An Efficient Rescaled Formulation for Tensor Green’s Function Computation in Cylindrical Multilayered Media

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Abstract—A robust formulation is presented to eliminate overflow, underflow, and convergence problems that arise the computation of tensor Green’s functions in cylindrical multilayered media under finite-precision arithmetic. This is done by first introducing a set of rescaled functions associated to the (canonical) cylindrical Bessel and Hankel functions representing standing waves and outgoing waves along the radial direction. The rescaled cylindrical functions are designed to avoid the poor scaling inherent to the canonical cylindrical functions for very small or very large arguments. In addition, a parabolic Sommerfeld integration path is constructed in the spectral complex plane to yield a numerical integration with good convergence properties for a wide range of parameter values.

Index Terms—Green’s function, borehole exploration, cylindrical eigenfunctions, multilayered media.

I. INTRODUCTION

Numerical computation of Green’s function in cylindrical multilayered media is of interest to a wide range of applications [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11]. The solution of this problem is well-known and can be expressed in terms of a spectral integral along the longitudinal wavenumber $k_z$ [12] involving a sum over discrete cylindrical harmonics. Challenges arise in the numerical computation because of the poor scaling, for extreme arguments, of the cylindrical Bessel and Hankel eigenfunctions that describe the radial dependency of the problem, as well as slow convergence of the spectral integral. Poor scaling leads to overflow and underflow problems in finite-precision arithmetic and can become especially deleterious when it is necessary to accommodate physical parameters, viz. frequency of operation, thicknesses, and constitutive properties of the cylindrical layers (such as conductivities), that may vary across several order of magnitude, such as in some geophysical exploration problems [13] [14] [15] [16]. This latter approach is predicated on the delineation of three argument types (small, moderate, and large) to define the associated set of rescaled cylindrical functions. Although numerically robust, this strategy requires the appropriate definition of such numerical thresholds to demarcate the three argument types.

In this work, we propose and combine two new ingredients to enhance the robustness of the numerical computation of tensor Green’s functions in cylindrical multilayered media. The first ingredient is a rescaled formulation with layer-adapted modified cylindrical functions that provides bounded results without the need for introducing numerical thresholds. The second ingredient is the use of a parabolic Sommerfeld integration path that yields fast convergence across different frequencies and multilayer properties, while requiring only a few control parameters and avoiding the need for problem-specific pole and branch-point singularity tracking that are otherwise necessary, for example, in (optimally-converging) integrations along the steepest-descent path.

II. GREEN’S FUNCTION REPRESENTATION

Fig. 1 shows the basic geometry of the problem. Here, $\omega = 2\pi f$ is the angular frequency and $j = \sqrt{-1}$ denotes imaginary unit. As usual, the time dependence $e^{j\omega t}$ is assumed and suppressed. The successive cylindrical layers are indicated by the index $\alpha = 1, \ldots, N$. In each layer, the inner radius (left boundary) is denoted as $a_{\alpha-1}$ and the outer radius...
(right boundary) as \( \alpha_n \). Moreover, \( \varepsilon_\alpha = \varepsilon_{\alpha \alpha} - j \frac{\omega_\alpha}{c} \) denotes the complex permittivity, and \( \mu_\alpha = \mu_{\alpha \alpha} - j \frac{\omega_\alpha}{c} \) denotes the complex permeability, with \( \varepsilon_{\alpha \alpha}, \mu_{\alpha \alpha} \) indicating the real permittivity and permeability, respectively, and \( \sigma_{\alpha}, \kappa_{\alpha \alpha} \) the electric and magnetic (effective) conductivities, respectively. The observation point is represented as \( r \) or \( (\rho, \phi, z) \) in cylindrical coordinates. Likewise \( r' \) or \( (\rho', \phi', z') \) denotes the source point. We let \( k_z \) denote longitudinal wavenumber along the \( z \) axis, so that the transverse wavenumber in layer \( \alpha \) is \( k_{\alpha p} = \sqrt{k_{\alpha}^2 - k_z^2} \), with \( k_{\alpha}^2 = \omega^2 \mu_{\alpha} \varepsilon_{\alpha} \). We let \( \alpha = s \) represent the layer where the dipole source resides.

![Fig. 1. Geometrical parameters of a cylindrical multilayered medium.](image)

For \( r \neq r' \), the tensor Green’s function (for the electric field due to a dipole) in a cylindrical multilayered media is written as [12, Ch. 7]

\[
G^{\alpha \alpha}(r, r') = \frac{j}{8\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} dk_z \left[ \mathbf{D}_{\alpha \mu} \cdot \left[ \mathbf{F}_n(\rho, \rho') \chi_n(z, \phi; z', \phi') \right] \cdot \mathbf{D}_{\alpha \mu}^T \right]
\]

with

\[
\chi_n(z, \phi; z', \phi') = e^{-jk_z(z-z')} e^{-jn(\phi-\phi')},
\]

and

\[
\mathbf{D}_{\alpha \mu} = \frac{1}{k_{\alpha p}} [\nabla \times \nabla \times \mathbf{a}_z, \quad -j\omega k_{\alpha} \nabla \times \mathbf{a}_z],
\]

with the latter operator acting on the primed coordinates to its left. The expressions for \( \mathbf{F}_n(\rho, \rho') \) are provided in Section III. From the \( e^{-jk_z(z-z')} e^{-jn(\phi-\phi')} \) dependence, \( \nabla = \mathbf{a}_z \frac{\partial}{\partial z} - \mathbf{a}_{\phi} \frac{\partial}{\partial \phi} - \mathbf{a}_\phi \frac{\partial}{\partial z} \) and \( \nabla' = \mathbf{a}_z \frac{\partial}{\partial z} + \mathbf{a}_{\phi} \frac{\partial}{\partial \phi} + \mathbf{a}_\phi \frac{\partial}{\partial z} \). Using electromagnetic duality, analytical expressions for the tensor Green’s function of the magnetic field due to a magnetic point source, \( \mathbf{G}^{\mu \mu}(r, r') \) can be similarly written.

III. RESCALED FORMULATION

A. Canonical integrand formulae

There are four possible expressions for the factor \( \mathbf{F}_n(\rho, \rho') \) appearing on (1a) according to the relative position of the observation and source points [16]:

Case 1: \( \rho \) and \( \rho' \) are in same layer and \( \rho \geq \rho' \)

\[
\mathbf{F}_n(\rho, \rho') = [H_n^{(2)}(k_{\alpha p}) \mathbf{I} + J_n(k_{\alpha p}) \mathbf{R}_{ss+1}^{-1}] \quad (2a)
\]

\[
\mathbf{M}_{ss+} \cdot [J_n(\rho) \mathbf{I} + H_n^{(2)}(\rho') \mathbf{R}_{ss+1}^{-1}],
\]

Case 2: \( \rho \) and \( \rho' \) are in same layer and \( \rho < \rho' \)

\[
\mathbf{F}_n(\rho, \rho') = [J_n(k_{\alpha p}) \mathbf{I} + H_n^{(2)}(k_{\alpha p}) \mathbf{R}_{ss-1}] \quad (2b)
\]

\[
\mathbf{M}_{ss-} \cdot [H_n^{(2)}(\rho') \mathbf{I} + J_n(\rho) \mathbf{R}_{ss-1}^{-1}],
\]

Case 3: \( \rho \) and \( \rho' \) are in different layers and \( \rho > \rho' \)

\[
\mathbf{F}_n(\rho, \rho') = [H_n^{(2)}(k_{\alpha p}) \mathbf{I} + J_n(k_{\alpha p}) \mathbf{R}_{ss+1}^{-1}] \quad (2c)
\]

\[
\mathbf{N}_{ss+} \cdot \mathbf{T}_{ss-} \cdot \mathbf{M}_{ss+} \cdot [J_n(\rho', \rho') \mathbf{I} + H_n^{(2)}(\rho') \mathbf{R}_{ss-1}^{-1}],
\]

Case 4: \( \rho \) and \( \rho' \) are in different layers and \( \rho < \rho' \)

\[
\mathbf{F}_n(\rho, \rho') = [J_n(k_{\alpha p}) \mathbf{I} + H_n^{(2)}(k_{\alpha p}) \mathbf{R}_{ss-1}] \quad (2d)
\]

\[
\mathbf{N}_{ss-} \cdot \mathbf{T}_{ss+} \cdot [H_n^{(2)}(\rho') \mathbf{I} + J_n(\rho') \mathbf{R}_{ss+1}^{-1}],
\]

In the above, \( J_n(\cdot) \) and \( H_n^{(2)}(\cdot) \) are Bessel and second-kind Hankel functions of order \( n \), \( \mathbf{I} \) is the \( 2 \times 2 \) identity matrix, and all other factors correspond to \( 2 \times 2 \) matrices, viz. \( \mathbf{M}_{ss+} = [\mathbf{I} - \mathbf{R}_{ss+1} \mathbf{R}_{ss+1}^{-1}]^{-1}, \mathbf{N}_{ss+} = [\mathbf{I} - \mathbf{R}_{ss+1} \mathbf{R}_{ss+1}^{-1}]^{-1} \) where \( \mathbf{R}_{ss+1} \) represents local reflection matrix between two adjacent cylindrical layers. Likewise \( \mathbf{R}_{ss+1} \) represents generalized reflection matrix between two adjacent layers, and \( \mathbf{T}_{ss+} \) represents generalized transmission matrix from the source layer to the observation layer. Such reflection and transmission matrices are determined from the boundary conditions at the interfaces. Explicit expressions thereof can be found in [12] and [16].

Even though (2) is analytically exact, it is often unsuited for computing accurate numerical results because of the poor scaling of Bessel and Hankel functions for extreme arguments. This is considered next.

B. Scaling behavior of cylindrical Bessel functions

The underflow and overflow problems that arise in the computation of \( J_n(z) \) and \( H_n^{(2)}(z) \) under finite-precision arithmetic can be briefly summarized as follow. When \( |z| \ll 1 \), \( H_n^{(2)}(z) \) assumes very large values which can produce overflow whereas \( J_n(z) \) assumes very small values which can produce underflow. In addition, when \( 3|m|n| > 1 \), \( H_n^{(2)}(z) \) assumes very small values which can produce underflow whereas \( J_n(z) \) assumes very large value which can produce overflow. One can thus identify three generic regions, dubbed here as “underflow”, “overflow”, and “normal-range” regions to classify the numerical behavior of \( J_n(z) \) and \( H_n^{(2)}(z) \), as illustrated in Fig. 2. This Figure delineates three regions, as yielded by assuming standard double-precision arithmetic and the numerical results produced by the standard freeware Bessel function library from Sandia National Laboratories [23]. The functions are computed along the path in complex plane set
by $z = x + jy, y = x, x = -10^p, p \in [-30, 30]$. The delineation of the three regions for $J_n(z)$ and $H_n^{(2)}(z)$ is similar, with complementary overflow and underflow domains. This complementarity is expected because the physical, hence bounded, quantities depend on their products. For both $J_n(z)$ and $H_n^{(2)}(z)$, their “normal-range” is narrowed in the argument domain for large orders $n$. Even though the “normal” range can widen with the use of higher precision arithmetic, i.e. quadruple-precision instead of double-precision, the basic overall delineation shape among these regions is the same because of the intrinsic poor-scaling behavior of cylindrical eigenfunctions for extreme arguments.

### C. Rescaled integrand formulae

In [16], a successful range-conditioned formulation to overcome the above underflow and overflow issues was developed based on the definition of three different types of arguments (small, moderate, and large) and associated numerical thresholds based on double-precision arithmetic. Here, we develop a more succinct rescaled form of $J_n(z)$ and $H_n^{(2)}(z)$ in each layer for the generic scenario depicted in Fig.1 that is independent of ad-hoc numerical thresholds and hence floating-point representation.

We denote $\hat{J}_n(z)$ and $\hat{H}_n^{(2)}(z)$ as the rescaled forms of $J_n(z)$ and $H_n^{(2)}(z)$ respectively. In the $\alpha$-th layer, $\hat{J}_n(z)$ and $\hat{H}_n^{(2)}(z)$ are defined as

$$\hat{J}_n(kp\rho) = \frac{J_n(kp\rho)}{J_n(kp\alpha)}, \quad \rho < \alpha, \tag{3a}$$

$$\hat{H}_n^{(2)}(kp\rho) = \frac{H_n^{(2)}(kp\rho)}{H_n^{(2)}(kp\alpha-1)}, \quad \rho > \alpha-1. \tag{3b}$$

Within the underflow or overflow ranges for $J_n(z)$ and $H_n^{(2)}(z)$, we can employ small or large arguments approximations, respectively, to analytically reduce (3) before any numerical computation, whereas in the normal range $\hat{J}_n(z)$ and $\hat{H}_n^{(2)}(z)$ can be computed directly from the above. The rescaled forms $\hat{J}_n(z)$ and $\hat{H}_n^{(2)}(z)$ are applicable for any range of arguments, but now become dependent on the (adjacent) layer interfaces. Using these rescaled functions, we can subsequently derive a rescaled formulation for computing $F_n(\rho, \rho')$ in a stable manner as follows:

Case 1: $\rho$ and $\rho'$ are in same layer and $\rho \geq \rho'$

$$F_n(\rho, \rho') = [\hat{H}_n^{(2)}(k\rho\rho') | \hat{J}_n(k\rho\rho') \hat{R}_{s\alpha+1} + \hat{M}_{s+1}] \hat{J}_n(k\rho\rho') \hat{H}_n^{(2)}(k\rho\rho'), \tag{4a}$$

Case 2: $\rho$ and $\rho'$ are in same layer and $\rho < \rho'$

$$F_n(\rho, \rho') = [\hat{J}_n(k\rho\rho') | \hat{H}_n^{(2)}(k\rho\rho') \hat{R}_{s\alpha+1}] \hat{J}_n(k\rho\rho') \hat{H}_n^{(2)}(k\rho\rho'), \tag{4b}$$

Case 3: $\rho$ and $\rho'$ are in different layers and $\rho > \rho'$

$$F_n(\rho, \rho') = [\hat{H}_n^{(2)}(k\rho\rho') | \hat{J}_n(k\rho\rho') \hat{R}_{s\alpha+1} \hat{N}_{\alpha+1} \cdot \hat{T}_{\alpha+1} \cdot \hat{M}_{s+1} \cdot \hat{M}_{s-1}] \hat{J}_n(k\rho\rho') \hat{H}_n^{(2)}(k\rho\rho'), \tag{4c}$$

Case 4: $\rho$ and $\rho'$ are in different layers and $\rho < \rho'$

$$F_n(\rho, \rho') = [\hat{J}_n(k\rho\rho') | \hat{H}_n^{(2)}(k\rho\rho') \hat{R}_{s\alpha+1}] \hat{J}_n(k\rho\rho') \hat{H}_n^{(2)}(k\rho\rho'), \tag{4d}$$

where $\hat{R}, \hat{T}, \hat{M}$ and $\hat{N}$ are rescaled versions of the prior $2 \times 2$ matrices $\hat{R}, \hat{T}, \hat{M}$ and $\hat{M}$. The generic relations between rescaled and original forms of the local reflection and transmission coefficients can be written as

$$\hat{R}_{\alpha+1} = \frac{J_n(k\rho\alpha \alpha) H_n^{(2)}(k\rho\alpha-1)}{H_n^{(2)}(k\rho\alpha \alpha)} R_{\alpha+1}, \tag{5a}$$

$$\hat{T}_{\alpha+1} = \frac{H_n^{(2)}(k\rho\alpha-1) H_n^{(2)}(k\rho\alpha \alpha)}{H_n^{(2)}(k\rho\alpha-1)} T_{\alpha+1}. \tag{5b}$$

Fig. 2. Numerical behavior under double-precision of Bessel and second-kind Hankel functions versus argument and order $n$. The functions are computed along a path in complex plane $z = u + jv$, with $u = -10^p$ and $p \in [-30, 30]$. 
\[ \hat{\textbf{R}}_{\alpha-1} = \frac{H_n^{(2)}(k_{\alpha}a_{\alpha-1})}{J_n(k_{\alpha}a_{\alpha})} \textbf{R}_{\alpha-1} , \]  
\[ \hat{\textbf{T}}_{\alpha-1} = \frac{J_n(k_{\alpha-1}a_{\alpha-1})}{J_n(k_{\alpha}a_{\alpha})} \textbf{T}_{\alpha-1} . \]  

The relation between the rescaled coefficients \( \hat{\textbf{R}}_{\alpha \pm 1} \) and \( \textbf{R}_{\alpha \pm 1} \) and the original ones also follows (5a-5c). The recursive expressions to obtain \( \hat{\textbf{T}}_{\beta,0} \) and \( \hat{\textbf{N}}_{\alpha \pm 1} \) in (5) remains the same as before; cf. [12], [16]. However, the expression for \( \hat{\textbf{M}}_{\pm, \pm} \) is modified to

\[ \hat{\textbf{M}}_{\pm, \pm} = \left[ I - \hat{H}_n^{(2)}(k_{\alpha} \rho') J_n(k_{\alpha} \rho') \hat{\textbf{R}}_{\alpha \pm 1} \right]^{-1} . \]

Formulas (3), (4), and (5) constitute the fully rescaled formulation to compute \( \textbf{F}_n(\rho, \rho') \) numerically. To compute \( \textbf{F}_n(\rho, \rho') \), it is necessary not only the numerical evaluation of \( J_n(z) \) and \( \hat{H}_n^{(2)}(z) \), but also that of \( zJ_n'(z) \) and \( z\hat{H}_n^{(2)}(z) \), as well as the product \( J_n(z)H_n^{(2)}(z) \). Fig. 3 displays the computed results of \( J_n(z) \), \( \hat{H}_n^{(2)}(z) \), \( zJ_n'(z) \), \( z\hat{H}_n^{(2)}(z) \), and \( J_n(z)H_n^{(2)}(z) \), along the same path in the complex plane as before for Fig. 2. For this plot, we set \( z/20 = 0.5 \) in \( J_n(z) = J_n(z)/J_n(0) \), and \( z/20 = 2.0 \) in \( \hat{H}_n^{(2)}(z) = H_n^{(2)}(z)/H_n^{(2)}(0) \).

The results from Fig. 3 indicate that the rescaled formulation does not exhibit any underflow and overflow problems in computation of the \( J_n(z) \), \( \hat{H}_n^{(2)}(z) \), \( zJ_n'(z) \), \( z\hat{H}_n^{(2)}(z) \), and \( J_n(z)H_n^{(2)}(z) \) factors present in the formulae, except at \( z = 0 \) for \( \hat{H}_n^{(2)}(z) \) and \( \hat{H}_n^{(2)}(z) \) due to the logarithmic singularity there. The magnitudes of \( J_n(z) \), \( \hat{H}_n^{(2)}(z) \), \( zJ_n'(z) \), \( z\hat{H}_n^{(2)}(z) \), and \( J_n(z)H_n^{(2)}(z) \) always decrease for the larger \( n \), which facilitates the choice of an adequate truncation of the azimuth series.

IV. PARABOLIC SOMMERFELD INTEGRATION PATH

Another important issue for the computation of the Green’s tensor in multilayered cylindrical media is how to evaluate the spectral integral in (1a) accurately and in a robust manner. In the complex \( k_z \) plane, the integrand factor \( \textbf{F}_n(\rho, \rho') \) and its derivatives have singularities at \( \rho = \pm \sqrt{k^2 - k_{\text{z}}^2} \) as well as a logarithmic branch-cut emanating from \( \rho = 0 \). Elsewhere, \( \textbf{F}_n(\rho, \rho') \) and its derivatives are analytic functions and integral value from \( -\infty \) to \( +\infty \) is invariant w.r.t. path deformations as long as those singularities are not crossed. Therefore, in principle there are many possible paths to perform the numerical integration; however, the numerical accuracy and convergence behavior can vary greatly among the possible choices. Generally the original integration path along the real \( k_z \) is undesirable because of nearby singularities and slow convergence properties. Ideally, the steepest-descent path (SDP) is the best option to yield fast convergence; however, it has the drawback of being problem-specific and necessitating detailed individual tracking of singularities as the problem parameters change. The Sommerfeld integration path (SIP), illustrated by the solid line in Fig. 4a, is usually employed to avoid singularities in a robust fashion under a variety of problem conditions. The SIP requires only three parameters for its specification: \( \delta_1 \), \( \delta_2 \) and \( \delta_3 \) as indicated, and is quite appropriate when \( z - z' \) is small. In contrast, when \( z - z' \) is large, the integrand factor \( e^{-jk_z(z-z')} \) becomes rapidly oscillatory along the SIP and hence more challenging to integrate numerically.

In this case, two deformed Sommerfeld integration paths: DSIP-I [3] [24] [7] and DSIP-II [16], indicated by the dotted-dashed line and the dashed line respectively in Fig. 4a, can be employed instead. The DSIP-I is useful for high-frequency (GHz range) problems, and it can be specified by three parameters: \( \delta_1 \), \( \delta_2 \), \( \delta_3 \) [3]. The DSIP-I parameter \( \delta_1 \) should be greater than the wavenumbers of all layers, and \( \delta_3 \) should be also large enough, as discussed in [24]. The \( \delta_2 \) parameter should be carefully chosen to avoid the physical poles and branch-cut. Along the triangular path, the factor \( e^{-jk_z(z-z')} = e^{-jRRe[k_z(z-z')]e^{-j3m[k_z(z-z')]}} \) decays exponentially with slow oscillations, which yields improved convergence. However, the extreme for this decaying factor is \( e^{-j\delta_2|z-z'|} \), so the convergence improvement with DSIP-I becomes limited for highly-oscillatory integrand case when \( |\rho-\rho'| = 0 \). The DSIP-II, which deforms the SIP along two vertical paths parallel to the imaginary axis, is specified by four parameters: \( \delta_1 \), \( \delta_2 \), \( \delta_3 \), and \( \delta_4 \), as indicated in Fig. 4a. Along the vertical segments, the factor \( e^{-jk_z(z-z')} \) becomes \( e^{-jRRe[k_z(z-z')]e^{-j3m[k_z(z-z')]}} \), i.e. exponentially decaying with decreased oscillations, thus leading to improved convergence. In [16], the DSIP-II has been used for frequencies of operation ranging from 0.01Hz to 10MHz. Nevertheless, an oscillatory behavior will remain present along the SIP portion of the DSIP-II. Moreover, the transition from SIP to DSIP-II is a discontinuous one, which requires a problem-dependent choice between SIP and DSIP-II.

To ameliorate these problems and utilize a path better applicable to both low and high frequency ranges, a parabolic Sommerfeld integration path (PSIP) is proposed here in lieu of the SIP and DSIP, as depicted in Fig. 4b. Due to its larger mean distance from the singular points, the PSIP yields a smoother integrand behavior and good convergence properties, while maintaining robustness against problem parameter variations. Since in actual computation, there can be two cases: \( \Im m[(k_{\alpha})_{\text{min}}] = 0 \) and \( \Im m[(k_{\alpha})_{\text{min}}] < 0 \), where \( \Im m[(k_{\alpha})_{\text{min}}] \) represents the minimum value among all branch points. We consider two PSIP accordingly, i.e. PSIP-I and PSIP-II, as indicated by the solid line and the dashed line, respectively, in Fig. 4b. The PSIP-I is described by three control parameters \( (\delta_2, \delta_3, \delta_4) \): the pair \( (\delta_1, \delta_2) \) representing the vertex coordinates of the parabola, which is also associated to another coordinate point on the parabola \( (\delta_3, \delta_{\text{PSIP-I}}) = 2\delta_1, \delta_{\text{PSIP-I}} \), and the integral truncation range \( \delta_4 \) in Fig. 4b. The PSIP-I parameter \( \delta_2 \) is chosen as in the DSIP-I case. The other PSIP-I parameters \( (\delta_3, \delta_{\text{PSIP-I}}) \) and \( \delta_4 \) are determined in the same manner as for PSIP-II, as explained next. The PSIP-II path is described by three control parameters \( (\delta_2, \delta_3, \delta_4) \): the pair \( (0, \delta_2) \) representing the vertex coordinates of the parabola, another coordinate point on the parabola \( (\delta_1, \delta_{\text{PSIP-II}}, \Im m[(k_{\alpha})_{\text{min}}]) \), and the integral truncation range \( \delta_4 \) indicated in Fig. 4b. The parameter \( \delta_2 \) is defined in the same manner as for SIP and DSIP-II in [16], i.e.
\( \delta_2 = \Im \{\Im(k_{\alpha})_{\text{min}}\}/2 \). The determination of \( \delta_3 \) is similar as DSIP-II, i.e. \( \delta_3 = 2 \Re \{\Im(k_{\alpha})_{\text{max}}\} \), and the truncation parameter \( \delta_4 \) is determined to ensure convergence. For the case \( |z - z'| = 0 \), we set \( |\delta_4| > a \Im \{\Im(k_{\alpha})_{\text{max}}\} \), where \( a \) is a control parameter chosen as \( a \geq 30 \) generally. For \( |z - z'| \neq 0 \), \( \delta_4 \) can be set by \( \delta_4,\text{DSIP-II} = \delta_4,\text{PSIP} = -\ln \gamma/|z - z'| \) as in [16], where \( \gamma \) is a tolerance parameter chosen as \( \gamma \leq 10^{-20} \).

To demonstrate the efficiency and robustness of the proposed PSIP, we consider the relative error of the Sommerfeld identity \( I_1 \) corresponding to a homogeneous media with \( \varepsilon_r = 1.0, \mu_r = 1.0, \) and \( \sigma = 0.0 \) or \( \sigma = 1.0 \) S/m with control parameters as defined above. The rescaled terms in the Sommerfeld identity \( I_1 \) are defined as

\[
I_{1,a} = \frac{e^{-jk_s|r-r'|}}{|r-r'|} \tag{6a}
\]

and

\[
I_{1,n} = -\frac{j}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_z e^{-jn(\phi-\phi')} e^{-jk_s(z-z')} f_n(\rho, \rho')
\]
with
\[ f_n(\rho, \rho') = J_n(k_\rho \rho') H_n^{(2)}(k_\rho \rho') \times \begin{cases} \hat{I}_n^{(2)}(k_\rho \rho), & \rho > \rho' \\ 1, & \rho = \rho' \\ J_n(k_\rho \rho'), & \rho < \rho' \end{cases} \]  
(6b)
where \( k^2 = \omega^2 \mu \varepsilon_c \), and \( \varepsilon_c \) is the complex permittivity. The relative error is computed as
\[ 10 \log_{10} \left| \frac{I_{1,n} - I_{1,a}}{I_{1,a}} \right| \]  
(7)
where \( I_{1,a} \) is closed-form (exact) value and \( I_{1,n} \) is the numerical integration result. In order to examine the convergence of the numerical integrals along the SIP and DSIP, we assign \( f = 6 \text{ GHz} \) as a representative “high” frequency and \( f = 20 \text{ KHz} \) as a representative “low” frequency. We assume \( n_{max} = 40 \) as the maximum azimuthal order and \( \phi - \phi' = 0 \). The relative error of the \( I_1 \) for a wide range of \( |z - z'| \) distances is shown in Fig. 5. It is seen that the relative error from PSIP is far less than the relative error from DSIP under otherwise same conditions. This result is particularly pronounced when \( |z - z'| \) becomes large, although for sufficiently large distances the results deteriorate using a fixed number of quadrature points (equal to 4,096 in this case). Fig. 6 shows the relative error of versus number of quadrature points for fixed \( \log_{10}(|\text{Re}[k_\rho]| |z - z'|) = -1 \). It is seen that the relative error from PSIP converges faster than the error from DSIP, with less quadrature points required for same accuracy. Figs. 5 and 6 also corroborate the fact that the PSIP is robust w.r.t. changes on the physical parameters (frequency of operation and medium conductivity). This is further verified by the numerical results presented in next section.

### V. Results

This section presents some results of practical significance for the modeling of well-logging tools in borehole geophysics. We assume both relative permittivity \( \varepsilon_r \) and relative permeability \( \mu_r \) equal to one, whereas the resistivity can exhibit large variations across the cylindrical layers. Because well-logging tools typically employ small coil antennas as both transmitters and receivers that can be well represented by magnetic dipoles, we consider here the Green’s tensor for the magnetic field produced by a magnetic source. The results are obtained using a double-precision C++ code running on a PC with a 2.6 GHz 2-core Opteron processor. First, we compare results for source and field points situated along \( z \) axis against numerical mode-matching method (NMM) results from [25]. Next, more general results of the present algorithm are compared against the finite element method (FEM) in [26] [27], and results from the range-conditioned cylindrical formulation (RCCF) considered in [16].

### A. Source and Field Points Along \( z \) Axis

We first compute the Green’s tensor in a cylindrical two-layer domain with inner resistivity \( R_{mud} \) (borehole mud) and outer resistivity \( R_\epsilon \) (Earth formation). This simple two-layer model is widely used for borehole effect corrections in forward simulations. Table I provides a comparison of Green’s tensor
component $\mathbf{G}_{zz}^{mm}$ computed by NMM and by the present algorithm, with frequency $f = 0.36$ KHz, fixed resistivity $R_t = 1.0$ Ω·m and varying resistivity $R_{mud} = 10^p$ Ω·m, with $p \in [-4, 5]$. The positions of transmitter and receiver are $\rho = \rho' = 0$, and such that $\phi - \phi' = 0$ and $z - z' = 40.16$ cm. The cylindrical layer interface is at $a_1 = 10.16$ cm. Note that when $R_{mud} = R_t = 1.0$ Ω·m, this scenario recover a homogeneous-domain problem, with a closed-form analytical expression equal to $(-0.17823718, -8.62681728)$ A/m. This value has better agreement with the present method than with NMM. Table I shows that the present results are in general good agreement with NMM results for a wide range of $R_{mud}$. The discrepancy is only pronounced for the imaginary part in the case $R_{mud} = 10^3$ Ω·m. Given the additional results presented below and the better match to the homogeneous-domain case, we have good reasons to believe that the results of present method provide better accuracy than the NMM results.

B. General Case

Corresponding to the four integrands in Eq. (2) and Eq. (4), we have computed 15 cylindrical multilayered scenarios with different number of layers and layer resistivities, as well as different source/field separations. The various case scenarios are summarized in Tables II and III. Note that for Cases 6, 7, 10, 11, and 12, there are 3 different subcases in each, involving field and source points in different layers. Table II lists the different cylindrical multilayered media considered, with respective resistivity values $R_\alpha$ and interface radii $a_\alpha$. Table III provides the additional parameters, i.e. operating frequency, field and source locations, and the component of Green’s tensor under evaluation, together with the cylindrical layered media considered. Note that the field and source positions are specified by $\rho, \rho', \Delta z = |z - z'|$, and we assume $\Delta \phi = |\phi - \phi'| = 0$. Table IV provides the computed results by FEM [26] [27], the RCCF approach [16], and present method. The Green’s tensor is expressed in phasor form (amplitude...
and phase) in Table IV. Note that there are closed-form analytical solutions for Case 1 and Case 2 in homogeneous media, equal to (4.1884±91.0681) and (8.3259±91.2105), respectively. Note also that the present method can compute all nine component values of Green’s tensor simultaneously, for example Case 1 and Case 2 together, or Case 4 and Case 5 together. Table IV illustrates the good agreement among FEM results, RCCF results, and the present method, and especially among the latter two. The latter two set of results also present better match to analytical solutions, Cases 1 and 2.

VI. CONCLUSION

We have developed an efficient rescaled formulation to compute the Green’s function due to sources in cylindrical multilayered media. Rescaled representations for cylindrical Bessel and Hankel functions, as well for analytical expressions for the integrand, are obtained to eliminate numerical overflows and underflows otherwise present in finite-precision computations based on the canonical expressions. In addition, a parabolic Sommerfeld integration path is proposed to enable accurate and fast-converging spectral integrations that are robust to variations on physical parameter values spanning several orders of magnitude. The suitability of the proposed algorithm has been verified in a number of scenarios relevant to borehole geophysics.

REFERENCES

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