}[u] + \|u\|^p_{L^p(\mathbb{R}^d)}}.$$

- We extend the notions of subcriticality/criticality also to $-\Delta_{A,p}$.

• The magnetic field (2 diff. form):
$B : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ smooth, $dB = 0$
, i.e. $\exists$ A with $dA = B$, $(B_{ij} = A_{j,x_i} - A_{i,x_j})$

• The choice of A does not matter to much...
If $A, \tilde{A} : \mathbb{R}^d \to \mathbb{R}^d$ s.t. $dA = d\tilde{A} = B$ then there exists a scalar field $\phi : \mathbb{R}^d \to \mathbb{R}$ such that $A - \tilde{A} = d\phi$. It is easy to see that

$$\mathcal{D}(h_{A,p}) = \mathcal{D}(h_{\tilde{A},p}) \quad \text{and } h_{A,p}[\psi] = h_{\tilde{A},p}[\psi e^{i\phi}], \quad \forall \psi \in C_c^\infty(\mathbb{R}^d). \tag{9}$$

The diamagnetic inequality/Kato's inequality

- Diamagnetic inequality:

$$|\nabla_A u(x)| \geq |\nabla |u|(x)| \quad \text{a.e. } x \in \mathbb{R}^d, \ \forall u \in \mathcal{D}(h_{A,p}). \tag{10}$$

Then

$$\int_{\mathbb{R}^d} |\nabla_A u|^p \, \mathrm{dx} \geq \int_{\mathbb{R}^d} |\nabla |u||^p \, \mathrm{dx}$$

- So, all the inequalities valid for the standard p-Laplacian transfer to the magnetic p-Laplacian.

- BUT can we improve them ?

Theorem (C.-Krejcirik-Lam-Laptev, NON 2024)

Let $p \geq d$ and B be a smooth and closed magnetic field with $B \neq 0$. Then there exists a constant $C_{B,p,d} > 0$ such that for any magnetic potential A with $dA = B$ we have

$$\int_{\mathbb{R}^d} |\nabla_A u|^p \, \mathrm{dx} \geq C_{B,p,d} \int_{\mathbb{R}^d} \rho(x) |u|^p \, \mathrm{dx}, \quad \forall u \in \mathcal{D}(h_{A,p}), \tag{11}$$

where

$$\rho(x) := \frac{1}{|x|^d \left(|\log |x||^p + |x|^{p-d} \right)}.$$

- $p \geq d \Rightarrow -\Delta_{A,p}$ is sub-critical ($-\Delta_p$ is critical) !

Previously known results (the case $p = d = 2$)

- $B \neq 0$, with $\rho(x) = \frac{1}{1+|x|^2|\log|x||^2}$ in [C.-Krejcirik 2016]
- $B \neq 0$, under the additional condition $\frac{1}{2\pi}\int_{\mathbb{R}^2} {}^\star B \, \mathrm{dx} \notin \mathbb{Z}$ where ${}^\star B := B_{12}$ it was proved with $\rho(x) = \frac{1}{1+|x|^2}$ in [Laptev-Weidl, 1998].
- $B \neq 0$ + compactly supported + unbounded ρ, done in [Cassano-Franceschi-Krejcirik-Prandi, 2023]
- For Aharonov-Bohm type $A(x) = \psi\left(\frac{x}{|x|}\right)\frac{(-x_2,x_1)}{|x|^2}$ it was shown with $\rho(x) = 1/|x|^2$ also in [Laptev-Weidl, 1998].

Sketch of proof $\int_{\mathbb{R}^d} |\nabla_A u|^p \, \mathrm{dx} \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p \, \mathrm{dx}$, $\rho(x) := \frac{1}{|x|^d(|\log|x||^p + |x|^{p-d})}$.

Step 1 If $p \geq d$ then for all $u \in C_c^\infty(B_{\tilde{R}}(0))$

$$\int_{B_{\tilde{R}}(0)} |\nabla u|^p \, \mathrm{dx} \geq \left(\frac{p-1}{p}\right)^p \frac{1}{\tilde{R}^{p-d}} \int_{B_{\tilde{R}}(0)} \frac{|u|^p}{|x|^d \left(\log \frac{\tilde{R}}{|x|}\right)^p} \, \mathrm{dx}.$$

Step 2 If $p \neq d$ then

$$\int_{B^c_{\tilde{R}}(0)} |\nabla u|^p \, \mathrm{dx} \geq \left|\frac{d-p}{p}\right|^p \int_{B^c_{\tilde{R}}(0)} \frac{|u|^p}{|x|^p} \, \mathrm{dx}, \quad \forall u \in C_c^\infty(B^c_{\tilde{R}}(0)).$$

Step 3 If $p = d$ then $\forall u \in C_c^\infty(B^c_{\tilde{R}}(0))$

$$\int_{B^c_{\tilde{R}}(0)} |\nabla u|^d \, \mathrm{dx} \geq \left(\frac{d-1}{d}\right)^d \int_{B^c_{\tilde{R}}(0)} \frac{|u|^d}{|x|^d \left(\log \frac{\tilde{R}}{|x|}\right)^d} \, \mathrm{dx}$$

Skecth of proof of $\int_{\mathbb{R}^d} |\nabla_A u|^p \, \mathrm{dx} \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p \, \mathrm{dx}$, $\rho(x) := \frac{1}{|x|^d(|\log|x||^p + |x|^{p-d})}$.

Lemma

Let $d \geq 2$ and $1 < p < \infty$. Assume also that $B \neq 0$ and let A be such that $B = dA$. Let $R > 1$ be fixed and consider the annular domain $\Omega_R := B_R(0) \setminus B_{\frac{1}{R}}(0)$. Then we define

$$\mu_B(R) := \inf_{u \in W^{1,p}(\Omega_R), u \neq 0} \frac{\int_{\Omega_R} |(\nabla + iA)u|^p \, \mathrm{dx}}{\int_{\Omega_R} |u|^p \, \mathrm{dx}}. \tag{12}$$

Then $\mu_B \neq 0$ on $(1, \infty)$.

PROOF = Steps 1-3 + Lemma + diamagnetic inequality + a localization argument.

What about the sub-criticality of $H_A := -\Delta_{A,p} - \mu_{p,d}\frac{|\cdot|^{p-2}\cdot}{|x|^p}$? when $p < d$?

Theorem (C-Krejcirik-Lam-Laptev, NON 2024)

Let $2 \leq p < d$ and B be a smooth and closed magnetic field with $B \neq 0$. Then there exists a constant $c(p) > 0$ such that for any vector field A with $dA = B$ we have

$$\int_{\mathbb{R}^d} |\nabla_A u|^p \,\mathrm{dx} - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \,\mathrm{dx} \geq c(p) \int_{\mathbb{R}^d} \left|\nabla_A (u|x|^{\frac{d-p}{p}})\right|^p |x|^{p-d} \,\mathrm{dx}, \tag{13}$$

The constant $c(p)$ in (13) is explicitly given by

$$c(p) := \inf_{(s,t)\in\mathbb{R}^2\setminus\{(0,0)\}} \frac{\left[t^2 + s^2 + 2s + 1\right]^{\frac{p}{2}} - 1 - ps}{\left[t^2 + s^2\right]^{\frac{p}{2}}} \in (0,1] \tag{14}$$

- The optimal value of the constant $c(p)$ is an interesting open problem.
- The case $1 < p < 2$ remains open.

Proof of Theorem (ineq (13))

(Step 1) First we prove the free-magnetic cas:

Corollary

For any $2 \leq p < d$ *there exists a positive constant* $c(p)$ *such that*

$$\int_{\mathbb{R}^d} |\nabla u|^p \,\mathrm{dx} - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \,\mathrm{dx} \geq c(p) \int_{\mathbb{R}^d} \left|\nabla (u|x|^{\frac{d-p}{p}})\right|^p |x|^{p-d} \,\mathrm{dx}, \quad \forall u \tag{15}$$

- For $p = 2$ (15) becomes an identity with $c(2) = 1$ ([Brezis-Vazquez 1997]):

$$\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{dx} - \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \,\mathrm{dx} = \int_{\mathbb{R}^d} \left|\nabla (u|x|^{\frac{d-2}{2}})\right|^2 |x|^{2-d} \,\mathrm{dx}$$

which gives rise to many applications...

• With the help of Corollary.... $H := -\Delta_p - \mu_{p,d} \frac{|.|^{p-2}.}{|x|^p}$ is subcritical in bounded domains, etc.

Proof of Theorem (ineq (13))

(15) is a consequence of the identity

$$\int_{\mathbb{R}^d} |\nabla u|^p \, dx - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} dx = \int_{\mathbb{R}^d} C_p\Big(\underbrace{\nabla u}_{\alpha}, \underbrace{|x|^{-\frac{d-p}{p}} \nabla \left(u \, |x|^{\frac{d-p}{p}} \right)}_{\beta} \Big) \, \mathrm{dx}, \quad (16)$$

where

$$C_p(\alpha, \beta) = |\alpha|^p - |\alpha - \beta|^p - p \, |\alpha - \beta|^{p-2} \operatorname{Re}(\alpha - \beta) \cdot \overline{\beta}$$

Then easily show that

$$C_p(\alpha, \beta) \geq c_p \, |\beta|^p \, , \alpha, \beta \in \mathbb{C}^d,$$

$$\text{where } c(p) := \inf_{(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{\left[t^2 + s^2 + 2s + 1\right]^{\frac{p}{2}} - 1 - ps}{\left[t^2 + s^2\right]^{\frac{p}{2}}} \in (0, 1]$$

Proof of Theorem (ineq. (13))

(Step 2)

$$|\nabla_A u|^p - C_p\left(\nabla_A u, |x|^{-\frac{d-p}{p}} \nabla_A\left(u\,|x|^{\frac{d-p}{p}}\right)\right)$$
$$= |\nabla u|^p - C_p\left(\nabla u, |x|^{-\frac{d-p}{p}} \nabla\left(u\,|x|^{\frac{d-p}{p}}\right)\right).$$

where $C_p(\alpha,\beta) = |\alpha|^p - |\alpha-\beta|^p - p\,|\alpha-\beta|^{p-2}\,\mathrm{Re}\,(\alpha-\beta)\cdot\overline{\beta}$

(Step 3) In view of (Step 2) + (16):

$$\int_{\mathbb{R}^d} |\nabla_A u|^p\,dx - \mu_{p,d}\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}dx$$
$$= \int_{\mathbb{R}^d} C_p\left(\nabla_A u, |x|^{-\frac{d-p}{p}} \nabla_A\left(u\,|x|^{\frac{d-p}{p}}\right)\right)\,\mathrm{dx}, \quad (17)$$

(Step 4) $C_p(\alpha,\beta) \geq c_p\,|\beta|^p + (17) =$ END OF PROOF.

Theorem (C.-Krejcirik-Lam-Laptev, NON 2024)

Let $2 \leq p < d$ and B be a smooth and closed magnetic field with $B \neq 0$. Then there exists a constant $C_{B,p,d} > 0$ such that for any vector field A with $dA = B$ we have

$$\int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, \mathrm{dx} \geq C_{B,p,d} \int_{\mathbb{R}^d} \rho(x) |u|^p \, \mathrm{dx}, \quad \forall u \in \mathcal{D}(h_{A,p}), \tag{18}$$

where

$$\rho(x) := \frac{1}{|x|^p \left(1 + |\log |x||^p\right)}.$$

- $2 \leq p < d \Rightarrow H_A = -\Delta_{A,p} - \mu_{p,d} \frac{|\cdot|^{p-2}\cdot}{|x|^p}$ is sub-critical ($-\Delta_p - \mu_{p,d} \frac{|\cdot|^{p-2}\cdot}{|x|^p}$ is critical !)
- This improves our previous result in [C.-Krejcirik, 2016, Thm. 1.1] from L^2 to the L^p setting by obtaining also an unbounded weight ρ.

Sketch of proof

$$\int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p} \, \mathrm{dx} \, \mathrm{dx} \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p \, \mathrm{dx},$$

$$\rho(x) := \frac{1}{|x|^d(|\log|x||^p+1)}.$$

Step 2 If If $p < d$ then

$$\int_{B_{\tilde{R}}(0)} \left|\nabla\left(u|x|^{\frac{d-p}{p}}\right)\right|^p |x|^{p-d} \, \mathrm{dx} \geq \left(\frac{p-1}{p}\right)^p \int_{B_{\tilde{R}}(0)} \frac{|u|^p}{|x|^p \left(\log \frac{\tilde{R}}{|x|}\right)^p} \, \mathrm{dx}$$

Step 3 If $p \neq d$ then

$$\int_{B^c_{\tilde{R}}(0)} \left|\nabla\left(u|x|^{\frac{d-p}{p}}\right)\right|^p |x|^{p-d} \, \mathrm{dx} \geq \left(\frac{p-1}{p}\right)^p \int_{B^c_{\tilde{R}}(0)} \frac{|u|^p}{|x|^p \left(\log \frac{\tilde{R}}{|x|}\right)^p} \, \mathrm{dx}$$

Sketch of proof of $\int_{\mathbb{R}^d} |\nabla_A u|^p - \mu_{p,d} \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^p}\,\mathrm{dx}\,\mathrm{dx} \geq C \int_{\mathbb{R}^d} \rho(x)|u|^p\,\mathrm{dx}$, $\rho(x) := \frac{1}{|x|^p(|\log|x||^p+1)}$.

Lemma

Let $d \geq 2$ and $1 < p < \infty$. Assume also that $B \neq 0$ and let A be such that $B = dA$. Let $R > 1$ be fixed and consider the annular domain $\Omega_R := B_R(0) \setminus B_{\frac{1}{R}}(0)$. Then we define

$$\mu_B(R) := \inf_{u \in W^{1,p}(\Omega_R), u \neq 0} \frac{\int_{\Omega_R} |(\nabla + iA)u|^p\,\mathrm{dx}}{\int_{\Omega_R} |u|^p\,\mathrm{dx}}. \qquad (19)$$

Then $\mu_B \neq 0$ on $(1, \infty)$.

PROOF = Steps 1-3 + Lemma + diamagnetic inequality + a localization argument

Finally, let us discuss the Aharonov–Bohm potential

$$A_\beta(x) = \beta \frac{(x_2, -x_1)}{|x|^2}, \quad \beta \in \mathbb{R}, \tag{20}$$

in the case of dimension $d = 2$.

Aharonov-Bohm potentials

Theorem (C.-Krejcirik-Lam-Laptev, NON 2024)

Let $d = 2$, $1 \leq p < 2$ and let A_β be given by (20). If $\beta \notin \mathbb{Z}$, then there exists a constant

$$\lambda_\beta(p) > \left(\frac{2-p}{p}\right)^p$$

such that

$$\int_{\mathbb{R}^2} |\nabla_{A_\beta} u|^p \, \mathrm{dx} \geq \lambda_\beta(p) \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \, \mathrm{dx}, \quad \forall u \in C_c^\infty(\mathbb{R}^2). \tag{21}$$

- This improves the sharp constant $\lambda_\beta(p)$ with respect to the non-magnetic case.
- The case $p = 2$: $\lambda(2) = \mathrm{dist}(\beta, \mathbb{Z})^2$ (for test functions $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ due to [Laptev-Weidl, 1998]

- Open problem: $\lambda_\beta(p) =?$

Sketch of proof of (21).

$$\lambda\left(\beta,p\right) := \inf_{u\in W^{1,p}(0,2\pi),u(0)=u(2\pi)} \frac{\int\limits_0^{2\pi} \left|\partial_\varphi u + i\beta u\right|^p d\varphi}{\int\limits_0^{2\pi} |u|^p \, d\varphi} \tag{22}$$

Then we have $\lambda\left(\beta,p\right) > 0$ if $\beta \notin \mathbb{Z}$.

- Open problem: $\lambda\left(\beta,p\right) =?$

$$\left(\int_{\mathbb{R}^2} |\nabla_A u|^p \ \mathrm{dx}\right)^{\frac{2}{p}} = \left(\int_0^\infty \int_0^{2\pi} \left[|\partial_r u|^2 + \frac{|\partial_\varphi u + i\beta u|^2}{r^2}\right]^{\frac{p}{2}} d\varphi r dr\right)^{\frac{2}{p}}$$

$$= \left\| |\partial_r u|^2 + \frac{|\partial_\varphi u + i\beta u|^2}{r^2} \right\|_{\frac{p}{2}}$$

$$\frac{p}{2} < 1 : \quad \geq \left\| |\partial_r u|^2 \right\|_{\frac{p}{2}} + \left\| \frac{|\partial_\varphi u + i\beta u|^2}{r^2} \right\|_{\frac{p}{2}}$$

$$= \| |\partial_r u|^p \|_1^{\frac{2}{p}} + \left\| \frac{|\partial_\varphi u + i\beta u|^p}{r^p} \right\|_1^{\frac{2}{p}}$$

$$(\text{Hardy+ } (22)) \geq \left[\left(\frac{2-p}{p}\right)^2 + \underbrace{\lambda(\beta,p)^{\frac{2}{p}}}_{>0}\right] \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \ \mathrm{dx}\right)^{\frac{2}{p}}$$

A mean value magnetic L^p inequality:

Theorem (cf. Thm. 2.1.1, Aermark, PhD Thesis Stockholm 2014)

Let $d = 2$, $1 < p < 2$ *and let* A_β *be given by* (20). *Then*

$$\left(\frac{\|\nabla_{A_\beta} u\|_{L^p(\mathbb{R}^2)} + \|\nabla_{A_\beta} \overline{u}\|_{L^p(\mathbb{R}^2)}}{2}\right)^p \geq \left(\frac{\sqrt{(2-p)^2 + \beta^2 p^2}}{p}\right)^p \int_{\mathbb{R}^2} \frac{|u|^p}{|x|^p} \tag{23}$$

for any $u \in C_c^\infty(\mathbb{R}^2)$.

- Notice that $|\nabla_{A_\beta} \bar{u}| = |\nabla_{-A_\beta} u|$, but not $|\nabla_{A_\beta} \bar{u}| = |\nabla_{A_\beta} u|$ in general, unless u is real valued test function. In this latter case inequality (23) reduces to (21) with

$$\lambda(p) = \left(\frac{\sqrt{(2-p)^2 + \beta^2 p^2}}{p}\right) > \left(\frac{2-p}{p}\right)^p, \text{provided } \beta \neq 0.$$

- Although this answers partially to our question the general case still remains open.

Some other recent developments on magnetic inequalities: [Fanelli-Krejcirik-Laptev-Vega 2020], [Lam-Lu, 2023], [Lu-Yang, 2024], [Fanelli-Kovarik], etc..

Thank you for your attention !