Algebraic geometry, analysis and combinatorics in the study of periodic graph operators

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A simple periodic graph

- $\mathcal{G} = \mathbb{Z}^d$
- Fix q_j, $j = 1, 2, \cdots, d$.
- Group: $G = q_1\mathbb{Z} \oplus q_2\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$
- Fundamental domain: $W = \mathbb{Z}^d / G = \{n = (n_1, n_2, \cdots, n_d) : 1 \leq n_j \leq q_j\}$.
- $\mathcal{G} = \bigcup_{g \in G}(g + W)$

Discrete periodic Schrödinger operators

- Adjacency matrices + periodic potentials
- Let Δ be the discrete Laplacian on $\mathbb{Z}^d$: $u(n), n \in \mathbb{Z}^d$,

$$(\Delta u)(n) = \sum_{||n'-n||=1} u(n'),$$

  where $||n|| = \sum_{i=1}^{d} |n_i|$ for $n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$.
- Periodic potentials V: $V(n+g) = V(n)$ for all $n \in \mathbb{Z}^d$ and $g \in G$.
- The discrete periodic Schrödinger operator $H_0 = \Delta + V$:

$$(H_0 u)(n) = (\Delta u)(n) + V(n)u(n), n \in \mathbb{Z}^d.$$

Periodic graphs and periodic graph operators

- $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$, $\mathcal{V}$: vertices, $\mathcal{E}$: edges
- Group: G
- Fundamental domain: $\mathcal{G}/G$ (finite).
- Invariants with respect to the group

An example: $d = 1$

- For a vector $u(n)$, $n \in \mathbb{Z}$,

$$(\Delta u)(n) = u(n+1) + u(n-1)$$

.

- $V = \{V(n)\}_{n\in\mathbb{Z}}$ is the potential.
- q_1 periodic: $V(n+q_1) = V(n)$ for any $n \in \mathbb{Z}$
-

$$\Delta + V = \begin{pmatrix} \ddots & \ddots & 0 & 0 & \cdots & 0 \\ \ddots & V_1 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & V_2 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & 0 & 1 & V_{q_1} & \ddots \\ 0 & \dots & 0 & 0 & \ddots & \ddots \end{pmatrix}$$

Floquet transform in one dimension: $k \in [0,1]$

- $H_0 = \Delta + V \cong \oplus_{k\in[0,1]} D_V(k)$, where

$$D_V(k) = \begin{pmatrix} V_1 & 1 & 0 & 0 & e^{-2\pi ik} \\ 1 & V_2 & 1 & \ddots & 0 \\ 0 & 1 & V_3 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ e^{2\pi ik} & \dots & 0 & 1 & V_{q_1} \end{pmatrix}$$

- $\Delta + V$ is on $\mathbb{Z}$ (Hilbert Space $\ell^2(\mathbb{Z})$)
- $\oplus_{k\in[0,1]} D_V(k)$
  Hilbert space $L^2(\mathbb{T}, \mathbb{C}^{q_1})$ ($\sum_{n=1}^{q_1} \int_{\mathbb{T}} |f(n,k)|^2 dk < \infty$); Operators: $D_V(k) f(\cdot, k)$

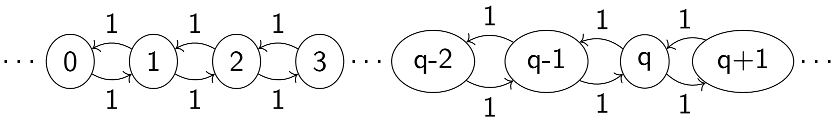
$D_V(k)$ in one dimension

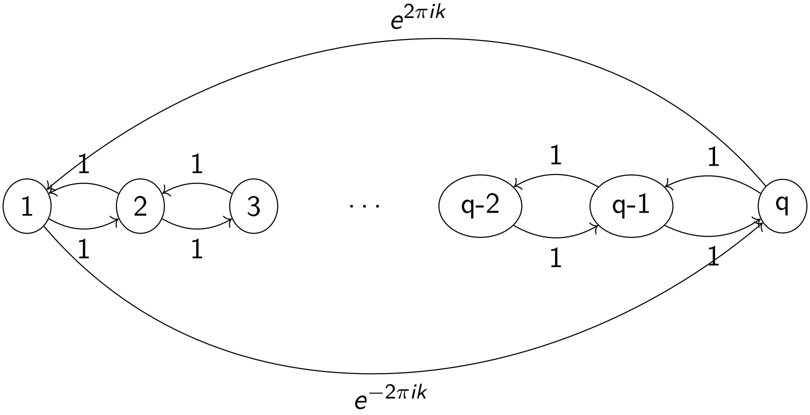
- Floquet-Bloch boundary condition

$$u(n+q_1)=e^{2\pi ik}u(n), n\in\mathbb{Z}. \tag{1}$$

- By writing out $H_0=\Delta+V$ on $\{u(n)\}, n=1,2,\cdots,q_1$, we obtain $D_V(k)$.

Combinatorics: Laplacian on q periodic lattice on $\mathbb{Z}$





Discrete Floquet transform

- Fundamental domain W:

$$W = \{n = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : 1 \leq n_j \leq q_j, j = 1, 2, \cdots, d\}.$$

- Cardinality of W: $Q = q_1 q_2 \cdots q_d$
- Floquet-Bloch boundary condition

$$u(n + q_j \mathbf{e}_j) = e^{2\pi i k_j} u(n), j = 1, 2, \cdots, d. \tag{2}$$

- By writing out $H_0 = \Delta + V$ as acting on the Q dimensional space $\{u(n), n \in W\}$, $\Delta + V$ with (2) translates into a $Q \times Q$ matrix $D_V(k)$.
- $\Delta + V$ is unitary equivalent to $\bigoplus_{k \in \mathbb{T}^d} D_V(k)$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

$D_V(k)$: $q_1\mathbb{Z} \oplus q_2\mathbb{Z}$ periodic lattice on $\mathbb{Z}^2$

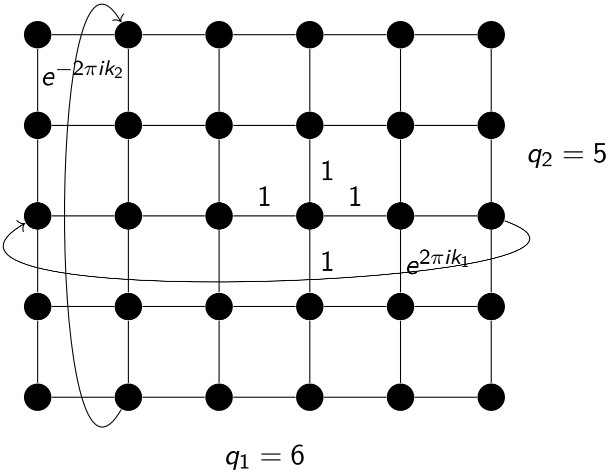


Figure: $\mathbb{Z}^2$

- Let $P_V(k,\lambda) = \det(D_V(k) - \lambda I)$ (characteristic function).
- Many problems related to periodic Schrödinger operators: study $P_V(k,\lambda)$

- Let $z_j = e^{2\pi i k_j}, j = 1, 2, \cdots, d$. $z = (z_1, z_2, \cdots, z_d)$ and $k = (k_1, k_2, \cdots, k_d)$.
- $$\mathcal{D}_V(z_1) = \begin{pmatrix} V_1 & 1 & 0 & 0 & z_1^{-1} \\ 1 & V_2 & 1 & \ddots & 0 \\ 0 & 1 & V_3 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ z_1 & \dots & 0 & 1 & V_{q_1} \end{pmatrix}$$
- $\mathcal{D}_V(z) = D_V(k)$.
- $\mathcal{P}_V(z, \lambda) = \det(\mathcal{D}_V(z) - \lambda I)$.
- $\mathcal{P}_V(z, \lambda)$ is a Laurent polynomial of λ and $z_1, z_2, \cdots, z_d$.

Algebraic properties vs spectral properties

- Real potentials V. Denote by $\lambda_V^j(k)$ the spectral band functions: eigenvalues of $D_V(k)$, $k \in [0,1]^d$:

$$\lambda_V^1(k) \leq \lambda_V^2(k) \leq \cdots \leq \lambda_V^Q(k)$$

- Flat band: $\lambda_V^j(k) \equiv \lambda_0$
- $\mathcal{P}_V(z,\lambda)$ has a factor of $\lambda - \lambda_0$
- λ_0 is an eigenvalue of $\Delta + V$, namely $\exists u \in \ell^2(\mathbb{Z}^d)$ such that $(\Delta + V)u = \lambda_0 u$

Irreducibility for lattices $q_1\mathbb{Z} \oplus \cdots \oplus q_d\mathbb{Z}$

Theorem 1 (L. GAFA 2022)

Let $d \geq 3$. Then for any $\lambda \in \mathbb{C}$, the Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z) is irreducible.

Theorem 2 (L. GAFA 2022)

Let $d = 2$. Then the Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z) is irreducible for any $\lambda \in \mathbb{C}$ except for $\lambda = [V]$. Moreover, $\mathcal{P}_V(z, [V])$ has at most two non-trivial irreducible factors.

When $d = 2$, for a constant function V, $\mathcal{P}_V(z, [V])$ has exactly two irreducible components.

Theorem 3 (L. GAFA 2022)

The Laurent polynomial $\mathcal{P}_V(z, \lambda)$ (as a function of z and λ) is irreducible.

Proof of two conjectures

- Bloch variety: $B(V) = \{(k, \lambda) \in \mathbb{C}^{d+1} : P_V(k, \lambda) = 0\}$
- Conjecture 1: Bloch variety is irreducible (modulo periodicity)
- Fermi variety: $F_\lambda(V) = \{k \in \mathbb{C}^d : P_V(k, \lambda) = 0\}$
- Conjecture 2: Fermi varieties $F_\lambda(V)$ are irreducible (modulo periodicity) for all λ but finitely many λ.
- The two conjectures have been mentioned in many articles [Knörrer-Trubowitz 1990, Bättig-Knörrer-Trubowitz 1991, Bättig 1992, Kuchment-Vainberg 2000, Kuchment 2016]

Previous results: $d = 2, 3$

- $d = 2$, the Bloch variety $(\mathcal{P}_V(z, \lambda))$ is irreducible [Bättig 1988].
- $d = 2$, the Fermi variety is irreducible except for finitely many values of λ [Gieseker-Knörrer-Trubowitz 1993]
- $d = 3$, the Fermi variety is irreducible for every λ [Bättig 1992].
- Previous approaches: construction of toroidal and directional compactifications of Fermi and Bloch varieties.

Further developments: more general lattices/polynomials

- Fillman-L.-Matos JFA 2022
- Fillman-L.-Matos JFA 2024
- Faust-Garcia preprint 2023

More about algebraic properties vs spectral properties

- (Ir)reducibility of Fermi variety is related the embedded eigenvalue problems [Kuchment-Vainberg CPDE 2000], [Kuchment-Vainberg CMP 2006], [Shipman CMP 2014], [L. GAFA 2022]
- Irreducibility of Bloch variety is related to quantum ergodicity, [L. JDE 2022 and Mckenzie-Sabri CMP 2023]
- properties of spectral band functions [L. GAFA 2022 and Filonov-Kachkovskiy CMP 2024]
- inverse problems: IDS [Gieseker-Knörrer-Trubotwitz 1993 Book], isospectrality [L. CPAM 2024 and CMP 2023] and Borg's Theorem [L. preprint 2023]

Spectral bands

Real potentials V

- Eigenvalues of $D_V(k)$, $k \in [0,1]^d$:

$$\lambda_V^1(k) \leq \lambda_V^2(k) \leq \cdots \leq \lambda_V^Q(k)$$

- Spectral band functions: $\lambda_V^m(k)$, $m = 1, 2, \cdots, Q$.
-

$$\sigma(\Delta + V) = \bigcup_{m=1}^{Q} [a_m^V, b_m^V], \tag{3}$$

  where $[a_m^V, b_m^V]$ is the range of $\lambda_V^m(k)$.

Applications: embedded eigenvalues

Perturbed periodic operators:

$$H = \Delta + V + v, \tag{4}$$

where V is a real periodic potential and v is a decaying function on $\mathbb{Z}^d$.
Spectral bands:

$$\sigma(\Delta + V) = \bigcup [a_m^V, b_m^V], \sigma_p(\Delta + V) = \emptyset.$$

Theorem 4 (L. GAFA 2022)

If there exist constants $C > 0$ and $\gamma > 1$ such that

$$|v(n)| \leq Ce^{-|n|^\gamma}, \tag{5}$$

then $H = \Delta + V + v$ does not have any embedded eigenvalues, i.e., for any $\lambda \in \bigcup (a_m^V, b_m^V)$, λ is not an eigenvalue of H.

Special case

Corollary 5

Assume $|v(n)| \leq Ce^{-|n|^\gamma}$ *for some* $C > 0$ *and* $\gamma > 1$*. Then* $\sigma_p(\Delta + v) \cap (-2d, 2d) = \emptyset$ *(no embedded eigenvalues).*

- $\sigma(\Delta) = [-2d, 2d]$.
- Compactly support v: [Isozaki-Morioka 2014]

Proof of Corollary 5

- Eigen-equation

$$(\Delta u)(n) + v(n)u(n) = \lambda u(n), n \in \mathbb{Z}^d.$$

- Prove by contradiction: $\lambda \in (-2d, 2d)$ and $u \in \ell^2(\mathbb{Z}^d)$

By the Fourier transform, one has that

$$h_0(x)u(x) + \psi(x) = \lambda u(x). \tag{6}$$

- $$h_0(x) = 2\sum_{j=1}^{d} \cos 2\pi x_j$$

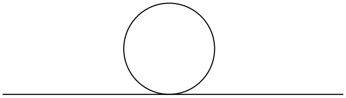
- $$\psi(x) = \sum_{n\in\mathbb{Z}^d} v(n)u(n)e^{-2\pi i n\cdot x}$$

- $$u(x) = \sum_{n\in\mathbb{Z}^d} u(n)e^{-2\pi i n\cdot x}$$

Proof of Corollary 5

- $u \in L^2(\mathbb{T}^d)$, $\psi(x)$ is an entire function with order $\frac{\gamma}{\gamma-1} + \varepsilon$.
- $$u(x) = \frac{\psi(x)}{(h_0(x) - \lambda)}. \tag{7}$$
- Claim: $u(x)$ is an entire function with order $\frac{\gamma}{\gamma-1} + \varepsilon$.
- Then $|u(n)| \leq Ce^{-|n|^{\gamma-\varepsilon}}$ which contradicts the unique continuation result.

Explanation: Proof of the claim



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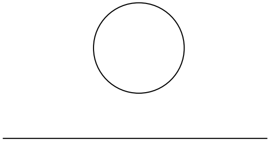

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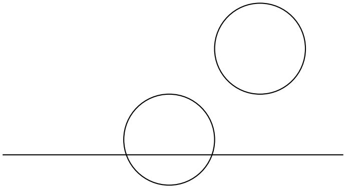

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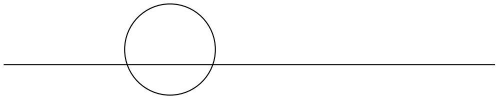

work

What do we need ?

- Unique continuation results (standard arguments)
- Real Fermi varieties have dimension $d - 1$ (standard arguments)
- Irreducibility of Fermi varieties ($\{x \in \mathbb{C}^d : h_0(x) - \lambda = 0\}$)

Classical Borg's theorem

Let $d = 1$. The following statements are equivalent:

1. The potential V is a constant function.
2. $\Delta + V$ has no spectral gaps.

Remark: for the constant potential $V \equiv K$, $\sigma(\Delta + V) = \bigcup_{m=1}^{Q} [a_m^V, b_m^V] = K + [-2d, 2d]$

Borg's theorem in higher dimension: Wrong

1. The analogue of Borg's theorem does not hold for $d \geq 2$.
2. Bethe-Sommerfeld Conjecture
3. Continuous: Karpeshina, Parnovski, Sobolev, Veliev ....
4. Discrete: Han-Jitomirskaya, Embree-Fillman, Filonov-Kachkovskiy

Geometric Borg's theorem

1. Denote by $B(V) \subset \mathbb{C}^d \times \mathbb{C}$ Bloch variety of $\Delta + V$:

$$B(V) = \{(k, \lambda) \in \mathbb{C}^d \times \mathbb{C} : \det(D_V(k) - \lambda I) = 0\}. \tag{8}$$

2. **Conjecture 3** [Kuchment, Knörrer-Trubowitz, Avron-Simon]
   The following statements are equivalent:
   1. The real potential V is a constant function.
   2. There exists an entire function $f(k)$ such that $(k, f(k)) \in B(V)$.
3. For $d = 1$, geometric Borg's theorem is equivalent to classical Borg's theorem [Knörrer-Trubowitz, Avron-Simon].

Geometric Borg's theorem for $d = 2$

1. Knörrer-Trubowitz (continuous case): Conjecture 3 holds for $d = 2$.

Main results: L. 2023

Theorem 6

Then the following statements are equivalent:

1. *The real potential V is a constant function.*
2. *There exists an entire function $f(k)$ such that $(k, f(k)) \in B(V)$.*

Main theorem: Characterize the complex potentials such that the graph of the Bloch variety contains an entire function

Conclusion: methods

- Focus on the study of $P_V(k,\lambda) = \det(D_V(k) - \lambda I)$ or $\mathcal{P}_V(z,\lambda)$.
- Analysis approaches to obtain the algebraic properties of $\mathcal{P}_V(z,\lambda)$.
- Math physics (spectral theory), complex analysis and combinatorics.

Thank you