Obstructions to Frame Vectors and Group Representations

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Let H a Hilbert space. A countable collection $\{f_n\}$ of vectors is called a frame if there are $A, B > 0$ such that for every vector $v \in H$ we have:

$$A||v||^2 \leqslant \sum_{n \in S} | < v, f_n > |^2 \leqslant B||v||^2 \tag{1}$$

If $A = B = 1$ the frame is called Parseval.

Introduced in R.J Duffin, A.C.Schaeffer *A class of nonharmonic Fourier series*. Trans. AMS. 72, 1952.

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Examples:

- An orthonormal basis $\{x_j\}$ is a Parseval frame:

$$||x||^2 = \sum_{n=1}^{\infty} |< x, x_n >|^2.$$

- Gabor Systems: Let $a, b > 0$ and $g \in L^2(\mathbb{R})$. A Gabor system:

$$g_{m,n}(\xi) = e^{2\pi i m b \xi} g(\xi - na), \quad m, n \in \mathbb{Z}.$$

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- Wavelets:

$$\Psi_{j,k}(s) = 2^{i/2}\psi(2^j - k), \quad \psi \in L^2(\mathbb{R}).$$

Many types: Haar, Daubechies (compactly supported), Morlet, etc.

One can define frame operator:

$$\theta : H \to \ell^2(S), \quad \theta(v)(m) = < v, f_m > .$$

θ is well-defined and one-to-one (isometry if frame is Parseval)

Theorem

If $\mathcal{A} = \{A_i\}_{i \in I}$ is a family of isometries on a Hilbert space H, having a finite, invariant (non-trivial) subspace K, then for every countable subset S of I and for every $v \in H$, the collection $\{A_i v\}_{i \in S}$ cannot be a frame.

Here invariant space means: $A_i^*(K) \subset K$.

Sketch of proof:
Notice that if H is finite dimensional one cannot have $||f_n|| = c > 0$ for all n.

Then if $\{A_i v\}$ is a frame we have:

$\{A_i(pv)\}_{i \in S}$ is a countable frame ($p : V \to K$ projection) with:

$$||A_i(pv)|| = ||pv|| = \text{constant}.$$

So $||pv|| = 0$ and hence $< A_i v, w >= 0, \forall w \in K$. The frame condition implies $w = 0$.

Definition

Let G be a countable discrete group G and $\pi : G \to U(H)$ a unitary representation. A vector $v \in H$ is called a frame vector if $\{\pi(g)v\}_{g \in G}$ is a frame. If this frame is Parseval, we will call v a Parseval frame vector.

One can consider projective representations too:

$$\pi_g \pi_h = \mu(g,h)\pi_{gh}, \quad \mu(g,h) \in \mathbb{C}, \quad |\mu(g,h)| = 1$$

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Corollary

Let G be a countable group and $\pi : G \to U(H)$ a projective representation such that $\{\pi_g\}_g$ has an eigenvector w. Then for every $v \in H$, the set $\{\pi_g v\}_{g\in G}$ cannot be a frame.

Examples:

- Let G locally compact acting through measure-preserving transformations on X.

$$\pi_g : L^2(X,\mu) \to L^2(X,\mu), \pi_g(\phi)(x) = \phi(g^{-1} \cdot x)$$

$\phi \equiv 1$ is an eigenvector $\Rightarrow$ No $\Psi \in L^2(X,\mu)$ is a frame vector.

- Let $G = SL_2(\mathbb{Z})$ acting on hyperbolic space $\mathbb{H}$:

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

Let $H_k = \{f : \mathbb{H} \to \mathbb{C} \mid f \text{ is holomorphic and } ||f||_k^2 < +\infty\}$,

$$||f||_k^2 = \int_{\mathbb{H}} |f(z)|^2 y^k \frac{dxdy}{y^2}.$$

If $j(g,z) = cz + d$,

$$\pi_k : SL_2(\mathbb{Z}) \times H_k \to H_k,$$
$$\pi_k(g)f(z) = j(g^{-1}, z)^{-k} f(g^{-1} \cdot z).$$

Fixed points form the (finite dimensional) space of modular forms of weight k:

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

For example the Eisenstein series:

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} (mz+n)^{-k}$$

Definition

A countable, discrete group G satisfies property (T) if for any unitary representation $\pi : G \to U(H)$ which has almost invariant vectors in H (i.e. a collection of $\{v_n\}$ with $||v_n|| = 1$ and

$$||\pi(g)v_n - v_n|| \to 0 \quad \forall g \in G), \tag{2}$$

there is a nonzero G invariant vector w, i.e. $\pi(g)w = w$ $\forall\, g \in G$.

If N is a subgroup of G and condition (2) implies $\pi(n)w = w$ for some $w \neq 0$ and for all $n \in N$, we will say that the pair (N, G) has relative property (T).

Examples:

- Compact topological groups: finite, $\mathbb{S}^1, \mathbb{Z}_p$;
- $SL_n(\mathbb{R})$ for $n \geqslant 3$ and their lattices;
- $SO(p, q)$ for $p > q \geqslant 2$, $Sp_{2n}(\mathbb{R})$;
- $(\mathbb{R}^n \rtimes SL_n(\mathbb{R}^n), \mathbb{R}^n)$ relative property (T);

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Recall left regular representation of a countable discrete group G:

$$\lambda : G \to U(l^2(G))$$
$$\lambda_g(\varphi)(h) = \varphi(g^{-1}h), \forall g, h \in G.$$

Definition

Let G be a countable, discrete group G. We say G is amenable if the left regular representation has almost invariant vectors (equivalent to the trivial representation is weakly contained in the left regular representation)(Dixmier's condition).

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Examples:

- Compact (finite) topological groups;
- All finitely generated abelian groups;
- All solvable groups;
- All groups of polynomial growth, subexponential growth;

Definition

A countable discrete group G has property **NFT** (non frame T) if any unitary representation which has almost invariant vectors **does not** have a frame vector.

If any unitary representation of G with almost invariant vectors **does not** have a Parseval frame vector then we say that G has property **NPT** (non Parseval T).

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Example: (N, G) has relative property (T) then it has **NFT**.

Theorem

Let G be a countable discrete group. Then the following are equivalent:
i) G is amenable;

ii) There is a unitary representation $\pi : G \to U(H)$ having an ONB vector and which has almost invariant vectors;

iii) There is a unitary representation $\pi : G \to U(H)$ having a Parseval frame vector and which has almost invariant vectors;

iv) There is a unitary representation $\pi : G \to U(H)$ having a frame vector and which has almost invariant vectors;

Idea of proof:
iv) implies i). Let $v_n \neq 0$, be almost invariant vectors.

Show $\theta(v_n)$ is almost invariant for λ_G.

$\theta(v_n) \neq 0$ since θ is injective.

$||\theta||||\pi_g v_n - v_n|| \geqslant ||\theta(\pi_g v_n) - \theta(v_n)|| = ||\lambda_G(\theta(v_n)) - \theta(v_n)|| \geqslant 0.$

Let X= topological space.
A kernel of positive type is is a continuous function

$\Phi : X \times X \to \mathbb{C}$ such that:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \bar{c}_j \phi(x_i, x_j) \geqslant 0, \text{for any } x_1, \ldots x_n \in X, \ c_1, \ldots c_n \in \mathbb{C}.$$

Example: If H is a Hilbert space and $f : X \to H$ continuous. Then:

$$\Phi(x, y) =< f(x), f(y) > .$$

is a kernel of positive type.

Theorem (Gelfand, Naimark,Segal construction)

Let Φ be a kernel of positive type on a topological space X.

Then there is a Hilbert space and a continuous mapping $f : X \to H$ with the following properties:

i) $\Phi(x, y) =< f(x), f(y) >$ for all x, y in X;

ii) The linear span of $\{f(x) \mid x \in X\}$ is dense in H.

Moreover the pair (H, f) is unique up to canonical isomorphism.

Definition

Le G be a discrete group. A function $\phi_G \to \mathbb{C}$ is said to be of positive type if the kernel defined by:

$$(g_1, g_2) \to \phi(g_2^{-1} g_1),$$

is of positive type.

Definition

Let G a countable, discrete group G. We say G has Haagerup property if there exists a sequence of positive definite functions $\phi_n : G \to \mathbb{C}$ such that:
i) $\forall\ n \in \mathbb{N}\ \phi_n(e) = 1$ (normalization), $\phi_n \to 1$ pointwise on G, and
ii) $\forall\ n \in \mathbb{N}\ \phi_n$ vanishes at infinity i.e., $\forall\ \epsilon > 0$ the set $\{g \in G \mid\ |\phi_n(g)| > \epsilon\}$ is finite.

Origin: Negation of (T)-property: whenever a sequence of continuous, normalized, positive definite functions on G converges to 1 uniformly on compact subsets, it converges to 1 uniformly on G.

For countable groups:

Amenable Groups $\subset$ Groups with Haagerup property.

Strict inclusion: F_n, $SL_2(\mathbb{Z})$, Thompson Groups T and V.

Han and Larson:

$$\rho : G \to U(H) \quad \text{with Parseval frame vectors}$$

Then:

$$\rho^{\infty} := \oplus_{n=1}^{\infty} \rho : \bigoplus_{n=1}^{\infty} H \to \bigoplus_{n=1}^{\infty} H \quad \text{no Parseval frame}$$

In particular λ_G^{∞} has no Parseval frame vectors on $\bigoplus_{n=1}^{\infty} \ell^2(G)$

(we don't know if they have frame vectors).

If G has Haagerup, the GNS construction applied to ϕ_n gives

$$\pi_n : G \to U(H_n), ||\xi_n|| = 1,$$

such that:

$$\phi_n(g) = < \pi_n(g)\xi_n, \xi_n > \quad \text{for every } g \text{ in } G$$

$$H_n = \overline{\text{span}\{\pi_n(g)\xi_n \mid g \in G\}}$$

One can define a new representation:

$$\pi : G \to \bigoplus_{n=1}^{\infty} H_n \quad \pi(g)(\oplus v_n) = \oplus_n \pi_n(g) v_n$$

Theorem

Let G be a countable, discrete group which is nonamenable and satisfies Haagerup property. Then G admits a non trivial representation without Parseval frame vectors which is not weakly equivalent to λ_G^∞. Moreover, if G is torsion free then this representation is faithful.

- $w_n = (0, \ldots, 0, \xi_n, 0, \ldots)$ is almost invariant:

$$\begin{aligned} &||\pi(g)w_n - w_n||^2 = ||\pi_n(g)\xi_n - \xi_n||^2 = \\ &= ||\xi_n||^2 + ||\pi_n(g)\xi_n||^2 - 2\,\mathrm{Re} < \pi_n(g)\xi_n, \xi_n >= \\ &= 2 - 2\,\mathrm{Re}\,\phi_n(g) \to 0 \end{aligned}$$

  Since G is nonamenable, our theorem implies representation has no Parseval frame vectors.
- If G is torsion free, then representation π is faithful.

Proof.

If $\pi_n(g) = Id =$ identity operator, then $\phi_n(g) = 1$, so $\phi_n(g^k) = 1$ for all k. But ϕ_n vanishes at infinity, so the set $\{g^k\}$ must be finite, which is not possible unless $g = 1$. □

- Representation π is not weakly equivalent to λ_G^∞.

Proof.

π had almost invariant vectors, so we would get that λ_G^∞ would have almost invariant vectors.

This implies λ_G would have almost invariant vectors which implies G was amenable, contradiction. □

Weak frame condition:

$$0 < \sum_{n \in S} | < v, f_n > |^2 < \infty. \tag{3}$$

Right hand side implies second frame condition from Uniform Boundedness Principle applied to operator:

$$\theta(x) = \sum_{n \in S} < x, f_n > f_n$$

What results remain true?

Definition

A countable discrete group G has property **NWFT**

(non weak frame T) if any unitary representation which has

almost invariant vectors **does not** have a weak frame vector.

Non-amenable $\subset$ **NWFT**

Proposition

If (N, G) has relative property (T) with N countable, then G has NWFT.

Questions:

- Is the above inclusion strict?
- Does this property has some hereditary properties?
- Can we apply these constructions for ”large” groups (groups of homeomorphisms)?

  Conjectures: Thompson’s group F is amenable? Is there an infinite group of homeomorphisms of the real line with property (T)?

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