Long time evolution of the Hénon-Heiles system for small energy

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The Hénon-Heiles system

1964: Michel Hénon and Carl Heiles introduced a simplified model for the planar motion of a star under the influence of a galactic center (a black hole). Their Hamiltonian is

$$h = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right) \tag{1}$$

Their numerical investigations found a highly intricate trajectory behavior:
"(1) there is an infinite number of islands (and of chains) ;
(2) the set of all the islands is dense everywhere ;
(3) but the islands do not cover the whole area since they become very small; there exists a "sea" between the islands and the ergodic trajectory is dense everywhere on the sea."

Equations

The equations of motion are

$$\frac{d^2x}{dt^2} = -x - 2xy,$$
$$\frac{d^2y}{dt^2} = -y + y^2 - x^2 \tag{2}$$

It is a four-dimensional ODE with a resonant fixed point at the origin. The behavior of trajectories at resonant fixed points is not yet fully understood.

The intricate behavior of their trajectories movitated numerous studies using a wide variety of methods: numerical methods, hamiltonian (non)integrability, integrability of near-by systems, fractal structures, chaos etc.

Our results

The Hamiltonian is conserved:

$$h = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right),$$

$$\begin{aligned} \frac{d^2x}{dt^2} &= -x - 2xy, \\ \frac{d^2y}{dt^2} &= -y + y^2 - x^2 \end{aligned} \tag{3}$$

For $h < \frac{1}{6}$ trajectories are confined within a triangular region.

We analyze the trajectories for small $(x, \dot{x}, y, \dot{y})$.

We find asymptotic formulas for solutions, valid for long times (much longer than those for which simple perturbation series are valid).

Idea: we find adiabatic invariants (i.e. quantities that remain approximately constant).

Features Revealed Numerically

Numerical calculations reveal several interesting features of the trajectories.

First, the plot the curve $(x(t), y(t))$ for a **numerically** obtained solution of the Hénon-Heiles system with initial conditions

$$x(0) = 0.1, \;\; y(0) = 0, \;\; \dot{x}(0) = 0.08, \;\; \dot{y}(0) = 0.1$$

(for which $h = 0.0132 < 1/6$).

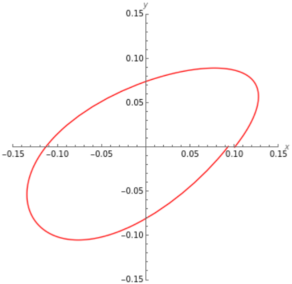

Figure: $(x(t), y(t))$ for $t \in [0, 6.27]$. Note: $6.27 \approx 2\pi$.

Over a short time, the trajectory is almost periodic.

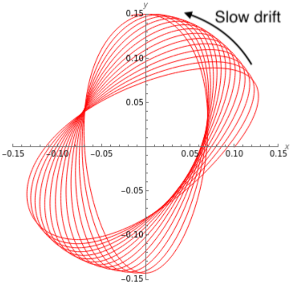

Figure: $(x(t), y(t))$ for $t < 90$

Each nearly closed trajectory drifts over time.

Features Revealed Numerically: very long time

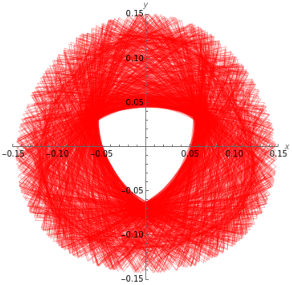

Figure: $(x(t), y(t))$ for $t < 4000$

Over a very long time, the trajectory appears to fill densely a domain.

Our results

We prove that:

- the first two pictures (behavior for short, up to long time) are correct,
- provide explicit formulas,
- and find error estimates.

Rescaling

Our investigation is for small variables: we rescale: $x \mapsto \epsilon x$ and $y \mapsto \epsilon y$ (hence $h \mapsto h/\epsilon^2$) introducing a small parameter ϵ.
The Hénon-Heiles system assumes a "perturbed" form:

$$\frac{d^2x}{dt^2} = -x - 2\epsilon xy, \qquad \frac{d^2y}{dt^2} = -y + \epsilon y^2 - \epsilon x^2. \tag{4}$$

with the Hamiltonian (conserved quantity)

$$h = \frac{1}{2}\left(x^2 + y^2 + \dot{x}^2 + \dot{y}^2\right) + \epsilon\, x^2 y - \epsilon\, \frac{y^3}{3} \tag{5}$$

This is a resonant perturbation of two harmonic oscillators.

Perturbation series solution?

It is natural to attempt solving the system using a perturbation series in ϵ. We obtain (by a straightforward calculation):

$$\begin{aligned} x(t) &= f_0(t) + \epsilon\, f_1(t) + \epsilon^2\, [f_3(t) + t\, f_4(t)] + O(\epsilon^3), \\ y(t) &= g_0(t) + \epsilon\, g_1(t) + \epsilon^2\, [g_3(t) + t\, g_4(t)] + O(\epsilon^3), \end{aligned} \tag{6}$$

where $f_k(t), g_k(t)$ are explicit, 2π-periodic functions.

Note the "almost periodic" nature of the motion: solutions initially $y(0) = y_0$ return to y_0 after time T determined explicitly from (6):

$$T = 2\pi + \epsilon^2 T_2 + O(\epsilon^3) \quad (\text{if } \dot{y}_0 \neq 0) \tag{7}$$

Note the appearance of secular term $t\epsilon^2$: the perturbation expansion holds only for times that are not too large, namely up to $t\epsilon^2 = O(1)$. Over this time scale the complexity of the evolution is not visible.

We overcome this limitation finding approximations that hold for sufficiently longer times.

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Slow Variables

We identify a variable $u := \Phi(x, \dot{x}, y, \dot{y})$ as "slow" if $\frac{d}{dt}\Phi = O(\epsilon)$.
For example, $x^2 + \dot{x}^2$, $y^2 + \dot{y}^2$, $xy + \dot{x}\dot{y}$, and $\dot{x}y - x\dot{y}$ are slow. Also h!

We use the slow variables

$$h, \quad v = y^2 + \dot{y}^2, \qquad w = \dot{x}\dot{y} + xy, \tag{8}$$

Let us observe the behavior of $v(t)$ and $w(t)$ numerically, and compare them with a non-slow variable.

The following graphs are obtained numerically for $\epsilon = 0.1$ and initial conditions $x_0 = \frac{\sqrt{3/5}}{2}, y_0 = 0, \dot{x}_0 = \frac{1}{5}, \dot{y}_0 = \frac{1}{10}$, (for which $h = 0.1$).

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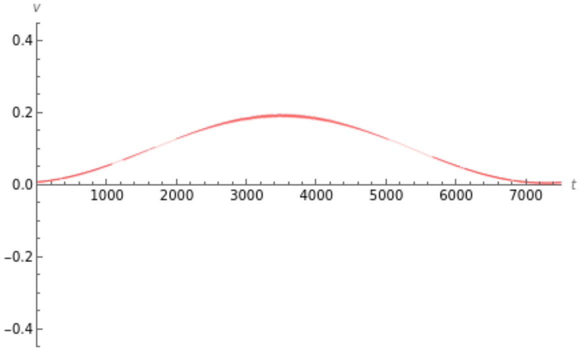
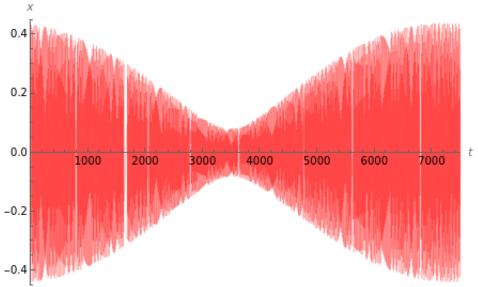
The slow variable $v = y^2 + \dot{y}^2$ as a function of t



$v(t)$ has a sinusoidal behavior, of period ≈ 7570.
The secular terms in the ϵ-expansion would have grown to 75.7 and the ϵ-expansion would have failed after a small fraction of that period.
Superimposed over it, we observe fast oscillations of small amplitude.

$w(t)$ is similar.

The fast variable $x(t)$



$x(t)$ oscillates rapidly within a larger range.

Main Theorem

Let $x(t), y(t)$ be solutions with IC $x_0 \neq 0,\ y_0 = 0,\ \dot{x}_0,\ \dot{y}_0$ in $\mathbb{R}$.

Let $u = h - \left(y^2 + \dot{y}^2\right)$ (also a slow variable!) and $w = \dot{x}\dot{y} + xy$.

Consider the iterated Poincaré map w.r.t. the manifold $\{y = 0\} \subset \mathbb{R}^4$: let u_n, w_n be the values of $u(t)$, $w(t)$ when $y = 0$ along the trajectory $(x(t), y(t), \dot{x}(t), \dot{y}(t))$ for the n^{th} time (increasingly).

We have:
(i) If $\dot{y}_0 \neq 0$, and $u_0 \neq 0$ or $w_0 \neq 0$, then there exist positive constants ϵ_0, K_0, and M, which depending only on the IC, such that...

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Main Theorem, continued

...for all n with $n\epsilon^3 \leq K_0$, and all $\epsilon \in [0, \epsilon_0]$, we have

$$u_n + iw_n = \sqrt{u_0^2 + w_0^2}\, e^{i\phi_n} \left(1 + n\epsilon^3 \eta_n\right) \tag{9}$$

with

$$\phi_n = \phi_0 + \frac{14\pi}{3}\epsilon^2 \sum_{k=0}^{n-1} \sqrt{h^2 - (u_k^2 + w_k^2)}, \tag{10}$$

where $e^{i\phi_0} = \dfrac{u_0 + iw_0}{\sqrt{u_0^2 + w_0^2}}$ and $|\eta_n| \leq M$.

(ii) In all other cases (i.e. $\dot{y}_0 = 0$, or $u_0 = w_0 = 0$) we have $u_n, w_n = O(n\epsilon^3)$.

Main Theorem, continued

For n slightly smaller, such that $n\epsilon^{5/2} \ll 1$, and for sufficiently small ϵ, formula for ϕ_n simplifies to

$$\phi_n = \phi_0 + \frac{14\pi}{3}\sqrt{k_0}\; n\epsilon^2 + n^2\epsilon^5\eta'_n, \qquad (\text{where } k_0 = h^2 - (u_0^2 + w_0^2)) \quad (11)$$

where η'_n is bounded by constants depending only on IC.
Since ϕ_n is real, separating the real and imaginary parts:

$$\begin{aligned} u_n &= \sqrt{u_0^2 + w_0^2}\cos\phi_n + n\epsilon^3\delta'_{1,n}, \\ w_n &= \sqrt{u_0^2 + w_0^2}\sin\phi_n + n\epsilon^3\delta'_{2,n}, \end{aligned} \quad (12)$$

where $\delta_{j,n}, \delta'_{j,n}$ are bounded by constants depending only on IC.

Note that n for which $n\epsilon^{5/2} \ll 1$ is significantly larger than the value at which secular terms become significant, as the latter corresponds to $n\epsilon^2 = O(1)$.

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Compare our approximation with numerical results: excellent agreement!

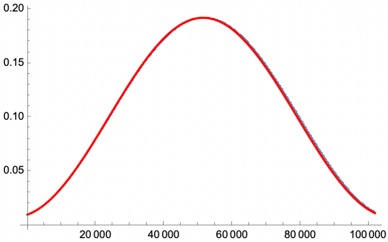

Figure: v_n obtained numerically (blue) and calculated from our Theorem (red).

(n corresponds to $t \approx 2n\pi$.)

w_n is similarly accurate.

Figure shows that the maximal error stays small for hundreds of thousands of oscillations.

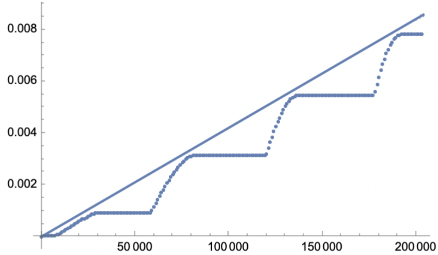

Figure: Error Analysis: maximum value of the difference between the numerical and theoretical value of v_n on the interval $[0, n]$ as a function of n.

Proof: main ideas

Our ODE system naturally extends to $\mathbb{C}$ and solutions are analytic.

We change variables (as analytic functions in $\mathbb{C}$):
$x, \dot{x}, y, \dot{y}$ (dependent), t (indep.) $\rightsquigarrow$ h, v, w, t (dependent), y (indep.)

The system becomes $\frac{dt}{dy} = \frac{1}{\sqrt{v - y^2}}$, $\frac{dh}{dy} = 0$,

$$\frac{dv}{dy} = -2\epsilon(x^2 - y^2), \qquad \frac{dw}{dy} = -\epsilon\left(\frac{x^2 - y^2}{\sqrt{v - y^2}}\dot{x} + 2xy\right)$$

where $x = x(y, v, w; \epsilon) = \frac{1}{v+2\epsilon y(v-y^2)}\left(wy + \sqrt{v - y^2}\sqrt{\mathcal{S}}\right)$ with

$$\mathcal{S} = 2\,hv - v^2 - w^2 + 2\epsilon y\left(2\,hv - v^2 - w^2\right) + \frac{4}{3}\epsilon^2 y^4 (v - y^2) - 4\,\epsilon\left(h - \frac{2}{3}v\right)y^3$$

and $\dot{x} = \dot{x}(y, v, w; \epsilon) = \frac{w - x(y,v,w;\epsilon)y}{\sqrt{v-y^2}}$

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Initial conditions (IC):

$v_0 := v|_{y=0} = \dot{y}_0^2, \quad w_0 := w|_{y=0} = \dot{x}_0\dot{y}_0, \quad t|_{y=0} = 0, \quad h|_{y=0} = h$

Turn the diff. system into a system of integral equations:

$$v(y) = v_0 + \epsilon \int_0^y F(s, v(s), w(s), \epsilon) ds, \; w(y) = w_0 + \epsilon \int_0^y G(s, v(s), w(s), \epsilon) ds$$

$$t(y) = \int_0^y \frac{ds}{\sqrt{v(s) - s^2}}, \qquad h(y) = h$$

Note: $v(y) = v_0 + O(\epsilon), \quad w(y) = w_0 + O(\epsilon)$. So

$$t(y) = \int_0^y \frac{ds}{\sqrt{v_0 - s^2}} + O(\epsilon)$$

Recall: $y(t) = periodic + O(\epsilon) \rightsquigarrow$

integrals must be on a closed path in $\mathbb{C}$ encircling the two singularities $-\sqrt{v_0}, \sqrt{v_0}$!

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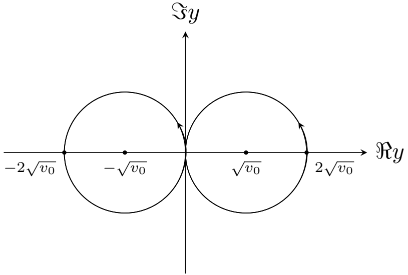
The path of integration $\mathcal{C}_{v_0}$, from 0, to 0.
Integration along it gives the Poincaré map w.r.t. $\{y = 0\}$.



After integration along one loop $t(y)$ becomes

$$T = \int_{\mathcal{C}_{v_0}} \frac{ds}{\sqrt{v_0 - s^2}} + O(\epsilon) = 2\pi + O(\epsilon)$$

Poincaré Iterated Map

Iterate: $v_{n+1} = v_n + \epsilon \oint_{\mathcal{C}_{v_0}} F(s, v(s), w(s), \epsilon) ds$

$$w_{n+1} = w_n + \epsilon \oint_{\mathcal{C}_{v_0}} G(s, v(s), w(s), \epsilon) ds$$

Expand in ϵ with remainder:
$v(y) = v_0 + \epsilon v^{[1]}(y) + \epsilon^2 v^{[2]}(y) + \epsilon^3 R(y; \epsilon)$ and
$w(y) = w_0 + \epsilon w^{[1]}(y) + \epsilon^2 w^{[2]}(y) + \epsilon^3 S(y, \epsilon)$

- where $v^{[j]}(y)$, $w^{[j]}(y)$ are calculated explicitly (long formulas) and
- $R(y; \epsilon), S(y; \epsilon)$ satisfy integral equations (obtained using the remainders $O(\epsilon^3)$ in F, G)

$$R(y) = \int_0^y [f_1(s) + \epsilon f_2(s, \epsilon R(s), \epsilon S(s))] ds := \mathcal{I}_1(R, S)(y)$$

$$S(y) = \int_0^y [g_1(s) + \epsilon g_2(s, \epsilon R(s), \epsilon S(s))] ds := \mathcal{I}_2(R, S)(y)$$

Integrate! We get:

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Iterated Poincaré map, expanded

$$v_{n+1} - v_n = \frac{14\pi}{3}\epsilon^2\, w_n \sqrt{2hv_n - v_n^2 - w_n^2} \ + \epsilon^3 R_n$$
$$w_{n+1} - w_n = \frac{14\pi}{3}\epsilon^2\, (h - v_n) \sqrt{2hv_n - v_n^2 - w_n^2} \ + \epsilon^3 S_n$$

where R_n, S_n are iterated Poincaré maps for $R(y;\epsilon), S(y;\epsilon)$.

Tasks: solve the leading order and estimate the remainders.

Solving the leading order

$$v_{n+1} - v_n = a\,\epsilon^2\, w_n \sqrt{2hv_n - v_n^2 - w_n^2} \; + \epsilon^3 R_n$$
$$w_{n+1} - w_n = a\,\epsilon^2\,(h - v_n)\sqrt{2hv_n - v_n^2 - w_n^2} \; + \epsilon^3 S_n, \quad (a = \frac{14\pi}{3})$$

Heuristics: solve

$$\frac{dV}{dn} = a\,\epsilon^2\, W\sqrt{2hV - V^2 - W^2}, \quad \frac{dW}{dn} = a\,\epsilon^2\,(h - V)\sqrt{2hV - V^2 - W^2}$$

Solutions

$$V(n) = h + A\sin(n\epsilon^2\beta + B), \quad W(n) = A\cos(n\epsilon^2\beta + B), \quad \text{where } \beta = a\sqrt{k_0}$$

where A, B and $k_0 = 2hV - V^2 - W^2$ are constants (determined from the initial conditions: $V(0) = v_0,\; W(0) = w_0$).

Rigorous proofs are based on careful (and painful) estimates and fixed point arguments. □

Estimating the remainders

Recall: $R(y;\epsilon), S(y;\epsilon)$ satisfy integral equations

$$R(y) = \int_0^y [f_1(s) + \epsilon f_2(s, \epsilon R(s), \epsilon S(s))] ds := \mathcal{I}_1(R,S)(y)$$

$$S(y) = \int_0^y [g_1(s) + \epsilon g_2(s, \epsilon R(s), \epsilon S(s))] ds := \mathcal{I}_2(R,S)(y)$$

Note: formulas are huge, but conceptually manageable.
Estimates and a fixed point argument on the operator $< \mathcal{I}_1, \mathcal{I}_2 > \rightsquigarrow$ $< R, S >$ exist and estimates are found. $\square$

Further questions:

- How can better approximations, and for longer times, be obtained? Perhaps by using a higher order truncation (but the formulas are already complicated here), or perhaps by using other slow variables.
- Determine, rigorously, the region that the trajectory fills densely, as seen in the numerical figures.
- Are there regions of order in the phase space, as suggested by numerical calculations?

Thank You!