On Calderón-Zygmund theory for the p-Laplacian

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Ohio State University Nov 14, 2024

[1]Funding: NSF Career DMS-2044898, Armin Schikorra is an Alexander-von-Humboldt-Fellow.

Classical Calderón-Zygmund estimates

Let[2]

$$\Delta u = \mathrm{div}(F) \quad \text{in } \mathbb{R}^n$$

then

$$\int_{\mathbb{R}^n} |\nabla u|^r \lesssim_r \int_{\mathbb{R}^n} |F|^r \quad \forall r \in (1, \infty).$$

[2]Recall: $\Delta u = \mathrm{div}\nabla u = \sum_{\alpha=1}^n \partial_\alpha \partial_\alpha u$, $\mathrm{div} F = \sum_{\alpha=1}^n \partial_\alpha F^\alpha$

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Proof.

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Classical Calderón-Zygmund estimates (handwavy)

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~~Proof~~ Intuition.

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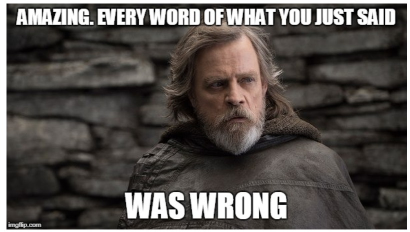

Counterexample:

$u(x) = x_1$, $F(x) = 0$.

$\blacksquare$

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Classical Calderón-Zygmund estimates

Theorem

Assume [a], $u \in W^{1,2}(\mathbb{R}^n)$ *solve*

$$\Delta u = \operatorname{div}(F) \quad \textit{in } \mathbb{R}^n \textit{ in distributional sense,}$$

$$\textit{then for } r \in (1,\infty) \quad \int_{\mathbb{R}^n} |\nabla u|^r \lesssim_r \int_{\mathbb{R}^n} |F|^r.$$

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[a] i.e. $\int_{\mathbb{R}^n} |u|^2 + |\nabla u|^2 < \infty$, in particular u is not a polynomial

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Proof.

$$\Delta u = \operatorname{div}(F)$$

$$\Leftrightarrow \quad u = -(-\Delta)^{-1}\operatorname{div}(F) \qquad (-\Delta \text{ is invertible operator})$$

$$\Leftrightarrow \quad \partial_\alpha u = -\sum_{\beta=1}^{n} \partial_\alpha (-\Delta)^{-1} \partial_\beta F^\beta$$

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Proof.

$$\partial_\alpha u = -\sum_{\beta=1}^{n} \partial_\alpha \partial_\beta (-\Delta)^{-1} F^\beta$$

Actually we have a precise formula for $(-\Delta)^{-1}$:

$$(-\Delta)^{-1} g(x) = c_n \int_{\mathbb{R}^n} |x-y|^{2-n} g(y)\, dy$$

so we have the representation formula for some 0-homogeneous $K_{\alpha\beta}(z)$

$$\partial_\alpha u(x) = -\sum_{\beta=1}^{n} \int_{\mathbb{R}^n} \frac{K_{\alpha\beta}(x-y)}{|x-y|^n} F^\beta(y)\, dy.$$

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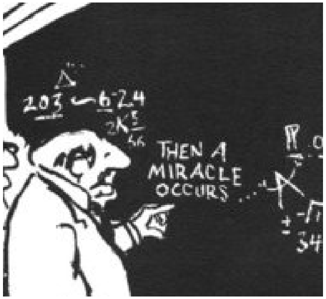
Proof.

$$\partial_\alpha u(x) = TF(x) := -\sum_{\beta=1}^{n} \int_{\mathbb{R}^n} \frac{K_{\alpha\beta}(x-y)}{|x-y|^n} F^\beta(y)\, dy.$$



Theorem (Calderón and Zygmund, 1952)

Operators like T are bounded linear operators from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, whenever $r \in (1,\infty)$.

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Calderón-Zygmund estimates on the ball

Theorem (Calderón-Zygmund)

Assume[3] $u \in W^{1,2}(\mathbb{B}^n)$ *solves in distributional sense*

$$\begin{cases} \Delta u = \mathrm{div}(F) & \text{in } \mathbb{B}^n \\ u = 0 & \text{on } \partial\mathbb{B}^n \end{cases}$$

then for $r \in (1, \infty)$ $\displaystyle\int_{\mathbb{B}^n} |\nabla u|^r \lesssim_r \int_{\mathbb{B}^n} |F|^r.$

[3] $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$

Calderón-Zygmund estimates on the ball

Theorem (Calderón-Zygmund)

Assume[3] $u \in W^{1,2}(\mathbb{B}^n)$ *solves in distributional sense*

$$\begin{cases} \mathrm{div}(A\nabla u) = \mathrm{div}(F) & \text{in } \mathbb{B}^n \\ u = 0 & \text{on } \partial\mathbb{B}^n \end{cases}$$

then for $r \in (1,\infty)$ $\quad \int_{\mathbb{B}^n} |\nabla u|^r \lesssim_{r,A} \int_{\mathbb{B}^n} |F|^r.$

if

- $A : \mathbb{B}^n \to \mathbb{R}^{n\times n}$ *symmetric, all eigenvalues bounded away from zero*
- *and* A *is Hölder continuous (easy)*
- *or* A *is BMO (fun)*

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Calderón-Zygmund estimates more general

Theorem (Calderón-Zygmund)

Assume $u \in W^{1,2}(\Omega)$ *solves in distributional sense*

$$\begin{cases} \operatorname{div}(A\nabla u) = \operatorname{div}(F) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\text{then for } r \in (1,\infty) \quad \int_\Omega |\nabla u|^r \lesssim_{r,A} \int_\Omega |F|^r.$$

if

- $A : \mathbb{B}^n \to \mathbb{R}^{n\times n}$ *symmetric, all eigenvalues bounded away from zero*
- *and* A *is Hölder continuous (easy)*
- *or* A *is BMO (fun)*
- *for some crazy set* $\Omega \subset \mathbb{R}^n$

Calderón-Zygmund estimates for the p-Laplace?

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Question (1980s)

What about Calderón-Zygmund*-theory for the p-Laplace:*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(F) & \text{in } \mathbb{B}^n \\ u = 0 & \text{on } \partial\mathbb{B}^n \end{cases}$$

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Intuition.

$$\begin{aligned} & \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(F) \\ \Leftrightarrow\quad & \cancel{\operatorname{div}}(|\nabla u|^{p-2}\nabla u) = \cancel{\operatorname{div}}(F) \\ \Rightarrow\quad & |\nabla u|^{p-1} \approx |F| \end{aligned}$$

Calderón-Zygmund estimates on the ball

Conjecture (Iwaniec 1983)

Let $p \in (1, \infty)$. Assume $u \in W^{1,p}(\mathbb{B}^n)$ solves in distributional sense

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Problem: nonlinear, so no good representation formula

Theorem (Iwaniec, Sbordone, diBenedetto, Manfredi, Mingione, and so many more)

There exists a small $\varepsilon > 0$ such that the above holds for $r \in (p - \varepsilon, \infty)$.

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For $p \neq 2$, the above does not hold for $r \approx p-1$

The Iwaniec conjecture is wrong.

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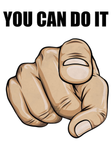

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Failure of Calderón-Zygmund estimates for p-Laplace

Theorem (S. 2024)

For any $p \in (1,\infty)$, $p \neq 2$, and any $\Lambda > 0$ and any r

$$\max\{1, p-1\} < r < \sup_{b \in (0,\infty)} \frac{p-1}{b^{p-1}+1} + \frac{b}{b+1} \quad (\text{if } p \neq 2,\ \exists \text{ such } r)$$

there exists $u, v \in W^{1,\infty}_0(\mathbb{B}^2)$ such that

$$\int_{\mathbb{B}^2} |\nabla u|^r > \Lambda \int_{\mathbb{B}^2} ||\nabla u|^{p-2}\nabla u - \nabla^\perp v|^{\frac{r}{p-1}}$$

Here

$$\nabla^\perp v = \begin{pmatrix} -\partial_y v \\ \partial_x v \end{pmatrix}$$

Thus

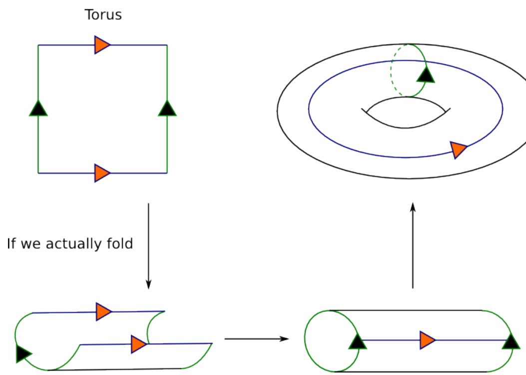
$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}(\underbrace{|\nabla u|^{p-2}\nabla u - \nabla^\perp v}_{=:F})$$

and thus we have

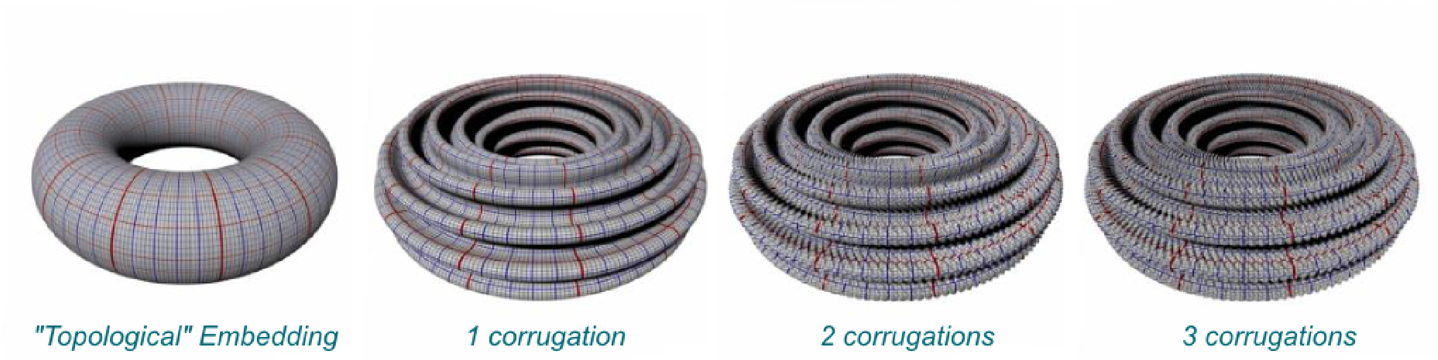
$$\int_{\mathbb{B}^n} |\nabla u|^r > \Lambda \int_{\mathbb{B}^n} |F|^{\frac{r}{p-1}}$$

Ingredients of the proof: wiggles from Convex Integration

The ideas of convex integration go back to NASH, KUIPER's isometric embedding theorem



Flat Torus topologically embedded



Flat torus isometrically embedded

Brief (incomplete!) history of Convex Integration

- The arguments by Nash and Kuiper (and Smale) were distilled by Gromov into the h-principle (and partial differential relations)
- In 2003, Müller-Šverák adapted Gromov's convex integration (and ideas from the compensated compactness theory of Tartar) to use convex integration technique to find solutions to Euler-Lagrange systems

$$\int_\Omega F(\nabla u)dx$$

  i.e. solutions to

$$\mathrm{div}(\nabla F(\nabla u)) = 0$$

  where $u : \mathbb{R}^2 \to \mathbb{R}^2$ is Lipschitz, but nowhere C^1 – even though F is strongly quasiconvex (dealing another blow to any hope for Hilbert's 19th problems for systems)
- Faraco refined those ideas, and convex integration is now a great tool to study counterexamples.

Example: Convex integration and Calderón-Zygmund

ASTALA, FARACO, SZEKELYHIDI used convex integration to find counterexamples to L^p-Calderón-Zygmund for bounded measurable coefficients.

Theorem (Positive: MEYERS 1963)

If a is bounded and measurable map $a(x) \in \{\frac{1}{K}, K\}$ *and* $u \in W^{1,2}(\mathbb{B}^2)$ *solves*

$$\mathrm{div}(a(x)\nabla u(x)) = 0 \quad \textit{in } \mathbb{B}^n$$

then $\nabla u \in L^{2+\varepsilon}_{loc}$.

Size of ε in 2D: LEONETTI-NESI, ASTALA, PETERMICHL-VOLBERG

Theorem (Negative: ASTALA, FARACO, SZEKELYHIDI, 2008)

There exists a measurable map $a(x) \in \{\frac{1}{K}, K\}$ *such that if* $u \in W^{1,2}(\mathbb{B}^2)$ *solves*

$$\mathrm{div}(a(x)\nabla u(x)) = 0 \quad \textit{in } \mathbb{B}^2$$

then

$$\int_B |\nabla u|^{\frac{2K}{K-1}} = +\infty \quad \forall \textit{balls } B \textit{ in } \mathbb{B}^2$$

The Colombo-Tione result for very weak solution

Conjecture (Iwaniec-Sbordone 1994)

For any $r > p - 1$ *assume* $u \in W^{1,r}(B)$ *solves in distributional sense*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \textit{(very weak solution) in } \mathbb{B}^n$$

then $u \in W^{1,p}_{loc}$.

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Theorem (Colombo-Tione 2022)

The above conjecture is wrong for $r \approx p-1$

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The above conjecture is wrong for $r \approx p-1$

however, we want variational solution.

Conjecture (Iwaniec 1983)

Let $p \in (1,\infty)$. Assume $u \in W^{1,p}(\mathbb{B}^n)$ solves in distributional sense

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then for $r \in (\max\{1, p-1\}, \infty)$

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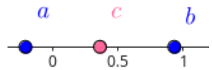
Proof: First, we wiggle in 1D

Lemma (one-dimensional wiggle)

Fix $a, b, c \in \mathbb{R}$ *and* $\lambda \in (0,1)$ *and assume that*

$$c = \lambda a + (1-\lambda) b.$$



Then *there is a piecewise affine map* $f : [0,1] \to \mathbb{R}$ *such that*

- $f(x) - cx = 0$ *for* $x = 0, 1$
- $|\{x \in [0,1] : f'(x) = a\}| = \lambda$
- $|\{x \in [0,1] : f'(x) = b\}| = 1 - \lambda$

Proof: First, we wiggle in 1D

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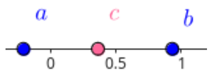

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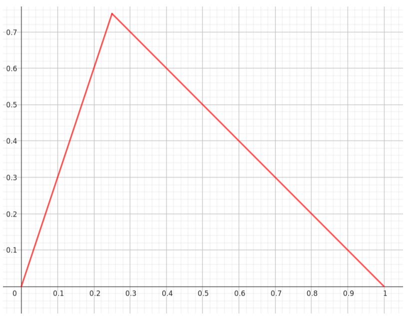
Proof.



$\lambda = \frac{1}{4}, \ c = 0, \ a = 3, \ b = -1$

∎

Proof: First, we wiggle in 1D

Lemma (one-dimensional wiggle)

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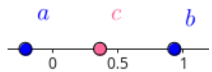
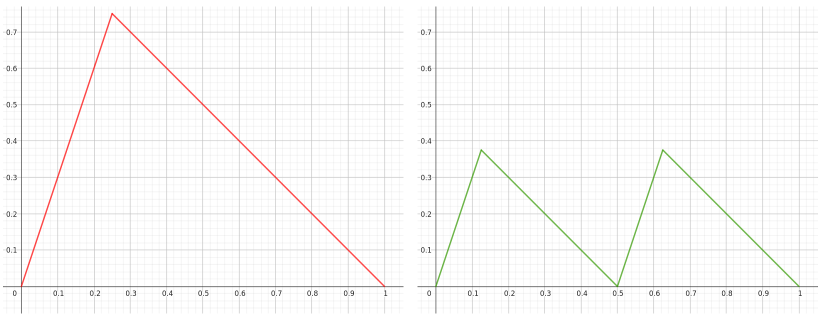


Then for any $\delta > 0$ *there is a piecewise affine map* $f : [0, 1] \to \mathbb{R}$ *such that*

- $f(x) - cx = 0$ *for* $x = 0, 1$
- $|\{x \in [0, 1] : f'(x) = a\}| = \lambda$
- $|\{x \in [0, 1] : f'(x) = b\}| = 1 - \lambda$
- $|f(x) - cx| < \delta$ *for all* x

Proof.



$$\lambda = \tfrac{1}{4},\ c = 0,\ a = 3,\ b = -1$$

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Now wiggle in nD

Lemma (one-dimensional wiggle)

Fix $a, b, c \in \mathbb{R}$ *and* $\lambda \in (0,1)$ *and assume that*

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Then for any $\delta > 0$ *there is a piecewise affine map* $f : [0,1] \to \mathbb{R}$ *such that*

- $f(x) - cx = 0$ *for* $x = 0, 1$, $|f(x) - cx| < \delta$ *for all* x
- $|\{x \in [0,1] : f'(x) = a\}| = \lambda$ *and* $|\{x \in [0,1] : f'(x) = b\}| = 1 - \lambda$

"essentially" setting $g(x) := f(\langle x, \vec{v}\rangle)\vec{w}$ and covering we get:

Lemma (n-dimensional wiggle)

Fix $A, B, C \in \mathbb{R}^{m\times n}$ *and* $\lambda \in (0,1)$ *and assume that*

$$C = \lambda A + (1-\lambda)B, \quad A - B = \vec{v} \otimes \vec{w}$$

Then for any $\delta > 0$, $\Omega \subset \mathbb{R}^n$ *there exists an p.w. affine map* $g : \Omega \to \mathbb{R}^m$ *s.t.*

- $g(x) - Cx = 0$ *for* $x \in \partial\Omega$, $|g(x) - Cx| < \delta$ *for all* x
- $|\{x \in \Omega : |\nabla g(x) - A| < \delta\}| = \lambda|\Omega|$
- $|\{x \in \Omega : |\nabla g(x) - B| < \delta\}| = (1-\lambda)|\Omega|$

Now wiggle some more

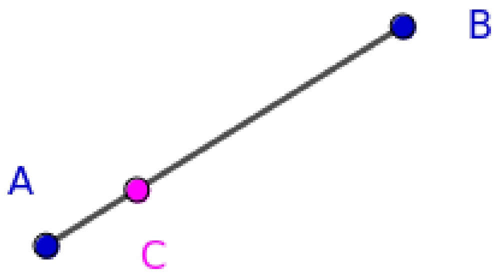
Lemma (n-dimensional wiggle)

Fix $A, B, C \in \mathbb{R}^{m\times n}$ and $\lambda \in (0,1)$ and assume that

$$C = \lambda A + (1-\lambda)B, \quad \mathrm{rank}(A-B) = 1$$

Then for any $\delta > 0$, $\Omega \subset \mathbb{R}^n$ there exists an p.w. affine map $g : \Omega \to \mathbb{R}^m$ s.t.

- $g(x) - Cx = 0$ *for* $x \in \partial\Omega$, $|g(x) - Cx| < \delta$ *for all* x
- $|\{x \in \Omega : |\nabla g(x) - A| < \delta\}| = \lambda|\Omega|$
- $|\{x \in \Omega : |\nabla g(x) - B| < \delta\}| = (1-\lambda)|\Omega|$



Now wiggle some more

Lemma (n-dimensional wiggle)

Fix $A, B, C \in \mathbb{R}^{m\times n}$ *and* $\lambda \in (0,1)$ *and assume that*

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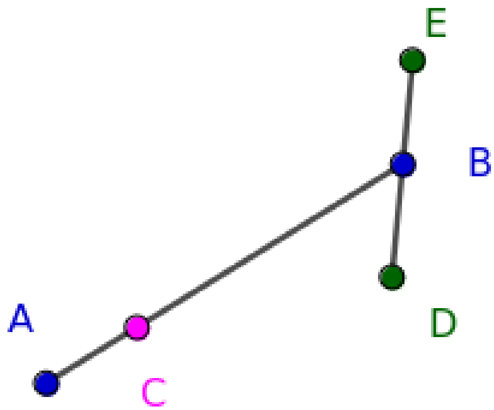

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- $|\{x \in \Omega : |\nabla g(x) - B| < \delta\}| = (1-\lambda)|\Omega|$

Lemma (iterated n-dimensional wiggle)

Fix $A, B, C, D, E \in \mathbb{R}^{m\times n}$ *and* $\lambda, \mu \in (0,1)$ *in addition to the above assume*

$$B = \mu D + (1-\mu)E, \quad \mathrm{rank}(E-D) = 1$$

Then for any $\delta > 0$, $\Omega \subset \mathbb{R}^n$ *there exists an p.w. affine map* $g : \Omega \to \mathbb{R}^m$ *s.t.*

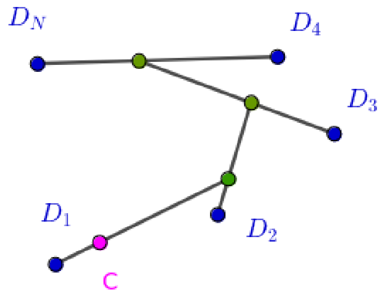
- $g(x) - Cx = 0$ *for* $x \in \partial\Omega$, $|g(x) - Cx| < \delta$ *for all* x
- $|\{x \in \Omega : |\nabla g(x) - A| < \delta\}| = \lambda|\Omega|$
- $|\{x \in \Omega : |\nabla g(x) - D| < \delta\}| = \mu(1-\lambda)|\Omega|$
- $|\{x \in \Omega : |\nabla g(x) - E| < \delta\}| = (1-\mu)(1-\lambda)|\Omega|$

Don't stop there, wiggle!

Proposition (The basic wiggle building block)

Fix a sequence $(D_k)_{k=1}^N \subset \mathbb{R}^{m\times n}$ *and* $(\lambda_i)_{i=1}^N \in (0,1)$, $\sum_{i=1}^N \lambda_i = 1$, *that are constructed via iteration rank-one convex decompositions as above, and*

$$C = \sum_{k=1}^N \lambda_k D_k$$



Then for any[3] $\delta > 0$, $\Omega \subset \mathbb{R}^n$ *there is a piecewise affine map* $g : \Omega \to \mathbb{R}^m$ *s.t.*

- $g(x) - Cx = 0$ *for* $x \in \partial\Omega$
- $|g(x) - Cx| < \delta$ *for all* x
- $|\{x \in \Omega : |\nabla g(x) - D_k| < \delta\}| = \lambda_k |\Omega|$

Such cascades of matrices are called a laminate of finite order. The process of decomposing a matrix into two (rank one connected) ones is called elementary splitting

[3] small

Step 2: Build your own laminate

Goal:

Build a laminate of finite order as above

$$0 = \sum_{k=1}^{N} \lambda_k D_k$$

such that

- **many** D_k are good 
- **few** D_k are bad 

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Observation:

If $\begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix} \in \left\{ D \approx \begin{pmatrix} z & 0 \\ 0 & -|z|^{p-2}z \end{pmatrix}, z \in \mathbb{R} \right\}$

then

$$\left| |\nabla u|^{p-2}\nabla u - \nabla^{\perp} v \right| \approx 0 \quad \ll \quad \left| |\nabla u|^{p-2}\nabla u \right|$$

So

$$D \text{ is good} \quad \Leftrightarrow \quad D \approx \begin{pmatrix} z & 0 \\ 0 & -|z|^{p-2}z \end{pmatrix}$$

Step 2: Build your own laminate

Fix $b \neq 1$. Consider sequences of $\mathbb{R}^{2\times 2}$-matrices

$$A_i = \begin{pmatrix} bi & 0 \\ 0 & i^{p-1} \end{pmatrix}, \quad i = \pm 1, \pm 2, \ldots$$

$$B_i = \begin{pmatrix} b(i-1) & 0 \\ 0 & -b^{p-1}(i-1)^{p-1} \end{pmatrix}, \quad C_i = \begin{pmatrix} -i & 0 \\ 0 & i^{p-1} \end{pmatrix}, \quad i = \pm 1, \pm 2, \ldots$$

Lemma

We can write

$$0 = \underbrace{\sum_{i=1}^{N} \lambda_i(\pm C_i) + \sum_{i=1}^{N} \mu_i(\pm B_i)} + \underbrace{\frac{1}{2}\Gamma_N A_N + \frac{1}{2}\Gamma_N(-A_N)}$$

is a laminate of finite order for any N.

The construction of the laminate

We can write (observe: this is a rank one decomposition!), since $-b < 0 < b$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix},$$

and since $-b^{p-1} < 0 < 1$

$$\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} b & 0 \\ 0 & -b^{p-1} \end{pmatrix}} + (1-\lambda) \underbrace{\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}}$$

$=A_1$

and similarly

$$\begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} -b & 0 \\ 0 & b^{p-1} \end{pmatrix}} + (1-\lambda) \underbrace{\begin{pmatrix} -b & 0 \\ 0 & -1 \end{pmatrix}}$$

$=-A_1$

for some computable $\lambda = \lambda_{b,p} \in (0,1)$.

Kill a bad guy, create a bigger bad guy

and since $-b^{p-1}i^{p-1} < i^{p-1} < (i+1)^{p-1}$

$$\underbrace{A_i} = \begin{pmatrix} bi & 0 \\ 0 & i^{p-1} \end{pmatrix} = \alpha \underbrace{\begin{pmatrix} bi & 0 \\ 0 & -b^{p-1}i^{p-1} \end{pmatrix}} + (1-\alpha) \begin{pmatrix} bi & 0 \\ 0 & (i+1)^{p-1} \end{pmatrix}$$

$=B_1$

and since $-(i+1) < bi < b(i+1)$

$$\begin{pmatrix} bi & 0 \\ 0 & (i+1)^{p-1} \end{pmatrix} = \beta \underbrace{\begin{pmatrix} -(i+1) & 0 \\ 0 & (i+1)^{p-1} \end{pmatrix}} + (1-\beta) \underbrace{\begin{pmatrix} b(i+1) & 0 \\ 0 & (i+1)^{p-1} \end{pmatrix}}$$

$=C_1$

$=A_{i+1}$

Recall: What does it mean to be good?

Recall that we have the **laminate of finite order**

$$0 = \underbrace{\sum_{i=1}^{N} \lambda_i(\pm C_i) + \sum_{i=1}^{N} \mu_i(\pm B_i)} + \underbrace{\frac{1}{2}\Gamma_N A_N + \frac{1}{2}\Gamma_N(-A_N)}$$

Apply the wiggle lemma, then we find u, v such that

$$\mathrm{dist}\left(\begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix}, \{\pm C_1, \ldots, \pm C_N, \pm B_1, \ldots, B_N, \pm A_N\} \right) \ll 1.$$

We remember that

$$\text{If } \begin{pmatrix} \partial_1 u & \partial_2 u \\ \partial_1 v & \partial_2 v \end{pmatrix} \approx$$

$$\ni \left\{ X = \begin{pmatrix} z & 0 \\ 0 & -|z|^{p-2}z \end{pmatrix}, |z| \gtrsim_b 1 \right\}$$

then

$$\left| |\nabla u|^{p-2}\nabla u - \nabla^{\perp} v \right| \approx 0 \quad \text{but} \quad \left| |\nabla u|^{p-2}\nabla u \right| \geq 1$$

We estimate

$$\int_{\mathbb{B}^2} \left| |\nabla u|^{p-2}\nabla u - \nabla^\perp v \right|^{\frac{r}{p-1}}$$

$$= \underbrace{\int_{\ } \left| |\nabla u|^{p-2}\nabla u - \nabla^\perp v \right|^{\frac{r}{p-1}}}_{\approx \delta} + \int_{\nabla(u,v)\approx \pm A_N} \Big| \underbrace{|\nabla u|^{p-2}\nabla u - \nabla^\perp v}_{\approx N^{p-1}} \Big|^{\frac{r}{p-1}}$$

$$\approx \delta + \Gamma_N \, N^r$$

where Γ_N is the prefactor/probability in front of A_N.

On the other hand

$$\int_{\mathbb{B}^2} \underbrace{|\nabla u|^r}_{\gtrsim_b 1} \gtrsim_b |\mathbb{B}^2|$$

For our (u, v) constructed from the laminate of finite order we have

$$\int_{\mathbb{B}^2} |\nabla u|^r \gtrsim_b |\mathbb{B}^2| \quad \text{and}$$

$$\int_{\mathbb{B}^2} ||\nabla u|^{p-2}\nabla u - \nabla^\perp v|^{\frac{r}{p-1}} \lesssim \delta + N^r \Gamma_N.$$

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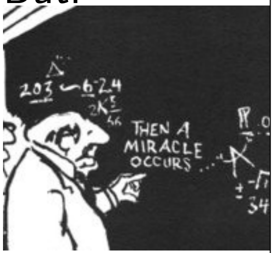

Lemma

[a] *Set*

$$q_0 := \sup_{b \in (0,\infty)} \frac{p-1}{b^{p-1}+1} + \frac{b}{b+1}$$

then for $b \in (0, \infty)$ *realizing this* sup*, the probability* Γ_N *of* $\pm A_N$ *is*

$$\Gamma_N \lesssim N^{-q_0} \quad \textit{as } N \to \infty$$

[a] This lemma is an explicit computation (essentially due to Colombo-Tione, streamlined in Kleiner-Müller-Székelyhidi-Xie)

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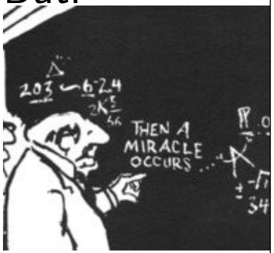

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Thus if $r < q_0$ and $\Lambda > 0$, for $N \gg 1$ we have

$$\int_{\mathbb{B}^2} |\nabla u|^r \not\lesssim \Lambda \int_{\mathbb{B}^2} ||\nabla u|^{p-2}\nabla u - \nabla^\perp v|^{\frac{r}{p-1}}$$

This disproves Iwaniec' conjecture. ∎

Thank you for your attention

Theorem (S. 2024)

For any $p \in (1,\infty)$, $p \neq 2$, and any $\Lambda > 0$ and any $r > \max\{1, p-1\}$ s.t.[4]

$$r < \sup_{b \in (0,\infty)} \frac{p-1}{b^{p-1}+1} + \frac{b}{b+1}$$

there exists $u, v \in W_0^{1,\infty}(\mathbb{B}^2)$ such that

$$\int_{\mathbb{B}^2} |\nabla u|^r > \Lambda \int_{\mathbb{B}^2} ||\nabla u|^{p-2}\nabla u - \nabla^\perp v|^{\frac{r}{p-1}}$$

- Colombo, Tione: Non-classical solutions of the p-Laplace equation (2022)
- Kleiner-Müller-Székelyhidi-Xie: Rigidity of Euclidean product structure: breakdown for low Sobolev exponents (2024)
- S.: Failure of L^r-Calderón-Zygmund estimates for the p-Laplace equation for small r (preprint 2024)

[4] observe: for $p = 2$ the supremum is 1, for $p \neq 2$ it is $> \max\{p-1, 1\}$.