Differential Forms, Metrics, and the Reflectionless Absorption of Electromagnetic Waves†

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Abstract

Two classes of formulations are prevalent on the perfectly matched layer (PML) concept for the reflectionless absorption of electromagnetic waves. In the first, additional degrees of freedom modify the curl and div operators of Maxwell's equations. This results in the so-called *non-Maxwellian PML*. The original Berenger formulation belongs to this class. The non-Maxwellian PML can be systematically derived by an analytic continuation of the coordinate space to a complex variables coordinate space (*complex-space*) which permits the extension of the PML to general geometries and media. In the second class of PML formulations, the additional degrees of freedom are entirely incorporated into modified constitutive tensors and the usual Maxwell's equations are recovered. This results in a *Maxwellian PML*. Interestingly enough, for *all* cases where the non-Maxwellian PML was derived, a Maxwellian PML was also later derived. This suggests a duality between the formulations and the possibility of a fundamental reason behind the existence of the Maxwellian PML.

In this work, we review the PML concept using the language of differential forms to (i) explain the deeper reason allowing for the ubiquitous presence of the Maxwellian PML; (ii) to provide the general framework which unifies the various PML formulations; and (iii) to show that, in principle, many other classes (hybrid) of PML formulations can be derived in the frequency-domain . This is done by introducing a novel, geometrical interpretation of the PML in terms of a change on the metric of space and exploring the metric independence of Maxwell's equations unfolded by such a language.

Short title - Forms, metrics, and reflectionless absorption of EM waves.

Index Terms - Differential forms, electromagnetic theory, perfectly matched layer.

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1. Introduction

The perfectly matched layer (PML) absorbing boundary condition (ABC) was introduced in [1] in the context of the finite difference time domain (FDTD) method to provide a reflectionless absorption of electromagnetic waves. Since then it has been proven to be a highly effective means to truncate the computational lattices in differential equation based numerical methods. Apart from its numerical efficiency, major advantages of the PML are its flexibility and relative ease in implementation.

In [2], the PML concept was shown to be equivalent to a complex coordinate stretching on the coordinate space of Maxwell's equations in the Fourier domain. The complex stretching introduces additional degrees of freedom to Maxwell's equations allowing for the reflectionless absorption of electromagnetic waves. The complex stretching is a mathematical model in the Fourier domain that can be viewed as a generalization of the Berenger PML formulation of [1]. This is because the Berenger formulation is equivalent to the time domain version of the complex stretching formulation when the electromagnetic fields are split to avoid convolutions in time. From the mathematical model furnished by the the complex stretching formulation, however, other time domain implementations are also possible, including ones without field splitting at all (at the expense of introducing time- and field- dependent sources) [3],[4].

In [5]-[8], the complex stretching interpretation was recognized as being equivalent to an analytic continuation of the coordinate space of Fourier domain Maxwell's equations to a complex variables coordinate space (*complex-space*). Under this analytic continuation, closed-form solutions for Maxwell's equations map continuously into corresponding closed-form solutions in the PML, with propagating modes (eigenfunctions) of Maxwell's equations being continuously mapped into exponentially decaying modes. This new interpretation of the PML made possible the systematic extension of the PML for more general geometries (cylindrical, spherical and conformal mesh terminations) [5]-[7] and for more general media (dispersive and/or bianisotropic) [8].

In the complex-space PML formulation, the added degrees of freedom modify Maxwell's equations in such a way that the resultant fields inside the PML cannot be associated with any possible solution of the original Maxwell's equations. It is also called, therefore, a *non-Maxwellian PML* formulation. In [9], however, it was shown that a dual, *Maxwellian PML* formulation exists in Cartesian coordinates. In such a formulation, the added degrees of freedom are entirely incorporated into the constitutive parameters and the familiar form of Maxwell's equations is retained. The resultant electromagnetic fields inside the PML can be associated with those of a medium with modified electric and magnetic constitutive parameters. This allows for the interpretation of the fields inside the PML as physical fields and for the straightforward extension of the PML to methods other than the finite-difference time-domain (FDTD) method, e.g., the finite-element method (FEM). As a mathematical model in the Fourier domain, the Maxwellian PML formulation also gives rise to a family of possible time domain Maxwellian implementations. A discussion on this and examples of time domain implementations are given, e. g., in [10], [11]. One particular time domain implementation with the goal to provide a physical basis for engineered artificial materials is presented in [12].

Later, it was shown [6]-[8] that the Maxwellian PML is derivable from the complex-space PML through some ad hoc field transformations. Furthermore, it was demonstrated that Maxwellian PML's also exist to match cylindrical, spherical, and conformal mesh terminations [6],[7], and to match bianisotropic and/or dispersive media [8]. This interesting fact suggests a duality between the formulations and the possible existence of a fundamental reason allowing for a Maxwellian PML in all cases.

In this work, we review the PML concept using the language of differential forms [13]-[19]. The main objectives are (i) to explain the deeper reason allowing for the ubiquitous existence of the Maxwellian PML ; (ii) to provide the compact, general framework which unifies the complex-space PML and the Maxwellian PML formulations; and (iii) to show that infinitely many other classes of PML formulations are indeed possible, giving rise to different mathematical models in the Fourier domain, as illustrated in Fig. 1.

A purely *geometric* interpretation is given for the PML as a modification on the *metric* of the space (this also clarifies the PML concept as being independent of the particular field equations and equally applicable to *any* linear wave phenomena) and the metric independence of Maxwell's equations (unfolded by the language of differential forms) is explored.

The analysis here relies heavily on the use of differential forms. In the usual vector calculus language, the metric independence of Maxwell's equations cannot be adequately appreciated. This turns out to be an useful example for the appreciation of the elegance of the differential forms language for electromagnetic theory. For conciseness, we will not digress here on the general advantages of its use. Excellent discussions on this topic and examples of applications of differential forms to electromagnetics can be found, e.g., in [13]-[19] and references therein.

Throughout the text, we assume linear media and work in the Fourier domain with the convention $e^{-i\omega t}$ assumed throughout.

2. The PML as a Change on the Metric of the Space

In the complex-space Cartesian PML formulation [2], [5], the spatial coordinates inside the PML are mapped to a complex variables domain as:

$$\zeta \to \tilde{\zeta} = \int_0^\zeta s_\zeta(\zeta')d\zeta', \tag{1}$$

where

$$s_\zeta(\zeta) = a_\zeta(\zeta) + i\sigma_\zeta(\zeta)/\omega \tag{2}$$

are the complex stretching variables, and ζ stands for the Cartesian coordinates (x^1, x^2, x^3). Since the complex stretching variables $s_\zeta(\zeta)$ are functions of ζ only, we can easily re-interpret such mapping on the coordinates as change on the metric of space that preserves orthogonality. From the Euclidean metric tensor

$$(ds)^2 = \sum_{i,j=1}^{3} g_{ij}dx^i dx^j = \sum_{i,j=1}^{3} \delta_{ij}dx^i dx^j \tag{3}$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

we are led to the following modified metric tensor (covariant components) within the PML

$$(ds)^2 \to (d\tilde{s})^2 = \sum_{i,j=1}^{3} \delta_{ij}d\tilde{x}^i d\tilde{x}^j = \sum_{i,j=1}^{3} \tilde{g}_{ij}dx^i dx^j \tag{5}$$

$$[g_{ij}(x^1,x^2,x^3)] \to [\tilde{g}_{ij}(x^1,x^2,x^3)] = \begin{bmatrix} (s_{x^1})^2 & 0 & 0 \\ 0 & (s_{x^2})^2 & 0 \\ 0 & 0 & (s_{x^3})^2 \end{bmatrix} \tag{6}$$

which can be recast as

$$[\tilde{g}_{ij}(x^1,x^2,x^3)] = [T_{ij}(x^1,x^2,x^3)] \cdot [\delta_{ij}] \cdot [T_{ij}(x^1,x^2,x^3)] \tag{7}$$

with

$$[T_{ij}(x^1,x^2,x^3)] = \begin{bmatrix} s_{x^1} & 0 & 0 \\ 0 & s_{x^2} & 0 \\ 0 & 0 & s_{x^3} \end{bmatrix} \tag{8}$$

This is the more general metric tensor for the Cartesian PML in three-dimensions, corresponding to a corner region where a simultaneous stretching is applied on all three coordinate variables. For single interface problems or two-dimensional corner interfaces, some of the metric coefficients remain unchanged.

These equations show that to work on the complex-space $(\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)$ with a real Euclidean metric $[g_{ij}] = [\delta_{ij}]$ is equivalent to work on the original real-space (x^1,x^2,x^3) but with a modified, complex metric $[\tilde{g}_{ij}]$ given as a function of the spatial coordinates in terms of the complex stretching variables. In the language of modern differential geometry, we say that given a chart $U = (x^1,x^2,x^3)$ on a three-dimensional differentiable manifold M with Euclidean metric $[g_{ij}] = [\delta_{ij}]$, the effect of an analytical continuation (complex coordinate stretching) as (1) on U: $U \to \tilde{U} = (\tilde{x}^1,\tilde{x}^2,\tilde{x}^3)$ is formally equivalent to a modification on the metric of $M : [g_{ij}(x^1,x^2,x^3)] \to [\tilde{g}_{ij}(x^1,x^2,x^3)]$, as given by (5),(6). This fact allows for a purely geometric interpretation of the PML concept.

In cylindrical and spherical coordinates, the metric tensors are given by

$$[g_{ij}(\rho)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

and

$$[g_{ij}(r,\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (r\sin\theta)^2 \end{bmatrix} \tag{10}$$

To achieve a PML on cylindrical and spherical coordinates, an analytic continuation, analogous to (1), is applied to the ρ and z (cylindrical case), and

r coordinates (spherical case) [5],[6]. Using the same reasoning as the Cartesian case, such analytic continuation is equivalent to a modification on corresponding metric. The modified metric tensors within the PML are given as:

$$[g_{ij}(\rho)] \to [\tilde{g}_{ij}(\rho, z)] = \begin{bmatrix} (s_\rho)^2 & 0 & 0 \\ 0 & (\tilde{\rho})^2 & 0 \\ 0 & 0 & (s_z)^2 \end{bmatrix} \tag{11}$$

$$[g_{ij}(r, \theta)] \to [\tilde{g}_{ij}(r, \theta)] = \begin{bmatrix} (s_r)^2 & 0 & 0 \\ 0 & (\tilde{r})^2 & 0 \\ 0 & 0 & (\tilde{r}\sin\theta)^2 \end{bmatrix} \tag{12}$$

They can be cast as:

$$[\tilde{g}_{ij}(\rho, z)] = [T_{ij}(\rho, z)] \cdot [g_{ij}(\rho)] \cdot [T_{ij}(\rho, z)] \tag{13}$$

and

$$[\tilde{g}_{ij}(r, \theta)] = [T_{ij}(r)] \cdot [g_{ij}(r, \theta)] \cdot [T_{ij}(r)] \tag{14}$$

with

$$[T_{ij}(\rho, z)] = \begin{bmatrix} s_\rho & 0 & 0 \\ 0 & \tilde{\rho}/\rho & 0 \\ 0 & 0 & s_z \end{bmatrix}, \tag{15}$$

$$[T_{ij}(r)] = \begin{bmatrix} s_r & 0 & 0 \\ 0 & \tilde{r}/r & 0 \\ 0 & 0 & \tilde{r}/r \end{bmatrix} \tag{16}$$

We use the arguments to distinguish the matrices $[T_{ij}]$ and $[g_{ij}]$ in the various coordinate systems. From (6), (11), and (12) we also see that the analytic continuation of the coordinate space preserves the orthogonality of the metric on these coordinate systems, being represented by a multiplication of the metric by the diagonal matrices of (8), (15) and (16). Note that the resultant metrics are not Riemannian (positive definite) anymore.

In a general orthogonal curvilinear system (u^1, u^2, u^3), if we choose u^3 to be analytically continued as

$$u^3 \to \tilde{u}^3 = \int_0^{u^3} s(\lambda) d\lambda, \tag{17}$$

then, the original metric tensor given in terms of the coefficients h_i

$$[g_{ij}(u^1, u^2, u^3)] = [{h_i}^2(u^1, u^2, u^3)\delta_{ij}] \tag{18}$$

$$[g_{ij}] = \begin{bmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & (h_3)^2 \end{bmatrix} \tag{19}$$

is mapped to

$$[\tilde{g}_{ij}] = \begin{bmatrix} (\tilde{h}_1)^2 & 0 & 0 \\ 0 & (\tilde{h}_2)^2 & 0 \\ 0 & 0 & (\tilde{h}_3)^2 \end{bmatrix} \tag{20}$$

where $\tilde{h}_i = h_i(u^1, u^2, \tilde{u}^3)$ for $i = 1, 2$, and $\tilde{h}_3 = sh_3(u^1, u^2, \tilde{u}^3)$. The new metric can be recast as

$$\left[\tilde{g}_{ij}(u^1, u^2, u^3)\right] = \left[T_{ij}(u^1, u^2, u^3)\right] \cdot \left[g_{ij}(u^1, u^2, u^3)\right] \cdot \left[T_{ij}(u^1, u^2, u^3)\right] \tag{21}$$

with

$$\left[T_{ij}(u^1, u^2, u^3)\right] = \begin{bmatrix} (\tilde{h}_1/h_1) & 0 & 0 \\ 0 & (\tilde{h}_2/h_2) & 0 \\ 0 & 0 & (\tilde{h}_3/h_3) \end{bmatrix} \tag{22}$$

If u^3 is the normal coordinate to the mesh termination defined by $u^3 = 0$, and if we set $h_3 = 1$, a conformal PML [7] can be built on *parallel surfaces* to the mesh termination (i.e., $u^3 = c$, $c \geq 0$) by an analytic continuation on u^3. The transverse coefficients h_i are given as $h_1 = (r_1 + u^3)/r_1$, $h_2 = (r_2 + u^3)/r_2$, where r_1, r_2 are the local radii of curvature and u^1, u^2, the principal directions on the termination surface. In this case, the original metric tensor

$$\left[g_{ij}(u^1, u^2, u^3)\right] = \begin{bmatrix} (h_1)^2 & 0 & 0 \\ 0 & (h_2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

is mapped to

$$\left[\tilde{g}_{ij}(u^1, u^2, u^3)\right] = \begin{bmatrix} (\tilde{h}_1)^2 & 0 & 0 \\ 0 & (\tilde{h}_2)^2 & 0 \\ 0 & 0 & s^2 \end{bmatrix} \tag{24}$$

with $\tilde{h}_1 = (r_1 + \tilde{u}^3)/r_1$, $\tilde{h}_2 = (r_2 + \tilde{u}^3)/r_2$. It can be recast as

$$\left[\tilde{g}_{ij}(u^1, u^2, u^3)\right] = \left[T_{ij}(u^1, u^2, u^3)\right] \cdot \left[g_{ij}(u^1, u^2, u^3)\right] \cdot \left[T_{ij}(u^1, u^2, u^3)\right] \tag{25}$$

with

$$\left[T_{ij}(u^1, u^2, u^3)\right] = \begin{bmatrix} (\tilde{h}_1/h_1) & 0 & 0 \\ 0 & (\tilde{h}_2/h_2) & 0 \\ 0 & 0 & s \end{bmatrix} \tag{26}$$

The Cartesian, cylindrical, and spherical PML can be viewed as special cases of the above, followed (possibly) by its successive application in orthogonal directions.

Finally, it should be pointed out that this purely geometric description of the PML is independent of the particular form of field equations. In other words, the concept of the PML as a change of the metric of space (or analytic continuation of the coordinate space) is not limited to electromagnetic problems and can be equally applied to any linear wave phenomena (such as acoustic or scalar [20], dispersive [21], and elastic [22]) to obtain the reflectionless absorption of waves in those cases.

However, a remarkable property peculiar to Maxwell's equations is that it allows any change on the metric of space to be entirely translated to a change on the constitutive parameters of the medium and, as a result, the original form of Maxwell's equations in the original metric can still be retained (Maxwellian PML formulation). This is the topic of the next section.

3. The Metric Independence of Maxwell's Equations

In the language of differential forms [13]-[19], Maxwell's equations are written as

$$dE = i\omega B \tag{27}$$

$$dH = -i\omega D + J_E \tag{28}$$

$$dD = \rho_E \tag{29}$$

$$dB = 0 \tag{30}$$

In the above, E and H are electric and magnetic field intensity 1-forms, D and B are electric and magnetic flux density 2-forms, J_E is the electric current density 2-form, and ρ_E is the electric charge density 3-form.

The operator d is the usual exterior derivative, which plays the role of the curl and div operators of vector calculus. The exterior derivative is an operator applicable to any differentiable manifold, even without a defined metric. In other words, such an operator is independent of the concept of distance. The Maxwell's equations in the above form (27)-(30) are metric independent and retain the same form when expressed in any of coordinate system. This is discussed, e.g., in [14], [15]. This is in marked contrast to the vector calculus operators, which depend on metric factors, and, therefore, have different expressions when written in different coordinate systems.

In the differential forms language, the constitutive parameters of a given medium relate the 1-forms E, H to the 2-forms D, B and are given in terms of *Hodge operators* $\star_e$ and $\star_h$ as [19]:

$$D = \star_e E \tag{31}$$

$$B = \star_h H \tag{32}$$

These relations close the Maxwell's system. In the case of a three-dimensional manifold, the Hodge operator establishes a natural $1:1$ map (isomorphism) between the space of 1-forms as E and H and the space of 2-forms as D and B. This isomorphism (a linear map) is usually called the Hodge duality map. The Hodge operators depend on a metric and, in the equations (27)-(32), all the information about the metric of space is contained in the constitutive relations (31)-(32). Any modification on the metric tensor preserves the form of (27)-(30). This observation leads to the the far-reaching conclusion that a change on the metric properties of space to achieve reflectionless absorption of electromagnetic waves (or for any other possible reason!) can *always* be entirely incorporated, in a dual formulation, as a modification on the constitutive tensors.

As expected, a duality relation also exists between the forms E, H, D, and B and the corresponding vector quantities $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$. The natural isomorphism between the 1-forms E, H and the vector fields $\mathbf{E}$, $\mathbf{H}$ (polar vectors) is established by the metric tensor, and can be extended to 2-forms D, B and the corresponding vector fields $\mathbf{D}$, $\mathbf{B}$ (axial vectors) through the Hodge duality. This is made explicit in the next section.

4. Unification of the PML Formulations

As mentioned in the previous section, the components of the electromagnetic fields in the vector representation and in the differential forms representation are isomorphic to each other, so that their components differ only by metric coefficients.

Since the modifications of the metric tensors, for the various PML's illustrated in the Section 2, are just special cases of $(h_1, h_2, h_3) \to (\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$, possibly followed by successive applications in orthogonal directions, we will restrict the discussion on this section to the general orthogonal curvilinear coordinates (u^1, u^2, u^3) case. In this case, the duality relation between forms and vectors can be entirely expressed in terms of the coefficients h_1, h_2, h_3. Given a general 1-form Ω expanded in terms of a basis of 1-forms $h_1 du^1$, $h_2 du^2$, $h_3 du^3$, the dual vector quantity under the natural isomorphism governed by metric tensor $[g_{ij}(u^1, u^2, u^3)] = [h_i^2 \delta_{ij}]$ will be given by

$$\Omega = \sum_{i=1}^{3} \Omega_i h_i du^i \rightarrow \mathbf{\Omega} = \sum_{i=1}^{3} \Omega_i \mathbf{u}^i \tag{33}$$

where $\mathbf{u}^i$ is the unit vector on the u^i direction. We call the basis $h_1 du^1$, $h_2 du^2$, $h_3 du^3$ an orthonormal basis, since its vector dual, $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$, constitutes an orthonormal vector basis.

In the case of a 2-form Φ expanded in terms of an orthonormal basis of 2-forms, $h_1 h_2 du^1 du^2$, $h_2 h_3 du^2 du^3$, $h_3 h_1 du^3 du^1$, the natural isomorphism under $[g_{ij}(u^1, u^2, u^3)]$ is

$$\Phi = \sum_{i=1}^{3} \Phi_i h_{[i+1]} h_{[i+2]} du^{[i+1]} \wedge du^{[i+2]} \rightarrow \mathbf{\Phi} = \sum_{i=1}^{3} \Phi_i \mathbf{u}^i \tag{34}$$

In the above, $[i] \equiv i \bmod 3$ for $i \neq 3$, and $[3] \equiv 3$. The wedge denotes exterior product and is usually suppressed. The 1- and 2-forms can be expanded in terms of a non-orthonormal basis, with analogous results. For instance, if Ω is expressed in terms of the coordinate basis du^1, du^2, du^3, then its vector dual is given by [18],[19]

$$\Omega = \sum_{i=1}^{3} \Omega_i du^i \rightarrow \mathbf{\Omega} = \sum_{i=1}^{3} \frac{1}{h_i} \Omega_i \mathbf{u}^i \tag{35}$$

If we express the electric and magnetic 1-forms in terms of the orthonormal basis of 1-forms:

$$E = \sum_{i=1}^{3} E_i h_i du^i \tag{36}$$

$$H = \sum_{i=1}^{3} H_i h_i du^i \tag{37}$$

then the operators $\star_e$ and $\star_h$ act on the electric and magnetic fields 1-forms to give the electric and magnetic fluxes 2-forms as follows [18],[19]

$$D = \star_e(\sum_{i=1}^{3} E_i h_i du^i) = \sum_{i,j=1}^{3} \epsilon_{ij} E_j h_{[i+1]} h_{[i+2]} du^{[i+1]} du^{[i+2]} \tag{38}$$

$$B = \star_h(\sum_{i=1}^{3} H_i h_i du^i) = \sum_{i,j=1}^{3} \mu_{ij} H_j h_{[i+1]} h_{[i+2]} du^{[i+1]} du^{[i+2]} \tag{39}$$

In the above, $[\epsilon_{ij}]$ and $[\mu_{ij}]$ are the material constitutive tensors expressed in terms of the local coordinate system (u^1, u^2, u^3). Here we consider the case of a medium with arbitrary electric and magnetic anisotropy, but such analysis can also be extended for the general bianisotropic case. The media may also present dispersive behavior, $[\epsilon_{ij}(\omega)]$, $[\mu_{ij}(\omega)]$. The dependence of the Hodge operators on the metric coefficients (coefficients h_1, h_2, h_3) is clear from these equations.

Under the change on the metric $[g_{ij}] \rightarrow [\tilde{g}_{ij}]$, the new Maxwell's equations inside the PML read:

$$d\tilde{E} = i\omega\tilde{B} \tag{40}$$

$$d\tilde{H} = -i\omega\tilde{D} \tag{41}$$

$$d\tilde{D} = 0 \tag{42}$$

$$d\tilde{B} = 0 \tag{43}$$

$$\tilde{D} = \tilde{\star}_e\tilde{E} \tag{44}$$

$$\tilde{B} = \tilde{\star}_h\tilde{H} \tag{45}$$

where no sources are assumed inside the PML. Eqns. (40)-(43) are the same as before. However, the new forms $\tilde{E}$, $\tilde{H}$, $\tilde{D}$, and $\tilde{B}$ are related through modified Hodge operators $\tilde{\star}_e$ and $\tilde{\star}_h$, defined by the modified metric $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$.

The PML in the differential forms language is unique and Eqns. (40)-(45) unify the various PML formulations. The different PML formulations in the vector calculus language can be derived from (40)-(45) by a simple choice on how to map these latter forms to the corresponding vector quantities inside the PML. This is done next.

4.1. The Maxwellian PML Formulation

In this case, the map from the 1-forms and 2-forms in (40)-(45) to the corresponding dual vector quantities is governed by the original metric tensor $[g_{ij}]$ (natural isomorphism under $[g_{ij}]$):

$$\tilde{E} = \sum_{i=1}^{3} \tilde{E}_i \tilde{h}_i du^i \xrightarrow{[g_{ij}]} \mathbf{E}^m = \sum_{i=1}^{3} E_i^m \mathbf{u}^i = \sum_{i=1}^{3} \frac{\tilde{h}_i}{h_i} \tilde{E}_i \mathbf{u}^i \tag{46}$$

$$\tilde{D} = \sum_{i,j=1}^{3} \epsilon_{ij} \tilde{E}_j \tilde{h}_{[i+1]} \tilde{h}_{[i+2]} du^{[i+1]} du^{[i+2]} \xrightarrow{[g_{ij}]}$$

$$\mathbf{D}^m = \sum_{i=1}^{3} D_i^m \mathbf{u}^i = \sum_{i,j=1}^{3} \frac{\tilde{h}_{[i+1]} \tilde{h}_{[i+2]}}{h_{[i+1]} h_{[i+2]}} \epsilon_{ij} \tilde{E}_j \mathbf{u}^i \tag{47}$$

and analogously for the H and B field relations. The superscript 'm' stands for Maxwellian fields. Since Eqns. (40)-(43) above preserve the form of Maxwell's equations (27)-(30), the resultant system in the vector calculus language also retains the form of Maxwell's equations on the original, real metric $[g_{ij}]$ (the metric factors appearing on curl and div operators are the usual ones associated with $[g_{ij}]$). In such a situation, the duals of the constitutive relations (44)-(45) in the vector calculus language incorporate all the effects on the change on the metric of space $[g_{ij}] \rightarrow [\tilde{g}_{ij}]$ within the PML. The added degrees of freedom in the PML are present only on those constitutive equations. Comparing $\mathbf{E}^m$ from (46) and $\mathbf{D}^m$ from (47) and after a little algebra, it is easy to show that the modified, general constitutive tensors that produce the Maxwellian PML are given by

$$\mathbf{D}^m = \overline{\boldsymbol{\epsilon}}_{PML} \cdot \mathbf{E}^m \tag{48}$$

with

$$(\epsilon_{PML})_{ij} = \left(\frac{\tilde{h}_i \tilde{h}_{[i+1]} \tilde{h}_{[i+2]}}{h_i h_{[i+1]} h_{[i+2]}} \right) \frac{h_i}{\tilde{h}_i} \epsilon_{ij} \frac{h_j}{\tilde{h}_j} \tag{49}$$

or

$$\overline{\boldsymbol{\epsilon}}_{PML} = \det\left([T_{ij}]\right) [T_{ij}]^{-1} \cdot \overline{\boldsymbol{\epsilon}} \cdot [T_{ij}]^{-1} \tag{50}$$

with $[T_{ij}] = [T_{ij}(u^1, u^2, u^3)]$ given as in (22), and analogous expressions for the magnetic case. Particular cases of this relation were derived before, in a 'pure' vector calculus context, using more involved algebra. For isotropic, dispersionless media, it was first derived in [9], and for the general bianisotropic and dispersive case, it was derived in [6].

4.2. The Complex-Space (non-Maxwellian) PML Formulation

In this case, the map from the 1-forms and 2-forms in (40)-(45) to the corresponding dual vector quantities is governed by the modified, complex metric tensor $[\tilde{g}_{ij}]$[1] (natural isomorphism under $[\tilde{g}_{ij}]$).

$$\tilde{E} = \sum_{i=1}^{3} \tilde{E}_i \tilde{h}_i du^i \xrightarrow{[\tilde{g}_{ij}]} \mathbf{E}^c = \sum_{i=1}^{3} E_i^c \mathbf{u}^i = \sum_{i=1}^{3} \tilde{E}_i \mathbf{u}^i \tag{51}$$

$$\tilde{D} = \sum_{i,j=1}^{3} \epsilon_{ij} \tilde{E}_j \tilde{h}_{[i+1]} \tilde{h}_{[i+2]} du^{[i+1]} du^{[i+2]} \xrightarrow{[\tilde{g}_{ij}]} \mathbf{D}^c = \sum_{i=1}^{3} D_i^c \mathbf{u}^i = \sum_{i,j=1}^{3} \epsilon_{ij} \tilde{E}_j \mathbf{u}^i \tag{52}$$

The superscript 'c' above stands for complex-space fields. The magnetic case follows similarly.

In this case, the resultant system in the vector calculus language does not retain the form of Maxwell's equations since the metric factors appearing in the vector language equivalent of equations (40)-(43) are not the usual ones associated with the original metric $[g_{ij}]$, but complex ones associated with $[\tilde{g}_{ij}]$. In the vector calculus language, the added degrees of freedom in the PML are entirely incorporated on the vector language equivalent of Eqs. (40)-(43). This reflects in the appearance of the complex stretching variables (or complex metric factors) altering the spatial derivative operators in Maxwell's equations. In the Cartesian case, the resulting modified Maxwell's equations in the Fourier domain are the same as those presented in [2], and if the electromagnetic fields are split to avoid convolutions in the time domain, then this formulation reduces to the original Berenger PML split-field, time domain scheme [1]. In contrast to the Maxwellian PML, the constitutive relations (44)-(45), when translated to the vector language in this formulation, appear as

$$\mathbf{D}^c = \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}^c \tag{53}$$

[1] This is the *canonical* isomorphism since its governing metric is the same metric which governs the resultant Maxwell's system (40)-(45) through Eqns. (44) and (45).

and similarly for the magnetic case. Therefore, the constitutive relations retain the same form as before the change of the metric, as opposed to Eqn. (48).

4.3. The General Relationship Between the Maxwellian PML and the Complex-Space PML

As noted previously, there is only one PML formulation in the differential forms language. The PML is governed by the change on the Hodge star operators induced by new complex metric: $\star_e \to \tilde{\star}_e$, $\star_h \to \tilde{\star}_h$. Via this change, propagating forms E, H, D, B are continuously mapped to exponentially decaying forms $\tilde{E}$, $\tilde{H}$, $\tilde{D}$, $\tilde{B}$. Since the isomorphism between forms and vectors is governed by simple metric factors, cf. Eqns. (46)-(47), (51)-(52), the corresponding vectors present the same attenuative behavior as the forms, differing only through metric factors.

Because of that, the vector fields in the Maxwellian and complex formulations are related by a simple general expression involving the ratio of the complex and real (original) metric factors. By comparing Eqns. (46) and (51), we get

$$E_i^m = \frac{\tilde{h}_i}{h_i} E_i^c \tag{54}$$

Or, simply

$$\mathbf{E}^m = [T_{ij}] \cdot \mathbf{E}^c \tag{55}$$

where $[T_{ij}]$ are the matrices given in (8), (15), (16), and (22) for the various coordinate systems PML's. The same is valid for the **H** field. For the **D** field, we compare Eqns. (47) and (52) to get

$$\mathbf{D}^m = \det\left([T_{ij}]\right) [T_{ij}]^{-1} \cdot \mathbf{D}^c \tag{56}$$

with an identical relation for the **B** field. This equation can also be simply obtained by substituting Eqns. (50), (53) and (55) into Eqn. (48). Since the metric tensors $[g_{ij}]$ and $[\tilde{g}_{ij}]$ coincide in the physical (non-PML) region ($\tilde{h}_i/h_i = 1$), these mappings give the same results in this region, as required.

At any point, if the analytic continuation (complex stretching) is taken along the u^3 coordinate, then the metric factors $h_1(u^1, u^2, u^3)$, $h_2(u^1, u^2, u^3)$, $\tilde{h}_1 = h_1(u^1, u^2, \tilde{u}^3)$, and $\tilde{h}_2 = h_2(u^1, u^2, \tilde{u}^3)$ are continuous. This is because the coefficients h_1, h_2 are continuous functions of u^3, and $\tilde{u}^3(u^3)$ itself is continuous from Eqn. (17). Therefore, from Eqn. (54), the *tangential* components of the fields in the two formulations, E_1^c, E_1^m, and E_2^c, E_2^m, differ by factors that are continuous everywhere and, as a result, they satisfy the same boundary conditions. In particular, the perfect matching condition of one formulation implies the same for the other formulation. However, the *normal* components E_3^c, E_3^m along the analytic continued coordinate satisfy, in general, different boundary conditions. If the original field component E_3 is continuous, then the E_3^c component is continuous along the u^3 direction, since it is just the analytic continuation of the original electric field, $E_3^c(u^1, u^2, u^3) = E_3(u^1, u^2, \tilde{u}^3)$, which preserves continuity (from the continuity of $\tilde{u}^3(u^3)$). However, the E_3^m component is not necessarily continuous since $\tilde{h}_3 = sh_3(u^1, u^2, \tilde{u}^3)$, so that a discontinuous complex stretching variable $s(u^3)$ in (17) induces a discontinuous complex metric factor $\tilde{h}_3$, and, consequently, and a discontinuous normal E_3^m component from Eqn. (46)[1,2]. This discontinuity might, in a numerical code, cause spurious reflection from the interface. The same is true for the $\mathbf{H}^c$ and $\mathbf{H}^m$ fields. For the $\mathbf{D}^c$, $\mathbf{D}^m$, $\mathbf{B}^c$, and $\mathbf{B}^m$ fields, a similar discussion also applies but with the continuity properties of the tangential and normal components of the $\mathbf{D}^m$ and and $\mathbf{B}^m$ fields exchanged from the above discussion, as revealed, for instance, by a comparison between Eqns. (55) and (56).

4.4. New Classes of PML Formulations

The analysis of the isomorphism between forms and vectors reveals that other choices of metrics $[\hat{g}_{ij}]$ are also possible to govern the isomorphism between forms and vectors, as long as they (1) recover the original real metric $[g_{ij}]$ in the physical domain and (2) preserve the perfect matching conditions. Each new choice for $[\hat{g}_{ij}]$ gives rise to a new PML formulation in the Fourier domain in terms of the non-splitted electromagnetic vector fields. The diagram in Fig. 1 illustrates this observation.

Obvious choices are hybridizations of the complex-space and Maxwellian

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[1] Note that h_3 itself is a continuous function of $\tilde{u}^3$ (and u^3).

[2] The complex stretching variable, s, need not to be continuous, but in practical numerical implementations, it is usually chosen so to minimize spurious reflections on the PML interface due to discretization.

formulations, as

$$[\hat{g}_{ij}] = \alpha[g_{ij}] + \beta[\tilde{g}_{ij}] \tag{57}$$

or

$$[\hat{g}_{ij}] = \sum_{k=1}^{3}[g_{ik}]^{\alpha}[\tilde{g}_{kj}]^{\beta} \tag{58}$$

where α and β are scalars such that $\alpha + \beta = 1$. Since those tensors are diagonal, the tensor product in Eqn. (58) is simply a product of the diagonal entries. From these equations, it is seen that an infinite number of PML formulations are possible.

The relation between the resulting electromagnetic vector fields of any of the formulations produced by (57)-(58) and the previous vector fields of the Maxwellian and complex-space formulations may be cast in terms of corresponding metric factors, analogous to (54)-(56). As observed before, the choice of isomorphism also affects the definition of the constituive tensors $\overline{\boldsymbol{\epsilon}}$ and $\overline{\boldsymbol{\mu}}$ that relate these vector quantities.

The modified Maxwell's equations for each of these new PML formulations can be easily written. In case of a general orthogonal system of curvilinear coordinates (u^1, u^2, u^3), the PML formulation derived from Eqn. (58) is simply determined by the substitution of h_i by $h_i^{\alpha}\tilde{h}_i^{\beta}$, $i = 1, 2, 3$ on the usual Maxwell's equations. Therefore, the resulting modified Maxwell's equations inside the PML are

$$\frac{1}{h_2^{\alpha}\tilde{h}_2^{\beta}h_3^{\alpha}\tilde{h}_3^{\beta}}\left[\frac{\partial}{\partial u^2}\left(h_3^{\alpha}\tilde{h}_3^{\beta}E_3^{(\alpha,\beta)}\right) - \frac{\partial}{\partial u^3}\left(h_2^{\alpha}\tilde{h}_2^{\beta}E_2^{(\alpha,\beta)}\right)\right] - i\omega B_1^{(\alpha,\beta)} = 0, \tag{59}$$

$$\frac{1}{h_3^{\alpha}\tilde{h}_3^{\beta}h_1^{\alpha}\tilde{h}_1^{\beta}}\left[\frac{\partial}{\partial u^3}\left(h_1^{\alpha}\tilde{h}_1^{\beta}E_1^{(\alpha,\beta)}\right) - \frac{\partial}{\partial u^1}\left(h_3^{\alpha}\tilde{h}_3^{\beta}E_3^{(\alpha,\beta)}\right)\right] - i\omega B_2^{(\alpha,\beta)} = 0, \tag{60}$$

$$\frac{1}{h_1^{\alpha}\tilde{h}_1^{\beta}h_2^{\alpha}\tilde{h}_2^{\beta}}\left[\frac{\partial}{\partial u^1}\left(h_2^{\alpha}\tilde{h}_2^{\beta}E_2^{(\alpha,\beta)}\right) - \frac{\partial}{\partial u^2}\left(h_1^{\alpha}\tilde{h}_1^{\beta}E_1^{(\alpha,\beta)}\right)\right] - i\omega B_3^{(\alpha,\beta)} = 0, \tag{61}$$

$$\frac{\partial}{\partial u^1}\left(h_2^{\alpha}\tilde{h}_2^{\beta}h_3^{\alpha}\tilde{h}_3^{\beta}D_1^{(\alpha,\beta)}\right) + \frac{\partial}{\partial u^2}\left(h_3^{\alpha}\tilde{h}_3^{\beta}h_1^{\alpha}\tilde{h}_1^{\beta}D_2^{(\alpha,\beta)}\right) + \frac{\partial}{\partial u^3}\left(h_1^{\alpha}\tilde{h}_1^{\beta}h_2^{\alpha}\tilde{h}_2^{\beta}D_3^{(\alpha,\beta)}\right) = 0, \tag{62}$$

and analogous equations for the remaining equations. Outside the PML, we have ($\tilde{h}_i = h_i$, $i = 1, 2, 3$), and these equations recover the original Maxwell's equations. The corresponding constitutive equations are found by comparing the expressions for the vectors $\mathbf{E}^{(\alpha,\beta)}$ and $\mathbf{D}^{(\alpha,\beta)}$, derived from the forms $\tilde{E}$ and $\tilde{D}$ under the isomorphism governed by the metric $[\hat{g}_{ij}] = [g_{ik}]^\alpha [\tilde{g}_{kj}]^\beta$:

$$\mathbf{E}^{(\alpha,\beta)} = \sum_{i=1}^{3} E_i^{(\alpha,\beta)} \mathbf{u}^i = \sum_{i=1}^{3} \frac{\tilde{h}_i^{1-\beta}}{h_i^\alpha} \tilde{E}_i \mathbf{u}^i \tag{63}$$

$$\mathbf{D}^{(\alpha,\beta)} = \sum_{i=1}^{3} D_i^{(\alpha,\beta)} \mathbf{u}^i = \sum_{i,j=1}^{3} \frac{\tilde{h}_{[i+1]}^{1-\beta} \tilde{h}_{[i+2]}^{1-\beta}}{h_{[i+1]}^\alpha h_{[i+2]}^\alpha} \epsilon_{ij} \tilde{E}_j \mathbf{u}^i \tag{64}$$

So that the electric constitutive relation within the PML should read as

$$\mathbf{D}^{(\alpha,\beta)} = \overline{\boldsymbol{\epsilon}}^{(\alpha,\beta)} \cdot \mathbf{E}^{(\alpha,\beta)} \tag{65}$$

with

$$\epsilon_{ij}^{(\alpha,\beta)} = \frac{\left(\tilde{h}_i \tilde{h}_{[i+1]} \tilde{h}_{[i+2]}\right)^{1-\beta}}{\left(h_i h_{[i+1]} h_{[i+2]}\right)^\alpha} \frac{h_i^\alpha}{\tilde{h}_i^{1-\beta}} \epsilon_{ij} \frac{h_j^\alpha}{\tilde{h}_j^{1-\beta}} \tag{66}$$

The magnetic constitutive relation follows similarly. Outside the PML, Eqn. (66) recovers $\epsilon_{ij}^{(\alpha,\beta)} = \epsilon_{ij}$. This gives a PML matched to an arbitrary anisotropic and/or dispersive media with constitutive parameters $[\epsilon_{ij}(\omega)]$ and $[\mu_{ij}(\omega)]$. Extension to the general bianisotropic case is straightforward. The corresponding equations for the usual Cartesian, cylindrical and spherical coordinate systems are a special case of above equations. It is clear also that Eqns. (59)-(66) recover the original complex-space PML formulation ($\alpha = 0$, $\beta = 1$) and the Maxwellian PML formulation ($\alpha = 1$, $\beta = 0$) as special cases. The diagram of Fig. 2 illustrates the relationship between the various PML formulations.

As opposed to the Maxwellian and complex-space PML formulations, which modify solely the consitutive relations *or* the Maxwell's equations, we note these new PML formulations modify both. Admittedly, this seems to result in more awkward PML formulations, and, possibly, of questionable aplicability for time-domain methods. However, some of these new hybrid PML's may have attractive characteristics for frequency-domain methods. For instance, the choice of (58) with $\alpha = \beta = 1/2$ is used for the Cartesian metric to provide a symmetric modified nabla operator for the Cartesian PML for use in the FEM [23]. Such symmetric operator leads to a symmetric FEM matrix. Moreover, numerical experiments [23] show that the condition number from the FEM matrix in this formulation is better that in the Maxwellian PML formulation of [9] ($\alpha = 1$, $\beta = 0$), thus requiring fewer iterations when using an iterative solver.

5. Conclusions

In this work, a general framework to unify the Maxwellian and non-Maxwellian (complex-space) PML formulations was described.

A simple geometric interpretation for the PML was given in terms of a change of the metric of space in the Fourier domain. In the PML, the original, real metric, $[g_{ij}]$, is mapped into a complex metric, $[\tilde{g}_{ij}]$, which induces a continuous mapping of propagating modes into exponentially decaying modes (reflectionless absorption of waves).

By exploring this new interpretation for the PML, it was shown that the differential forms language provides the appropriate means to unify the PML formulations. This is because the metric independence of Maxwell's equations is manifested in such language. A modification in the metric is the same as a modification on the Hodge star operators (constitutive relations), and the distinction between the Maxwellian and the complex-space PML formulations is simply non existent in terms of differential forms quantities. The existence of the Maxwellian PML is a simple consequence of the metric independence of Maxwell's equations.

The different PML formulations in the vector calculus language arise from different choices on how to map the *same* differential forms quantities into corresponding vector quantities. This map corresponds to a natural isomorphism between forms and vectors and it is governed by a metric. If the real metric $[g_{ij}]$ is chosen to govern such a map, then the Maxwellian PML formulation is recovered. Alternatively, if the complex metric $[\tilde{g}_{ij}]$ is chosen, then the complex-space PML formulation is recovered. Finally, the analysis also revealed that the isomorphism between forms and vectors can also be governed by other metrics. In this case, infinitely many new classes of PML formulations are obtained.

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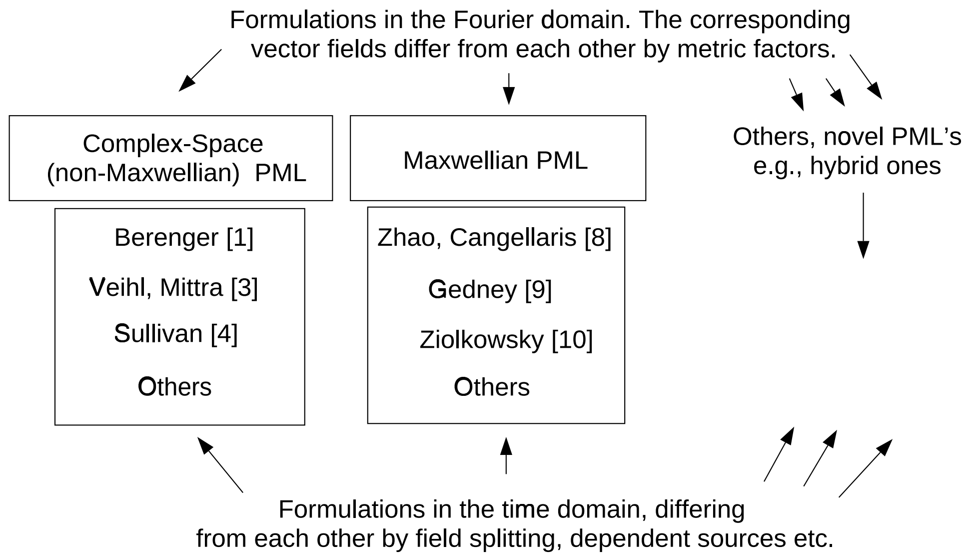

Fig. 1 - Diagram illustrating the different PML formulations in time and Fourier domains.

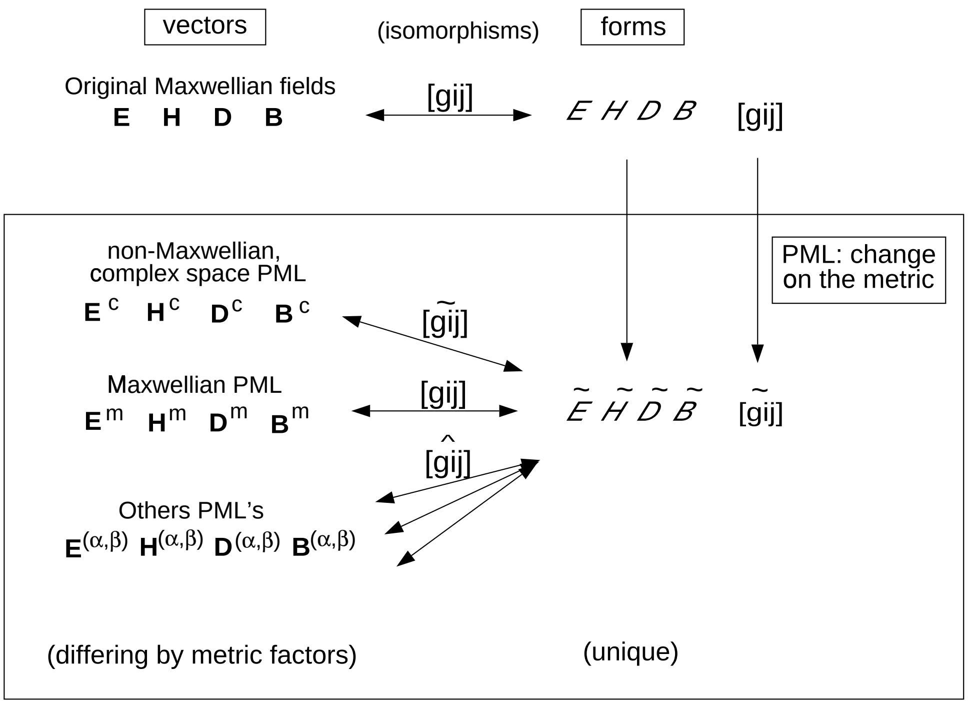

Fig. 2 - Diagram illustrating the relationship among the various PML formulations in the vector calculus language and in the differential forms language.