A finer limit circle/limit point classification for Sturm–Liouville operators

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Based on joint work with

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Setting the stage with the Jacobi differential equation

Consider the Jacobi differential expression

$$\tau_{\alpha,\beta} = -(1-x)^{-\alpha}(1+x)^{-\beta}(d/dx)\big((1-x)^{\alpha+1}(1+x)^{\beta+1}\big)(d/dx), \qquad \alpha,\beta\in\mathbb{R},\ x\in(-1,1).$$

When $\alpha,\beta\in(-1,0)$ (the regular case since coefficient functions are integrable), the spectrum of any self-adjoint problem $\tau_{\alpha,\beta}y = zy$ consists of **discrete eigenvalues** which satisfy the Weyl asymptotics $\lambda_n \sim n^2$.

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In fact, this is true for the limit circle case $\alpha, \beta \in [0, 1)$ as well, which can be seen by using the regularization of Niessen and Zettl for any LCNO endpoint.

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What about the remaining so-called limit point case? Generically, the spectrum can have wildly different behavior now, though the above is still true for this example!

Singular Sturm–Liouville operators

Hypothesis 1

Let $(a,b) \subseteq \mathbb{R}$ and suppose that p, q, r are (Lebesgue) measurable functions on (a,b) such that the following items (i)–(iii) hold:

(i) $r > 0$ a.e. on (a,b), $r \in L^1_{loc}((a,b); dx)$.

(ii) $p > 0$ a.e. on (a,b), $1/p \in L^1_{loc}((a,b); dx)$.

(iii) q is real-valued a.e. on (a,b), $q \in L^1_{loc}((a,b); dx)$.

Given Hypothesis 1, we now consider Sturm–Liouville operators associated with the possibly **singular** (regular= L^1 above) differential expression

$$\tau = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q(x)\right] \quad \text{for a.e. } x \in (a,b) \subseteq \mathbb{R},$$

where one defines maximal (T_{max}), pre-minimal ($T_{min,0}$), and minimal (T_{min}) operators in the usual way.

Next, we recall Weyl's alternative:

Theorem (Weyl's Alternative)

Assume Hypothesis 1. Then the following alternative holds:

(i) **Limit circle:** For every $z \in \mathbb{C}$, all solutions u of $(\tau - z)u = 0$ are in $L^2((a,b); rdx)$ near b (resp., near a).

(ii) **Limit point:** For every $z \in \mathbb{C}$, there exists at least one solution u of $(\tau - z)u = 0$ which is not in $L^2((a,b); rdx)$ near b (resp., near a). In this case, for each $z \in \mathbb{C}\backslash\mathbb{R}$, there exists precisely one solution u_b (resp., u_a) of $(\tau - z)u = 0$ (up to constant multiples) which lies in $L^2((a,b); rdx)$ near b (resp., near a).

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(Think limit circle needs boundary conditions to choose the correct solution!)

When both endpoints are in the **limit circle nonoscillatory** case, the minimal operator is bounded from below and all self-adjoint extensions have spectrum consisting of **discrete eigenvalues** which satisfy the Weyl asymptotics

$$\lambda_n \sim \pi^2 n^2 \left(\int_a^b \sqrt{\frac{r(t)}{p(t)}} dt \right)^{-2}.$$

Principal and nonprincipal solutions

It is known that if the equation

$$\tau y = \lambda y, \qquad \lambda \in \mathbb{R} \tag{1}$$

is nonoscillatory near a (resp., b), then there exists an up to constant multiples unique solution u_a (resp., u_b) of (1) satisfying

$$\lim_{x \to a^+} \frac{u_a(x)}{v_a(x)} = 0 \qquad \left(\text{resp., } \lim_{x \to b-} \frac{u_b(x)}{v_b(x)} = 0\right)$$

for any linearly independent solution v_a (resp., v_b) of (1). In this case u_a (resp., u_b) is called the **principal solution** (think small) of (1) at a (resp., b), and v_a (resp. v_b) is called a **nonprincipal solution** (think large) of (1).

Main hypothesis

Hypothesis 2

Assume that the equation $\tau y = \lambda y$ is nonoscillatory at the endpoint a for **some** $\lambda \in \mathbb{R}$, and that for some $c \in (a, b)$

$$\int_a^c |u_a(x)v_a(x)r(x)|dx < \infty$$

where u_a, v_a are any principal, resp. nonprincipal solutions of $\tau y = \lambda y$ near the endpoint $x = a$.

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Example: Consider the Jacobi differential expression again, but only for $\alpha, \beta \in (0, \infty) \backslash \mathbb{N}$ and take $\lambda = 0$. Then the hypothesis holds at $x = -1$ as $r(x) = (1-x)^\alpha (1+x)^\beta$ and

$$u_{-1;\alpha,\beta}(0, x) = 1, \quad v_{-1;\alpha,\beta}(0, x) \underset{x \to -1}{\propto} (1+x)^{-\beta}, \quad \alpha, \beta \in (0, \infty) \backslash \mathbb{N}.$$

Nonexample: Laguerre differential expression

$$\tau_\alpha = -x^{1-\alpha} e^x \frac{d}{dx} x^\alpha e^{-x} \frac{d}{dx}, \qquad \alpha \in (0,\infty)\backslash\mathbb{N},\ x \in (0,\infty).$$

One can verify using confluent hypergeometric functions that (for $c \in (0,\infty)$)

$$u_{\infty;\alpha}(\lambda,x) \underset{x\to\infty}{\propto} x^\lambda \in L^2((c,\infty); x^{\alpha-1}e^{-x}dx),$$
$$v_{\infty;\alpha}(\lambda,x) \underset{x\to\infty}{\propto} x^{-\lambda-\alpha}e^x \notin L^2((c,\infty); x^{\alpha-1}e^{-x}dx),$$

so the hypothesis does not hold at ∞ since multiplying the lead behaviors by $r(x) = x^{\alpha-1}e^{-x}$ gives x^{-1}, which is not integrable near ∞.

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Note that associated self-adjoint problems still have discrete eigenvalues, but they now grow like n.

Trace class property

We say that τ satisfies the *trace class property at* $x = a$ (resp., at $x = b$) if and only if every self-adjoint realization T of $\tau|_{(a,c)}$ (resp., $\tau|_{(c,b)}$) for some (hence any) $c \in (a, b)$ has trace class resolvent $(T - zI)^{-1}$ for some (hence any) z in the resolvent set $\rho(T)$.

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We can show that Hypothesis 2 is surprisingly equivalent to the following:

Trace Class Hypothesis

Assume that τ satisfies the trace class property at a (resp., b) and that self–adjoint realizations of $\tau|_{(a,c)}$ (resp., $\tau|_{(c,b)}$) with $c \in (a, b)$ are semibounded.

Spectral parameter power series solution

We now construct a solution $\varphi(z, x)$ which is principal at $x = a$ and entire in $z \in \mathbb{C}$ via the infinite power series given by

$$\varphi(z, x) = \sum_{n=0}^{\infty} \varphi_n(x)(z - \lambda)^n, \qquad x \in (a, b),\ \lambda \in \mathbb{R},$$

and explore properties of the functions $\varphi_n(x)$ (and the implications). Note that $\varphi_n(x)$ clearly **depends on the choice** of λ, but it will turn out that the choice of $\lambda \in \mathbb{R}$ does not play any significant role.

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Assuming Hypothesis 2 holds at $x = a$, we define $\varphi_0(x) = u_a(x)$, where u_a is a principal solution to begin by constructing the series **about a point** $\lambda \in \mathbb{R}$ such that $\tau y = \lambda y$ is nonoscillatory. We then define iteratively

$$\varphi_n(x) = \int_a^x [u_a(t)v_a(x) - v_a(t)u_a(x)]\varphi_{n-1}(t)r(t)dt, \qquad x \in (a, b),$$

where v_a is a nonprincipal solution satisfying $W(v_a, u_a) = 1$.

Spectral parameter power series solution

Proposition

The infinite series constructed converges for all $x \in (a, b)$, $z \in \mathbb{C}$, and defines a function $\varphi(z, x)$ which is entire in z and satisfies

$$\tau\varphi(z, x) = z\varphi(z, x).$$

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The proof of this result involves carefully estimating the φ_n by using the previous integrability assumptions and then extending using uniqueness results.

One also needs the following important technical lemma:

Lemma

Let $D = U \times (c,d)$, where $a < c < d < b$ and $U \subseteq \mathbb{C}$ is open. Denote by $T_{max}^{(c,d)}$ the maximal operator associated with $\tau|_{(c,d)}$.

1. *Let $y \colon D \to \mathbb{C}$ be given such that $y(z,\,\cdot\,) \in dom(T_{max}^{(c,d)})$ for all $z \in U$. Moreover, assume that $\tau y(z,x) = zy(z,x)$ for $(z,x) \in D$, with $y(z,x)$ being holomorphic in z. Then the mapping $z \mapsto y(z,\,\cdot\,)$ is an $L^2((c,d); r(x)dx)$-valued holomorphic mapping and y has locally around $z_0 \in U$ a series expansion (where each y_n is in $dom(T_{max}^{(c,d)})$)*

$$y(z,x) = \sum_{n \geqslant 0} y_n(z_0,x)(z - z_0)^n, \tag{2}$$

$$(\tau - z_0)y_0 = 0, \qquad (\tau - z_0)y_n = y_{n-1}, \quad n > 0. \tag{3}$$

2. *Assume that $y \colon D \to \mathbb{C}$ has locally the series representation (2) in the space $L^2((c,d); r(x)dx)$ with $y_n \in dom(T_{max}^{(c,d)})$ satisfying (3) (in particular $z \to y(z,\,\cdot\,)$ is an $L^2((c,d); r(x)dx)$-valued analytic mapping). Then $y(z,\,\cdot\,)$ for $z \in U$ is in $dom(T_{max}^{(c,d)})$ and satisfies $\tau y(z,x) = zy(z,x)$ for $(z,x) \in D$.*

We now note some implications of the construction of φ.

Corollary

Assume Hypothesis 1 and that Hypothesis 2 holds at $x = a$. Then φ defined by the constructed series with iteratively defined coefficients satisfies

$$\lim_{x \to a^+} \frac{\varphi(z_1, x)}{\varphi(z_2, x)} = 1, \qquad z_1, z_2 \in \mathbb{C}.$$

In particular,

$$\tau y = \lambda y$$

is nonoscillatory **for all** $\lambda \in \mathbb{R}$ *and $\varphi(z, x)$ is principal at a for all $z \in \mathbb{R}$. Moreover, Hypothesis 2 is independent of $\lambda \in \mathbb{R}$, that is, if it holds for one $\lambda \in \mathbb{R}$ it will hold for all $\lambda \in \mathbb{R}$.*

This allows one to extend the constructed series to any $\lambda \in \mathbb{R}$.

Second linearly independent solution

Corollary

Assume Hypothesis 1 holds and let φ satisfy the normalization

$$\lim_{x\to a^+} \frac{\varphi(z_1,x)}{\varphi(z_2,x)} = 1, \qquad z_1, z_2 \in \mathbb{C}.$$

Then any entire solution θ satisfying $W(\theta(z,\,\cdot\,),\varphi(z,\,\cdot\,)) = 1$ will also satisfy

$$\lim_{x\to a^+} \frac{\theta(z_1,x)}{\theta(z_2,x)} = 1, \qquad \lim_{x\to a^+} W(\theta(z_2,x),\varphi(z_1,x)) = 1, \qquad z_1, z_2 \in \mathbb{R}.$$

Corollary

Consider the power series expansion of θ with respect to z,

$$\theta(z,x) = \sum_{n=0}^{\infty} \theta_n(x)(z-\lambda)^n, \quad \lambda \in \mathbb{R}.$$

Then

$$\lim_{x \to a^+} \frac{\theta_n(x)}{\theta_0(x)} = 0, \qquad \textit{for all } n \geqslant 1.$$

By the previous technical lemma, we know that $(\tau - \lambda)\theta_0(x) = 0$ and $(\tau - \lambda)\theta_n(x) = \theta_{n-1}(x)$ for $n \geqslant 1$, hence,

$$\begin{aligned} \theta_n(x) &= A_n \varphi_0(x) \\ &\quad + \theta_0(x) \int_a^x \varphi_0(t)\theta_{n-1}(t)r(t)dt + \varphi_0(x) \int_x^c \theta_0(t)\theta_{n-1}(t)r(t)dt. \end{aligned}$$

Definition (Regularization index)

Assume Hypothesis 1 and that Hypothesis 2 holds at $x = a$. Let φ be given via the constructed series and take any entire nonprincipal solution $\theta(z, x)$ satisfying $W(\theta(z,\cdot), \varphi(z,\cdot)) = 1$ which is real for $z \in \mathbb{R}$. Then we define the **regularization index** $\boldsymbol{\ell_a} \in \mathbb{N}_0 \cup \{+\infty\}$ of τ at the endpoint $x = a$ to be smallest nonnegative integer ℓ_a such that

$$\lim_{x\to a^+} \frac{\varphi_0(x)}{\theta_n(x)} = 0, \qquad \text{for all} \quad n \in \{0, \dots, \ell_a\}$$
$$\lim_{x\to a^+} \frac{\varphi_0(x)}{\theta_{\ell_a+1}(x)} \neq 0,$$

in case such an integer exists. Otherwise

$$\lim_{x\to a^+} \frac{\varphi_0(x)}{\theta_n(x)} = 0, \qquad \text{for all} \quad n \geqslant 0,$$

and we set $\ell_a = +\infty$.

The regularization index ℓ_a is independent of the choice of $\lambda \in \mathbb{R}$ in the series.

Some results for the regularization index

Theorem

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Proposition

Assume Hypothesis 1, that Hypothesis 2 holds at $x = a$, and $\ell_a < \infty$. Then $\theta_{\ell_a}(x) \in L^2((a, c); r(x)dx)$ for any $c \in (a, b)$. (In fact, $\theta_{\ell_a}\theta_0 \in L^1((a, c); r(x)dx)$.)

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Interestingly, it can happen that $\theta_n(x) \in L^2((a, c); r(x)dx)$ for $n < \ell_a < \infty$.

Question

If $\ell_a < \infty$, can one characterize for what $n < \ell_a$ one has $\theta_n \in L^2((a, c); r(x)dx)$?

Returning to the Jacobi example

Let's once again consider the Jacobi differential expression for $\alpha, \beta \in (0, \infty)\backslash\mathbb{N}$:

$$\varphi_{0;\alpha,\beta}(0, x) \underset{x\to-1}{\propto} 1,$$
$$\theta_{n;\alpha,\beta}(0, x) \underset{x\to-1}{\propto} (1 + x)^{-\beta+n}, \quad \alpha, \beta \in (0, \infty)\backslash\mathbb{N}.$$

Thus, one concludes that $\ell_{-1} = \lfloor\beta\rfloor$. Similarly, $\ell_{+1} = \lfloor\alpha\rfloor$. Therefore, we recover the well-known fact that the Jacobi equation is in the limit-circle case at both endpoints if $\alpha, \beta \in (0, 1)$ since $\ell_{-1} = 0 = \ell_{+1}$ for these choices.

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Furthermore,

$$\theta_{n;\alpha,\beta} \in L^2((-1,1);(1-x)^\alpha(1+x)^\beta dx) \text{ for } n > \max\{(\alpha-1)/2, (\beta-1)/2\}$$

(though at each endpoint one only needs the inequality for one parameter).

A perturbation result

Theorem

Assume Hypothesis 1 and that Hypothesis 2 holds at $x = a$. Choose u_a, v_a to be principal resp. nonprincipal solutions of $\tau f = \lambda f$, $\lambda \in \mathbb{R}$ satisfying $W(v_a, u_a) = 1$ and define the perturbed Sturm–Liouville differential expression

$$\tau^{per} = \frac{1}{r(x)}\left[-\frac{d}{dx}p(x)\frac{d}{dx} + q^{per}(x)\right] \text{ for a.e. } x \in (a, b) \subseteq \mathbb{R},$$

where the potential $q^{per} = q + q_0$ satisfies $q_0 \in L^1_{loc}((a, b); dx)$ and

$$\int_a^b |u_a(x)v_a(x)q_0(x)|dx < \infty.$$

Then τ^{per} will satisfy Hypothesis 2 and the regularization indices of τ and τ^{per} at $x = a$ coincide, that is, $\ell_a^{per} = \ell_a$.

The previous perturbation condition generalizes the condition in Kostenko, Sakhnovich, and Teschl '10 for perturbed spherical Schrödinger operators. In fact, the unperturbed case treated there corresponds to

$$\tau = H_l = -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2}, \qquad x \in (0,1), \quad l \geqslant -1/2.$$

Entire principal and nonprincipal solutions are given in terms of Bessel functions and satisfy

$$\varphi(z,x) \propto x^{l+1}, \quad x \to 0^+,$$
$$\theta(z,x) \propto \begin{cases} x^{-l}, & l > -1/2 \\ x^{1/2}\ln(x), & l = -1/2 \end{cases}, \quad x \to 0^+.$$

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Thus for $\varphi\theta q_0$ to be integrable we need to require $q_0 \in L^1_{loc}((0,1); dx)$ and

$$\begin{cases} xq_0(x) \in L^1((0,1); dx), & l > -1/2, \\ x\ln(x)q_0(x) \in L^1((0,1); dx), & l = -1/2. \end{cases}$$

This allows for the inclusion of the classical Coulomb case $q_0(x) = C/x$.

Canonical normalization

Definition

Assume Hypothesis 1 and that Hypothesis 2 holds at $x = a$. We call the system of entire solutions φ, θ of $\tau y = zy$ **canonically normalized** (at $x = a$) if and only if

1. $\varphi(z, \cdot)$ is principal at $x = a$ for all $z \in \mathbb{R}$ and $\lim_{x\to a^+} \frac{\varphi(z_1,x)}{\varphi(z_2,x)} = 1$ for $z_1, z_2 \in \mathbb{R}$;
2. $W(\theta(z, \cdot), \varphi(z, \cdot)) = 1$ for all $z \in \mathbb{R}$;
3. $\lim_{x\to a^+} \frac{\theta_n(x)}{\varphi_0(x)} = 0$ for all $n > \ell_a$.

The canonical normalization condition is equivalent to the recursion

$$\theta_n(x) = \int_a^x \left[\theta_0(x)\varphi_0(t) - \varphi_0(x)\theta_0(t)\right]\theta_{n-1}(t)r(t)dt, \qquad \text{for all} \quad n > \ell_a.$$

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Our notion of canonically normalized generalizes with the notion of 'Frobenius solution' introduced in Kostenko and Teschl '11.

Theorem

Assume Hypothesis 1, that Hypothesis 2 holds at $x = a$ and let φ, θ be canonically normalized. Then for all $k \geqslant 0$ we have

$$\lim_{x \to a^+} \frac{\varphi_k(x)}{\theta_{\ell_a+k}(x)} = 0, \qquad \lim_{x \to a^+} \frac{\theta_{\ell_a+k+1}(x)}{\varphi_k(x)} = 0,$$

and

$$\lim_{x \to a^+} \frac{\varphi_{k+1}(x)}{\varphi_k(x)} = 0, \qquad \lim_{x \to a^+} \frac{\theta_{k+1}(x)}{\theta_k(x)} = 0.$$

This can be summarized as stating there is interlacing growth such that

$$|\theta_0(x)| \gg \cdots \gg |\theta_{\ell_a}(x)| \gg |\varphi_0(x)| \gg |\theta_{\ell_a+1}(x)| \gg |\varphi_1(x)| \gg \ldots,$$

where $f \gg g$ is shorthand for $\lim_{x \to a^+} \frac{g}{f} = 0$.

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This can be summarized as stating there is interlacing growth such that

$$|\theta_0(x)| \gg \cdots \gg |\theta_{\ell_a}(x)| \gg |\varphi_0(x)| \gg |\theta_{\ell_a+1}(x)| \gg |\varphi_1(x)| \gg \ldots,$$

where $f \gg g$ is shorthand for $\lim_{x\to a^+} \frac{g}{f} = 0$.

The last ratios in the theorem being zero show both series are ‘well-behaved’ in the sense that later terms can be viewed small corrections for $x \to a^+$.

Liouville and Darboux transformations

Let us now turn to an application of the regularization index. For this we will make additional regularity assumptions on our Sturm–Liouville differential expression.

Hypothesis 3

In addition to Hypotheses 1, assume further that $(pr), (pr)'/r \in AC_{loc}((a,b))$ and $(pr)\big|_{(a,b)} > 0$.

With these assumptions the Sturm–Liouville differential expression can be transformed into an equivalent Schrödinger differential expression given by

$$-\frac{d^2}{dX^2} + Q(X), \qquad X \in (A,B) \subseteq \mathbb{R},$$

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To avoid unnecessary notation, we will for the rest of this talk assume that $p \equiv 1$ and $r \equiv 1$, so that $\tau = -\frac{d^2}{dx^2} + q$ is already in Schrödinger form.

Let us now assume that ψ is a positive solution of $\tau y = \lambda y$, meaning that

$$\tau\psi = \lambda\psi, \quad \text{with} \quad \psi(x) > 0, \quad x \in (a, b).$$

Such ψ, often called the seed function, exists if and only if τ is nonoscillatory at both endpoints, which we will assume from now on. Then as a formal differential expression, τ can be factorized as follows

$$\begin{aligned}\tau = -\frac{d^2}{dx^2} + q &= \Big(\frac{d}{dx} + \frac{\psi'}{\psi}\Big)\Big(-\frac{d}{dx} + \frac{\psi'}{\psi}\Big) + \lambda \\ &= B_\psi A_\psi + \lambda.\end{aligned}$$

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We define the associated Darboux transformed differential expression by

$$\begin{aligned} \widehat{\tau} = A_\psi B_\psi + \lambda &= \Big(-\frac{d}{dx} + \frac{\psi'}{\psi}\Big)\Big(\frac{d}{dx} + \frac{\psi'}{\psi}\Big) + \lambda \\ &= -\frac{d^2}{dx^2} + \widehat{q}, \end{aligned}$$

where

$$\widehat{q} = q - 2\frac{d}{dx}\Big(\frac{\psi'}{\psi}\Big),$$

and say that $\widehat{\tau}$ is obtained from τ via a Darboux transform with seed function ψ.

Regularization index after Darboux transform

Theorem

Assume Hypothesis 1 and that Hypothesis 2 holds for τ at the endpoint $x = a$. Then Hypothesis 2 holds for $\widehat{\tau}$ at the endpoint $x = a$ as well. Moreover, if ℓ_a is the regularization index of τ at $x = a$ then the regularization index $\widehat{\ell}_a$ of $\widehat{\tau}$ at $x = a$ satisfies

1. $\widehat{\ell}_a = \ell_a + 1$ *if the seed function ψ is principal at $x = a$,*
2. *and*

$$\widehat{\ell}_a = \begin{cases} 0 & \textit{if } \ell_a = 0, \\ \ell_a - 1 & \textit{if } \ell_a > 0, \end{cases}$$

   if the seed function ψ is nonprincipal at $x = a$.

Here, we interpret $\infty \pm 1$ as ∞.

The appropriateness of the terminology ‘regularization index’

The following quantifies how far a limit-point endpoint is away from being transformed to a limit-circle endpoint (corresponding to $\ell_a = 0$).

Theorem

Assume Hypothesis 1 and let τ *be a Schrödinger differential expression which is nonoscillatory at both endpoints. Then* τ *can be transformed via a finite series of Darboux transform to a Schrödinger differential expression* $\widetilde{\tau}$ *which is in the limit-circle case at* $x = a$ *if and only if Hypothesis 2 holds at* $x = a$ *and* $\ell_a < \infty$.

In this case, the minimal number of Darboux transform is ℓ_a *and is achieved if the seed functions are always chosen to be nonprincipal at* $x = a$.

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Jacobi example once more: Recall that in the Jacobi example we had $\ell_{-1} = \lfloor |\beta| \rfloor$ and $\ell_{+1} = \lfloor |\alpha| \rfloor$. This implies the problem can be regularized after a Liouville transform and $\max\{\lfloor |\alpha| \rfloor, \lfloor |\beta| \rfloor\}$ Darboux transforms.

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This means it has the same spectrum (almost) as an associated regular problem and Weyl asymptotics hold just as in the (quasi-)regular case!

Weyl asymptotics without additional assumptions

While our proof of the previous results regarding Weyl asymptotics relies on using the Liouville and Darboux transforms (hence requires Hypothesis 3), if the regularization index is finite, the Weyl asymptotics will still hold without Hypothesis 3 by results of Langer and Woracek '23.

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In fact, they introduce an index for canonical systems that can be compared to ours by answering the previous question about when $\theta_n \in L^2((a, c); r(x)dx)$.

Final remarks and applications

The Trace Class Hypothesis is also the most general condition under which the series constructed for φ is well-behaved in the sense that without it, higher order terms $\widetilde{\varphi}_n$ cannot be viewed as small corrections for $x \to a^+$, despite the series being convergent and φ being entire.

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Thanks!

A last example

Consider the half-line Schrödinger differential expression

$$\tau_\alpha = -\frac{d^2}{dx^2} + x^\alpha, \quad \alpha > 0, \; x \in (0, \infty).$$

Note $\tau_\alpha y = zy$ is LCNO for all $\lambda \in \mathbb{R}$ at $x = \infty$, and has solutions

$$\varphi_{0;\alpha}(0,x) = x^{1/2} K_{\frac{1}{\alpha+2}}\left(\left[2x^{(2+\alpha)/2}\right]/(\alpha+2)\right) \underset{x\to\infty}{\propto} x^{-\alpha/4} e^{-x^{(2+\alpha)/2}},$$

$$\theta_{0;\alpha}(0,x) = -(2/(a+2))x^{1/2} I_{\frac{1}{\alpha+2}}\left(\left[2x^{(2+\alpha)/2}\right]/(\alpha+2)\right) \underset{x\to\infty}{\propto} x^{-\alpha/4} e^{x^{(2+\alpha)/2}},$$

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Thus Hypothesis 2 holds if and only if $\alpha > 2$. Furthermore, when iteratively constructing $\theta_\alpha(z,x)$, the terms $\theta_{n;\alpha}(0,x)$ will always include an exponentially growing term for $x \to \infty$, so $\ell_\infty = \infty$ for $\alpha > 2$.

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Note the eigenvalue asymptotics for the problem $\tau_\alpha y = zy$ are

$$\lambda_n \underset{n\to\infty}{\sim} \left[\frac{2\pi^{1/2}\alpha\Gamma\left(\frac{3}{2}+\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\right]^{2\alpha/(\alpha+2)} n^{2\alpha/(\alpha+2)}, \quad \alpha > 0,$$

which satisfies any growth n^γ for any $\gamma \in (1, 2)$ when $\alpha > 2$.

Regularization of Niessen and Zettl

Many regular Sturm-Liouville problems meet these assumptions and some singular problems can be regularized with the new associated regular problem meeting these assumptions using the following regularization result from Niessen and Zettl:

For some $\lambda_0 \in \mathbb{R}$ there exists a nonprincipal solution $\widehat{u}(\lambda_0, \cdot\,) \in \operatorname{dom}(T_{max})$ such that the nonoscillatory quasi-regular problem,

$$-(p(x)y'(z,x))' + q(x)y(z,x) = zr(x)y(z,x)$$

on (a, b) satisfying Hypothesis 1 can be transformed into the regular problem,

$$-(P(x)v'(z,x))' + Q(x)v(z,x) = zR(x)v(z,x)$$

with

$$P(x) = [\widehat{u}(\lambda_0,x)]^2 p(x), \quad R(x) = [\widehat{u}(\lambda_0,x)]^2 r(x),$$
$$Q(x) = r(x)\widehat{u}(\lambda_0,x)(\tau\widehat{u})(\lambda_0,x), \text{ satisfying } 1/P, Q, R \in L^1((a,b);dx).$$

In particular, if $y(z, \cdot\,)$ is a solution of the quasi-regular problem, then one has that $v(z, \cdot\,) = y(z, \cdot\,)/\widehat{u}(\lambda_0, \cdot\,)$ is a solution of the associated regular problem.