Potential dependence of the density of states for Schrödinger operators

Analysis and Operator Theory Seminar
(Oct 17, 2024)

Chris Marx, Oberlin College

based on joint work w/
Peter D. Hislop (Univ. of Kentucky)

I. Motivation -

Density of states (outer) measure

Density of states

... characterizes averaged spectral properties of quantum systems in macroscopic limit

crystal in $\mathbb{Z}^d$ (square lattice)

Hamiltonian (energy operator):

$$H_V : \ell^2(\mathbb{Z}^d) \circlearrowleft$$

discrete model

pts of $\mathbb{Z}^d$ ≙ lattice sites (vertices)

$$\boxed{H_V = \Delta + V}$$

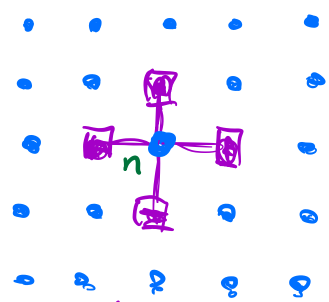
... discrete Schrödinger operator

$$(\Delta\psi)(n) = \sum_{j\sim n} \psi(j)$$

discrete Laplacian on $\mathbb{Z}^d$

2d ... shifts to nearest neighbors of site n

$\mathbb{Z}^2$:



$j \sim n :\Leftrightarrow \|j-n\|_1 = 1$

$$(V\psi)(n) = V(n)\cdot\psi(n)$$

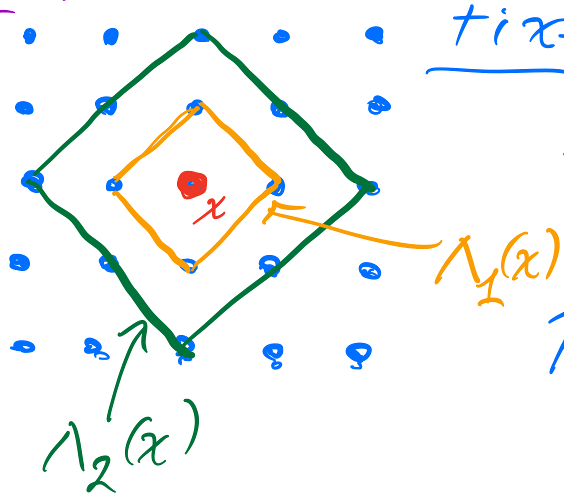
$$V : \mathbb{Z}^d \to \mathbb{R} \; ; \; \ell^\infty$$

potential

$\Rightarrow$ H_V is bdd. self adjoint op.

measurement of energy-dependent observable $f: \mathbb{R} \to \mathbb{C}$, $\mathcal{B}_b$ bdd, Borel

$\mathbb{Z}^2$:



Fixed $x \in \mathbb{Z}^d$, $L \in \mathbb{N}$:

$\Lambda_L(x)$ ... ball w/ radius L at x

$P_L(x)$ ... orthogonal projection in $\ell^2(\mathbb{Z}^d)$ onto sites in $\Lambda_L(x)$

$$\mathrm{tr}(P_L(x)) = |\Lambda_L(x)| \sim L^d$$

measure f in finite-volume $\Lambda_L(x)$:

$$\frac{1}{|\Lambda_L(x)|} \mathrm{tr}\{P_L(x) f(H_V) P_L(x)\}$$ ... finite volume spectral avarage

macroscopic "limit:"

$$n^*_{V;x}(f) := \limsup_{L \to \infty} \frac{1}{|\Lambda_L(x)|} \mathrm{tr}\{P_L(x) f(H_V) P_L(x)\}$$

local quantity (at x!)

$$n^{*}_{V;x}(f) := \limsup_{L\to\infty} \frac{1}{|\Lambda_L(x)|} \operatorname{tr}\{P_L(x) f(H_V) P_L(x)\}$$

remove locality ("bulk" quantity):

$$n^{*}_{V}(f) := \limsup_{L\to\infty} \left\{ \sup_{x\in\mathbb{Z}^d} \frac{1}{|\Lambda_L(x)|} \operatorname{tr}\{P_L(x) f(H_V) P_L(x)\} \right\}$$

... can do this for each observable

$f \in \mathcal{B}_b(\mathbb{R})$

$\Rightarrow$ obtain outer (SUBadditive) Borel measures (on $\mathbb{R}$)

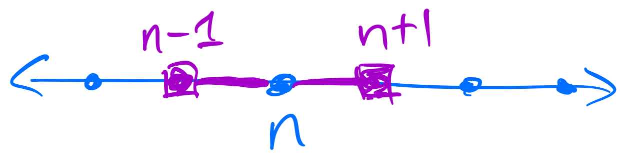
... density of states outer measure(s) (DOSoM)

$f \longmapsto n^{*}_{V;x}(f)$	...	local DOSoM (at x)
$f \longmapsto n^{*}_{V}(f)$	...	DOSoM (uniform/non-local)

NOTE: IF $\limsup_{L\to\infty}$ is a LIMIT (e.g. for random or periodic potentials) local DOSoM becomes a MEASURE, called the density of states measure (DOSM).

Example: free Laplacian:

for $\mathbb{Z}^1$ $(d=1)$:



$$(H_0\psi)(n) = (\Delta\psi)(n) = \psi(n-1) + \psi(n+1)$$

$\uparrow$ no potential

on $\ell^2(\mathbb{Z})$

Spectral resolution provided by Fourier series:

$$\begin{cases} (\mathcal{F}\psi)(\theta) = \sum\limits_{n\in\mathbb{Z}} \hat{\psi}_n e^{2\pi i n\theta} \\ \mathcal{F}: \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}/2\pi\mathbb{Z}) \end{cases}$$

$$\Rightarrow (\mathcal{F} H_0 \mathcal{F}^{-1})\hat{\psi}(\theta) = (e^{2\pi i\theta} + e^{-2\pi i\theta})\hat{\psi}(\theta)$$

$$= 2\cos(2\pi\theta)\cdot\hat{\psi}(\theta)$$

thus for observable $f \in \mathcal{B}_b(\mathbb{R})$:

$$f(H_0)\psi = \mathcal{F}^{-1}\Big(f(2\cos(2\pi\theta))\cdot\hat{\psi}(\theta)\Big)$$

Compute (local) DOSoM at $x \in \mathbb{Z}$:

Given $L \in \mathbb{N}$: $\frac{1}{|\Lambda_L(x)|} \operatorname{tr}\{P_L(x) f(H_0) P_L(x)\} =$

$$= \frac{1}{(2L+1)} \sum_{j=x-L}^{x+L} \underbrace{\langle \delta_j , f(H_0) \delta_j \rangle}_{= \int_0^1 \overline{e^{2\pi i j\theta}} f(2\cos(2\pi\theta)) e^{2\pi i\theta} d\theta}$$

j-th std. basis vector in $\ell^2(\mathbb{Z})$

$$= \frac{1}{\cancel{(2L+1)}} \cancel{(2L+1)} \cdot \int_0^1 f(2\cos(2\pi\theta))\, d\theta$$

$\Rightarrow$ limit $L \to \infty$ exists (yields a MEASURE) and is x-independent:

$$n_0(f) = \int_0^1 f(2\cos(2\pi\theta))\, d\theta \quad \underline{\underline{E = 2\cos(2\pi\theta)}}$$

$$= \int_{\mathbb{R}} f(E) \underbrace{\frac{1}{2\pi} \frac{1}{\sqrt{1-(\frac{E}{2})^2}} \chi_{(-2,2)}(E)\, dE}$$

$=: \varrho_{d=1}^{(0)}(E)\, dE$ ... DOS Measure of the free Laplacian on $\mathbb{Z}^1$

Remark: For $d \geq 2$, obtain DOSM through convolution:

$$dn_0(E) = (\underbrace{\rho_{d=1}^{(0)} * \ldots * \rho_{d=1}^{(0)}}_{d\text{-fold}})(E)\, dE$$

Question: What happens when potential is turned on?

- Expect: Continuous behavior in V
- in particular, as "$V \to 0$:"

  special case: for fixed $v \in \ell^\infty$

  $$V = \lambda \cdot v, \ \lambda \in \mathbb{R}$$

  expect that: $n^*_{\lambda \cdot v} \xrightarrow{\lambda \to 0^+} n_0$ (in some sense)

  (weak-coupling limit)

$$\boxed{\text{Main result: Quantify (general) potential-dependence } (\ell^\infty\text{-norm}) \text{ of DOSoM (weak-top.)}}$$

I. Main result –

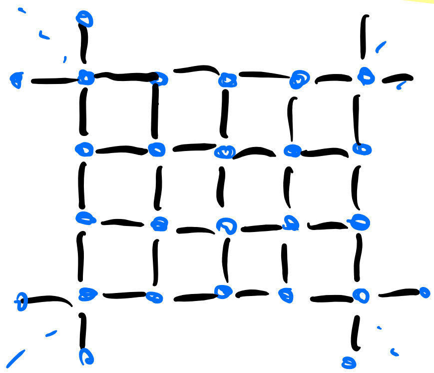
Quantitative continuity of the DOSoM in the potential

Set-up: Discrete Schrödinger operators on graphs

Connected (infinite) graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ — vertices, edges

Examples:

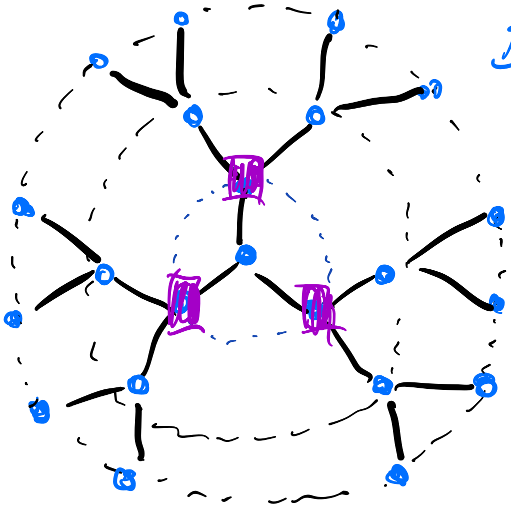
- $\mathbb{Z}^d$ (so far) (d = 2: square lattice)



$\forall x \in \mathbb{Z}^d,\ L \in \mathbb{N}$:
$$|\Lambda_L(x)| \sim L^d \quad (\text{large } L)$$

- other $\mathbb{Z}^2$ lattices e.g.: triangular/hexagonal

- Bethe lattice, $\mathbb{B}_k$ ($k \geq 3$):



Bethe lattice $\mathbb{B}_3$ w/ coordination number $k=3$

$\forall x \in \mathbb{B}_k,\ L \in \mathbb{N}$:
$$|\Lambda_L(x)| \sim (k-1)^L \quad (\text{large } L)$$

Characterize growth of balls in $\mathbb{G} = (\mathcal{V}, \mathcal{E})$

by a **uniform growth fcn.**:

$\gamma_{\mathbb{G}} : [1,\infty) \circlearrowleft$, strictly $\nearrow$, such that $\forall n \in \mathbb{N}$

$$\limsup_{L\to\infty} \left\{ \sup_{x\in\mathcal{V}} \frac{|\Lambda_{L+n}(x)|}{|\Lambda_L(x)|} \right\} \leq \gamma_{\mathbb{G}}(n) \qquad \underline{(*)}$$

Examples:

① $\mathbb{Z}^d$:

$\forall x \in \mathbb{Z}^d,\ L \in \mathbb{N}$:
$|\Lambda_L(x)| \sim L^d$ (large L)

polyn. growth of balls

$\Rightarrow$ LHS of $\underline{(*)} = 1$, thus $\boxed{\text{any}}$ $\gamma_{\mathbb{G}}$ works!

② Bethe lattice $\mathbb{B}_k$:

$\forall x \in \mathbb{B}_k,\ L \in \mathbb{N}$:
$|\Lambda_L(x)| \sim (k-1)^L$ (large L)

$\Rightarrow$ LHS of $\underline{(*)} = (k-1)^n =: \gamma_{\mathbb{B}_k}(n)$

... reflects exponential growth of balls

Goal: Obtain modulus of continuity for

$\ell^\infty(G;\mathbb{R}) \ni V \longmapsto n_V^*$ ← outer measure, compactly supp on $\mathbb{R}$

Topology for codomain: Generalization of weak topology to outer measures

For outer measures μ^*, ν^* on $[-M,M] \subseteq \mathbb{R}$, define (pseudo) metric:

$$\boxed{d_\omega(\mu^*,\nu^*) := \sup\{ |\mu^*(f) - \nu^*(f)| : f \in \mathrm{Lip}([-M,M]) \text{ w/ } \|f\|_{\mathrm{Lip}} \leq 1\}}$$

NOTE that: For measures, d_ω is known to be equivalent to the weak topology (Fortet-Mourier metric)

i.e. for (μ_n), μ measures on $[-M,M]$

$$d_\omega(\mu_n,\mu) \xrightarrow{n\to\infty} 0 \iff \forall f \in \mathcal{C}([-M,M]),\ \mu_n(f) \xrightarrow{n\to\infty} \mu(f)$$

Main theorem: Given $\mathbb{G} = (V, \mathcal{E})$, a connected infinite graph, admitting a uniform growth fcn. $\gamma_{\mathbb{G}}$.

Then, $\exists$ constants c_3, $\varrho_{\mathbb{G}} > 0$ s.t. for every $C > 0$ the map

$$\ell^\infty(\mathbb{G}; [-C, C]) \ni V \longmapsto n_V^* \quad \ldots \text{ outer measure supp on } [-\varrho_{\mathbb{G}} - C, \varrho_{\mathbb{G}} + C]$$

is continuous w/ modulus:

For all $0 < \xi < 1$, one has

$$\boxed{d_\omega(n_V^*, n_W^*) \leq 4\,(\varrho_{\mathbb{G}} + C)\,c_3 \cdot \frac{1}{\gamma_{\mathbb{G}}^{-1}\left(\left(\frac{1}{\|V - W\|_\infty}\right)^{\xi}\right)} + \left(\|V - W\|_\infty\right)^{1-\xi}}$$

, for all $V, W \in \ell^\infty(\mathbb{G}; [-C, C])$, $\|V - W\|_\infty < 1$.

Remarks:

1. Modulus of continuity captures the geometry of $\mathbb{G}$ through the uniform growth fcn.

$$\gamma_{\mathbb{G}} : [1,+\infty) \circlearrowleft \ , \text{ strictly } \nearrow$$

s.t. $$\limsup_{L\to\infty} \left\{ \sup_{x\in V} \frac{|\Lambda_{L+n}(x')|}{|\Lambda_L(x)|} \right\} \leq \gamma_{\mathbb{G}}(n) \ , \ \forall n \in \mathbb{N}$$

2. Parameter $0<\xi<1$ in thm. allows to optimize.

3. An analogous result holds for the local DOSoM at all vertex points

1. Application to $\mathbb{Z}^d$:

polyn. growth of balls $\Rightarrow$ $\limsup_{L\to\infty} \left\{ \sup_{x\in\mathbb{Z}^d} \frac{|\Lambda_{L+n}(x')|}{|\Lambda_L(x)|} \right\} = \underline{\underline{1}}$

$\Rightarrow$ Freedom to use ANY growth function

For $0<\xi,\alpha$, take $\boxed{\gamma_{\mathbb{Z}^d}(n) = n^{\xi\cdot\alpha}}$.

Then, given potentials $V, W \in \ell^\infty(\mathbb{Z}^d; [-C,C])$ w/ $0<\varepsilon := \|V-W\|_\infty < 1$, the theorem yields:

$$d_\omega(n_V^*, n_W^*) \leq 4(2d+C)_{\xi}\cdot \varepsilon^{1/\alpha} + \varepsilon^{1-\xi}$$

Optimize first taking $\alpha \to 0^+$ (1st term drops) and then letting $\xi \to 0^+$:

$$\boxed{d_\omega(n_V^*, n_W^*) \lesssim \varepsilon}$$

Lipschitz continuity!

Thm 1 (potential dependence of DOSoM for $\mathbb{Z}^d$, $d \in \mathbb{N}$):

For each fixed $C>0$ the map

$$\ell^\infty(\mathbb{Z}^d; [-C,C]) \ni V \longmapsto n_V^* \text{ (weak topology)}$$

is Lipschitz continuous.

In particular, for $V = \lambda \cdot \vartheta$, $\lambda \in \mathbb{R}$, "coupling parameter", this implies:

Thm 2 (weak coupling limit for $\mathbb{Z}^d$, $d \in \mathbb{N}$):

Let $C>0$ and $\vartheta \in \ell^\infty(\mathbb{Z}^d; [-C,C])$, then:

$$\boxed{d_w(n^*_{\lambda\cdot\vartheta}, n_0) \leq \lambda \cdot \|\vartheta\|_\infty, \quad 0 \leq |\lambda| < 1}.$$

Here, n_0 ... DOSM of free Laplacian on $\mathbb{Z}^d$.

As before, analogous results hold for the local DOSoM.

Can formulate results in terms of cumulative distribution of DOSoM:

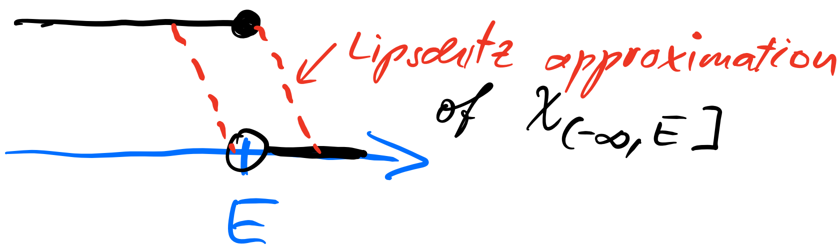
$$N_V^*(E) := n_V^*(\chi_{(-\infty,E]}) \ , \ E \in \mathbb{R}$$

... integrated outer density of states (IoDS)

Translating our results for potential dep. of the DOSoM (thm 1):

$$| n_V^*(f) - n_W^*(f) | \leq \|f\|_{Lip} \cdot \|V-W\|_\infty ,$$

to the IoDS requires information about the E-dep. for fixed potential



Known (cf. $\mathbb{Z}^d$): Log-Hölder continuous in E, for fixed $V \in L^\infty(\mathbb{Z}^d;[-c,c])$.

(J. Bourgain, A. Klein - 2013)

Using their, our result in thm 1 implies Log-Hölder behavior in the POTENTIAL for fixed $E \in \mathbb{R}$:

Thm: For $\mathbb{Z}^d$, $d \in \mathbb{N}$, and each $C > 0$, $\exists$ $K_0 = K_0(C, d)$: $\forall$ $V, W \in \ell^\infty(\mathbb{Z}^d; [-C, C])$ w/ $\|V - W\|_\infty \leq 1$ one has

$$|N_V^*(E) - N_W^*(E)| \leq \frac{K_0}{\log\left(\frac{1}{\|V - W\|_\infty}\right)}, \ \forall E \in \mathbb{R}.$$

For the weak coupling limit on $\mathbb{Z}^d$:

$V = \lambda \cdot v$, $\lambda \to 0$,

the explicit formula for the DOSM of the free Laplacian implies

$$E \mapsto N_0(E) \begin{cases} 1/2\text{-Hölder}, & d = 1, \\ \text{Lipschitz}, & d \geq 2, \end{cases}$$

which improves the λ-dependence:

Thm: For $v \in \ell^\infty(\mathbb{Z}^d; [-C, C])$, $C > 0$:

$$|N_{\lambda \cdot v}(E) - N_0(E)| \leq c_0 \cdot \lambda^r, \ E \in \mathbb{R},$$

w/ $r = \begin{cases} 1/3, & d = 1, \\ 1/2, & d \geq 2. \end{cases}$

Context for weak coupling limit; $\mathbb{Z}^d$:

So far available results were limited to the IDS of random potentials, most of which were for $d=1$:

- $d=1$: Bovier-Klein, Campanino-Klein, Speis (late 1980s, early 1990s)

- $d \geq 2$:

  * Hislop-Klopp-Schenker (2005), Schenker (2004) — certain random potentials, $\sim \lambda^{1/8}$-dep.

  * Hislop-Marx (2018): ALL random potentials, $\sim \lambda^{\frac{1}{1+2d} \cdot \frac{1}{2}}$-dep.

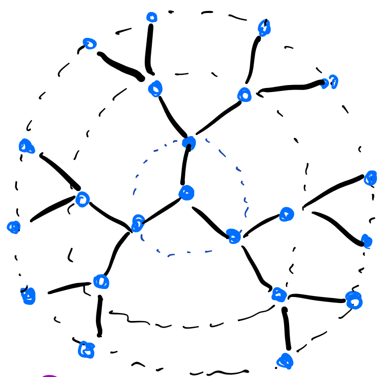
  * M. Shamis (2019); I. Kachkovskiy (2019): Lipschitz-dep., ALL random potentials

Our work improves these results to deterministic potentials on graphs (which admit a uniform growth fcn.).

2. Application to the Bethe lattice, $\mathbb{B}_k$

$$\boxed{\begin{array}{l}\forall x \in \mathbb{B}_k,\ L \in \mathbb{N}: \\ |\Lambda_L(x)| \sim (k-1)^L \\ \qquad\qquad (\text{large } L)\end{array}}$$

exponential growth of volume of balls $\Longrightarrow$



$\mathbb{B}_k$ for $k=3$

$$\limsup_{L\to\infty}\left\{\sup_{x\in\mathbb{B}_k}\frac{|\Lambda_{L+n}(x')|}{|\Lambda_L(x)|}\right\} = (k-1)^n =: \gamma_{\mathbb{B}_k}(n)$$

∴ Main theorem thus yields log-Hölder continuity for the potential dependence of the DOSM:

thm: Given $\mathbb{B}_k$, $k \geq 3$, and $C>0$. There exist $\alpha_k>0$ s.t. $\forall V,W \in \ell^\infty(\mathbb{B}_k;[-C,C])$, $\|V-W\|_\infty<1$:

$$d_\omega(n_V^*, n_W^*) \leq \frac{\alpha_k}{\log\left(\frac{1}{\|V-W\|_\infty}\right)}.$$

III. Elements of the proof –

(of the main theorem)

The proof of the main theorem characterizing the potential dependence of the DOSoM relies on two key steps:

Step 1 — finite-range reduction:

$$\frac{1}{|\Lambda_L(x)|}\operatorname{tr}\{P_L(x)f(H_V)P_L(x)\}$$

depends on the potential on infinitely many vertices!

Reduce variation of the potential at infinitely many vertices of the graph to a FINITE subgraph and quantify the resulting error term

Step 2 — Single site variations:

Quantify the effects of single-site variations of the potential on finite volume spectral averages

(Lipschitz property)

Step 1 – finite-range reduction

Given two potentials $V, W \in \ell^\infty(\mathbb{G}; [-C,C])$, an observable $f \in \mathrm{Lip}([-C-\rho_\Delta, C+\rho_\Delta])$. Fix $x \in \mathcal{V}$.

<u>Goal</u>: Quantify variations of the potential in spectral averages:

$$\frac{1}{|\Lambda_L(x)|}\Big|\mathrm{tr}\{P_L(x) f(H_V) P_L(x)\} - \mathrm{tr}\{P_L(x) f(H_W) P_L(x)\}\Big|$$

<u>Idea</u>: Vary potential locally in a ball $\Lambda_R(x)$, for <u>appropriate $R>0$</u>

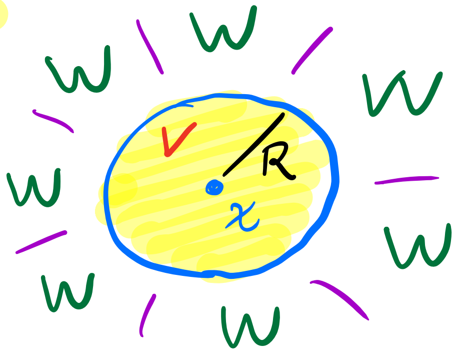
$\to$ A-priori estimate of error resulting from DROPPING contributions from $\mathbb{G} \setminus \Lambda_R(x)$

<u>Consider</u>: $V_W^{(R;x)} \in \ell^\infty(\mathbb{G})$,

the (R,W)-modification of V at x

... the potential obtained by changing V to W on $\mathbb{G} \setminus \Lambda_R(x)$

Thus: $V_W^{(R;x)}$ and W differ only inside ball $\Lambda_R(x)$ (FINITELY many vertices)



$\Rightarrow$ Quantify LOSS of "throwing away" contributions outside $\Lambda_R(x)$:

Key: The finite-difference structure of the Laplacian

$$(\Delta\psi)(x) = \sum_{y \sim x} \psi(y)$$

nearest neighbors only!

$\Rightarrow$ For polynomial p,

$$V \longmapsto \operatorname{tr}\{P_L(x)\, p(H_V)\, P_L(x)\}$$

depends only on potential at FINITELY many vertices inside ball $\Lambda_{L+\lfloor \frac{\deg p}{2} \rfloor}(x)$

enlarge radius by $\sim \deg p$

Quantify loss of throwing away all but finitely many contributions to varying the potential through polynomial approximation:

Lemma (finite-range reduction):

Given error $\eta > 0$ and observable $f \in \mathcal{C}(\mathbb{R})$. Suppose p is a polynomial s.t.

$$\|f - p\|_{\infty;[-C-\rho_{\mathbb{G}}, C+\rho_{\mathbb{G}}]} < \frac{1}{2}\eta .$$

Then, letting $R := L + \lfloor \frac{\deg p}{2} \rfloor$, one has:

$$\frac{1}{|\Lambda_2(x)|}\left| tr\{P_L(x) f(H_V) P_L(x)\} - tr\{P_L(x) f(H_{V_W^{(R;x)}}) P_L(x)\}\right| \leq \eta .$$

Remark: For Lipschitz observables f, use approximation by Jackson polyn.

$$\left(\|J_n[f] - f\|_\infty \leq \frac{c_J}{n} \right)$$

$\leftarrow$ nth-Jackson poly

for which rate of convergence optimally inherits the modulus of contin. of f:

$$\|f - p\|_\infty < \frac{\eta}{2} \iff \deg p \sim \eta^{-1}$$

Step 2 – Single site variations

- Step 1 reduces varying the potential at the finitely many vertices inside $\Lambda_{L+\lfloor \frac{\deg p}{2} \rfloor}(x)$
- Variation at only one vertex $x \in V$ at a time, leads to one-parameter family:

$$\begin{cases} \lambda \longmapsto H_\lambda := H_{V_x^\perp} + \lambda \cdot \pi_x \,, \quad \lambda \in [-C, C] \\ \text{where } \pi_x(\cdot) = \langle \delta_x, \cdot \rangle \delta_x \end{cases}$$

... orthogonal proj. onto std. basis vector δ_x

By the defn. of our "metric" for outer measures, it thus suffices to show that for given $f \in Lip_c(\mathbb{R})$ and $z \in V$, the map

$$\boxed{\lambda \longmapsto \mathcal{F}_f(\lambda) := \langle \delta_z, f(H_\lambda) \delta_z \rangle}$$

satisfies:

Prop. ("Lipschitz property"): The map $\lambda \longmapsto \mathcal{F}_f(\lambda)$ is Lipschitz and bdd. in λ w/

$$|\mathcal{F}_f(\lambda_1) - \mathcal{F}_f(\lambda_2)| \leq |\lambda_1 - \lambda_2| \cdot \|f\|_{Lip} \,, \quad \text{all } \lambda_1, \lambda_2 \in \mathbb{R}$$

Proof uses the almost analytic extensions & Helffer–Sjöstrand functional calculus

Step 3 – Finishing up...

Given $V, W \in \ell^\infty(\mathcal{G}; [-C, C])$, $\|V-W\|_\infty =: \varepsilon < 1$
and $f \in \mathrm{Lip}([-C-\rho_G, C+\rho_G])$.

Let $\eta : (0, +\infty) \circlearrowleft$ TBD ("modulus fcn.")
s.t. $\eta(y) \searrow 0$ as $y \searrow 0^+$

1. For $L \in \mathbb{N}$, take the cut-off radius R according to the finite range reduction (Step 1): $R \sim L + \left\lfloor \frac{1}{\eta(\varepsilon)} \right\rfloor$

2. Apply the Lipschitz property (Step 2) to each vertex inside $\Lambda_R(x)$:

$$\frac{1}{|\Lambda_L(x)|}\left| tr\{P_L(x) f(H_V) P_L(x)\} - tr\{P_L(x) f(H_W) P_L(x)\}\right|$$

$$\leq \frac{1}{|\Lambda_L(x)|}\left| tr\{P_L(x) f(H_V) P_L(x)\} - tr\{P_L(x) f(H_{V_W^{(R;x)}}) P_L(x)\}\right|$$

$$+ \frac{1}{|\Lambda_L(x)|}\left| tr\{P_L(x) f(H_{V_W^{(R;x)}}) P_L(x)\} - tr\{P_L(x) f(H_W) P_L(x)\}\right|$$

$$\leq \eta(\varepsilon) + \|f\|_{Lip} \cdot \frac{|\Lambda_R(x)|}{|\Lambda_L(x)|} \cdot \varepsilon$$

$$\frac{1}{|\Lambda_L(x)|}\left| tr\{P_L(x) f(H_V) P_L(x)\} - tr\{P_L(x) f(H_W) P_L(x)\}\right|$$

$$\leq \eta(\varepsilon) + \|f\|_{Lip} \cdot \frac{|\Lambda_R(x)|}{|\Lambda_L(x)|} \cdot \varepsilon ,$$

, where $R \sim L + \left\lfloor \frac{L}{\eta(\varepsilon)} \right\rfloor$

3. By the defn. of the uniform growth fcn.

$$\limsup_{L\to\infty} \sup_{x\in V} \left\{ \frac{|\Lambda_R(x)|}{|\Lambda_L(x)|} \right\} \leq \gamma_G(R-L),$$

thus conclude:

$$|n_V^*(f) - n_W^*(f)| \leq$$

$$\eta(\varepsilon) + \|f\|_{Lip} \cdot \gamma_G\left(\left\lfloor \frac{L}{\eta(\varepsilon)} \right\rfloor\right) \cdot \varepsilon \qquad \underline{(*)}$$

Balancing two terms in $(*)$ by choosing:

$$\eta(y) \sim \|f\|_{Lip} \cdot \frac{1}{\gamma_G^{-1}\left(\frac{1}{y^{\xi}}\right)} \quad , \text{ for } \xi > 0$$

obtain final result for the modulus of continuity. $\boxed{}$

Thank you!

P.D. Hislop, C.A. Marx,

Journal of Functional Analysis 281 (2021), 109186.