# Density of States

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February 13, 2018

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# 1 Preface

In these notes, the expressions $V_d$ denotes the volume of a $d$-dimensional sphere of radius $R$ and $\Omega_{d-1}$ denotes the solid angle a $d$-dimensional sphere (or equivalently, the surface area of a $d$-dimensional sphere of unit radius).[1] These expressions are given by

$$V_d = \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})} R^d \,, \qquad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \,. \tag{1}$$

For example, for $d = 1, 2$, and 3, these are

$$\begin{aligned} d = 1 : \quad & V_1 = 2R \,, && \Omega_0 = 2 \,, \\ d = 2 : \quad & V_2 = \pi R^2 \,, && \Omega_1 = 2\pi \,, \\ d = 3 : \quad & V_3 = \frac{4}{3}\pi R^3 \,, && \Omega_2 = 4\pi \,. \end{aligned}$$

[1] The $d-1$ subscript refers to the fact that the fact that the surface area of a $d$-dimensional sphere is $(d-1)$-dimensional. It's also useful to use $d-1$ as the subscript, since the surface area of a $d$-dimensional sphere of radius $R$ is given by $S_{d-1} = \Omega_{d-1} R^{d-1}$.

# 2 Summation definition

Intensive quantities $A$ can generally be expressed in the form

$$A = \frac{1}{V_d} \sum_i a(\epsilon_i)\,, \tag{2}$$

where $V_d$ is the $d$-dimensional volume of the system, the sum is over all possible single-particle states $i$, and $\epsilon_i$ is the energy of the single-particle state $i$. In the continuum limit (thermodynamic limit), we can similarly define intensive quantities through

$$A = \int_{-\infty}^{\infty} a(\epsilon) g(\epsilon)\, d\epsilon\,, \tag{3}$$

where $g(\epsilon)$ is called the *density of states* (DOS). Setting Eqs. 2 and 3 equal to each other, we obtain

$$\frac{1}{V_d} \sum_i a(\epsilon_i) = \int_{-\infty}^{\infty} a(\epsilon) g(\epsilon)\, d\epsilon\,, \tag{4}$$

which implies that the DOS is given by

$$\boxed{g(\epsilon) = \frac{1}{V_d} \sum_i \delta(\epsilon - \epsilon_i)}\,, \tag{5}$$

as we can verify by inserting this into the right-hand side of Eq. 4:

$$\begin{aligned}
\int_{-\infty}^{\infty} a(\epsilon) g(\epsilon)\, d\epsilon &= \int_{-\infty}^{\infty} a(\epsilon) \left[ \frac{1}{V_d} \sum_i \delta(\epsilon - \epsilon_i) \right] d\epsilon \\
&= \frac{1}{V_d} \sum_i \int_{-\infty}^{\infty} a(\epsilon) \delta(\epsilon - \epsilon_i)\, d\epsilon \\
&= \frac{1}{V_d} \sum_i a(\epsilon_i)\,. \quad \checkmark
\end{aligned}$$

## 2.1 Examples of Eqs. 2 and 3 for a fermionic system

As concrete examples of Eqs. 2 and 3, let's consider a fermionic system, whose distribution is the Fermi–Dirac distribution

$$n_F(\epsilon) = \frac{1}{e^{(\epsilon - \mu)/k_B T} + 1}\,. \tag{6}$$

We can describe the system's number density $n \equiv N/V_d$ and energy density $u \equiv U/V_d$ (where $U = E_{\text{tot}}$) as

$$n = \frac{1}{V_d} \sum_i n_F(\epsilon_i) = \int_{-\infty}^{\infty} n_F(\epsilon) g(\epsilon)\, d\epsilon\,, \tag{7}$$

$$u = \frac{1}{V_d} \sum_i \epsilon_i n_F(\epsilon_i) = \int_{-\infty}^{\infty} \epsilon n_F(\epsilon) g(\epsilon)\, d\epsilon\,. \tag{8}$$

## 2.2 Example: Ideal Fermi gas

Consider an ideal, nonrelativistic Fermi gas comprised of electrons of mass $m$ with an energy–momentum relation

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m}\,. \tag{9}$$

The DOS is given by Eq. 5, which reads

$$g(\epsilon) = \frac{1}{V_d} \sum_i \delta(\epsilon - \epsilon_i)\,. \tag{10}$$

We can rewrite the sum over single-particle states as a sum over single-particle momenta

$$\sum_i \to 2 \sum_{\mathbf{k}_i}\,, \tag{11}$$

where the factor of 2 is due to the fact that there are two different single-particle states corresponding to a given momentum $\mathbf{k}_i$ (spin up and spin down). In the continuum limit, we use the prescription

$$\boxed{\sum_{\mathbf{k}_i} \to V_d \int \frac{d^d k_i}{(2\pi)^d}} \tag{12}$$

to rewrite Eq. 10 as

$$g(\epsilon) = 2 \int \delta(\epsilon - \epsilon_i)\, \frac{d^d k_i}{(2\pi)^d} = \frac{2}{(2\pi)^d} \int \delta(\epsilon - \epsilon')\, d^d k'\,. \tag{13}$$

We can use rotational invariance to rewrite the integration element as $d^d k' = \Omega_{d-1} {k'}^{d-1}\, dk'$, so we obtain

$$g(\epsilon) = \frac{2\Omega_{d-1}}{(2\pi)^d} \int_0^{\infty} \delta(\epsilon - \epsilon') {k'}^{d-1}\, dk' = \frac{2\Omega_{d-1}}{(2\pi)^d} \int_0^{\infty} \delta(\epsilon - \epsilon') {k'}^{d-1}\, dk'\,.$$

Changing the integral to an energy integral through

$$k' = \frac{\sqrt{2m\epsilon'}}{\hbar}\,, \quad dk' = \frac{1}{\hbar} \sqrt{\frac{m}{2\epsilon'}}\, d\epsilon'\,, \tag{14}$$

we obtain

$$
\begin{aligned}
g(\epsilon) &= \frac{2\Omega_{d-1}}{(2\pi)^d}\int_0^\infty \delta(\epsilon-\epsilon')\left(\frac{\sqrt{2m\epsilon'}}{\hbar}\right)^{d-1}\frac{1}{\hbar}\sqrt{\frac{m}{2\epsilon'}}\,d\epsilon' \\
&= \frac{m^{d/2}\Omega_{d-1}}{2^{d/2}\pi^d\hbar^d}\epsilon^{(d-2)/2}\theta(\epsilon) \\
&= \frac{m^{d/2}}{2^{d/2-1}\pi^{d/2}\hbar^d\Gamma(\frac{d}{2})}\epsilon^{(d-2)/2}\theta(\epsilon) \\
&= \frac{m^{d/2}d}{2^{d/2}\pi^{d/2}\hbar^d\Gamma(1+\frac{d}{2})}\epsilon^{(d-2)/2}\theta(\epsilon)\,.
\end{aligned}
$$

We thus find that the DOS for an ideal $d$-dimensional Fermi gas is given by

$$
g(\epsilon) = \frac{m^{d/2}d}{(2\pi)^{d/2}\hbar^d\Gamma(1+\frac{d}{2})}\epsilon^{(d-2)/2}\theta(\epsilon)\,. \tag{15}
$$

# 3 Derivative definition

For a $d$-dimensional thermodynamic system of volume $V_d$, the DOS $g(\epsilon)$ is defined by

$$
\boxed{g(\epsilon) \equiv \frac{dn}{d\epsilon}}\,. \tag{16}
$$

Then, $g(\epsilon)\,d\epsilon$ is the number of states per unit volume in the energy interval $(\epsilon, \epsilon + d\epsilon)$.

## 3.1 Example: Ideal Fermi gas

Consider an ideal, nonrelativistic Fermi gas comprised of electrons of mass $m$ with an energy–momentum relation

$$
\epsilon(k) = \frac{\hbar^2 k^2}{2m} \tag{17}
$$

in a $d$-dimensional box of volume $V_d = L^d$. Since all of the energies are nonnegative, we will multiply $g(\epsilon)$ by the step function $\theta(\epsilon)$ to make sure that we only integrate from $\epsilon = 0$ to $\infty$ if we use the DOS inside an integral:

$$
g(\epsilon) = \frac{dn}{d\epsilon}\theta(\epsilon)\,. \tag{18}
$$

We can rewrite this using the chain rule as

$$
g(\epsilon) = \frac{dn}{dk}\frac{dk}{d\epsilon}\theta(\epsilon)\,. \tag{19}
$$

The derivative on the right can simply be obtained from Eq. 17:

$$\frac{dk}{d\epsilon} = \frac{d}{d\epsilon}\frac{\sqrt{2m\epsilon}}{\hbar} = \frac{\sqrt{2m}}{2\hbar\sqrt{\epsilon}} = \frac{1}{\hbar}\sqrt{\frac{m}{2}}\epsilon^{-1/2}\,. \tag{20}$$

For the other derivative, we need to find an expression for $n$. In this $d$-dimensional box of volume $L^d$, $n$ is given by

$$n = \frac{N}{L^d}\,. \tag{21}$$

We need to find $N$ now, which we will find by looking at the problem in $d$-dimensional $\mathbf{k}$-space.

For a given value of $\mathbf{k}$, we can consider a corresponding sphere of radius $k \equiv |\mathbf{k}|$ in $d$-dimensional $\mathbf{k}$-space whose volume is

$$V_d(k) = \frac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})}k^d\,. \tag{22}$$

We now imagine dividing this sphere into cubic cells of length $k_{\text{cell}}$ and volume $k_{\text{cell}}^d$. In the large $k$ limit ($k \gg k_{\text{cell}}$), the number of cells that fit inside the sphere is therefore

$$N_{\text{cells}} = \frac{V_d(k)}{k_{\text{cell}}^d}\,. \tag{23}$$

Now, we know that we can only have two particles (with opposite spins) of momentum $\mathbf{k}$ per $\mathbf{k}$-space volume $(2\pi/L)^d$, which means that for this problem we can use

$$k_{\text{cell}} = \frac{2\pi}{L} \tag{24}$$

to find the number of particles that fit in a $d$-dimensional $\mathbf{k}$-space sphere of radius $k$:

$$N = 2N_{\text{cells}} = \frac{2V_d(k)}{k_{\text{cell}}^d} = \frac{2\pi^{d/2}k^d}{\Gamma(1+\frac{d}{2})}\left(\frac{L}{2\pi}\right)^d = \frac{L^d k^d}{2^{d-1}\pi^{d/2}\Gamma(1+\frac{d}{2})}\,. \tag{25}$$

Inserting this into Eq. 21, we find

$$n = \frac{k^d}{2^{d-1}\pi^{d/2}\Gamma(1+\frac{d}{2})}\,. \tag{26}$$

We can now compute the $k$ derivative of $n$:

$$\frac{dn}{dk} = \frac{k^{d-1}d}{2^{d-1}\pi^{d/2}\Gamma(1+\frac{d}{2})}\,. \tag{27}$$

In terms of $\epsilon$, this is

$$\frac{dn}{dk} = \frac{d}{2^{d-1}\pi^{d/2}\Gamma(1+\frac{d}{2})}\left(\frac{2m\epsilon}{\hbar^2}\right)^{(d-1)/2} = \frac{m^{(d-1)/2}d}{2^{(d-1)/2}\pi^{d/2}\hbar^{d-1}\Gamma(1+\frac{d}{2})}\epsilon^{(d-1)/2}\,. \tag{28}$$

Using this derivative and the one we computed in Eq. 20, we have

$$\begin{aligned}
g(\epsilon) &= \frac{m^{(d-1)/2}d}{2^{(d-1)/2}\pi^{d/2}\hbar^{d-1}\Gamma(1+\frac{d}{2})}\epsilon^{(d-1)/2}\frac{1}{\hbar}\sqrt{\frac{m}{2}}\epsilon^{-1/2}\theta(\epsilon) \\
&= \frac{m^{d/2}d}{2^{d/2}\pi^{d/2}\hbar^{d}\Gamma(1+\frac{d}{2})}\epsilon^{(d-2)/2}\theta(\epsilon)\,.
\end{aligned}$$

We thus find that the DOS for an ideal $d$-dimensional Fermi gas is given by

$$\boxed{g(\epsilon) = \frac{m^{d/2}d}{2^{d/2}\pi^{d/2}\hbar^{d}\Gamma(1+\frac{d}{2})}\epsilon^{(d-2)/2}\theta(\epsilon)}\,, \tag{29}$$

which agrees with Eq. 15.

# 4 Useful expressions

The DOS is useful for computing thermodynamic quantities, such as the following:

$$n \equiv \frac{N}{V_d} = \frac{1}{V_d}\sum_i n_{F/B}(\epsilon_i) = \int_{-\infty}^{\infty} n_{F/B}(\epsilon)g(\epsilon)\,d\epsilon\,, \tag{30}$$

$$u \equiv \frac{U}{V_d} = \frac{1}{V_d}\sum_i \epsilon_i n_{F/B}(\epsilon_i) = \int_{-\infty}^{\infty} \epsilon n_{F/B}(\epsilon)g(\epsilon)\,d\epsilon\,, \tag{31}$$

where $U = E_{\text{tot}}$ is the system's total energy and $n_{F/B}(\epsilon)$ denotes the system's distribution, i.e. the Fermi–Dirac distribution $n_F(\epsilon)$ or the Bose–Einstein distribution $n_B(\epsilon)$.

For a fermionic system at $T = 0$, the Fermi–Dirac distribution is just $n_F(\epsilon) = \theta(-\epsilon)\theta(\epsilon-\mu)$. The chemical potential at $T = 0$ is called the Fermi energy:

$$E_F \equiv \mu(T=0) = \frac{\hbar^2 k_F^2}{2m}\,, \tag{32}$$

so the integral expressions for $n$ and $u$ become

$$n = \int_0^{E_F} g(\epsilon)\,d\epsilon\,, \tag{33}$$

$$u = \int_0^{E_F} \epsilon g(\epsilon)\,d\epsilon\,. \tag{34}$$