NLS equations with periodic BCs: asymptotics, elliptic finite-band potentials and inverse scattering

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Outline

Introduction: Nonlinear wave equations and integrable systems

- In the 20th century, many nonlinear analogues of the classical PDEs of mathematical physics (Laplace's equation, heat equation, wave equation) were derived.
- Like their linear counterparts, these equations are also universal, i.e., they arise in many disparate physical contexts (water waves, optics, acoustics, condensed matter, cosmology, etc.)
- Some of these equations are also completely integrable infinite-dimensional Hamiltonian systems, and they possess a deep mathematical structure.
- The study of these equations has ties with many different areas of pure and applied mathematics, and offers a unique combination of interesting mathematics and practical applications.
- This talk: nonlinear Schrödinger (NLS) equation,

$$iq_t + q_{xx} - 2s|q|^2 q = 0\,,$$

$q = q(x,t) :\ \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$.

$s = +1$: defocusing NLS equation;
$s = -1$: focusing NLS equation.

NLS, Lax pair, scattering problem

- Nonlinear Schrödinger (NLS) equation:

$$iq_t + q_{xx} - 2s|q|^2 q = 0\,,$$

  $s = -1$: focusing case; $s = +1$: defocusing case.

- Universal model for the dynamics of nonlinear dispersive wave trains.
  (Arises in optics, water waves, plasmas, acoustics, Bose-Einstein condensates…)

- Lax pair: [Zakharov-Shabat 1972]

$$v_x = X\,v, \quad v_t = T\,v,$$

$$X = -iz\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ sq^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = [2iz^2 + 2zQ - i(Q^2 + Q_x)]\sigma_3.$$

- Compatibility condition: $v_{xt} = v_{tx} \;\Leftrightarrow\; X_t - T_x + [X,T] = 0 \;\Leftrightarrow\; \mathrm{NLS}[q] = 0.$
- The first half of Lax pair is the Zakharov-Shabat (ZS) scattering problem.
- The ZS scattering problem is equivalent to the eigenvalue problem for a zero-mass one-dimensional Dirac operator:

$$Lv = zv, \qquad L = i\sigma_3(\partial_x - Q)\,.$$

  z: eigenvalue, or scattering parameter,
  q: scattering potential.

IST for defocusing NLS on the line with ZBC - Direct problem

- IST:
  1. Direct problem: map q into a suitable set of "scattering data" using the 1st half of the Lax pair (i.e., the scattering problem).
  2. The time evolution of the scattering data is computed using the 2nd half of the Lax pair (easy, just like w/ FTs).
  3. Inverse problem: recover the potential from the scattering data.
- Direct problem: define Jost eigenfunctions, simultaneous solutions of the scattering problem that tend to pure Fourier modes as $x \to \pm\infty$:

$$\phi_\pm(x,t,z) = \mathrm{e}^{-izx\sigma_3} + o(1), \quad x \to \pm\infty.$$

- Continuous spectrum of the scattering problem: $\Sigma(q) = \{z \in \mathbb{C} \text{ s.t. } \phi_\pm \text{ bounded}\}$. Here $\Sigma(q) = \mathbb{R}$.
- Scattering relation: $\phi_-(x,t,z) = \phi_+(x,t,z)\,A(z,t), \quad z \in \mathbb{R}$.
  $A(z,t)$: scattering matrix.
- Analyticity: Each column of $\phi_\pm$ can be extended to either $z \in \mathbb{C}^+$ or $z \in \mathbb{C}^-$.
- Omitted for brevity: symmetries, discrete eigenvalues, norming constants, asymptotics as $z \to \infty \ldots$

Inverse problem: Matrix Riemann-Hilbert problem

- Inverse problem: recover the potential q from the scattering data.
- Rearrange the scattering relation $\phi_- = \phi_+ A$ by analyticity of the columns:

$$M(x,t,z) = \begin{cases} \left(\phi_{-,1}\,, \phi_{+,2}/a_{22}\right) \mathrm{e}^{izx\sigma_3}, & z \in \mathbb{C}^+, \\ \left(\phi_{+,1}/a_{11}\,, \phi_{-,2}\right) \mathrm{e}^{-izx\sigma_3}, & z \in \mathbb{C}^-. \end{cases}$$

- M is discontinuous across $z \in \mathbb{R}$.
- Jump condition: $M^+ = M^- V, \quad z \in \mathbb{R}$.

  $M^\pm$: projection of μ to $z \in \mathbb{R}$ from above/below.

  Jump matrix: $V(x,t,z) = \mathrm{e}^{i\theta\sigma_3} \begin{pmatrix} 1-\rho\tilde{\rho} & -\tilde{\rho} \\ \rho & 1 \end{pmatrix} \mathrm{e}^{-i\theta\sigma_3}$,

  Phase function: $\theta(x,t,z) = zx - 2z^2 t$.

  Reflection coefficients: $\rho(z) = a_{21}/a_{11}|_{t=0}, \quad \tilde{\rho}(z) = a_{12}/a_{22}|_{t=0} = -\rho^*(z)$.
- Riemann-Hilbert problem (RHP):

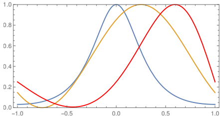
  find a sectionally analytic function from its known jump across a given contour.
- To ensure the RHP has a unique solution, one also needs:

  - normalization condition: $M(x,t,z) = I + O(1/z)$ as $z \to \infty$.

  - residue conditions if the discrete spectrum is non-empty.
- Once M is known, $q(x,t) = 2i \lim_{z\to\infty} (z M_{12}(x,t,z))$.

Outline

Semiclassical periodic focusing NLS [B-Oregero, SAPM2020]



- Focusing NLS in semiclassical scaling:

$$i\varepsilon q_t + \varepsilon^2 q_{xx} + 2|q|^2 q = 0.$$

- Real, periodic single-lobe potentials:
  there exist a single max and a single min within a period.

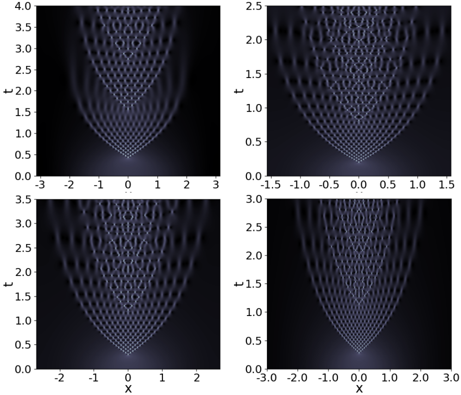
- Examples:

$$q_{\cos}(x,0) = (1+\cos x)/2,$$
$$q_{\text{expsin}}(x,0) = \mathrm{e}^{-\sin^2 x},$$
$$q_{\text{dn}}(x,0) = \mathrm{dn}(x;m).$$

- Semiclassical dynamics of periodic BCs and real line are remarkably similar.

- The focusing Zakharov-Shabat problem is non-self-adjoint.

- The spectrum, the dynamics and the analysis are complicated.



Top left: raised cosine. Top right: expsin.
Bottom left: dn. Bottom right: sech.

Focusing NLS: Lax pair, periodic BC

- Lax pair of focusing NLS in semiclassical scaling:

$$\varepsilon\phi_x = X\phi\,, \quad \varepsilon\phi_t = T\phi\,, \qquad X = -iz\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

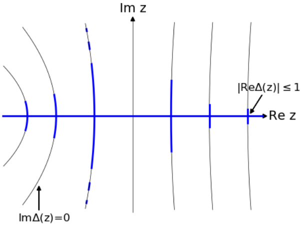
- Here: focusing NLS with periodic BC: $q(x+l,t) = q(x,t)$.
- Recall the focusing ZS problem is equivalent to the Dirac eigenvalue problem

$$L\phi = z\phi\,, \qquad L = i\sigma_3(\varepsilon\partial_x - Q)\,.$$

- Direct and inverse spectral theory for periodic KdV & periodic NLS:
  - Novikov 1974; Dubrovin 1975; Its-Matveev 1975, Its-Kotlyarov 1976; Lax 1974–1976;
  - Kac & van Moerbeke 1975; McKean & van Moerbeke 1975; McKean-Trubowitz 1976;
  - Date-Tanaka 1976; Flaschka-McLaughlin 1976; Ma-Ablowitz 1981;
  - Flaschka-Forest-McLaughlin 1980; Forest-Lee 1986; McLaughlin-Overman 1995;
  - Gesztesy-Weikard 1995–1998; Grébert-Kappeler-Mityagin, 1998–2001; ...
- The IVP for periodic NLS and periodic KdV is usually formulated in terms of the so-called algebro-geometric or finite-genus method.
- Here, no finite-genus machinery.
  Instead: Bloch-Floquet theory and Riemann-Hilbert problems.

Periodic focusing NLS: Bloch-Floquet theory

- Bloch-Floquet (normal) solutions of ZS: $\psi(x+l,z) = \rho\,\psi(x,z)$.
- Monodromy matrix: $M(z)$ s.t. $\Phi(x+l,z) = \Phi(x,z)\,M(z)$.
  $\Phi(x,z)$ = any fundamental matrix solution of ZS.
- ρ: Floquet multipliers, eigenvalues of M, satisfy
  $\rho^2 - 2\Delta(z)\,\rho + 1 = 0$.
- Floquet discriminant: $\Delta(z) = \frac{1}{2}\,\mathrm{tr}\,M(z)$.
- Bounded solutions of ZS $\Leftrightarrow$ $|\rho| = 1$,
  $\phi(x,z) = \mathrm{e}^{i\nu x}\,w(x,z)$, $w(x,z)$ l-periodic,
  ν: Floquet exponent ($\rho = \mathrm{e}^{i\nu l}$).

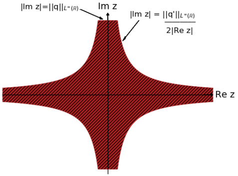


- Floquet spectrum: $\Sigma_\nu(L) = \{z \in \mathbb{C} : \Delta(z) = \cos(2\nu l)\}$.
- Lax spectrum: $\Sigma(L) = \cup_{\nu\in\mathbb{R}}\Sigma_\nu(L) = \{z \in \mathbb{C} : (\mathrm{Im}\,\Delta(z) = 0) \wedge (\mathrm{Re}\,\Delta(z) \in [-1,1])\}$.
- Can rigorously define bands and gaps, as in a self-adjoint problem.
  But here bands are not restricted to $z \in \mathbb{R}$, and can intersect.
- Band edges:
  - periodic spectrum, $\Sigma_+(L)$: $z \in \mathbb{C}$ s.t. $\Delta(z) = 1$ $\Leftrightarrow$ $\nu = 2n\pi/l$.
  - antiperiodic spectrum, $\Sigma_-(L)$: $z \in \mathbb{C}$ s.t. $\Delta(z) = -1$ $\Leftrightarrow$ $\nu = (2n+1)\pi/l$.

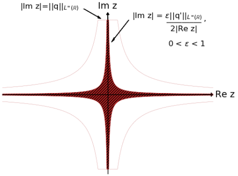
Spectrum localization in semiclassical periodic focusing NLS

[B-Oregero-Tovbis, JST2022]

- Recall:
  - The Floquet discriminant $\Delta(z)$ is an entire function.
  - Schwarz symmetry: $\Delta(z^*) = \Delta^*(z)\ \forall z \in \mathbb{C}$.
  - The real z-axis is an infinitely long band.
  - If $q \in C(\mathbb{R})$, $\Delta(z) = \cos(zl) + o(1)$, $z \to \infty$, $|\operatorname{Im} z| \leq \|q\|_\infty$.

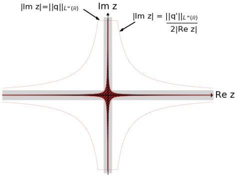


- Spectral bands can intersect at most once, and they cannot form closed curves.



- The resolvent set $\rho = \mathbb{C} \setminus \sigma(L)$ has two connected components.
- Lemma: if $q \in L^\infty([0,L])$ & $q' \in L^\infty([0,L])$,

  $$|\operatorname{Im} z| \leq \|q\|_\infty,$$
  $$|\operatorname{Re} z \operatorname{Im} z| \leq \tfrac{1}{2}\varepsilon \|q'\|_\infty.$$



- Theorem: As $\varepsilon \to 0$, the spectrum localizes to the cross $\Sigma_\infty = \mathbb{R} \cup i[-\|q\|_\infty, \|q\|_\infty]$.
  - Explicitly: $\forall d > 0\ \exists \varepsilon_* > 0$ s.t. $\Sigma(L) \subset N_d(\Sigma_\infty)\ \forall \varepsilon < \varepsilon_*$.

    (But note $\varepsilon \to 0$ is a singular limit.)

Periodic focusing NLS: Further results

- Any band intersecting the real axis is called a spine.
- Theorem:
  $\exists N_* > 0$ s.t. all bands outside R_{N_*} are spines.
  $R_N = [-N, N] \times i[-\|q\|_\infty, \|q\|_\infty] \subset \mathbb{C}$.
- Corollary:
  The number of gaps is always finite; i.e., every potential is finite-gap.
- ◦ Thus, in the focusing case, the meaningful distinction is between finite-band versus infinite-band potentials.
- Theorem:
  q is a finite-band potential iff $\exists N_* > 0$ s.t. $\Sigma(L) \setminus \mathbb{R} \subset R_{N_*}$.
- Corollary:
  If $\Sigma_+(L) \cup \Sigma_-(L) \subset \mathbb{R} \cup i\mathbb{R}$, q is a finite-band potential.
  (This will turn out to be very useful later.)
- Lemma:
  If q is real/even/odd, $\Sigma(L)$ is symmetric with respect to the imaginary z-axis, and $\operatorname{Im}\Delta(z) = 0$ for $z \in i\mathbb{R}$.

Hill's equation with complex potential, WKB [B-Oregero, SAPM2020]

- If q is real, the ZS problem is equivalent to a complexified Hill's equation:

$$-\varepsilon^2 v'' + V_\pm(x)\, v = \lambda v, \qquad \lambda = z^2,$$

$$V_\pm(x) = \pm i\varepsilon q' - q^2. \qquad (v_\pm = \phi_1 \pm i\phi_2)$$

- If q is also even, $V(x)$ is PT-symmetric: $V(-x) = V^*(x)$.
- Want to characterize the asymptotic properties of the spectrum as $\varepsilon \to 0$.
- The eigenvalue problem becomes formally self-adjoint as $\varepsilon \to 0$.
- Can also use Fourier-Hill's method to numerically compute the spectrum: Indeed, the Lax spectrum localizes to the real and imag z-axes as $\varepsilon \to 0$.
- We will use WKB with $\lambda \in \mathbb{R}$.
- Result: the real λ-axis is divided into three regions.
  $\lambda \in (-\infty, -q_{\rm max}^2)$: [no turning points] no spectrum;
  $\lambda \in (-q_{\rm min}^2, \infty)$: [no turning points] one infinitely long band;
  $\lambda \in (-q_{\rm max}^2, -q_{\rm min}^2)$: [turning points] bands and gaps, determined by

$$\Delta(\lambda) = \cosh(S_2(\lambda)/\varepsilon)\, \cos(S_1(\lambda)/\varepsilon),$$

$$S_1(\lambda) = \int_{-l/2}^{l/2} \sqrt{(q^2(x)+\lambda)_+}\, \mathrm{d}x, \qquad S_2(\lambda) = \int_{-l/2}^{l/2} \sqrt{(q^2(x)+\lambda)_-}\, \mathrm{d}x,$$

Semiclassical periodic focusing NLS: WKB results

- Recall $\Delta(\lambda) = \cosh(S_2/\varepsilon)\cos(S_1/\varepsilon)$,

$$S_1(\lambda) = \int_{-l/2}^{l/2} \sqrt{(q^2(x)+\lambda)_+}\,\mathrm{d}x, \quad S_2(\lambda) = \int_{-l/2}^{l/2} \sqrt{(q^2(x)+\lambda)_-}\,\mathrm{d}x,$$

- The above expression enables us to do asymptotics:

  - number of bands:

$$N_\varepsilon = \lfloor S_1(-q_{\min}^2)/(\pi\varepsilon) + \tfrac{1}{2}\rfloor.$$

  - band widths and gap widths,

  - number of effective solitons:

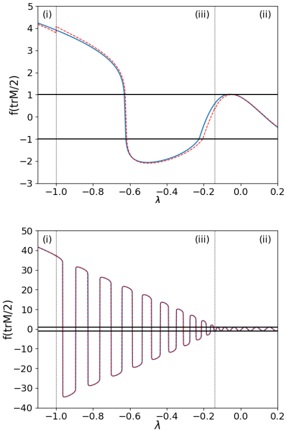
$$N_s = \lfloor S_1(\lambda_s)/(\pi\varepsilon) + \tfrac{1}{2}\rfloor,$$

$$\lambda_s = S_2^{-1}\big(\tfrac{1}{2}\varepsilon\ln[8/(\pi\kappa)]\big),$$

  κ: threshold relative bandwidth for effective solitons.

- $\lambda_s \to -q_{\min}^2$ as $\varepsilon \to 0$.
  Thus, like with KdV and defocusing NLS,
  all excitations become effective solitons as $\varepsilon \to 0$.

- Crucial difference: Here all solitons have zero velocity.
  Thus they form a coherent soliton condensate.



$$f(\Delta) = \begin{cases} \Delta, & |\Delta| \le 1, \\ \operatorname{sign}(\Delta)\log(1+|\Delta|), & |\Delta| > 1 \end{cases}$$

Red: WKB. Blue: numerical ZS.
Top: $\varepsilon = 0.3$. Bottom: $\varepsilon = 0.0255$.

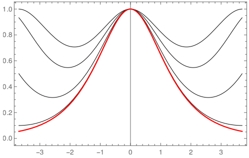
Outline

Two-parameter family of elliptic potentials

[B-Luo-Oregero-Tovbis, Adv. Math. 2023]

- The next part of the talk will focus on a two-parameter family of elliptic potentials:

$$q(x) = A\,\mathrm{dn}(x;m)\,, \qquad 0 < m < 1\,.$$



$\mathrm{dn}(x;m)$; $m = 0.5, 0.7, 0.9, 0.99, 0.999$.

- ◦ (Real) spatial period: $l = 2K(m)$. WLOG can take $A > 0$.
- Distinguished limits: $m = 0$ and $m = 1$.
- $m = 0$: $q(x) = A$, constant background. Then $\Sigma(L) \equiv \mathbb{R} \cup [-iA, iA]$.
- $m = 1$: $q(x) = A\,\mathrm{sech}\,x$. (Singular limit, b/c $l \to \infty$ as $m \to 1$.)
- ◦ Satsuma-Yajima, 1974 (using the IST on the infinite line)

  Transformation from ZS to hypergeometric ODE; scattering coeff $a(z)$ expressed in terms of Gamma functions; $b(z) = i\sin(\pi A)\,\mathrm{sech}(\pi z)$.
- ◦ Iff $A \in \mathbb{N}$, $q(x) = A\,\mathrm{sech}\,x$ is a reflectionless potential, with discrete spectrum $z_n = i(A - n + \frac{1}{2})$, $n = 1, \ldots, A$.
- But what about $m \in (0,1)$?
- Theorem: Iff $A \in \mathbb{N}$, $q(x) = A\,\mathrm{dn}(x;m)$ is a finite-band potential.
  In fact, iff $A \in \mathbb{N}$, $\Sigma(L) \subset \mathbb{R} \cup (-iA, iA)$, and q is a genus $2A - 1$ potential.

Context for the results

- The problem can be formulated in semiclassical scaling by letting $A = 1/\varepsilon$.
- Recall: $q(x)$ is real, so $v_\pm = \phi_1 \pm i\phi_2$ maps ZS into Hill's equation,
$$-y'' + V_\pm(x)\,y = \lambda\,y,$$
with $\lambda = z^2$ and the complex potential $V_\pm(x) = q^2 \pm iq'$.
- Note $V_\pm(x)$ is PT-symmetric: $\overline{V_\pm(x)} = V_\pm(-x)$.
- Thus, the results provide an example of a PT-symmetric potential of Hill's equation with purely real spectrum.
- The above ODE is also a complex deformation of Lamé's equation.
[Note $\mathrm{dn}^2(x;m) = 1 - m\,\mathrm{sn}^2(x;m)$. Lamé potentials: $u(x) = C\,\mathrm{sn}^2(x;m)$.]
- Theorem [Lamé, 1837]: $u(x)$ is finite-band iff $C = n(n+1)\,m$, $n \in \mathbb{N}$.
- Thus, the results provide the analogue of Lamé potentials for the focusing ZS operator.
- Finally, the results also provide an example of a solvable connection problem for Heun's ODE.

Trigonometric ODE; three-term recurrence relation

- The change of independent variable $x \mapsto t = 2\,\mathrm{am}(x;m)$ maps Hill's ODE into the trigonometric ODE

$$4(1-m\sin^2(t/2))\,y'' - m\sin t\,y' + \big(\lambda + A^2(1-m\sin^2(t/2)) + \tfrac{i}{2}Am\sin t\big)\,y = 0$$

(i.e., the transformation eliminates all elliptic functions and square roots).

- This trigonometric ODE is a complex deformation of Ince's equation, and is the starting point for the analysis.
- Floquet theorem: All bounded solutions have the form $y(t) = \mathrm{e}^{i\nu t}\,w(t)$, w/ $w(t+2\pi) = w(t)$ and $\nu \in \mathbb{R}$ (Floquet exponent)
- Thus, consider the Fourier series expansion

$$y(t) = \mathrm{e}^{i\nu t} \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{int}\,.$$

- The Fourier coefficients c_n satisfy the three-term recurrence relation

$$a_{n,n-1}c_{n-1} + (a_{n,n}-\lambda)c_n + a_{n,n+1}c_{n+1} = 0\,, \qquad n\in\mathbb{Z}\,,$$

$$a_{n,n-1} = -\tfrac{1}{4}m[A-(2n+2\nu-2)][A+(2n+2\nu-1)]\,,$$

$$a_{n,n} = \tfrac{1}{2}(2-m)[(2n+2\nu)^2 - A^2]\,,$$

$$a_{n,n+1} = -\tfrac{1}{4}m[A-(2n+2\nu+2)][A+(2n+2\nu+1)]\,.$$

Periodic/antiperiodic spectrum and half-infinite Fourier series

- The ZS eigenvalue problem is equivalent to the eigenvalue problem
$$B_\nu\, c = \lambda c,$$
with $c = \{c_n\}_{n\in\mathbb{Z}}$ and B_ν = unbounded tridiagonal operator
($a_{n,n'} = O(n^2)$ as $|n| \to \infty$).
- Recall: a tridiagonal matrix is "reducible" when there exists a zero along the subdiagonal or the superdiagonal.
- Note: $\nu \in \mathbb{Z}$ $\Leftrightarrow$ periodic eigenvalues; $\nu \in \frac{1}{2} + \mathbb{Z}$ $\Leftrightarrow$ antiperiodic eigenvalues.
- If $A \in \mathbb{N}$, for $\nu \in \mathbb{Z}$ and for $\nu \in \frac{1}{2} + \mathbb{Z}$ there exists a zero along both a superdiagonal and a subdiagonal.
- Lemma: If $A \in \mathbb{N}$, for each periodic/antiperiodic eigenvalue of B_ν there exists an associated eigenfunction generated by either an ascending or a descending half-infinite Fourier series.
- Thus, we can decompose the periodic/antiperiodic eigenvalue problem into the union of 2 eigenvalue problems for 2 half-infinite matrices.
- Note: in contrast to Lamé's equation, here there are no eigenfunctions given by finite Fourier sums.

Transformation to Heun's ODE; Heun connection problem

- Next, perform the further change of independent variable $t \mapsto \zeta = \mathrm{e}^{it}$.
- This maps the trigonometric ODE into Heun's ODE:

$$\zeta^2 F(\zeta)\, y'' + \zeta\, G(\zeta)\, y' + H(\zeta)\, y = 0\,,$$

primes now denote derivatives w.r.t. ζ,

$$F(\zeta) = -m\,\zeta^2 + (2m-4)\,\zeta - m\,, \quad G(\zeta) = -\tfrac{3}{2}\,m\,\zeta^2 + (2m-4)\,\zeta - \tfrac{1}{2}\,m\,,$$
$$H(\zeta) = \tfrac{1}{4}\,mA(A+1)\,\zeta^2 + (\lambda + A^2 - \tfrac{1}{2}A^2 m)\,\zeta + \tfrac{1}{4}Am\,(A-1)\,.$$

- ◦ Recall: Heun's ODE is a linear 2nd-order ODE with four regular singular points.
- The singular points are: $0,\ \zeta_\pm = \frac{1}{m}\big(m - 2 \pm \sqrt{1-m}\big),\ \infty,\ \ \zeta_\mp = 1/\zeta_\mp < 0.$
- Look for Frobenius series solutions near $\zeta = 0$ & $\zeta = \infty$:

$$y(\zeta) = \zeta^r \sum_{n=0}^\infty y_n\, \zeta^n\,, \qquad y(\zeta) = \zeta^{-\tilde r} \sum_{n=0}^\infty \tilde y_n\, \zeta^{-n}\,,$$

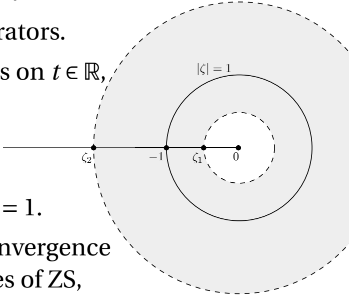
  - Frobenius exponents at $\zeta = 0$: $r_\pm = A/2,\ (1-A)/2$,
  - Frobenius exponents at $\zeta = \infty$: $\tilde r_\pm = -A/2,\ (1+A)/2$.
- If $A \in \mathbb{N}$, one of the Frobenius exponents is integer and the other half-integer, leading to periodic/antiperiodic eigenfunctions, respectively.

Augmented convergence and Perron's theorem

- The Frobenius series coefficients are given by three-term recurrence relations that are equivalent to those for the Fourier coefficients:

$$b_o y_o + c_o y_1 = 0\,, \qquad a_n y_{n-1} + b_n y_n + c_n y_{n+1} = 0\,, \quad n \in \mathbb{N}\,,$$

- Let $T_o^\pm \,\&\, T_\infty^\pm$ be the corresponding tridiagonal operators.
- An eigenfunction is a Fourier series that converges on $t \in \mathbb{R}$, i.e., a Frobenius series that converges on $|\zeta| = 1$.
- But in general, Frobenius series converge to the nearest singular point. (Recall $|\zeta_-| < 1 < |\zeta_+|$.) Neither one is sufficient to extend the series to $|\zeta| = 1$.



- The λ's for which a series has a larger radius of convergence are precisely the periodic/antiperiodic eigenvalues of ZS, which are also the eigenvalues of $T_o^\pm \,\&\, T_\infty^\pm$.
- Perron's theorem: The eigenvalues λ are given by the solutions of

$$\frac{b_o}{c_o} = \cfrac{a_1}{b_1 - \cfrac{a_2 c_1}{b_2 - a_3 c_2/(b_3 - \cdots)}}\,.$$

Truncated Heun matrices and reality of their eigenvalues

- Let $T^{\pm}_{o,N}$ & $T^{\pm}_{\infty,N}$ be the finite truncations of the half-infinite Heun matrices.
- Lemma: $T^{\pm}_{\infty,N}$ and $T^{-}_{o,N}$ are similar to real symmetric matrices:

$$T^{-}_{o,N} = \begin{pmatrix} b_o & c_o & & & \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{pmatrix} \quad \Leftrightarrow \quad J_N = \begin{pmatrix} b_o & f_1 & & & \\ f_1 & b_1 & f_2 & & \\ & \ddots & \ddots & \ddots & \\ & & f_{N-1} & b_{N-1} & f_N \\ & & & f_N & b_N \end{pmatrix},$$

$J_N = D_N^{-1}\,T^{-}_{o,N} D_N =$ unbounded Jacobi matrix, $D_N = \mathrm{diag}(1, d_1, \ldots, d_N)$,

$$f_n = \sqrt{a_n c_{n-1}}\,, \quad d_n = \sqrt{\frac{a_1 \cdots a_n}{c_o \cdots c_{n-1}}}\,, \quad n = 1, \ldots, N.$$

Similarly for $T^{\pm}_{\infty,N}$.

- Corollary: $T^{\pm}_{\infty,N}$ and $T^{-}_{o,N}$ all have real simple eigenvalues.
- This approach does not work for $T^{+}_{o,N}$ (b/c one of the products $a_n c_{n-1}$ is negative).
- Lemma: $T^{+}_{o,N}$ is an irreducibly diagonally dominant matrix.
  As a result, $T^{+}_{o,N}$ also has real simple eigenvalues. [cf. Veselic 1979]

Convergence of eigenvalues as $N \to \infty$

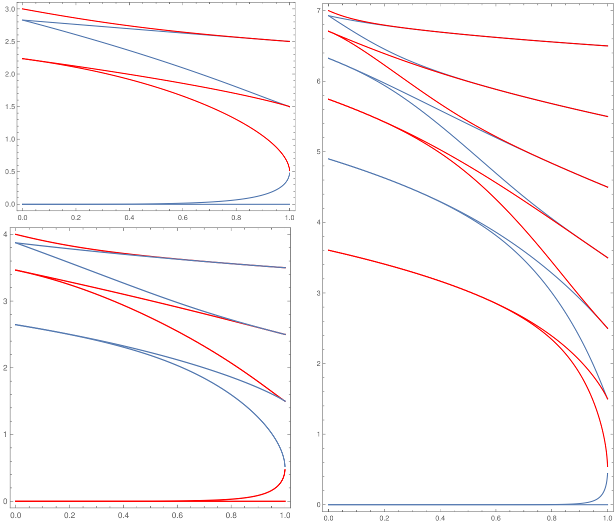
- Recall: An $N\times N$ tridiagonal matrix M is irreducibly diagonally dominant if
  (i) M is irreducible; i.e., all entries along the upper and lower diagonals are nonzero;
  (ii) M is diagonally dominant; i.e., $|m_{ii}| \ge \sum_{j\ne i} |m_{ij}|\ \forall i = 1,\dots,N$.
  (iii) There exists an $i_* \in [1,N]$ s.t. the above inequality is strict.
- The last step is to prove convergence of the eigenvalues in the limit.
- To do this, we use the concept of *generalized convergence* of closed operators between Banach spaces. [Kato]
- Theorem [Kato]: Let T & $\{T_n\}_{n\in\mathbb{N}}$ be closed operators between Banach spaces, and suppose T^{-1} exists. Then $T_n \to T$ in the generalized sense if and only if T_n^{-1} exists and is bounded for sufficiently large n and $\| T_n^{-1} - T^{-1} \| \to 0$.
- Theorem: T_o^+ is a closed operator with compact resolvent, and the geometric multiplicity of each eigenvalue is 1.
- Let $T_o^+ = T_{\rm diag} + T_{\rm off}$ & $T_n = T_{\rm diag} + P_n T_{\rm off}$, $P_n = n\times n$ truncation operator.
- Theorem: $T_n \to T_o^+$ in the generalized sense. [cf. Volkmer 2008]
- Corollary: For any fixed $M \in \mathbb{N}$, the eigenvalues $\lambda_{1,N},\dots,\lambda_{M,N}$ of $T_{o,N}^+$ converge to the eigenvalues $\lambda_1 \dots,\lambda_M$ of T_o^+ as $N\to\infty$. [cf. Volkmer 2008]
- Similar arguments apply for T_o^- and $T_\infty^\pm$.

More detailed results about the spectrum of $q(x) = A\,\mathrm{dn}(x,m)$

- Theorem: If $A \in \mathbb{N}$, then $\forall m \in (0,1)$:
  1. There are $2A$ symmetric bands on $(-iA, iA)$ separated by $2A-1$ open gaps.
  2. $\mathbb{R} \subset \Sigma(L)$ contains infinitely many interlaced periodic and antiperiodic eigenvalues, symmetrically located with respect to $z=0$, each with geometric multiplicity 2.
  3. $z=0$ is a periodic eigenvalue if $A=2n$ and an antiperiodic one if $A=2n+1$.
  4. Each real periodic/antiperiodic eigenvalue in $\mathbb{R}$ has geometric multiplicity 2 and is also an immovable Dirichlet eigenvalue.
  5. Each periodic/antiperiodic eigenvalue in $(-iA, iA) \setminus \{0\}$ has geometric multiplicity 1.
  6. There is exactly one movable Dirichlet eigenvalue in each of the $2A-1$ gaps.
  7. The point $z=0$ is a Dirichlet eigenvalue with algebraic multiplicity ≥ 2.

- Theorem: If $A \notin \mathbb{N}$, then $\forall m \in (0,1)$:
  1. There are no real periodic/antiperiodic eigenvalues.
  2. There are infinitely many spines, arising at the critical points of the Floquet discriminant.
  3. Each periodic/antiperiodic eigenvalue has geometric multiplicity 1.

Spectrum: $A \in \mathbb{N}$

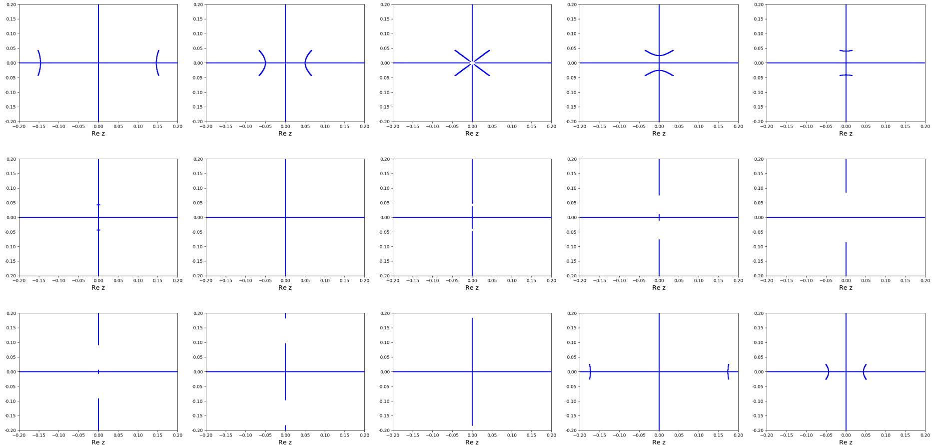


Top left: $A = 3$.
Bottom left: $A = 4$.
Right: $A = 7$.

(Eigenvalues are computed via the finite truncations.)

Periodic (red) and antiperiodic (blue) eigenvalues (vertical axis) along the imaginary z-axis as a function of m (horizontal axis).

Spectrum: $A \notin \mathbb{N}$



Lax spectrum in the complex z-plane for $m = 0.9$ and increasing values of A.
From top left to bottom right: (Eigenvalues are computed with Floquet Hill's method.)

$A = 3.99$, 3.9975, 3.9985, 3.99875, 3.99915,
3.999245, 3.999249, 3.99926, 3.9997, **4.0**,
4.0002, 4.007, 4.02, 4.99, 4.999.

(Outside the plot window: infinitely other spines off $z \in \mathbb{R}$, shrinking to zero when $A \in \mathbb{N}$.)

Outline

1. Introduction: Nonlinear wave equations and integrable systems

2. Semiclassical periodic focusing NLS: Spectral theory basics, WKB, applications

3. Elliptic finite-band potentials of the non-self-adjoint Dirac operator

4. Inverse spectral theory for periodic ZS and IST for periodic defocusing NLS
   - Direct spectral theory for periodic self-adjoint ZS
   - Inverse spectral theory for the periodic self-adjoint ZS via RHP
   - Initial value problem for periodic defocusing NLS via RHP

Introduction and motivation

- The IST on the line uses Riemann-Hilbert problem (RHP) formalism.
  But the IST with periodic BC uses the finite-genus formalism.

- Is there a way to solve the periodic problem in a way that's more similar to the IST formalism on the line?

- Not just a theoretical question, since rigorous and powerful methods of asymptotic analysis exist fo RHPs (e.g., Deift-Zhou).

- Recently, McLaughlin-Nabelek formulated the IST for Hill's equation†
  via a RHP approach and used it to solve the IVP for KdV with periodic BC.
  [K. McLaughlin and P. Nabelek, Int. Math. Res. Not. **2021**, 1288–1352 (2021)]
  (Recall: Hill's equation = 1D time-independent Schrödinger equation, $v_{xx} + (u + \lambda)\, v = 0$, with periodic potentials, which is the 1st half of the Lax pair for the (KdV) equation, $u_t + uu_x + u_{xxx} = 0$.)

- Here we generalize the approach to the ZS problem and use it to solve the IVP for the defocusing NLS equation with periodic BC.
  (See also Deconinck-Fokas-Lenells for an alternative RHP-based approach.)

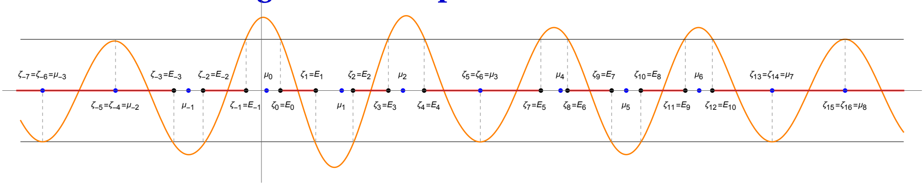
Periodic defocusing NLS: Bloch-Floquet theory

- Consider now defocusing NLS, with $\varepsilon = 1$ and periodic BC: $q(x+l,t) = q(x,t)$.
- Monodromy matrix: $M(z)$ s.t. $Y(x+l,z) = Y(x,z)\,M(z)$,
  $Y(x,z)$ is any fundamental matrix solution of ZS.
- Bloch-Floquet (normal) solutions: solutions of ZS s.t. $\psi(x+l,z) = \rho\,\psi(x,z)$.
- ρ: Floquet multipliers = eigenvalues of M, given by $\rho^2 - 2\Delta(z)\,\rho + 1 = 0$.
- Floquet discriminant: $\Delta(z) = \frac{1}{2}\operatorname{tr} M(z)$.
- For each $z \in \mathbb{R}$:
  (i) $\Delta(z)^2 > 1 \Rightarrow |\rho| \gtrless 1 \Rightarrow$ the two Bloch eigenfunctions are unbounded;
  (ii) $\Delta(z)^2 < 1 \Rightarrow |\rho| = 1 \Rightarrow$ the two Bloch eigenfunctions are bounded;
  (iii) $\Delta(z)^2 = 1 \Rightarrow \rho = \pm 1 \Rightarrow$ one of the Bloch eigenfunctions is periodic or antiperiodic.
- Lax spectrum $\Sigma(q)$: set of values $z \in \mathbb{C}$ s.t. bdd solutions exist

$$\Sigma(L) = \{ z \in \mathbb{C} :\ \Delta \in [-1,1] \}.$$

- In the defocusing case $(s = 1)$, L is self-adjoint $\Rightarrow \Sigma(q) \subset \mathbb{R}$.
- The Lax spectrum $\Sigma(q)$ comprises a countable union of spectral bands separated by (possibly degenerate) spectral gaps.

Periodic defocusing NLS: Main spectrum



- Main spectrum: set $\{\zeta_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}$, s.t. $\Delta^2(\zeta) - 1 = 0$.
  Decomposes into:
  - periodic eigenvalues:
    $z \in \mathbb{R}$ s.t. $\Delta(z) = 1 \Rightarrow \exists\, \phi(x,z)$ s.t. $\phi(x+l,z) = \phi(x)$.
  - antiperiodic eigenvalues:
    $z \in \mathbb{R}$ s.t. $\Delta(z) = -1 \Rightarrow \exists\, \phi(x,z)$ s.t. $\phi(x+l,z) = -\phi(x)$.
- The periodic/antiperiodic eigenvalues are the edges of the spectral bands.
- The main spectrum divides into:
  - degenerate band edges: $\hat{\zeta}_j$ s.t. $\zeta_j = \zeta_{j+1}$ or ζ_{j-1},
  - nondegenerate band edges: $E_j = \zeta_j$ s.t. $\zeta_j \neq \zeta_{j\pm1}$.
- However, knowledge of the main spectrum is not sufficient to reconstruct the potential. Additional information must be provided.

Dirichlet spectrum; finite-genus trace formulae

- Dirichlet BC with base point x_o: $[v = (v_1, v_2)^T]$

$$\mathrm{BC_{Dir}}(x_o): \quad v_1(x_o,z) + v_2(x_o,z) = v_1(x_o+l,z) + v_2(x_o+l) = 0\,.$$

- Dirichlet spectrum, $\{\mu_j(x_o)\}_{j\in\mathbb{Z}}$: $z\in\mathbb{C}$ s.t. $\exists\, v(x,z)$ satisfying $\mathrm{BC_{Dir}}(x_o)$.
- Properties of the Dirichlet spectrum:
  - All Dirichlet eigenvalues are real; each lies inside a spectral gap.
  - There is exactly one Dirichlet eigenvalue of each type in each spectral gap.
  - Dirichlet eigenvalues in the open spectral gaps are movable, depend on x_o.
  - All other Dirichlet eigenvalues are fixed, are at a band edge, independent of x_o.
- Finite-genus potentials: q s.t. there is only a finite number g (genus) of non-degenerate spectral gaps.
- Finite-genus trace formulae:

$$q(x) + q^*(x) = 2\sum_{j=1}^{g}(E_{2j} + E_{2j+1} - 2\mu_j(x))\,,$$

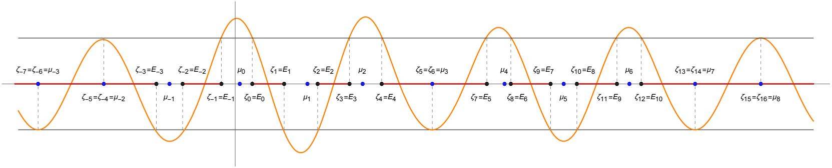
$$q(x) - q^*(x) = -2i\sum_{j=1}^{g}(E_{2j} + E_{2j+1} - 2\eta_j(x))\,.$$

- Auxiliary Dirichlet spectrum, $\{\eta_j(x_o)\}_{i\in\mathbb{Z}}$: $z\in\mathbb{C}$ s.t. $\exists\, v(x,z)$ satisfying $\mathrm{BC_{Dir,*}}(x_o)$,

$$\mathrm{BC_{Dir,*}}(x_o): \quad v_1(x_o,z) + iv_2(x_o,z) = v_1(x_o+l,z) + iv_2(x_o+l,z) = 0.$$

Direct spectral theory for periodic ZS: spectral data

- Hereafter, let $\{\mu_j\}_{j\in\mathbb{Z}}$ be the main Dirichlet spectrum with base point $x_o = 0$.
- The sequence of $\{\zeta_j\}_{j\in\mathbb{Z}}$ extends to infinity in both directions.



(For Hill's Eq. & KdV, the spectrum is bounded from below along the real λ axis.)

- Denote g_- the largest integer s.t. $\zeta_{2j-1} = \zeta_{2j}$ for all $j < g_-$
  & g_+ the smallest integer s.t. $\zeta_{2j-1} = \zeta_{2j}$ for all $j > g_+$.
  (Either or both g_+ and g_- can be finite or infinite.)
- Equivalent representation of the Lax spectrum:

$$\Sigma(L) = (-\infty, E_{2g_- -1}] \cup [E_{2g_+}, \infty) \cup \left(\bigcup_{j=g_-}^{g_+ -1} [E_{2j}, E_{2j+1}] \right).$$

- Define the spectral data associated to the potential q as:

$$S(q) = \{E_{2j-1}, E_{2j}, \mu_j, \nu_j\}_{j=g_-,\ldots,g_+},$$

  with ν_k given below.
- Claim: $S(q)$ is sufficient to uniquely reconstruct the potential.

Modified ZS, Bloch-Floquet eigenfunctions

- $Y(x,z) = (y_1, y_2)$: fundamental matrix solution of ZS s.t. $Y(0,z) = I$.
- $Y(x,z)$ is an entire function of z. Also, $M(z) = Y(l,z)$, so $\Delta(z) = \frac{1}{2}\,\mathrm{tr}\,Y(l,z)$.
- Can rigorously compute the asymptotic behavior of $Y(x,z)$ as $z \to \infty$.
- Problem: The asymp. behavior of $Y(x,z)$ contains $q(0)$ and $\overline{q(0)}$, which are unknown. Therefore $Y(x,z)$ is not useful to reconstruct the potential. (This is a fundamental but typical obstacle in BVPs.)
- Introduce a unitary transformation to define modified Bloch-Floquet solutions:

$$\tilde{Y}(x,z) = U\,Y(x,z)\,U^{-1}, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

- $\tilde{Y}(x,z)$ solves a modified ZS problem:

$$\tilde{y}_x = U(-iz\sigma_3 + Q)\,U^{-1}\,\tilde{y}.$$

- Lemma: The μ_j are the values $\mu \in \mathbb{R}$ s.t. $\tilde{y}_{12}(l,\mu) = 0$. (Note $\tilde{y}_{12}(0,z) = 0\ \forall z \in \mathbb{C}$.)
- Def: $\nu_k = -\,\mathrm{sign}(\log|\tilde{y}_{22}(l,\mu_k)|)$. Explicitly:
  $\nu_k = \pm 1$ if $\tilde{y}_2(x,\mu_k)$ is associated with the Floquet multiplier $\rho^{\pm 1}(\mu_k)$;
  $\nu_k = 0$ if μ_k lies on an edge of a (degenerate or non-degenerate) spectral gap.

Modified Bloch-Floquet eigenfunctions

- Define the modified Bloch-Floquet eigenfunctions $\psi^{\pm}(x,z) = (\psi_1^{\pm},\psi_2^{\pm})^T$ as the solutions of modified ZS s.t.

$$\psi^{\pm}(x+L,z) = \rho_{\pm}(z)\psi^{\pm}(x,z)\,, \qquad \psi_1^{\pm}(0,z) = 1.$$

- Proposition:

$$\psi^{\pm}(x,z) = \tilde{y}_1(x,z) + \frac{\rho^{\pm 1}(z) - \tilde{y}_{11}(l,z)}{\tilde{y}_{12}(l,z)}\tilde{y}_2(x,z)\,,$$

- Also define a matrix Bloch-Floquet eigenfunction:

$$\Psi(x,z) = \begin{cases} (\psi^-,\psi^+), & z\in\mathbb{C}^+\,, \\ (\psi^+,\psi^-), & z\in\mathbb{C}^+\,. \end{cases}$$

- Can obtain the asymptotic behavior of $\Psi(x,z)$ as $z\to\infty$ from that of $Y(x,z)$.
- Result: $\Psi(x,z)$ has "nice" asymptotics as $z\to\infty$, unlike $Y(x,z)$ or $\tilde{Y}(x,z)$.
- Reconstruction formula:

$$Q(x) = \lim_{z\to\infty} 2iz\,[\sigma_3, U^{-1}\Psi(x,z)\,\mathrm{e}^{izx\sigma_3}]\,.$$

- Since $\tilde{y}_{12}(l,z) = 0$ at $z = \mu_j$, either $\psi^{\pm}(x,z)$ has a pole at $z = \mu_j$ unless $\nu_j = 0$.
- Ψ is also discontinuous on $z\in\mathbb{R}$.

Inverse spectral theory via RHP: Preliminaries

- Goal: define a RHP for the matrix Bloch-Floquet function $\Psi(x,z)$.
- Problem #1: $\Psi(x,z)$ has a possibly infinite number of poles.
  Problem #2: $\det\Psi(x,z)\neq 1$.
- We deal with both issues by performing a further change of dependent variable.
- Take $E_{-1}\le 0\le E_0$ for concreteness, and $\Delta(E_{-1})=\Delta(E_0)=1$. If $\sigma_0=-1$, let

$$f^0(z)=\tfrac{1}{2}\prod_{\nu_j=0}\frac{l}{\pi j}(\mu_j-z)\,,\quad f^+(z)=\prod_{\nu_j=1}\frac{l}{\pi j}(\mu_j-z)\,,\quad f^-(z)=l(z-\mu_0)\prod_{\substack{j\neq 0\\ \nu_j=-1}}\frac{l}{\pi j}(\mu_j-z)\,.$$

  (If $\sigma_0=1$ or $\sigma_0=0$, the factor $l(z-\mu_0)$ appears respectively in f^+ or f^0 instead of f^-.)
- Then let $B(z)$ be the 2×2 matrix-valued function

$$B(z)=\begin{cases}\dfrac{i\sqrt{f^0(z)}}{\sqrt[4]{\Delta(z)^2-1}}\,\mathrm{diag}(f^-,f^+), & z\in\mathbb{C}^+,\\[2ex] \dfrac{\sqrt{f^0(z)}}{\sqrt[4]{\Delta(z)^2-1}}\,\mathrm{diag}(f^+,f^-), & z\in\mathbb{C}^-,\end{cases}$$

- Lemma: $B(z)$ is completely determined by just the spectral data $S(q)$.
  (Explicit expression omitted for brevity.)

Inverse spectral theory: Formulation of the RHP

- Sectionally meromorphic 2×2 matrix $\Phi(x,z)$:

$$\Phi(x,z) = \Psi(x,z)\,B(z)\,\mathrm{e}^{izx\sigma_3}\,, \quad z \in \mathbb{C}\,.$$

- Jump matrix:

$$V(x,z) = \begin{cases} (-1)^{n+m(z)}\,i\mathrm{e}^{-2izx\sigma_3}\sigma_1\,, & z \in (E_{2n-1}, E_{2n})\,, \\ (-1)^{n+m(z)} \begin{pmatrix} f^-(z)/f^+(z) & 0 \\ 0 & f^+(z)/f^-(z) \end{pmatrix}, & z \in (E_{2n-2}, E_{2n-1})\,. \end{cases}$$

- Counting function $m(z)$:
  - If $\nexists \mu_j$ with $\nu_j = 0$, let $m(z) = 0$.
  - If $\exists \mu_j$ with $\nu_j = 0$, let $\mu_* =$ value of μ_j closest to zero with $\nu_j = 0$. Then

$$m(z) = \begin{cases} \big|\{\mu_j:\ z < \mu_j < \mu_* \;\&\; \nu_j = 0\}\big|, & z < \mu_*\,, \\ \big|\{\mu_j:\ \mu_* \le \mu_j \le z \;\&\; \nu_j = 0\}\big|, & z \ge \mu_*\,. \end{cases}$$

- Also needed:
  - $R = \max_{g^- \le k \le g^+} \{E_{2k} - E_{2k-1}\}$,
  - $D_j =$ open disc of radius R centered at $c_j = \frac{1}{2}(\zeta_{2j} + \zeta_{2j-1})$,
  - $\mathscr{D} = \mathbb{R} \cup \bigcup_{j=g^-}^{g^+} \bar{D}_j$.

Inverse spectral theory: RHP

Theorem

The 2×2 *matrix-valued function* $\Phi(x,z)$ *solves the following RHP:*

RHP

Find a 2×2 *matrix-valued function* $\Phi(x,z)$ *such that*

1. $\Phi(x,z)$ *is an analytic function of* z *for* $z\in\mathbb{C}\setminus\mathbb{R}$.
2. $\Phi^{\pm}(x,z)$ *are continuous functions of* z *for* $z\in\mathbb{R}\setminus\{E_k\}$, *and have at worst quartic root singularities on* $\{E_k\}$.
3. $\Phi^{\pm}(x,z)$ *satisfy the jump relation*
$$\Phi^{+}(x,z)=\Phi^{-}(x,z)\,V(x,z),\qquad z\in\mathbb{R}.$$
4. *As* $z\to\infty$ *with* $z\in\Omega_s$ *for fixed* $s>0$, $\Phi(x,z)$ *has the following asymptotics:*
$$\Phi(x,z)=\tilde{U}\,(I+O(1/z))\,B(z)\,.$$
5. *There exist positive constants* c *and* M *such that* $|\phi_{ij}(x,z)|\le M\mathrm{e}^{c|z|^2}\ \forall z\in\mathbb{C}\setminus\mathscr{D}$.

The proofs of Conditions 1–4 are straightforward.
Condition 5 uses the Phragmen-Lindelöf theorem (as in McLaughlin-Nabelek).

Inverse spectral theory: RHP (continued)

Lemma

If $\Phi(x,z)$ *solves the RHP, then* $\det\Phi(x,z) \equiv 1$.

Theorem

For any fixed $x \in \mathbb{R}$*, the solution of the above RHP is unique.*

Corollary

The potential matrix $Q(x)$ *of the ZS problem is obtained from the solution* $\Phi(x,z)$ *of the above RHP as*

$$Q(x) = \lim_{z\to\infty} 2iz\,[\sigma_3, U^{Z}{-}1\Phi(x,z)B^{-1}(z)]\,.$$

IVP: time-dependent Bloch-Floquet eigenfunctions

- Next: apply the formalism to solve the IVP for defocusing NLS with periodic BC.
- $\Delta(z)$, and thus the ζ, are time-independent. However, the μ are not.
- An equivalent formulation to the zero-curvature condition is the Lax equation:

$$L_t = [A, L]\,, \qquad A = 2i\sigma_3\partial_x^2 + 2iQ\sigma_3\partial_x - i(Q^2 - Q_x)\sigma_3\,.$$

- Lemma: the Dirichlet eigenvalues satisfy

$$\frac{\partial\mu_n}{\partial t} = \sigma_n(t)c_1(\mu_n)\left(\rho(\mu_n) - \rho^{-1}(\mu_n)\right)/\tilde{y}'_{12}(L,t,\mu_n),$$

$$c_1(z) = -2z^2 - iz(q^*(0,t) - q(0,t)) - |q(0,t)|^2 - \tfrac{1}{2}(q_x(0,t) + q_x^*(0,t))\,.$$

- Lemma: The time-dependent Bloch-Floquet solutions $\psi^\pm(x,t,z)$ satisfy the ODE

$$\psi_t^\pm(x,t,z) + \alpha^\pm(t,z)\psi^\pm(x,t,z) = \tilde{A}\psi^\pm(x,t,z),$$

$$\alpha^\pm(t,z) = 2zq_{\mathrm{re}}(0,t) + q_{x,\mathrm{im}}(0,t) + \left(2z^2 + 2zq_{\mathrm{im}}(0,t) + |q(0,t)|^2 + q_{x,\mathrm{re}}(0,t)\right)\psi_2^\pm(0,t,z).$$

- As $z\to\infty$, $\alpha^\pm = \pm 2iz^2 + O(1/z)$, $\quad z\in\Omega_s^+$; $\quad \alpha^\pm = \mp 2iz^2 + O(1/z)$, $\quad z\in\Omega_s^-$.
- Let $e^\pm(t,z) = \exp\left[\int_0^t \alpha^\pm(t',z)\,\mathrm{d}t'\right]$.
- $e^\pm(t,z)$ have various analyticity and asymptotic properties (omitted for brevity).

Initial value problem: time-dependent RHP

- Define the time-dependent Bloch-Floquet solutions as
$$\check{\psi}^{\pm}(x,t,z) = \psi^{\pm}(x,t,z)\, e^{\pm}(t,z)\,.$$
- Lemma: $\check{\psi}^{\pm}(x,t,z)$ satisfy the modified Lax pair
$$\tilde{L}\check{\psi}^{\pm} = z\check{\psi}^{\pm}\,, \qquad \check{\psi}^{\pm}_t = \tilde{A}\check{\psi}^{\pm}\,,$$
where $\tilde{L} = ULU^{-1}$ and $\tilde{A} = UAU^{-1}$.
- $e^{\pm}(t,z)$ "fix" the singularities of $\check{\psi}^{\pm}(x,t,z)$ to be those of $\psi^{\pm}(x,0,z)$.
- Define $\check{V}(x,t,z)$ like $V(x,z)$, but with $\mathrm{e}^{-2izx\sigma_3}$ replaced by $\mathrm{e}^{-2i(zx+2z^2t)\sigma_3}$.
- Define $\check{\Psi}(x,t,z)$ like $\Psi(x,z)$, but with $\psi^{\pm}_j$ replaced by $\check{\psi}^{\pm}_j$.
- Define $\check{\Phi}(x,t,z)$ as $\check{\Phi}(x,t,z) = \check{\Psi}(x,t,z)\,B(z)\,\mathrm{e}^{i(zx+2z^2t)\sigma_3}$.

Theorem

Let $q(x,t)$ be the solution to defocusing NLS with initial data $q(x,0)$.

There exists a solution $\check{\Phi}(x,t,z)$ to the RHP obtained by replacing $V(x,z)$ with $V(x,t,z)$.

The potential matrix $Q(x,t)$ of ZS is obtained in terms of any solution $\check{\Phi}(x,t,z)$ of the above RHP by the same expression as before.

Conclusions

- Related results (omitted for brevity):
  - Relation with matrix Baker-Akhiezer eigenfunctions and the theta-function representation of the solution of the NLS equation.
  - One could use the second set of Dirichlet eigenvalues instead.
    It works just as well.
  - RHP formulation of periodicity condition in space and time.
  - Applications to soliton gases.

- Many related open problems:
  - Rigorous validation of the WKB approximations.
  - IST via RHP for periodic focusing NLS and other integrable PDEs.
  - Further applications.
  - Lots of other open questions and possible extensions/applications.

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Thank you for your attention!