Biorthogonal measures associated with polymer partition functions

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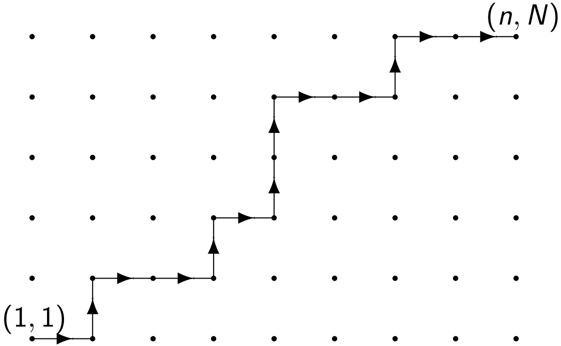
Outline

1. Exactly solvable polymer models and random matrix models
2. Fredholm determinant identities
3. Biorthogonal measures
4. Large deviations

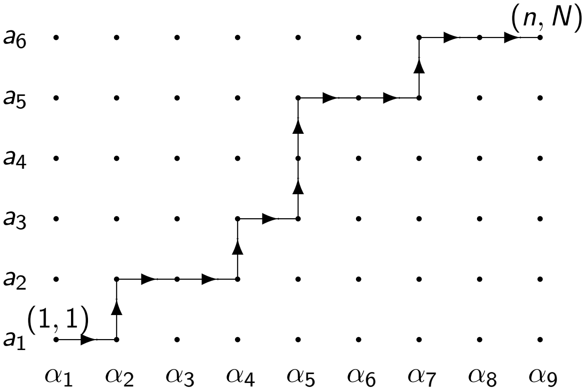
Log-Gamma Polymer

- The **log-Gamma Polymer** is a probability measure on directed up-right lattice paths on a rectangular lattice of size $n \times N$, introduced by *Seppalainen* in 2012.



Log-Gamma Polymer

- To each column $j = 1, \ldots, n$, we assign a parameter α_j, and to each row $k = 1, \ldots, N$, we assign a parameter a_k, and they are such that $\alpha_{j,k} := \alpha_j - a_k > 0$ for all j, k.
- The **homogeneous lattice** corresponds to $\alpha_j = -a_k = \theta > 0$.



Log-Gamma Polymer

- We assign an independent **inverse-Gamma distributed random weight** $d_{i,j} \geq 0$ to each vertex in the lattice,

$$\mathbb{P}(d_{j,k} \leq y) = \frac{1}{\Gamma(\alpha_{j,k})} \int_0^y x^{-\alpha_{j,k}-1} \mathrm{e}^{-1/x} \mathrm{d}x.$$

- The weight of an up-right path π connecting $(1,1)$ with (n,N) is the product of the weights of the vertices in the path, $\prod_{(j,k)\in\pi} d_{j,k}$, and the **partition function** of the model is the **random variable**

$$Z_{n,N}(\vec{\alpha},\vec{a}) := \sum_{\pi:(1,1)\nearrow(n,N)} \prod_{(j,k)\in\pi} d_{j,k},$$

where the sum is over all up-right paths between $(1,1)$ and (n,N).

- We will now restrict to the **homogeneous** $(\alpha_j = -a_k = \theta > 0)$ **square** $(n = N)$ **lattice**, and write $Z_n(\theta)$ for the partition function.

Zero temperature limit

- As $\theta \to 0$, the random variables $u_{i,j}(\theta) := 2\theta \log d_{i,j}(\theta)$ converge in distribution to **independent exponential random variables** $u_{i,j}$, i.e., $\mathbb{P}(u_{i,j} \leq s) = 1 - e^{-u_{i,j}}$.
- It follows that

$$2\theta \log Z_n(\theta) \longrightarrow F_n^{\mathrm{LPP}} := \max_{\pi:(1,1)\to(n,n)} \sum_{(i,j)\in\pi} u_{i,j}$$

weakly as $\theta \to 0$ with fixed n. The right hand side is the maximum additive weight of an up-right path in random exponential environment. This model is known as **last passage percolation** or **corner growth** with exponential weights.

- The distribution of F_n^{LPP} is identical to that of the **largest eigenvalue of an LUE random matrix** *(Johansson '00)*.

An exact Fredholm determinant identity

- The model is **exactly solvable**: the **Laplace transform of the partition function** *(Borodin-Corwin-Remenik '13)* is given by

$$\mathbb{E}(e^{-uZ_n(\theta)}) = \det(I + K_n^{u,\theta})_{L^2(\Sigma)},$$

where $K_n^{u,\theta} : L^2(\Sigma) \to L^2(\Sigma)$ is the integral kernel operator with kernel

$$K_n^{u,\theta}(v, v') = \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} dw \frac{\pi u^{w-v}}{\sin \pi(v - w)} \frac{W(w)}{W(v)} \frac{1}{w - v'}, \quad W(z) = \frac{\Gamma(\theta - z)^n}{\Gamma(\theta + z)^n},$$

with Σ a circle of radius $r < \theta$ around $-\theta$.

- The determinant is a **Fredholm determinant** given by the **Fredholm series**

$$\det(I + K)_{L^2(\gamma)} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\gamma^k} \det\left(K(x_i, x_j)\right)_{i,j=1}^{k} dx_1 \cdots dx_k.$$

- **Theorem** *(Borodin-Corwin-Remenik '13, Krishnan-Quastel '18, Barraquand-Corwin-Dimitrov '21)*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z_n(\theta) + 2n\psi(\theta)}{(-\psi''(\theta)n)^{1/3}} \leq r\right) = F_{\mathrm{TW}}(r), \qquad r \in \mathbb{R},$$

where $F_{\mathrm{TW}}(r)$ is the $\beta = 2$ **Tracy-Widom distribution**

$$F_{\mathrm{TW}}(r) = \det\left(1 - K^{\mathrm{Ai}}\right)_{L^2(r,\infty)}, \; K^{\mathrm{Ai}}(x, x') = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(x') - \mathrm{Ai}(x')\mathrm{Ai}'(x)}{x - x'}.$$

- The **zero-temperature analogue** of this is a well-known result *(Johansson '00)*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{F_n^{\mathrm{LPP}} - 4n}{2^{4/3}n^{1/3}} \leq r\right) = F_{\mathrm{TW}}(r), \qquad r \in \mathbb{R},$$

for last passage percolation (or for the largest eigenvalue in LUE).

Log-Gamma Polymer

- An important open problem is to understand **tail probabilities and large deviations**: how does

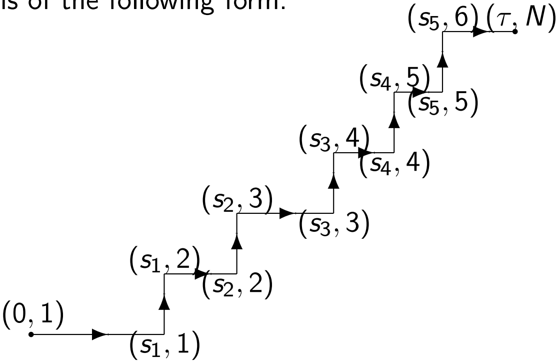
$$\mathbb{P}\left(\frac{\log Z_{n,N}(\alpha) - nh(\alpha, N/n)}{n^{1/3}\sigma(\alpha, N/n)} \leq s\right)$$

behave as $s \to \pm\infty$ together with $n \to \infty$?

  - The upper tail $s \to +\infty$ is not so hard to analyze and well understood.
  - The **lower tail** $s \to -\infty$ is not understood, and understanding it is our main challenge and motivation.

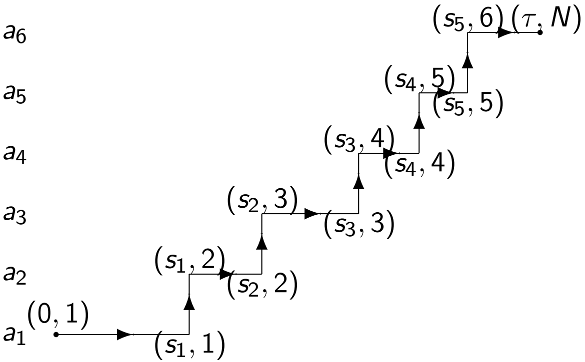
O'Connell-Yor Polymer

- Another exactly solvable polymer model is the **O'Connell-Yor Polymer** or **semi-discrete directed polymer**. Here the lattice is discrete in the vertical direction but continuous in the horizontal, and we have paths of the following form.



O'Connell-Yor Polymer

- Each row receives a real-valued parameter a_j, $j = 1, \ldots, N$, and an independent time-dependent random weight $B_j(t)$, which is a **Brownian motion with drift** a_j.



O'Connell-Yor Polymer

- The **energy of a path** $\pi(s_1, \ldots, s_N)$ is the sum of the Brownian increments between the jump points,

$$E(\pi) = B_1(s_1) + (B_2(s_2) - B_2(s_1)) + \ldots + (B_N(\tau) - B_N(s_{N-1})).$$

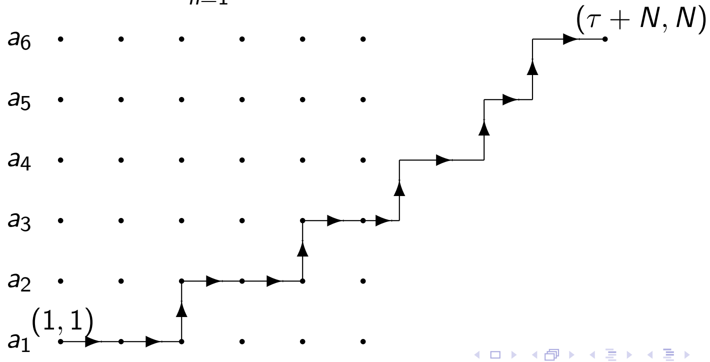
The O'Connell-Yor polymer **partition function** is then defined as

$$Z_N^{\mathrm{OY}}(\vec{a}, \tau) := \int_{0<s_1<\ldots<s_{N-1}<\tau} \mathrm{e}^{E(\pi(s_1,\ldots,s_N))} \mathrm{d}s_1 \cdots \mathrm{d}s_{N-1}.$$

Mixed Polymer

- The **Mixed Polymer** or **O'Connell-Yor Polymer with boundary sources** *(Borodin-Corwin-Ferrari-Vető '15)* is a combination of the log-Gamma Polymer and the O'Connell-Yor Polymer. The **partition function** in this model is given by

$$Z_N^{\mathrm{Mixed}}(\vec{\alpha}, \vec{a}, \tau) = \sum_{n=1}^{N} Z_{n,N}^{\mathrm{Log}\Gamma}(\vec{\alpha}, a_1, \ldots, a_n) Z_{N-n}^{\mathrm{OY}}(a_{n+1}, \ldots, a_N, \tau).$$



Fredholm determinant identities

- Similarly to the log-Gamma Polymer:
  - Zero-temperature limit of the O'Connell-Yor Polymer is a corner growth model, related to GUE random matrices.
  - There is a Fredholm determinant identity for the Laplace transform of the O'Connell-Yor Polymer and the Mixed Polymer partition functions.
  - Fluctuations of Z_N^{OY} and Z_N^{Mixed} converge to the **Tracy-Widom distribution** *(Borodin-Corwin-Ferrari '14, Borodin-Corwin-Ferrari-Vető '15, Talyigás-Vető '20)*.

Polymers and random matrices

Zero-Temperature Polymer model	Random matrix model
Last passage percolation with exp. weights	Laguerre Unitary Ensemble
Brownian last passage percolation	Gaussian Unitary Ensemble
Brownian LPP with sources	?

Finite-Temperature Polymer model	?
Log-Gamma Polymer	?
O'Connell-Yor Polymer	?
Mixed Polymer	?

- We will see that all these models can be interpreted in terms of **signed biorthogonal measures**.

Biorthogonal measures

- A **biorthogonal ensemble** is a **probability measure** on $\mathbb{R}^N$ of the form

$$d\mu_N(x_1,\ldots,x_N) = \frac{1}{Z_N}\det\left(f_m(x_k)\right)_{k,m=1}^N \det\left(g_m(x_k)\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k,$$

for certain sets of functions $f_1,\ldots,f_N$ and $g_1,\ldots,g_N$.

  - For $f_m(x)=x^{m-1}$ and $g_m(x)=x^{m-1}w(x)$, we obtain the **orthogonal polynomial ensemble**

$$\frac{1}{Z_N}\prod_{1\le j<k\le N}(x_k-x_j)^2\prod_{k=1}^N w(x_k)\mathrm{d}x_k.$$

    - GUE eigenvalue distribution if $w(x)=e^{-x^2/2}$.
    - LUE eigenvalue distribution if $w(x)=x^\nu e^{-x}$.

- We will consider measures with this biorthogonal structure which are **not necessarily positive**.

Biorthogonal measures

- A **biorthogonal ensemble** admits a correlation kernel of the form

$$L_N(x, x') = \sum_{k=1}^{N} \psi_k(x)\phi_k(x'),$$

where $\psi_1, \ldots, \psi_N$ have the same linear span as $f_1, \ldots, f_N$, $\phi_1, \ldots, \phi_N$ have the same linear span as $g_1, \ldots, g_N$, and they satisfy the **biorthogonality conditions**

$$\int_{\mathbb{R}} \psi_j(x)\phi_k(x)dx = \delta_{jk}.$$

- The k-**point correlation function** is given by

$$\rho_k(x_1, \ldots, x_k) = \det\left(L_N(x_i, x_j)\right)_{i,j=1}^{k}.$$

Biorthogonal measures

- In our cases of interest, the kernel has another useful representation as a **double contour integral**:

$$L_N(x, x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \frac{W_N(v)}{W_N(u)} \frac{e^{-vx+ux'}}{v-u},$$

where there exists a vertical strip $\mathcal{S}_N := \{z \in \mathbb{C} : \alpha < \operatorname{Re} z < \beta\}$ such that

1. $W_N : \mathcal{S}_N \to \mathbb{C}$ is analytic, not identically zero and such that $W_N(z) = O(|z|^{-\epsilon})$ as $z \to \infty$ in $\mathcal{S}_N$, for some $\epsilon > 0$;
2. ℓ_N is a vertical line inside $\mathcal{S}_N$, oriented upwards, i.e. $\ell_N = c + \mathrm{i}\mathbb{R}$ with $\alpha < c < \beta$;
3. Σ_N is a closed positively oriented curve in $\mathcal{S}_N$ without self-intersections, lying at the left of ℓ_N, which has N (not necessarily distinct) zeros of W_N in its interior, and none on its image; we denote these zeros as $a_1, \ldots, a_N$.

Biorthogonal measures

Theorem (*C-Cafasso '24*)

$\mathrm{d}\mu_N(\vec{x}) = \frac{1}{N!}\det\left(L_N(x_i,x_j)\right)_{i,j=1}^N dx_1 \dots dx_N$ *is a biorthogonal measure.*

1. *If all zeros $a_1, \dots, a_N$ are distinct, we have*

$$\mathrm{d}\mu_N(\vec{x}) = \frac{1}{N!}\det\left(\mathrm{e}^{a_m x_k}\right)_{k,m=1}^N \det\left(\psi_m(\mathrm{e}^{x_k})\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k,$$

with $\psi_m(y) := \frac{1}{2\pi\mathrm{i}W_N'(a_m)}\int_{\ell_N}\frac{W_N(v)y^{-v}}{v-a_m}\mathrm{d}v.$

2. *If $a_1 = \cdots = a_N = a$,*

$$\mathrm{d}\mu_N(\vec{x}) = \frac{1}{Z_N}\prod_{1\leq j<k\leq N}(x_k - x_j)\det\left(\phi_m(\mathrm{e}^{x_k})\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k,$$

where $\phi_m(y) := \frac{y^a}{2\pi\mathrm{i}}\int_{\ell_N}\frac{W_N(v)y^{-v}}{(v-a)^{N-m+1}}\mathrm{d}v.$

Biorthogonal measures

- An **average multiplicative statistic** with respect to the biorthogonal measure is a quantity like

$$\mu_N[\sigma] := \int_{\mathbb{R}^n} \prod_{k=1}^{N} (1 - \sigma(x_k)) \mathrm{d}\mu_N(x_1, \ldots, x_N).$$

- Average multiplicative statistics are **Fredholm determinants**:

$$\mathbb{E} \prod_{j=1}^{N} (1 + h(x_j)) = \det(1 + hL_N)_{L^2(\mathbb{R})}.$$

- These Fredholm determinants can be related to the ones appearing in polymer models. As a consequence, we can **characterize the three polymer partition functions in terms of such a biorthogonal measure**.

Biorthogonal measures

Theorem (C-Cafasso '24)

We have the identities

$$\mathbb{E}\left[e^{-e^t Z_N(\vec{\alpha},\vec{a},\tau)}\right] = \mu_N[\sigma_t], \quad \sigma_t(x) = \frac{1}{1+\mathrm{e}^{-x-t}},$$

1. *with* $W_N(z) = \frac{\prod_{j=1}^n \Gamma(\alpha_j - z)}{\prod_{k=1}^N \Gamma(z-a_k)}$*, if* Z_N *is the log-Gamma polymer partition function;*
2. *with* $W_N(z) = \frac{e^{\tau z^2/2}}{\prod_{k=1}^N \Gamma(z-a_k)}$*, if* Z_N *is the O'Connell-Yor polymer partition function;*
3. *with* $W_N(z) = \frac{e^{\tau z^2/2}\prod_{j=1}^N \Gamma(\alpha_j - z)}{\prod_{k=1}^N \Gamma(z-a_k)}$*, if* Z_N *is the Mixed Polymer partition function.*

Polymers and biorthogonal measures

- This can be seen as the finite temperature analogue of similar identities relating last passage percolation models to random matrix models like GUE and LUE *(Johansson '00, Borodin-Péché '08)*).
- For the **homogeneous case** $a_j = a$ of the **O'Connell-Yor Polymer**, this identity was already proved by *Imamura-Sasamoto '16*.
- For the **Mixed Polymer**, the Fredholm determinant identity of *Imamura-Sasamoto '17* hints at a biorthogonal structure, and it is possible that they were already aware of it.
- Our class of biorthogonal measures, with functional dependence on W_N, contains several **random matrix ensembles**, and moreover some of these arise as **scaling limits of the polymer biorthogonal measures**.

Random matrices and biorthogonal measures

Random matrix models with biorthogonal measures as eigenvalue distributions.

Model	$W_N(z)$	f_m	g_m
GUE	$z^N e^{z^2/2}$	x^{m-1}	$x^{m-1} e^{-x^2/2}$
GUE+ext source	$\prod_{m=1}^{N}(z-a_m)\ e^{z^2/2}$	x^{m-1}	$e^{a_m x} e^{-x^2/2}$
LUE	$\frac{z^N}{(z-1)^{N+\nu}}$	x^{m-1}	$x^{m-1+\nu} e^{-x}$
LUE+ext source	$\frac{\prod_{m=1}^{N}(z-a_m)}{(z-1)^{N+\nu}}$	x^{m-1}	$e^{a_m x} x^{\nu} e^{-x}$
Ginibre products	$\frac{\prod_{k=0}^{n}\Gamma(1+\nu_k-z)}{\Gamma(1-N-z)}$	y^{m-1}	$G_{0,n}^{n,0}\left({- \atop \vec{\nu}} \middle\vert 1/y \right)$
Muttalib-Borodin	$\frac{\Gamma(1-z)\Gamma(\nu+1-\theta z)}{\Gamma(1-N-z)}$	y^{m-1}	$y^{\theta(m-1)} e^{-y}$
Trunc. unitary	$\frac{\prod_{k=0}^{n}\Gamma(1+\nu_k-z)}{\prod_{k=0}^{n}\Gamma(1+\ell_k-N-z)}$	y^{m-1}	$G_{n,n}^{n,0}\left({\vec{\mu}+\vec{\nu} \atop \vec{\nu}} \middle\vert 1/y \right)$

Polymer models and associated biorthogonal measures.

Model	$W_N(z)$	f_m	g_m
Log Γ	$\frac{\prod_{j=1}^{n}\Gamma(\alpha_j - z)}{\prod_{k=1}^{N}\Gamma(z-a_k)}$	$e^{a_m x}$	$G^{n,0}_{0,n+N}\left(\begin{matrix} - \\ \vec{\alpha};\vec{1}+\vec{a}-\vec{e}_m \end{matrix} \middle\vert\, e^x\right)$
Confl. Log Γ	$\frac{\Gamma(\alpha-z)^n}{\Gamma(z-a)^N}$	x^{m-1}	$G^{n,0}_{0,n+N}\left(\begin{matrix} - \\ \vec{\alpha};(a+1)_{m-1};\vec{a} \end{matrix} \middle\vert\, e^x\right)$
OY	$e^{\tau z^2/2}\frac{1}{\prod_{k=1}^{N}\Gamma(z-a_k)}$	$e^{a_m x}$	
Confl. OY	$e^{\tau z^2/2}\frac{1}{\Gamma(z-a)^N}$	x^{m-1}	
Mixed	$e^{\tau z^2/2}\frac{\prod_{j=1}^{N}\Gamma(\alpha_j - z)}{\prod_{k=1}^{N}\Gamma(z-a_k)}$	$e^{a_m x}$	
Confl. Mixed	$e^{\tau z^2/2}\frac{\Gamma(\alpha-z)^N}{\Gamma(z-a)^N}$	x^{m-1}	

Polymers and random matrices

Zero-Temperature Polymer model	Random matrix model
Last passage percolation with exp. weights	Laguerre Unitary Ensemble
Brownian last passage percolation	Gaussian Unitary Ensemble
Brownian LPP with sources	(LUE+ext. source)+GUE

Finite-Temperature Polymer model	Biorthogonal measure
Log-Gamma Polymer	✓
O'Connell-Yor Polymer	✓
Mixed Polymer	✓

Large deviations for the log-Gamma polymer

- The associated biorthogonal measures help to understand the large N asymptotics for polymer partition functions. As an illustration, we have a **conjecture about the large deviations of the log-Gamma polymer partition function**.
- For $0 < s < -\theta\psi(\theta)$, define $b = b(s;\theta) > 0$ as the unique positive number solving the equation

$$\int_0^1 \left(\theta\psi(\theta + iu\theta b/2) + \theta\psi(\theta - iu\theta b/2) + 2s\right) \frac{du}{\pi\sqrt{1-u^2}} = 0,$$

where $\psi = \Gamma'/\Gamma$ is the di-Gamma function.
- Define

$$f(s;\theta) = b^2 \int_0^1 \left(\theta\psi(\theta + iu\theta b/2)) + \theta\psi(\theta - iu\theta b/2) + 2s\right) \sqrt{1-u^2}\frac{du}{2\pi},$$

and

$$F(s;\theta) = \int_0^s f(t;\theta)dt.$$

Large deviations for the log-Gamma polymer partition function

Conjecture (C-Mauersberger '24)

There exists $\theta_0 > 0$ *such that uniformly for* $0 < \theta < \theta_0$ *and* $\epsilon < s \leq -\theta\psi(\theta)$ *for any* $\epsilon > 0$*, we have*

$$\lim_{n\to\infty} \frac{-1}{n^2} \log \mathbb{P}\left[\log Z_n(\theta) \leq \frac{2n}{\theta} s\right] = F(s;\theta).$$

- One can check that

$$\lim_{\theta\to 0} f(s;\theta) = 2 - s - \frac{1}{s}, \quad \lim_{\theta\to 0} F(s;\theta) = \frac{s^2}{2} + \frac{3}{2} - 2s + \log s,$$

  which is precisely the large deviation rate function for last passage percolation.
- We prove this result rigorously, except for one step for which we only have heuristic evidence.

Conclusion

- We defined a **class of biorthogonal measures** and investigated **Fredholm determinant identities** for multiplicative statistics.
- Special cases of these measures characterize **Polymer Partition Functions**, other special cases are eigenvalue distributions in **random matrix models**.
- This unified framework of biorthogonal measures allows us to easily identify random matrix ensembles with polymer models and their zero-temperature limits.
- We hope that the biorthogonal structure and the double integral structure of the kernel L will help to study **asymptotic behavior of Polymer Partition functions**.

Thank you for your attention!