Robust Algorithms for Electromagnetic Field Computation with Conduction Currents and Kinetic Charge-Transport Models

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

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2015

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Abstract

Numerical simulations of electromagnetic phenomena can guide, complement, and even replace real measurements and experiments in many areas such as radar scattering, geophysical exploration, device modeling, and laser-plasma interactions. This dissertation introduces several numerical algorithms for two classes of electromagnetic problems involving charge transport. The first class of problems is associated to geophysical exploration using borehole sensors, where the underlying response can be modelled as those of dipoles or current electrodes embedded within cylindrically-stratified and anisotropic layers. This problem exhibits both wave and diffusive phenomena, the latter substantiated by macroscopic currents comprising the net flow of charged particles in conductive earth formations. Historically, several notorious numerical challenges exist to solve this problem because the numerical computation under finite machine precision is neither stable nor accurate when using the canonical mathematical expressions. In this dissertation, we develop a robust mathematical formulation that is amenable to numerical implementation in finite (double) precision under a vast range of parameters, such as operating frequencies ranging from 0.01 Hz to hundreds of MHz, layer conductivities spanning about ten orders of magnitude, large number of cylindrical layers, and varying layer thicknesses. The second class of problems considered here deals with particle-in-cell algorithms to solve for the electromagnetic fields and kinetic charge transport in plasma-related applications. One
long-outstanding challenge for particle-in-cell algorithms on unstructured grids has been the gradual deterioration of accuracy caused by the violation of charge conservation even though those grids are often necessary to accurately model complex geometries and devices. In this dissertation, we introduce a new particle-in-cell algorithm that yields exact charge and energy conservation properties. The proposed algorithm is based on the exterior calculus representation of the various dynamical objects (field, currents, charges) as differential forms of various degrees and their consistent interpolation from the grid to continuum space.
To my family and friends
Acknowledgments

First and foremost, I would like to gratefully and sincerely thank my advisor, Prof. Fernando L. Teixeira, for his encouragement, guidance, support, and even patience during my graduate studies at The Ohio State University. I have been amazingly fortunate to have invaluable discussions with him on many subjects and to receive his advice as his doctoral student. I would never have been able to complete a Ph.D. degree without his endless enthusiasm and help.

Also, I would like to express my sincere appreciation to all committee members, Prof. Joel T. Johnson, Prof. Ronald M. Reano, and Prof. Roberto Rojas-Teran for their continuous support and feedback, which enabled me to complete the research presented here.

I am also indebted to many of my ESL colleagues, who are Kamalesh Sainath, Seungho Doo, WoonGi Yeo, Henry Vo, and Dongyeop Na.

Last but not least, I offer my sincerest regards to my beloved wife, Jooyoung Jang, who always supports me. Also, I would like to thank my parents, who trust and encourage me forever.

Financial support from National Science Foundation grant ECCS-1305838, Halliburton Technology Research, Ohio Supercomputer Center grants PAS-0061 and PAS-0110, and The Ohio State University Presidential Fellowship Program are gratefully acknowledged.
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Chapter 1

Introduction

1.1 Background and Motivation

This dissertation focuses on the development of new algorithms for the simulation of electromagnetic problems involving charge transport. The contributions are two-pronged.

First, we develop a robust methodology for computing fields and potentials due to dipole sources or current electrodes in cylindrically-stratified and anisotropic media where the charge transport is affected by conduction or eddy currents. This type of geometry is of particular importance to a number of technological applications including geophysical borehole exploration, where well-logging tools are deployed to determine the electromagnetic properties of nearby earth formations [1, 2, 3]. Figure 1.1 depicts the basic geometry of such cylindrically-stratified problems. In particular, the conductivity of the surrounding formations around the borehole wherein well-logging tools are lowered is one of the key parameters of interest for identifying hydrocarbon-bearing zones. Two basic types of well-logging tools can be utilized for this purpose. The first type is induction logging [4, 5, 6, 7, 8, 9], where several coil antennas wrapped around a metallic mandrel concentric to the borehole are used
to excite and measure eddy (or Foucault) currents in the earth formations. The
induced eddy current increases as the formation conductivity increases. Induction
tools are well-suited for cases where the drilling of the borehole utilizes oil-based mud
(with low conductivity) as lubricant. The second type of logging is electrode logging
[10, 11, 12, 13, 14, 15], in which lower frequency current electrodes–instead of coil
antennas–are utilized to directly inject conduction currents into the formation. In
electrode logging, the potential differences between a set of transmit and receive elec-
trodes are measured to determine the conductivity of the surrounding formations.
Electrode tools are well-suited for cases where the drilling of the borehole utilizes
water-based mud (with high conductivity) as lubricant. Historically, several notori-
ous numerical challenges exist to solve this type of problems because of the extremely
wide range of physical parameters encountered in practice. That is because, when
using the canonical mathematical formulation for this problem (based on cylindrical
Bessel and Hankel eigenfunctions together with a spectral integration along the axial wavenumber), numerical computation under finite machine precision is neither stable nor accurate for extreme parameters. In this dissertation, we develop and implement a robust mathematical formulation that is amenable to numerical implementation in finite (double) precision under a vast range of input parameters, such as operating frequencies ranging from 0.01 Hz to hundreds of MHz, layer conductivities spanning more than ten orders of magnitude, large number of cylindrical layers, varying layer thicknesses, and arbitrary anisotropy ratios.

The second part of this dissertation focuses on the problem of simulating charge transport based on a kinetic model where the movement and mutual interaction of many individual particles (space charges) are considered in the presence of external electromagnetic fields. This type of model finds application in problems where the collision rate among charged particles is small or negligible and the movement of each particle, rather than the macroscopic net current distribution as considered in the first part of the dissertation, needs to be modelled. These conditions are encountered in many plasma-related applications. Particle-in-cell (PIC) algorithms are typically used in this case [16]. Conventional PIC algorithms use a marching-on-time approach, where each time step consists of four substeps: field-update, gather, particle-update, and scatter, as schematically illustrated in Figure 1.2. During the field-update substep, the electromagnetic field values are updated from a finite-difference, finite-volume or finite-element discretization of Maxwell’s equations on a mesh (grid). In this dissertation, we utilize a finite-element discretization on unstructured (i.e., generally irregular) grids because they are highly desirable to better model complex
Figure 1.2: Four substeps of particle-in-cell algorithms.

geometries and devices. In the gather substep, the field values thus obtained are interpolated at the positions of particles. Then, in the particle-update substep, kinetic attributes of each particle such as position and velocity are updated using Newton’s second law and Lorentz law. Finally, in the scatter substep, the updated particle attributes are used to determine induced charge and current densities and to interpolate those onto mesh elements (nodes and edges, respectively), for use in the next field-update substep. One long-outstanding challenge for PIC simulations on unstructured grids has been the gradual deterioration of accuracy caused by the violation of charge conservation. This dissertation introduces a new particle-in-cell algorithm that yields exact charge and energy conservation properties on unstructured grids. The proposed algorithm is based on the exterior calculus representation of the various dynamical objects (field, currents, charges) as differential forms of various degrees and their consistent interpolation from the grid to continuum space.
1.2 Organization of this Dissertation

This dissertation is organized as follows. In rough lines, Chapter 2 and Chapter 3 introduce robust formulations for induction logging in isotropic and anisotropic formations, respectively. Chapter 4 does the same for electrode logging. Chapter 5 introduces the new charge conserving PIC algorithm.

More specifically, Chapter 2 provides a semi-analytical algorithm for the numerical computation of electromagnetic fields due to Hertzian dipoles in cylindrically stratified and isotropic media. The tensor Green’s function for this problem has the generic form [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]:

$$\sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \Phi_n(\rho, \rho'),$$

where the integrand factor $\Phi_n(\rho, \rho')$ incorporates products of standard cylindrical (Bessel and Hankel) functions. Even though several brute-force methodologies such as finite-elements [30, 31, 32, 33, 34, 35, 36], finite-differences [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47], or finite-volumes [48, 49] can in principle be used to solve the problem, they lack optimality for this specific geometry, typically demanding computational resources that are two or more orders of magnitude larger than those of the semi-analytical methods considered here. This property makes brute-force methods less suited for solving the inverse problem (i.e., to determine the formation conductivity from a given the well-logging tool response) based on iterative algorithms demanding many repeated forward solutions. As noted before, the semi-analytical formulation developed here is designed to remain stable and yield accurate results under finite machine precision for any number of layers and layer thickness values, and across the
entire range of possible layer conductivities and frequencies of operation, which can both span several orders of magnitude.

In Chapter 3, the semi-analytical algorithm discussed in Chapter 2 is extended for cylindrical layers with uniaxial anisotropy. In borehole geophysics, anisotropy is quite common and may result from geological factors affecting the various earth layers such as salt water penetrating porous fractured formations and thereby increasing the conductivity in the direction parallel to the fracture and/or the presence of clay and sand laminates with directionally dependent resistivities. Here, for generality, we assume each layer to be doubly uniaxial, i.e., both the complex permittivity tensor $\tilde{\epsilon}$ (which includes the conductivity tensor) and the permeability tensor $\tilde{\mu}$ are independently uniaxial, which facilitates the analysis of equivalent problems using electromagnetic duality.

In Chapter 4, we present a robust semi-analytical formulation now to tackle the numerical computation of electric potentials due to current electrodes in cylindrically-stratified and anisotropic media. The motivation and construction of the algorithm is similar, but not identical, to those in the Chapter 2 and Chapter 3. In particular, as the mathematical properties of the involved spectral integral are different in this case, other numerical integration algorithms, based on extrapolation techniques, are considered.

In Chapter 5, we describe a charge-conserving scatter-gather algorithm for particle-in-cell simulations on unstructured grids. Charge conservation is obtained from first principles, i.e., without the need for any post-processing or correction steps. Some unique ingredients of the new algorithm are (1) the use of (discrete) differential forms of various degrees to represent fields, currents, and charged particles and to provide
localization rules for the degrees of freedom thereof on the various corresponding grid elements (nodes, edges, facets), (2) the use of Whitney forms as basic interpolants from discrete differential forms to continuum space, and (3) the use of a Galerkin formula for the discrete Hodge star operators (i.e., “mass matrices” incorporating the metric datum of the grid) applicable to generally irregular, unstructured grids. The expressions obtained for the scatter charges and scatter currents are very concise and do not involve numerical quadrature rules. Appropriate fractional areas within each grid element are found to exactly represent these scatter charges and scatter currents within the element. Also, a simple geometric representation for the (exact) charge conservation mechanism is obtained by such identification. The field update is based on the coupled first-order Maxwell’s curl equations to avoid spurious modes with secular growth (otherwise present in formulations that discretize the second-order wave equation). A number of examples with practical interest are provided to verify the preservation of discrete Gauss’ law for all times.

Finally, Chapter 6 provides several concluding remarks.
Chapter 2

Electromagnetic Fields from Dipole Sources in Cylindrically Stratified and Isotropic Media

In this chapter, the computation of electromagnetic fields due to arbitrarily-oriented Hertzian dipoles in cylindrically stratified media is discussed. Similar discussion can be found in [50]. When all cylindrical layers are isotropic, there exist the analytical formulations based on canonical cylindrical eigenfunctions such as Bessel and Hankel functions in the transverse ($\rho$ and $\phi$) domain and their azimuthal magnitudes along with harmonic functions along the axial ($z$) direction [17]. The essence of the formulations is the analytical description of multiple reflections and transmissions obtained by appropriate boundary conditions. As the eigenfunctions can be expressed by a continuum spectrum in the spectral domain (Green’s function), a Fourier-type integral with respect to the axial spectral wavenumber $k_z$ is utilized for valid electromagnetic solutions [51, 52, 53, 54, 55, 56, 57, 58, 59]. Unfortunately, the numerical implementation of the formulations under finite machine precision suffers from undesired numerical challenges such as underflow, overflow, and/or round-off errors. These challenges are often occurred in most practical problems and extremely intractable. The underlying reason is the poor scaling of the eigenfunctions for very small and large arguments, and high azimuthal orders. Furthermore, the convergence
issues associated with the evaluation of the spectral integral and truncation of an infinite series over azimuthal modes aggravate the numerical challenge. It should be assumed that these challenges are always confronted because relevant physical parameters such as medium properties (resistivities, permittivities, and permeabilities), layer thicknesses, the transverse and longitudinal separation between source and observation locations, and the source frequency can have extreme values [60]. There have been several efforts to handle these types of challenges in the past. Swathi and Tong [61] developed a stable numerical algorithm for plane-wave scattering by highly absorbing layers based upon several scaled cylindrical functions. Jiang et al. [62] tried to stabilize the numerical computation for a large number of cylindrical layers and disparate radii. Mie scattering from multilayered spheres [63, 64] was computed successfully by avoiding the unstable behavior of Bessel functions of various orders with continued fractions [65], logarithmic derivatives [66], or arbitrary-precision arithmetic [67].

A salient feature of the present algorithm is that it is not limited to particular scenarios of interest but constructed systematically for general purposes. The algorithm is initially inspired by various sets of range-conditioned cylindrical functions in place of standard Bessel and Hankel functions. These new functions are defined on three non-overlapping subdomains for stable numerical computations under double-precision arithmetics. The algorithm is then improved by two robust integration schemes in the complex $k_z$ plane in order to yield fast convergence for any physical scenarios.

This chapter is organized as follows. In Section 2.1, the existing formulations for the described problem above are reviewed. In Section 2.2, the new set of functions
are introduced with several relevant numerical parameters. The modifications (range-conditioning) of the existing formulations are discussed in Section 2.3, where all types of coefficients are modified using the new functions. After that, the integrand is differently modified for all three components of electromagnetic fields. In Section 2.4, the two robust integration schemes and proper truncation strategies needed to deal with an improper integral and infinite series are discussed. In Section 2.5, the results using the new algorithm for geophysical problems with layer resistivities ranging from \(10^{-8} \, \Omega \cdot m\) to the orders of \(10^3 \, \Omega \cdot m\), frequencies from 10 MHz down to 0.01 Hz, and for various layer thicknesses are compared to the finite element method (FEM) data for validation. Throughout this chapter, all cylindrical layers are assumed to be isotropic and circular concentric, and all values are represented in a phasor form with \(e^{-i\omega t}\) convention except Section 2.5.

### 2.1 Analytical Formulations

The expressions of electromagnetic fields due to arbitrarily-oriented Hertzian dipoles in cylindrically stratified and isotropic media are provided in this section. Detailed discussions can be found in [17]. Since the direct computation of \(\rho\) and \(\phi\)-components of the fields is too inefficient, the best approach is to derive these components from the \(z\)-components. Let us first take a look at the derivation of the \(z\)-components. To begin with, the scalar Green’s function is the solution to

\[
(\nabla^2 + k^2)g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),
\]

where

\[
g(\mathbf{r}, \mathbf{r}') = g(\mathbf{r} - \mathbf{r}') = \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}.
\]
Using the scalar Green’s function, the $z$-components of the electric and magnetic field can be obtained by solving the vector wave equation. For the Hertzian electric dipole source of the form

$$J(r) = II\alpha'\delta(r - r'),$$

(2.3)

the $z$-components are

$$E_z = \frac{II}{\omega \epsilon} \left[ \hat{z} \cdot \hat{\alpha}' k^2 + \frac{\partial}{\partial z'} \nabla' \cdot \hat{\alpha}' \right] \frac{e^{ik|r-r'|}}{4\pi |r-r'|};$$

(2.4a)

$$H_z = -\hat{z} \cdot \nabla' \times \hat{\alpha}' II \frac{e^{ik|r-r'|}}{4\pi |r-r'|}.$$  

(2.4b)

If the source exists in cylindrically stratified media, the expansion of the scalar Green’s function in terms of cylindrical wave functions is necessary. With the aid of Sommerfeld identity, (2.2) can be expanded as

$$e^{ik|r-r'|}/|r-r'| = \sum_{n=-\infty}^{\infty} \frac{i e^{in(\phi-\phi')}}{2} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} J_n(k_\rho \rho') H_n^{(1)}(k_\rho \rho').$$

(2.5)

The $z$-components in a homogeneous medium are

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{II}{4\pi \omega \epsilon} D' \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} J_n(k_\rho \rho') H_n^{(1)}(k_\rho \rho'),$$

(2.6)

where

$$D' = \frac{i}{2} \left[ (\hat{z}k^2 + \frac{\partial}{\partial z'} \nabla') \cdot \hat{\alpha}' \right].$$

(2.7)

When the medium is cylindrically stratified, reflections and transmissions at each interface occur. In order to take these into account, (2.6) is modified to

$$\begin{bmatrix} E_{jz} \\ H_{jz} \end{bmatrix} = \frac{II}{4\pi \omega \epsilon_j} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left\{ J_n(k_{j\rho} \rho) H_n^{(1)}(k_{j\rho} \rho) \hat{I} + H_n^{(1)}(k_{j\rho} \rho) \hat{a}_{jn}(\rho') + J_n(k_{j\rho} \rho) \hat{B}_{jn}(\rho') \right\} \cdot \hat{D}'_j,$$

(2.8)
where

\[
\vec{D}'_j = \frac{i}{2} \left[ \left( \hat{z}k_j^2 - ik_z \nabla' \right) \cdot \hat{\alpha}' \right].
\]  \tag{2.9}

Equation (2.8) is only valid when a field point of interest is within a source layer. That is the reason why the subscript \( j \) is used above. The factors \( \vec{a}_{jn}(\rho') \) and \( \vec{b}_{jn}(\rho') \) depend on generalized reflection coefficients, to be determined. \( \vec{D}'_j \) described in (2.9) is an operator acting on the primed variables at the left, with \( \hat{\alpha}' \) being a unit vector corresponding to the dipole orientation. When a field at a point outside the source layer is of interest, generalized transmission coefficients as well as generalized reflection coefficients are required. As a result, the computation of the \( z \)-components can be divided into four different cases in terms of the relative position of a source point and a field point. With the assumption that \( \rho \) is the distance between the origin and a field point and \( \rho' \) is the distance between the origin and a source point, the \( z \)-components for the four cases are

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{iIl}{4\pi\omega} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z - z')} \mathbf{F}_n(\rho, \rho') \cdot \vec{D}'_j, \tag{2.10}
\]

where

for Case 1 : \( \rho \) and \( \rho' \) are in the same layer. \( (\rho \geq \rho') \)

\[
\mathbf{F}_n(\rho, \rho') = \left[ H_n^{(1)}(k_{jp}\rho)\mathbf{I} + J_n(k_{jp}\rho)\tilde{\mathbf{R}}_{j,j+1} \right] \cdot \tilde{\mathbf{M}}_{j+}
\]

\[
\cdot \left[ J_n(k_{jp}'\rho')\mathbf{I} + H_n^{(1)}(k_{jp}'\rho')\tilde{\mathbf{R}}_{j,j-1} \right], \tag{2.11a}
\]

for Case 2 : \( \rho \) and \( \rho' \) are in the same layer. \( (\rho < \rho') \)

\[
\mathbf{F}_n(\rho, \rho') = \left[ J_n(k_{jp}\rho)\mathbf{I} + H_n^{(1)}(k_{jp}\rho)\tilde{\mathbf{R}}_{j,j-1} \right] \cdot \tilde{\mathbf{M}}_{j-}
\]

\[
\cdot \left[ H_n^{(1)}(k_{jp}'\rho)\mathbf{I} + J_n(k_{jp}'\rho)\tilde{\mathbf{R}}_{j,j+1} \right], \tag{2.11b}
\]
for Case 3: $\rho$ and $\rho'$ are in different layers. ($\rho > \rho'$)

$$
\mathbf{F}_n(\rho, \rho') = \left[ H_n^{(1)}(k_{i\rho}) \mathbf{I} + J_n(k_{i\rho}) \tilde{R}_{i,i+1} \right] \cdot \mathbf{N}_{i+} \cdot \tilde{T}_{ji} \cdot \tilde{M}_{j+} \\
\quad \cdot \left[ J_n(k_{j\rho'}) \mathbf{I} + H_n^{(1)}(k_{j\rho'}) \tilde{R}_{j,j-1} \right],
$$

(2.11c)

for Case 4: $\rho$ and $\rho'$ are in different layers. ($\rho < \rho'$)

$$
\mathbf{F}_n(\rho, \rho') = \left[ J_n(k_{i\rho}) \mathbf{I} + H_n^{(1)}(k_{i\rho}) \tilde{R}_{i,i-1} \right] \cdot \mathbf{N}_{i-} \cdot \tilde{T}_{ji} \cdot \tilde{M}_{j-} \\
\quad \cdot \left[ H_n^{(1)}(k_{j\rho'}) \mathbf{I} + J_n(k_{j\rho'}) \tilde{R}_{j,j+1} \right].
$$

(2.11d)

In (2.11a) – (2.11d), $\mathbf{I}$ is a 2×2 identity matrix, $\tilde{R}_{j,j\pm 1}$ and $\tilde{R}_{i,i\pm 1}$ are generalized reflection coefficients defined in Section 2.3.2, $\tilde{T}_{ji}$ are generalized transmission coefficients defined in Section 2.3.3, and $\tilde{M}_{j\pm}$ and $\tilde{N}_{i\pm}$ are auxiliary coefficients defined in Section 2.3.4. It should be noted that $\tilde{M}_{i\pm}$ in [17, p. 176] for Case 3 and Case 4 are incorrect and replaced with $\tilde{N}_{i\pm}$ as shown in (2.11c) and (2.11d).

As mentioned above, the transverse components can be derived from the $z$-components. The transverse electric and magnetic fields can be expressed as [68]

$$
\mathbf{E}_s(\mathbf{r}) = \int_{-\infty}^{\infty} dk_z \tilde{E}_s(k_z, \mathbf{r}),
$$

(2.12a)

$$
\mathbf{H}_s(\mathbf{r}) = \int_{-\infty}^{\infty} dk_z \tilde{H}_s(k_z, \mathbf{r}),
$$

(2.12b)

where $\tilde{E}_s(k_z, \mathbf{r})$ and $\tilde{H}_s(k_z, \mathbf{r})$ are the corresponding spectral components and the subscript $s$ simply indicates transverse-to-$z$ components. For each $k_z$ value, these spectral components can be determined from the longitudinal components $E_z$ and $H_z$.
by using Maxwell’s equations. They are expressed as

\[
\tilde{E}_s(k_z, r) = \frac{1}{k_p^2} \left[ ik_z \nabla_s E_z - i \omega \mu \hat{z} \times \nabla_s H_z \right]
\]

\[
= \frac{1}{k_p^2} \left[ ik_z \left( \hat{\rho} \frac{\partial E_z}{\partial \rho} + \hat{\phi} \frac{n \omega \mu}{\rho} E_z \right) - i \omega \mu \hat{z} \times \left( \frac{\partial H_z}{\partial \rho} + \frac{\partial}{\partial \rho} H_z \right) \right]
\]

\[
= \hat{\rho} \left[ \frac{1}{k_p^2} \left( ik_z \frac{\partial E_z}{\partial \rho} - \frac{n \omega \mu}{\rho} H_z \right) \right] + \hat{\phi} \left[ \frac{1}{k_p^2} \left( -\frac{n k_z}{\rho} E_z - i \omega \mu \frac{\partial H_z}{\partial \rho} \right) \right], \quad (2.13a)
\]

\[
\tilde{H}_s(k_z, r) = \frac{1}{k_p^2} \left[ ik_z \nabla_s H_z + i \omega \epsilon \hat{z} \times \nabla_s E_z \right]
\]

\[
= \frac{1}{k_p^2} \left[ ik_z \left( \frac{\partial H_z}{\partial \rho} + \hat{\phi} \frac{n \omega \epsilon}{\rho} H_z \right) + i \omega \epsilon \hat{z} \times \left( \frac{\partial E_z}{\partial \rho} + \frac{\partial}{\partial \rho} E_z \right) \right]
\]

\[
= \hat{\rho} \left[ \frac{1}{k_p^2} \left( ik_z \frac{\partial H_z}{\partial \rho} + \frac{n \omega \epsilon}{\rho} E_z \right) \right] + \hat{\phi} \left[ \frac{1}{k_p^2} \left( -n k_z \frac{H_z}{\rho} + i \omega \epsilon \frac{\partial E_z}{\partial \rho} \right) \right]. \quad (2.13b)
\]

The matrix representations of (2.13a) and (2.13b) for the \(\rho\) and \(\phi\)-components are

\[
\begin{bmatrix}
E_{\rho} \\
H_{\rho}
\end{bmatrix} = \frac{1}{k_p^2} \left[ ik_z \frac{\partial}{\partial \rho} - \frac{n \omega \mu}{\rho} \right] \begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{1}{k_p^2} \mathbf{B}_n \begin{bmatrix}
E_z \\
H_z
\end{bmatrix}, \quad (2.14)
\]

\[
\begin{bmatrix}
E_{\phi} \\
H_{\phi}
\end{bmatrix} = \frac{1}{k_p^2} \left[ -\frac{n k_z}{\rho} \frac{\partial}{\partial \rho} - i \omega \mu \frac{\partial}{\partial \rho} \right] \begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{1}{k_p^2} \mathbf{C}_n \begin{bmatrix}
E_z \\
H_z
\end{bmatrix}, \quad (2.15)
\]

where

\[
\mathbf{B}_n = \begin{bmatrix}
ik_z \frac{\partial}{\partial \rho} - \frac{n \omega \mu}{\rho} \\
\frac{n \omega \mu}{\rho} & i k_z \frac{\partial}{\partial \rho}
\end{bmatrix} = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}, \quad (2.16)
\]

\[
\mathbf{C}_n = \begin{bmatrix}
-\frac{n k_z}{\rho} \frac{\partial}{\partial \rho} - i \omega \mu \frac{\partial}{\partial \rho} \\
\frac{\partial}{\partial \rho} & -n k_z \frac{1}{\rho}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} \frac{\partial}{\partial \rho} \\
C_{21} \frac{\partial}{\partial \rho} & C_{22}
\end{bmatrix}. \quad (2.17)
\]

2.2 Range-Conditioned Cylindrical Functions

The computation of the \(z\)-components of electromagnetic fields in cylindrically stratified media includes the product of the Bessel function of the first kind \(J_n\) and the Hankel function of the first kind \(H_n^{(1)}\) for all the cases described in (2.11a) – (2.11d). In addition, when the \(\rho\)- and \(\phi\)-components are required to fully describe
electromagnetic fields, the product of derivatives of these functions also needs to be evaluated. In fact, those products sometimes involve very small or very large values due to the behavior of $J_n$ and $H_n^{(1)}$. For example, when $|z| << 1$, $H_n^{(1)}(z)$ has very large values whereas $J_n(z)$ has very small values. This disparity becomes more extreme for higher order modes. On the other hand, when the imaginary part of $z$ is much greater than one, i.e., $\Im m[z] >> 1$, $H_n^{(1)}(z)$ has very small values while $J_n(z)$ has very large values. In numerical computations, such disparate values can lead to unreliable results.

Furthermore, for cylindrically stratified media, different layers might yield small or large arguments even for the same integration point due to different medium properties. Since the magnitude of the argument depends on the variable of integration $k_z$, large variations in the magnitude might happen as $k_z$ varies as well. Therefore, significant round-off errors or inaccurate computation will occur in the course of numerical integration unless one carefully examines the inherent characteristics of the functions beforehand. As the large variation of function values stems from the exponential behavior of $J_n$ and $H_n^{(1)}$, exponential factors should be factored out whenever possible for robust and efficient numerical integration. In the sense, modified functions, for which the range of possible values is conditioned, are defined and denoted as “range-conditioned cylindrical functions.”

In the followings, the definition of range-conditioned cylindrical functions is provided in Section 2.2.1. In Section 2.2.2, threshold values to distinguish three different argument types for the functions are discussed.
2.2.1 Definition

When an argument \( z \) is sufficiently small compared to unity, \( J_n(z) \) and \( H_n^{(1)}(z) \) can be evaluated through the small argument approximations for \( n > 0 \) [17, p. 15].

\[
J_n(z) \approx \frac{1}{n!} \left( \frac{z}{2} \right)^n, \quad (2.18a)
\]

\[
J'_n(z) \approx \frac{1}{2(n-1)!} \left( \frac{z}{2} \right)^{n-1}, \quad (2.18b)
\]

\[
H_n^{(1)}(z) \approx -\frac{i(n-1)!}{\pi} \left( \frac{2}{z} \right)^n, \quad (2.18c)
\]

\[
H'_n^{(1)}(z) \approx \frac{i(n)!}{2\pi} \left( \frac{2}{z} \right)^{n+1}. \quad (2.18d)
\]

The argument of interest is the product of a radial wavenumber in the \( i \)-th layer and a radial distance, \( z = k_i \rho a_i \). Therefore, (2.18a) – (2.18d) can be rearranged as

\[
J_n(k_i \rho a_i) = \frac{1}{n!} \left( \frac{k_i \rho a_i}{2} \right)^n = \frac{1}{n!} \left( \frac{k_i \rho}{2} \right)^n a_i^n \cdot 1 = G_i a_i^n \hat{J}_n(k_i \rho a_i), \quad (2.19a)
\]

\[
J'_n(k_i \rho a_i) = \frac{1}{2(n-1)!} \left( \frac{k_i \rho a_i}{2} \right)^{n-1} = \frac{1}{n!} \left( \frac{k_i \rho}{2} \right)^n a_i^n \cdot \frac{n}{k_i \rho a_i} = G_i a_i^n \hat{J}'_n(k_i \rho a_i), \quad (2.19b)
\]

\[
H_n^{(1)}(k_i \rho a_i) = -\frac{i(n-1)!}{\pi} \left( \frac{2}{k_i \rho a_i} \right)^n = n! \left( \frac{2}{k_i \rho} \right)^n a_i^{-n} \cdot \left( -\frac{i}{n\pi} \right) = G_i^{-1} a_i^{-n} \hat{H}_n^{(1)}(k_i \rho a_i), \quad (2.19c)
\]

\[
H'_n^{(1)}(k_i \rho a_i) = \frac{i(n)!}{2\pi} \left( \frac{2}{k_i \rho a_i} \right)^{n+1} = n! \left( \frac{2}{k_i \rho} \right)^n a_i^{-n} \cdot \left( \frac{i}{\pi k_i \rho a_i} \right) = G_i^{-1} a_i^{-n} \hat{H}'_n^{(1)}(k_i \rho a_i), \quad (2.19d)
\]

where

\[
G_i = \frac{1}{n!} \left( \frac{k_i \rho}{2} \right)^n. \quad (2.20)
\]
In (2.19a) – (2.19d), \( \hat{J}_n, \hat{J}_n', \hat{H}^{(1)}_n, \) and \( \hat{H}^{(1)}_n' \) are called range-conditioned cylindrical functions for small arguments and designated by a caret to distinguish from the original functions. Note that the multiplicative factor associated with \( \hat{J}_n(k_i\rho a_i) \) and \( \hat{J}_n'(k_i\rho a_i) \) is reciprocal to the one associated with \( \hat{H}^{(1)}_n(k_i\rho a_i) \) and \( \hat{H}^{(1)}_n'(k_i\rho a_i) \).

When an argument \( z \) is large compared to unity, the large argument approximations can be used [69, p. 131, 132, 207, 236].

\[
J_n(z) = \sqrt{\frac{2}{\pi z}} \left[ P(n, z) \cos \chi - Q(n, z) \sin \chi \right], \quad (2.21a)
\]

\[
Y_n(z) = \sqrt{\frac{2}{\pi z}} \left[ P(n, z) \sin \chi + Q(n, z) \cos \chi \right], \quad (2.21b)
\]

\[
I_n(z) = e^z \left[ \frac{1}{\sqrt{2\pi z}} \left( 1 - \frac{1}{1!(8z)} + \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \cdots \right) \right], \quad (2.21c)
\]

\[
K_n(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{1}{1!(8z)} + \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \cdots \right], \quad (2.21d)
\]

\[
H^{(1)}_n(z) = \frac{2}{\pi} (-i)^{n+1} K_n(-iz), \quad (2.21e)
\]

where

\[
\chi = k_i\rho a_i - \frac{n\pi}{2} - \frac{\pi}{4}, \quad (2.22a)
\]

\[
P(n, z) \approx 1 - \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \frac{(\mu - 1)(\mu - 9)(\mu - 25)(\mu - 49)}{4!(8z)^3} - \cdots, \quad (2.22b)
\]

\[
Q(n, z) \approx \frac{(\mu - 1)}{1!(8z)} - \frac{(\mu - 1)(\mu - 9)(\mu - 25)}{3!(8z)^2} + \frac{(\mu - 1)(\mu - 81)}{5!(8z)^3} - \cdots, \quad (2.22c)
\]

\[
\mu = 4n^2. \quad (2.22d)
\]

From (2.21a) – (2.21e), it can be understood that the imaginary part of an argument \( k_i\rho a_i \) determines the magnitude of the function values while the real part is responsible for the phase. Two trigonometric functions of \( \chi \) shown in (2.21a) and (2.21b) can be decomposed into two terms and approximated by one of the two terms if \( \chi'' \) is large.
enough compared to unity such that

\[ \cos \chi = \cos(\chi' + i\chi'') \]
\[ = \frac{1}{2} \left[ e^{i(\chi' + i\chi'')} + e^{-i(\chi' + i\chi'')} \right] \]
\[ = \frac{1}{2} \left[ e^{-\chi''} e^{i\chi'} + e^{\chi''} e^{-i\chi'} \right] \]
\[ \approx \frac{1}{2} e^{\chi''} e^{-i\chi'}, \tag{2.23} \]

\[ \sin \chi = \sin(\chi' + i\chi'') \]
\[ = \frac{1}{2i} \left[ e^{i(\chi' + i\chi'')} - e^{-i(\chi' + i\chi'')} \right] \]
\[ = \frac{1}{2i} \left[ e^{-\chi''} e^{i\chi'} - e^{\chi''} e^{-i\chi'} \right] \]
\[ \approx \frac{i}{2} e^{\chi''} e^{-i\chi'} . \tag{2.24} \]

Therefore, using (2.23) and (2.24), the range-conditioned cylindrical functions for large arguments can be defined as

\[ J_n(k_i a_i) = \sqrt{\frac{2}{\pi k_i a_i}} \left[ P(n, k_i a_i) \cos \chi - Q(n, k_i a_i) \sin \chi \right] \]
\[ = \sqrt{\frac{2}{\pi k_i a_i}} \left[ P(n, k_i a_i) \frac{1}{2} e^{\chi''} e^{-i\chi'} - Q(n, k_i a_i) \frac{i}{2} e^{\chi''} e^{-i\chi'} \right] \]
\[ = e^{\chi''} \sqrt{\frac{2}{\pi k_i a_i}} \left[ P(n, k_i a_i) e^{-i\chi'} - Q(n, k_i a_i) \frac{i e^{-i\chi'}}{2} \right] \]
\[ = e^{k_i^* a_i} \sqrt{\frac{2}{\pi k_i a_i}} \left[ P(n, k_i a_i) \frac{e^{-i\chi'}}{2} - Q(n, k_i a_i) \frac{i e^{-i\chi'}}{2} \right] \]
\[ = e^{k_i^* a_i} \hat{J}_n(k_i a_i), \tag{2.25} \]
\[ H_n^{(1)}(k_{iρ}a_i) = \frac{2}{π}(-i)^{n+1}K_n(-ik_{iρ}a_i) \]
\[ = \frac{2}{π}(-i)^{n+1} \sqrt{\frac{π}{2z}} e^{-k''_{iρ}a_i} e^{ik'_{iρ}a_i} \left[ 1 + \frac{(μ - 1)}{1!(8z)} + \frac{(μ - 1)(μ - 9)}{2!(8z)^2} + \cdots \right] \]
\[ = e^{-k''_{iρ}a_i} \hat{H}_n^{(1)}(k_{iρ}a_i). \]  
(2.26)

Again, the associated multiplicative factors are reciprocal to each other. The derivatives of the range-conditioned cylindrical functions for large arguments can be derived through recursive formulas.

\[ J_n'(k_{iρ}a_i) = J_{n-1}(k_{iρ}a_i) - \frac{n}{k_{iρ}a_i} J_n(k_{iρ}a_i) \]
\[ = e^{k''_{iρ}a_i} J_{n-1}(k_{iρ}a_i) - e^{k''_{iρ}a_i} \frac{n}{k_{iρ}a_i} J_n(k_{iρ}a_i) \]
\[ = e^{k''_{iρ}a_i} \hat{J}_n'(k_{iρ}a_i), \]  
(2.27)

\[ H_n^{(1)}(k_{iρ}a_i) = H_{n-1}^{(1)}(k_{iρ}a_i) - \frac{n}{k_{iρ}a_i} H_n^{(1)}(k_{iρ}a_i) \]
\[ = e^{-k''_{iρ}a_i} \hat{H}_n^{(1)}(k_{iρ}a_i) - e^{-k''_{iρ}a_i} \frac{n}{k_{iρ}a_i} \hat{H}_n^{(1)}(k_{iρ}a_i) \]
\[ = e^{-k''_{iρ}a_i} \hat{H}_n^{(1)}(k_{iρ}a_i). \]  
(2.28)

When an argument \( z \) is neither too small nor too large, range-conditioned cylindrical functions are defined in a similar manner to small and large arguments.

\[ J_n(k_{iρ}a_i) = P_{ii} \hat{J}_n(k_{iρ}a_i), \]  
(2.29a)

\[ J'_n(k_{iρ}a_i) = P_{ii} \hat{J}'_n(k_{iρ}a_i), \]  
(2.29b)

\[ H_n^{(1)}(k_{iρ}a_i) = P_{ii}^{-1} \hat{H}_n^{(1)}(k_{iρ}a_i), \]  
(2.29c)

\[ H'_n^{(1)}(k_{iρ}a_i) = P_{ii}^{-1} \hat{H}'_n^{(1)}(k_{iρ}a_i), \]  
(2.29d)
Table 2.1: Definition of the range-conditioned cylindrical functions for isotropic media for all types of arguments.

<table>
<thead>
<tr>
<th>Function</th>
<th>Small Arguments</th>
<th>Moderate Arguments</th>
<th>Large Arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_n(k_i \rho a_i)$</td>
<td>$G_i a_i^n J_n(k_i \rho a_i)$</td>
<td>$P_{ii} J_n(k_i \rho a_i)$</td>
<td>$e^{k_i'</td>
</tr>
<tr>
<td>$J'_n(k_i \rho a_i)$</td>
<td>$G_i a_i^n J'_n(k_i \rho a_i)$</td>
<td>$P_{ii} J'_n(k_i \rho a_i)$</td>
<td>$e^{k_i'</td>
</tr>
<tr>
<td>$H_n^{(1)}(k_i \rho a_i)$</td>
<td>$G_i^{-1} a_i^{-n} \hat{H}_n^{(1)}(k_i \rho a_i)$</td>
<td>$P_{ii}^{-1} \hat{H}_n^{(1)}(k_i \rho a_i)$</td>
<td>$e^{-k_i'</td>
</tr>
<tr>
<td>$H_n^{(1)}(k_i \rho a_i)$</td>
<td>$G_i^{-1} a_i^{-n} \hat{H'}_n^{(1)}(k_i \rho a_i)$</td>
<td>$P_{ii}^{-1} \hat{H'}_n^{(1)}(k_i \rho a_i)$</td>
<td>$e^{-k_i'</td>
</tr>
</tbody>
</table>

where $P_{ii}$ is determined in Section 2.2.2 and its first and second subscript correspond to the radial wavenumber and radial distance (layer radius), respectively.

Therefore, the argument for range-conditioned cylindrical functions can be categorized into three types in terms of magnitude; small, moderate, and large. Depending on the type, the associated multiplicative factors to the range-conditioned cylindrical functions are different. The summary of the range-conditioned cylindrical functions are given in Table 2.1.

In fact, in calculation of electromagnetic fields in cylindrical coordinates, the cylindrical functions, $J_n$ and $H_n^{(1)}$, are used to define some $2 \times 2$ matrices because reflection and transmission coefficients in cylindrical coordinates are vectors in nature [17, p. 163]. Therefore, those matrices should be redefined in the same manner. For small
arguments,
\[ \mathbf{J}_n(k_i\rho a_i) = \frac{1}{k_i^{2} a_i} \left[ i\omega \varepsilon(k_i\rho a_i)J_n(k_i\rho a_i) - nk_z J_n(k_i\rho a_i) \right. \\
- nk_z J_n(k_i\rho a_i) - i\omega \mu k_i\rho a_iJ'_n(k_i\rho a_i) \right] \\
= G_i a_i^{-n} \mathbf{J}_n(k_i\rho a_i), \quad (2.30a) \]
\[ \mathbf{H}_n^{(1)}(k_i\rho a_i) = \frac{1}{k_i^{2} a_i} \left[ i\omega \varepsilon(k_i\rho a_i)H_n^{(1)}(k_i\rho a_i) - nk_z H_n^{(1)}(k_i\rho a_i) \right. \\
- nk_z H_n^{(1)}(k_i\rho a_i) - i\omega \mu k_i\rho a_iH'_n(k_i\rho a_i) \right] \\
= G_i^{-1} a_i^{-n} \mathbf{H}_n^{(1)}(k_i\rho a_i). \quad (2.30b) \]

For large arguments,
\[ \mathbf{J}_n(k_i\rho a_i) = \frac{1}{k_i^{2} a_i} \left[ i\omega \varepsilon(k_i\rho a_i)J_n(k_i\rho a_i) - nk_z J_n(k_i\rho a_i) \right. \\
- nk_z J_n(k_i\rho a_i) - i\omega \mu k_i\rho a_iJ'_n(k_i\rho a_i) \right] \\
= e^{i\omega \varepsilon(k_i\rho a_i)} \mathbf{J}_n(k_i\rho a_i), \quad (2.31a) \]
\[ \mathbf{H}_n^{(1)}(k_i\rho a_i) = \frac{1}{k_i^{2} a_i} \left[ i\omega \varepsilon(k_i\rho a_i)H_n^{(1)}(k_i\rho a_i) - nk_z H_n^{(1)}(k_i\rho a_i) \right. \\
- nk_z H_n^{(1)}(k_i\rho a_i) - i\omega \mu k_i\rho a_iH'_n(k_i\rho a_i) \right] \\
= e^{-i\omega \varepsilon(k_i\rho a_i)} \mathbf{H}_n^{(1)}(k_i\rho a_i). \quad (2.31b) \]

For moderate arguments,
\[ \mathbf{J}_n(k_i\rho a_i) = P_{ii} \mathbf{J}_n(k_i\rho a_i), \quad (2.32a) \]
\[ \mathbf{H}_n^{(1)}(k_i\rho a_i) = P_{ii}^{-1} \mathbf{H}_n^{(1)}(k_i\rho a_i). \quad (2.32b) \]

It should be noted that the multiplicative factors in (2.30a) – (2.32b) are the same as the ones for the range-conditioned cylindrical functions, which provides an useful simplification for the modification of the integrand.
2.2.2 Thresholds for Small, Moderate, and Large Arguments

The range-conditioned cylindrical functions are loosely defined in the previous section in terms of the argument type. In any computation, the actual crossovers between small and moderate arguments, and between moderate and large arguments should be determined. In addition, the functions with some moderate arguments still have very small or large values for higher order modes, which leads to inaccurate calculation of reflection coefficients. Hence, it is necessary to define another threshold for moderate arguments. In this section, these three thresholds are provided for double precision arithmetic.

As discussed in Section 2.2.1, expressions of the range-conditioned cylindrical functions for small and large arguments are based on approximations. Therefore, it should be examined how good these approximations are. The approximated values are compared with true values and relative error defined in (2.33) is used for the comparison.

\[
\text{relative error} = \frac{|F(z)_{\text{true}} - F(z)_{\text{approx.}}|}{|F(z)_{\text{true}}|},
\]  

(2.33)

where \( F(z) \) is either \( J_n(z) \) or \( H_n^{(1)}(z) \). To see the accuracy of the small argument approximations, it is assumed that a complex argument is

\[
z = |z|e^{\frac{\pi}{4}}.
\]  

(2.34)

Tables 2.2 and 2.3 show the relative errors for different magnitudes of the argument and orders. Here, the “true values” are obtained from Matlab routines. It should be noted that the relative error decreases remarkably when the magnitude becomes smaller. When \( n \) increases, the relative error also decreases but little. The threshold for small arguments can be any value depending on the desired accuracy. However,
Table 2.2: Relative errors of $J_n(z)$ using the small argument approximations.

| $|z|$ | $n = 1$ | $n = 5$ | $n = 10$ | $n = 20$ | $n = 30$ |
|------|---------|---------|----------|----------|----------|
| $10^{-1}$ | 1.2491×10^{-3} | 4.1658×10^{-4} | 2.2724×10^{-4} | 1.1904×10^{-4} | 8.0642×10^{-5} |
| $10^{-3}$ | 1.2499×10^{-7} | 4.1667×10^{-8} | 2.2727×10^{-8} | 1.1905×10^{-8} | 8.0645×10^{-9} |
| $10^{-5}$ | 1.2499×10^{-11} | 4.1636×10^{-12} | 2.2833×10^{-12} | 1.1860×10^{-12} | 8.3446×10^{-13} |
| $10^{-7}$ | 2.3161×10^{-14} | 7.0166×10^{-15} | 3.3853×10^{-14} | 3.5975×10^{-14} | 1.7873×10^{-14} |

Table 2.3: Relative errors of $H_n^{(1)}(z)$ using the small argument approximations.

| $|z|$ | $n = 1$ | $n = 5$ | $n = 10$ | $n = 20$ | $n = 30$ |
|------|---------|---------|----------|----------|----------|
| $10^{-1}$ | 1.4879×10^{-2} | 6.2510×10^{-4} | 2.7780×10^{-4} | 1.3158×10^{-4} | 8.6209×10^{-5} |
| $10^{-3}$ | 3.7647×10^{-6} | 6.2500×10^{-8} | 2.7778×10^{-8} | 1.3158×10^{-8} | 8.6207×10^{-9} |
| $10^{-5}$ | 6.0662×10^{-10} | 6.2500×10^{-12} | 2.7784×10^{-12} | 1.3190×10^{-12} | 8.6646×10^{-13} |
| $10^{-7}$ | 8.3734×10^{-14} | 4.3195×10^{-16} | 2.6175×10^{-16} | 4.4286×10^{-16} | 4.0776×10^{-15} |

it should not be a very small value because when any argument is a little greater than it, overflow and/or underflow might occur for higher order modes. In addition, the threshold should not be applied to lower orders because the approximation is not valid for the zeroth mode. Threshold values between $10^{-5}$ and $10^{-3}$ for $n \geq 5$ are recommended for small arguments.

To obtain the threshold for large arguments, let us examine (2.21a). Two trigonometric functions mostly determine the magnitude of $J_n(z)$ and can be decomposed
into two terms as shown in (2.23) and (2.24). When $\chi'' = k''_{\mu a_i} = 30$, the ratio of the two terms is

$$\frac{e^{30}}{e^{-30}} = 1.1420 \times 10^{+26}. \quad (2.35)$$

If double precision is assumed for numerical computation, the threshold of 30 for large arguments is sufficient to ignore one of the two terms since double precision supports up to 16 or 17 digits.

Unlike small and large arguments, $J_n(z)$ and $H_n^{(1)}(z)$ and their derivatives for moderate arguments do not depend on approximations due to their accuracy. When moderate arguments are close to the threshold for small arguments, the functions still have very small or large values for higher order modes. Furthermore, it is true that the effect of higher order modes becomes significant when the field point of interest is close to a source. When higher order modes play an important role for any case, reflection coefficients might not be expressible for moderate arguments whereas the functions are. Table 2.4 shows the expressible range of the functions, where the negative values represent $\log_{10} |J_n(z)|$ and positive values represent $\log_{10} |H_n^{(1)}(z)|$. To understand the phenomenon, consider $z = 10^{-3}$ with $n = 50$ and assume the threshold for small argument is smaller than $10^{-3}$. $J_n(z)$ and $H_n^{(1)}(z)$ can be expressed by double precision but $R_{12}$ is roughly proportional to the square of $H_n^{(1)}(z)$, which is about $10^{+460}$. Therefore, the threshold of function magnitudes for moderate arguments is necessary to avoid this undesired numerical underflow and overflow. Since the supported range of values by double precision is about from $10^{-300}$ to $10^{+300}$, let us define the threshold for moderate argument as

$$T_{\text{moderate}} = 10^{+100}, \quad (2.36)$$
Table 2.4: Expressible range of \( J_n(z) \) and \( H_n^{(1)}(z) \) for double precision.

<table>
<thead>
<tr>
<th></th>
<th>( n = 40 )</th>
<th>( n = 50 )</th>
<th>( n = 60 )</th>
<th>( n = 70 )</th>
<th>( n = 80 )</th>
<th>( n = 90 )</th>
<th>( n = 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z = 10^{-1} )</td>
<td>-100</td>
<td>-130</td>
<td>-160</td>
<td>-192</td>
<td>-223</td>
<td>-255</td>
<td>-288</td>
</tr>
<tr>
<td></td>
<td>+100</td>
<td>+130</td>
<td>+160</td>
<td>+191</td>
<td>+223</td>
<td>+255</td>
<td>+288</td>
</tr>
<tr>
<td>( z = 10^{-2} )</td>
<td>-140</td>
<td>-180</td>
<td>-220</td>
<td>-261</td>
<td>-303</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
</tr>
<tr>
<td></td>
<td>+141</td>
<td>+181</td>
<td>+221</td>
<td>+262</td>
<td>+304</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
</tr>
<tr>
<td>( z = 10^{-3} )</td>
<td>-180</td>
<td>-230</td>
<td>-280</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
</tr>
<tr>
<td></td>
<td>+182</td>
<td>+232</td>
<td>+282</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
</tr>
<tr>
<td>( z = 10^{-4} )</td>
<td>-220</td>
<td>-280</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
</tr>
<tr>
<td></td>
<td>+223</td>
<td>+283</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
</tr>
<tr>
<td>( z = 10^{-5} )</td>
<td>-260</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
<td>-( \infty )</td>
</tr>
<tr>
<td></td>
<td>+264</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
<td>+( \infty )</td>
</tr>
</tbody>
</table>

where the exponent of +100 instead of +150 is chosen for giving sufficient margin to the calculation of reflection coefficients. Using \( T_{\text{moderate}} \), the multiplicative factor \( P_{ii} \) associated with the range-conditioned cylindrical functions for moderate arguments is defined as

\[
\text{If } |J_n(k_i \rho a_i)|^{-1} < T_{\text{moderate}}, \quad P_{ii} = 1, \quad (2.37a)
\]
\[
\text{If } |J_n(k_i \rho a_i)|^{-1} \geq T_{\text{moderate}}, \quad P_{ii} = |J_n(k_i \rho a_i)|. \quad (2.37b)
\]

It should be noted that \( P_{ii} \) is defined from \( J_n(k_i \rho a_i) \), not \( H_n^{(1)}(k_i \rho a_i) \) in order to be consistent with the definition of the multiplicative factor for small arguments. Hence,
the range-conditioned cylindrical functions for moderate arguments are

\[
\hat{J}_n(k_i \rho a_i) = P_i^{-1} J_n(k_i \rho a_i), \quad (2.38a)
\]

\[
\hat{J}_n'(k_i \rho a_i) = P_i^{-1} J_n'(k_i \rho a_i), \quad (2.38b)
\]

\[
\hat{H}_n^{(1)}(k_i \rho a_i) = P_i H_n^{(1)}(k_i \rho a_i), \quad (2.38c)
\]

\[
\hat{H}_n^{(1)'}(k_i \rho a_i) = P_i H_n^{(1)'}(k_i \rho a_i). \quad (2.38d)
\]

It is worthwhile to note that the region of underflow and overflow \((-\infty \text{ or } +\infty\) shown in Table 2.4 can only be expressed by the small argument approximation at the cost of accuracy.

In summary, the three thresholds for double precision are

\[
10^{-3} \leq T_{\text{small}} \leq 10^{-5}, \quad \text{for } n \geq 5, \quad (2.39a)
\]

\[
T_{\text{moderate}} = 10^{+100}, \quad (2.39b)
\]

\[
T_{\text{large}} = 30, \quad \text{for all orders.} \quad (2.39c)
\]

Note that \(T_{\text{small}}\) and \(T_{\text{large}}\) are related to an argument itself whereas \(T_{\text{moderate}}\) is related to a function value. Figure 2.1 schematically describes the three types of arguments for the range-conditioned cylindrical functions with the defined thresholds. In the white space, the range-conditioned cylindrical functions are the same as the original cylindrical functions. For moderate arguments, \(P_{ii} = |J_n(k_i \rho a_i)|\) for the green region where the function magnitude threshold applies, otherwise \(P_{ii} = 1\).

### 2.3 Range-Conditioning

Using the new set of range-conditioned cylindrical functions, existing formulations for various coefficients and integrand are modified (range-conditioned) accordingly.
2.3.1 Reflection and Transmission Coefficients

When a medium consists of two cylindrical layers, (local) reflection and transmission coefficients at boundary $\rho = a_1$ depicted in Figure 2.2 are obtained using the boundary conditions of electromagnetic fields. When one or two layers are associated with small or large arguments, the expression of those coefficients are modified using the range-conditioned cylindrical functions. Therefore, with two cylindrical layers, there are total nine cases (see Table 2.5). The first group (Case 1, 2, 3) only incorporates small and moderate arguments. The second group (Case 4, 5, 6) only incorporates large and moderate arguments, and the third group (Case 7, 8, 9) consists
of the remaining cases. The coefficients will be redefined for all the nine cases. The two-layer configuration is shown in Figure 2.2. Note that the index of the interface follows the left (inner) layer.

For Case 1, where $k_{1\rho}a_1$ and $k_{2\rho}a_1$ are small, all coefficients are redefined as shown below.

$$
\bar{D} = H_n^{(1)}(k_{2\rho}a_1) \bar{J}_n(k_{1\rho}a_1) - J_n(k_{1\rho}a_1) \bar{H}_n^{(1)}(k_{2\rho}a_1)
= G_1 a_1^n G_2^{-1} a_1^{-n} \left[ \hat{H}_n^{(1)}(k_{2\rho}a_1) \hat{J}_n(k_{1\rho}a_1) - \hat{J}_n(k_{1\rho}a_1) \hat{H}_n^{(1)}(k_{2\rho}a_1) \right]
= G_1 G_2^{-1} \hat{\bar{D}},
$$

(2.40a)

$$
\bar{D}^{-1} = G_1^{-1} G_2 \hat{\bar{D}}^{-1},
$$

(2.40b)
Table 2.5: All possible cases for two cylindrical layers.

<table>
<thead>
<tr>
<th></th>
<th>(k_1a_1)</th>
<th>(k_2a_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Small</td>
<td>Small</td>
</tr>
<tr>
<td>Case 2</td>
<td>Small</td>
<td>Moderate</td>
</tr>
<tr>
<td>Case 3</td>
<td>Moderate</td>
<td>Small</td>
</tr>
<tr>
<td>Case 4</td>
<td>Large</td>
<td>Large</td>
</tr>
<tr>
<td>Case 5</td>
<td>Large</td>
<td>Moderate</td>
</tr>
<tr>
<td>Case 6</td>
<td>Moderate</td>
<td>Large</td>
</tr>
<tr>
<td>Case 7</td>
<td>Small</td>
<td>Large</td>
</tr>
<tr>
<td>Case 8</td>
<td>Large</td>
<td>Small</td>
</tr>
<tr>
<td>Case 9</td>
<td>Moderate</td>
<td>Moderate</td>
</tr>
</tbody>
</table>

\[
\hat{R}_{12} = \hat{D}^{-1} \cdot \left[H_n^{(1)}(k_1a_1)\hat{H}_n^{(1)}(k_2a_1) - H_n^{(1)}(k_2a_1)\hat{H}_n^{(1)}(k_1a_1)\right] \\
= \left(G_1^{-1}G_2\hat{D}^{-1}\right)G_1^{-1}a_1^{-n}G_2^{-1}a_1^{-n} \cdot \left[H_n^{(1)}(k_1a_1)\hat{H}_n^{(1)}(k_2a_1) - H_n^{(1)}(k_2a_1)\hat{H}_n^{(1)}(k_1a_1)\right] \\
= G_1^{-2}a_1^{-2n}\hat{D}^{-1} \cdot \left[H_n^{(1)}(k_1a_1)\hat{H}_n^{(1)}(k_2a_1) - H_n^{(1)}(k_2a_1)\hat{H}_n^{(1)}(k_1a_1)\right] \\
= (G_1^{-1}a_1^{-n})^2 \hat{R}_{12}, \tag{2.40c}
\]

\[
\hat{R}_{21} = \hat{D}^{-1} \cdot \left[J_n(k_1a_1)\hat{J}_n(k_2a_1) - J_n(k_2a_1)\hat{J}_n(k_1a_1)\right] \\
= \left(G_1^{-1}G_2\hat{D}^{-1}\right)G_1a_1^nG_2a_1^n \cdot \left[J_n(k_1a_1)\hat{J}_n(k_2a_1) - J_n(k_2a_1)\hat{J}_n(k_1a_1)\right] \\
= G_2^2a_1^{2n}\hat{D}^{-1} \cdot \left[J_n(k_1a_1)\hat{J}_n(k_2a_1) - J_n(k_2a_1)\hat{J}_n(k_1a_1)\right] \\
= (G_2^n a_1^n)^2 \hat{R}_{21}, \tag{2.40d}
\]
\[ T_{12} = \frac{2\omega}{\pi k_{1p}^2 a_1} \hat{D}^{-1} \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix} \]
\[ = G_1^{-1} G_2 \frac{2\omega}{\pi k_{2p}^2 a_1} \hat{D}^{-1} \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix} \]
\[ = G_1^{-1} G_2 \hat{T}_{12}, \quad (2.40e) \]

\[ \bar{T}_{21} = \frac{2\omega}{\pi k_{2p}^2 a_1} \hat{D}^{-1} \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix} \]
\[ = G_1^{-1} G_2 \frac{2\omega}{\pi k_{2p}^2 a_1} \hat{D}^{-1} \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix} \]
\[ = G_1^{-1} G_2 \hat{T}_{21}. \quad (2.40f) \]

For Case 2, where \( k_{1p}a_1 \) is small but \( k_{2p}a_1 \) is moderate, all coefficients are redefined as shown below.

\[ \hat{D} = H_n^{(1)}(k_{2p}a_1) \hat{J}_n(k_{1p}a_1) - J_n(k_{1p}a_1) \hat{H}_n^{(1)}(k_{2p}a_1) \]
\[ = G_1 a_1^n P_{21}^{-1} \left[ \hat{H}_n^{(1)}(k_{2p}a_1) \hat{J}_n(k_{1p}a_1) - \hat{J}_n(k_{1p}a_1) \hat{H}_n^{(1)}(k_{2p}a_1) \right] \]
\[ = G_1 a_1^n P_{21}^{-1} \hat{D}, \quad (2.41a) \]

\[ \hat{D}^{-1} = G_1^{-1} a_1^{-n} P_{21} \hat{D}^{-1}, \quad (2.41b) \]

\[ \bar{R}_{12} = \hat{D}^{-1} \cdot [H_n^{(1)}(k_{1p}a_1) \hat{H}_n^{(1)}(k_{2p}a_1) - H_n^{(1)}(k_{2p}a_1) \hat{H}_n^{(1)}(k_{1p}a_1)] \]
\[ = \left( G_1^{-1} a_1^{-n} P_{21} \hat{D}^{-1} \right) G_1^{-1} a_1^{-n} P_{21}^{-1} \cdot [H_n^{(1)}(k_{1p}a_1) \hat{H}_n^{(1)}(k_{2p}a_1) - \hat{H}_n^{(1)}(k_{2p}a_1) \hat{H}_n^{(1)}(k_{1p}a_1)] \]
\[ = G_1^{-2} a_1^{-2n} \hat{D}^{-1} \cdot [H_n^{(1)}(k_{1p}a_1) \hat{H}_n^{(1)}(k_{2p}a_1) - \hat{H}_n^{(1)}(k_{2p}a_1) \hat{H}_n^{(1)}(k_{1p}a_1)] \]
\[ = (G_1^{-1} a_1^{-n})^2 \hat{R}_{12}, \quad (2.41c) \]
\[ \overline{R}_{21} = \hat{D}^{-1} \cdot [J_n(k_1\rho a_1) \overline{J}_n(k_2\rho a_1) - J_n(k_2\rho a_1) \overline{J}_n(k_1\rho a_1)] \]

\[ = \left( G_1^{-1} a_1^{-n} P_{21} \hat{D}^{-1} \right) G_1 a_1^n P_{21} \cdot \left[ \dot{J}_n(k_1\rho a_1) \overline{J}_n(k_2\rho a_1) - \dot{J}_n(k_2\rho a_1) \overline{J}_n(k_1\rho a_1) \right] \]

\[ = P_{21}^2 \hat{D}^{-1} \cdot \left[ \dot{J}_n(k_1\rho a_1) \overline{J}_n(k_2\rho a_1) - \dot{J}_n(k_2\rho a_1) \overline{J}_n(k_1\rho a_1) \right] \]

\[ = P_{21}^2 \hat{R}_{21}, \quad (2.41d) \]

\[ T_{12} = \frac{2\omega}{\pi k_1^2 a_1} \hat{D}^{-1} \cdot \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix} \]

\[ = G_1^{-1} a_1^{-n} P_{21} \frac{2\omega}{\pi k_1^2 a_1} \hat{D}^{-1} \cdot \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix} \]

\[ = G_1^{-1} a_1^{-n} P_{21} \hat{T}_{12}, \quad (2.41e) \]

\[ T_{21} = \frac{2\omega}{\pi k_2^2 a_1} \hat{D}^{-1} \cdot \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix} \]

\[ = G_1^{-1} a_1^{-n} P_{21} \frac{2\omega}{\pi k_2^2 a_1} \hat{D}^{-1} \cdot \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix} \]

\[ = G_1^{-1} a_1^{-n} P_{21} \hat{T}_{21}. \quad (2.41f) \]

For Case 3, where \( k_1\rho a_1 \) is moderate but \( k_2\rho a_1 \) is small, all coefficients are redefined as shown below.

\[ \hat{D} = H_n^{(1)}(k_2\rho a_1) \overline{J}_n(k_1\rho a_1) - J_n(k_1\rho a_1) \overline{H}_n^{(1)}(k_2\rho a_1) \]

\[ = G_2^{-1} a_1^{-n} P_{11} \left[ \dot{H}_n^{(1)}(k_2\rho a_1) \overline{J}_n(k_1\rho a_1) - \dot{J}_n(k_1\rho a_1) \overline{H}_n^{(1)}(k_2\rho a_1) \right] \]

\[ = G_2^{-1} a_1^{-n} P_{11} \hat{D}, \quad (2.42a) \]

\[ \hat{D}^{-1} = G_2 a_1^n P_{11}^{-1} \hat{D}^{-1}, \quad (2.42b) \]
\( \mathbf{R}_{12} = \mathbf{D}^{-1} \cdot [H_n^{(1)}(k_{1\rho}a_1)\mathbf{H}_n^{(1)}(k_{2\rho}a_1) - H_n^{(1)}(k_{2\rho}a_1)\mathbf{H}_n^{(1)}(k_{1\rho}a_1)] \)

\[
= \left( G_2^{a_1^n}P_{11}^{-1} \mathbf{D}^{-1} \right) G_2^{-1}a_1^{-n}P_{11}^{-1} \cdot \left[ \hat{H}_n^{(1)}(k_{1\rho}a_1)\mathbf{H}_n^{(1)}(k_{2\rho}a_1) - \hat{H}_n^{(1)}(k_{2\rho}a_1)\mathbf{H}_n^{(1)}(k_{1\rho}a_1) \right]
\]

\[
= P_{11}^{-2} \mathbf{D}^{-1} \cdot \left[ \hat{H}_n^{(1)}(k_{1\rho}a_1)\mathbf{H}_n^{(1)}(k_{2\rho}a_1) - \hat{H}_n^{(1)}(k_{2\rho}a_1)\mathbf{H}_n^{(1)}(k_{1\rho}a_1) \right]
\]

\[
= P_{11}^{-2} \mathbf{R}_{12}, \quad (2.42c)
\]

\( \mathbf{R}_{21} = \mathbf{D}^{-1} \cdot [J_n(k_{1\rho}a_1)\mathbf{J}_n(k_{2\rho}a_1) - J_n(k_{2\rho}a_1)\mathbf{J}_n(k_{1\rho}a_1)] \)

\[
= \left( G_2^{a_1^n}P_{11}^{-1} \mathbf{D}^{-1} \right) G_2^{a_1^n}P_{11} \cdot \left[ \hat{J}_n(k_{1\rho}a_1)\mathbf{J}_n(k_{2\rho}a_1) - \hat{J}_n(k_{2\rho}a_1)\mathbf{J}_n(k_{1\rho}a_1) \right]
\]

\[
= G_2^{a_1^n}\mathbf{D}^{-1} \cdot \left[ \hat{J}_n(k_{1\rho}a_1)\mathbf{J}_n(k_{2\rho}a_1) - \hat{J}_n(k_{2\rho}a_1)\mathbf{J}_n(k_{1\rho}a_1) \right]
\]

\[
= (G_2^{a_1^n})^2 \mathbf{R}_{21}, \quad (2.42d)
\]

\[
\mathbf{T}_{12} = \frac{2\omega}{\pi k_{1\rho}^2 a_1} \mathbf{D}^{-1} \cdot \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix}
\]

\[
= G_2^{a_1^n}P_{11}^{-1} \frac{2\omega}{\pi k_{1\rho}^2 a_1} \mathbf{D}^{-1} \cdot \begin{bmatrix} \epsilon_1 & 0 \\ 0 & -\mu_1 \end{bmatrix}
\]

\[
= G_2^{a_1^n}P_{11}^{-1} \mathbf{\hat{T}}_{12}, \quad (2.42e)
\]

\[
\mathbf{T}_{21} = \frac{2\omega}{\pi k_{2\rho}^2 a_1} \mathbf{D}^{-1} \cdot \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix}
\]

\[
= G_2^{a_1^n}P_{11}^{-1} \frac{2\omega}{\pi k_{2\rho}^2 a_1} \mathbf{D}^{-1} \cdot \begin{bmatrix} \epsilon_2 & 0 \\ 0 & -\mu_2 \end{bmatrix}
\]

\[
= G_2^{a_1^n}P_{11}^{-1} \mathbf{\hat{T}}_{21}. \quad (2.42f)
\]

For the second group (Case 4, 5, 6) and the third group (Case 7, 8, 9), the redefinition of the coefficients is basically similar to the first group but the associated
coefficients are different. The redefined coefficients for all three groups are summarized in Table 2.6.

Table 2.6: Redefined reflection and transmission coefficients using the range-conditioned cylindrical functions for two cylindrical layers.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \mathcal{R}_{12} )</th>
<th>( \mathcal{R}_{21} )</th>
<th>( \mathcal{T}_{12} )</th>
<th>( \mathcal{T}_{21} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>( G_1^{-2} a_1^{-2n} \hat{R}_{12} )</td>
<td>( G_2 a_1^{2n} \hat{R}_{21} )</td>
<td>( G_1^{-1} a_1^{-n} G_2 a_1^n \hat{T}_{12} )</td>
<td>( G_1^{-1} a_1^{-n} G_2 a_1^n \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 2</td>
<td>( G_1^{-2} a_1^{-2n} \hat{R}_{12} )</td>
<td>( P_{21} \hat{R}_{21} )</td>
<td>( G_1^{-1} a_1^{-n} P_{21} \hat{T}_{12} )</td>
<td>( G_1^{-1} a_1^{-n} P_{21} \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 3</td>
<td>( P_{11}^{-2} \hat{R}_{12} )</td>
<td>( G_2 a_1^{2n} \hat{R}_{21} )</td>
<td>( G_2 a_1^{n} P_{11}^{-1} \hat{T}_{12} )</td>
<td>( G_2 a_1^{n} P_{11}^{-1} \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 4</td>
<td>( e^{-2k''<em>{i_v} a_1} \hat{R}</em>{12} )</td>
<td>( e^{2k''<em>{i_v} a_1} \hat{R}</em>{21} )</td>
<td>( e^{-k''<em>{i_v} a_1} e^{k''</em>{i_v} a_1} \hat{T}_{12} )</td>
<td>( e^{-k''<em>{i_v} a_1} e^{k''</em>{i_v} a_1} \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 5</td>
<td>( e^{-2k''<em>{i_v} a_1} \hat{R}</em>{12} )</td>
<td>( P_{21} \hat{R}_{21} )</td>
<td>( e^{-k''<em>{i_v} a_1} P</em>{21} \hat{T}_{12} )</td>
<td>( e^{-k''<em>{i_v} a_1} P</em>{21} \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 6</td>
<td>( P_{11}^{-2} \hat{R}_{12} )</td>
<td>( e^{2k''<em>{i_v} a_1} \hat{R}</em>{21} )</td>
<td>( e^{k''<em>{i_v} a_1} P</em>{11}^{-1} \hat{T}_{12} )</td>
<td>( e^{k''<em>{i_v} a_1} P</em>{11}^{-1} \hat{T}_{21} )</td>
</tr>
<tr>
<td>Case 7</td>
<td>( G_1^{-2} a_1^{-2n} \hat{R}_{12} )</td>
<td>( e^{2k''<em>{i_v} a_1} \hat{R}</em>{21} )</td>
<td>( G_1^{-1} a_1^{-n} e^{k''<em>{i_v} a_1} \hat{T}</em>{12} )</td>
<td>( G_1^{-1} a_1^{-n} e^{k''<em>{i_v} a_1} \hat{T}</em>{21} )</td>
</tr>
<tr>
<td>Case 8</td>
<td>( e^{-2k''<em>{i_v} a_1} \hat{R}</em>{12} )</td>
<td>( G_2 a_1^{2n} \hat{R}_{21} )</td>
<td>( e^{-k''<em>{i_v} a_1} G_2 a_1^{n} \hat{T}</em>{12} )</td>
<td>( e^{-k''<em>{i_v} a_1} G_2 a_1^{n} \hat{T}</em>{21} )</td>
</tr>
<tr>
<td>Case 9</td>
<td>( P_{11}^{-2} \hat{R}_{12} )</td>
<td>( P_{21} \hat{R}_{21} )</td>
<td>( P_{11}^{-1} P_{21} \hat{T}_{12} )</td>
<td>( P_{11}^{-1} P_{21} \hat{T}_{21} )</td>
</tr>
</tbody>
</table>

It should be noted that there is a relationship between the original cylindrical functions and the range-conditioned cylindrical functions for all the nine cases. Therefore, the associated multiplicative factors can be generalized for two adjacent cylindrical layers with arbitrary indexing. Reflection and transmission coefficients shown in Fig-
Figure 2.3: Reflection and transmission coefficients for two adjacent arbitrarily-indexed cylindrical layers in the $\rho z$-plane.

Figure 2.3 can be succinctly expressed as

\begin{align}
\bar{R}_{i,i+1} &= \alpha_{ii}^2 \hat{R}_{i,i+1}, \\
\bar{R}_{i+1,i} &= \beta_{i+1,i}^2 \hat{R}_{i+1,i}, \\
\bar{T}_{i,i+1} &= \alpha_{ii} \beta_{i+1,i} \hat{T}_{i,i+1}, \\
\bar{T}_{i+1,i} &= \alpha_{ii} \beta_{i+1,i} \hat{T}_{i+1,i},
\end{align}

where two types of multiplicative factors are observed: $\alpha_{ii}$ is the function of $k_{i\rho} a_i$ and $\beta_{i+1,i}$ is the function of $k_{i+1\rho} a_i$. The first subscript of $\alpha$ and $\beta$ stands for the radial wavenumber and the second subscript stands for the radial distance. The values for $\alpha_{ij}$ and $\beta_{ij}$ are summarized in Table 2.7 for the three types of arguments. Note that two different indices, $i$ and $j$, are used to indicate that they are independent to each other. There are two important properties of $\alpha$ and $\beta$.  

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Table 2.7: Definition of the multiplicative factors, $\alpha_{ij}$ and $\beta_{ij}$, for electromagnetic fields in isotropic media.

<table>
<thead>
<tr>
<th>Argument Type</th>
<th>$\alpha_{ij}$</th>
<th>$\beta_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>$G_i^{-1}a_j^{-n}$</td>
<td>$G_i^na_j^n$</td>
</tr>
<tr>
<td>Moderate</td>
<td>$P_{ij}^{-1}$</td>
<td>$P_{ij}$</td>
</tr>
<tr>
<td>Large</td>
<td>$e^{-k''_{i\rho}a_j}$</td>
<td>$e^{k''_{i\rho}a_j}$</td>
</tr>
</tbody>
</table>

1. Reciprocity: $\alpha$ and $\beta$ are reciprocal to each other when their two subscripts are the same.

$$\alpha_{ii} = \frac{1}{\beta_{ii}}.$$ (2.44)

2. Boundness: When the radial wavenumbers of $\alpha$ and $\beta$ are the same but the radial distance for $\alpha$ is larger than that for $\beta$, the absolute value of the product of $\alpha$ and $\beta$ is always less than or equal to unity.

$$|\beta_{ii} \alpha_{ij}| \leq 1, \quad \text{for } a_i < a_j.$$ (2.45)

Finally, redefined reflection and transmission coefficients using the range-conditioned cylindrical functions for two cylindrical layers are

$$\hat{R}_{i,i+1} = \hat{D}^{-1} \cdot \left[ \hat{H}_n^{(1)}(k_{i\rho}a_i)\hat{H}_n^{(1)}(k_{i+1,\rho}a_i) - \hat{H}_n^{(1)}(k_{i+1,\rho}a_i)\hat{H}_n^{(1)}(k_{i\rho}a_i) \right], \quad (2.46a)$$

$$\hat{R}_{i+1,i} = \hat{D}^{-1} \cdot \left[ \hat{J}_n(k_{i\rho}a_i)\hat{J}_n(k_{i+1,\rho}a_i) - \hat{J}_n(k_{i+1,\rho}a_i)\hat{J}_n(k_{i\rho}a_i) \right], \quad (2.46b)$$

$$\hat{T}_{i,i+1} = \frac{2\omega}{\pi k_{i\rho}^2 a_i} \hat{D}^{-1} \cdot \left[ \epsilon_i \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \mu_i \right], \quad (2.46c)$$

$$\hat{T}_{i+1,i} = \frac{2\omega}{\pi k_{i+1,\rho}^2 a_i} \hat{D}^{-1} \cdot \left[ \epsilon_{i+1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \mu_{i+1} \right], \quad (2.46d)$$

$$\hat{D}^{-1} = \left[ \hat{H}_n^{(1)}(k_{i+1,\rho}a_i)\hat{J}_n(k_{i\rho}a_i) - \hat{J}_n(k_{i\rho}a_i)\hat{H}_n^{(1)}(k_{i+1,\rho}a_i) \right]^{-1}. \quad (2.46e)$$
2.3.2 Generalized Reflection Coefficients

When more than two cylindrical layers are present, reflection and transmission coefficients are the combination of those for two layers because of multiple reflections and transmissions at each interface. Using the redefined reflection and transmission coefficients for two layers, (2.43a) – (2.43d), generalized reflection coefficients can be redefined through the range-conditioned cylindrical functions. The redefinition of generalized transmission coefficients will be discussed in Section 2.3.3. Figure 2.4a and 2.4b describe the generalized reflection coefficients for the outgoing-wave case and standing-wave case.

For the outgoing-wave case, the generalized reflection coefficient at $a_1$ is defined as

$$\tilde{R}_{12} = R_{12} + T_{21} \cdot R_{23} \cdot [I - R_{21} \cdot R_{23}]^{-1} \cdot T_{12}, \quad (2.47)$$
where $\mathbf{I}$ is a $2 \times 2$ identity matrix. If all coefficients in (2.47) are replaced with their range-conditioned-versions,

\[
\tilde{\mathbf{R}}_{12} = \mathbf{R}_{12} + \mathbf{T}_{21} \cdot \mathbf{R}_{23} \cdot \left[ \mathbf{I} - \mathbf{R}_{21} \cdot \mathbf{R}_{23} \right]^{-1} \cdot \mathbf{T}_{12}
\]

\[
= \alpha_{11}^2 \tilde{\mathbf{R}}_{12} + \left( \alpha_{11} \beta_{21} \tilde{T}_{21} \right) \cdot \left( \alpha_{22} \tilde{\mathbf{R}}_{23} \right) \cdot \left[ \mathbf{I} - \left( \beta_{21} \tilde{\mathbf{R}}_{21} \right) \cdot \left( \alpha_{22} \tilde{\mathbf{R}}_{23} \right) \right]^{-1} \cdot \left( \alpha_{11} \beta_{21} \tilde{T}_{12} \right)
\]

\[
= \alpha_{11}^2 \left\{ \tilde{\mathbf{R}}_{12} + \beta_{21}^2 \alpha_{22} \tilde{T}_{21} \cdot \tilde{\mathbf{R}}_{23} \cdot \left[ \mathbf{I} - \beta_{21}^2 \alpha_{22} \tilde{\mathbf{R}}_{21} \cdot \tilde{\mathbf{R}}_{23} \right]^{-1} \cdot \tilde{T}_{12} \right\}
\]

\[
= \alpha_{11}^2 \tilde{\mathbf{R}}_{12}. \quad (2.48)
\]

Therefore, the generalized reflection coefficient for the outgoing-wave case is redefined as

\[
\hat{\tilde{\mathbf{R}}}_{12} = \tilde{\mathbf{R}}_{12} + \beta_{21}^2 \alpha_{22} \tilde{T}_{21} \cdot \tilde{\mathbf{R}}_{23} \cdot \left[ \mathbf{I} - \beta_{21}^2 \alpha_{22} \tilde{\mathbf{R}}_{21} \cdot \tilde{\mathbf{R}}_{23} \right]^{-1} \cdot \tilde{T}_{12}. \quad (2.49)
\]

Note that $|\beta_{21} \alpha_{22}| \leq 1$ due to the boundness property (refer to (2.45)). This condition makes the evaluation of the range-conditioned-version of the generalized reflection coefficient numerically stable.

For the standing-wave case, the generalized reflection coefficient at $a_2$ is defined as

\[
\tilde{\mathbf{R}}_{32} = \mathbf{R}_{32} + \mathbf{T}_{23} \cdot \mathbf{R}_{21} \cdot \left[ \mathbf{I} - \mathbf{R}_{21} \cdot \mathbf{R}_{23} \right]^{-1} \cdot \mathbf{T}_{32}. \quad (2.50)
\]

Similarly, the generalized reflection coefficient for the standing-wave case can be re-defined as

\[
\tilde{\mathbf{R}}_{32} = \mathbf{R}_{32} + \mathbf{T}_{23} \cdot \mathbf{R}_{21} \cdot \left[ \mathbf{I} - \mathbf{R}_{21} \cdot \mathbf{R}_{23} \right]^{-1} \cdot \mathbf{T}_{32}
\]

\[
= \beta_{32}^2 \tilde{\mathbf{R}}_{32} + \left( \alpha_{22} \beta_{32} \tilde{T}_{23} \right) \cdot \left( \beta_{21} \tilde{\mathbf{R}}_{21} \right) \cdot \left[ \mathbf{I} - \left( \alpha_{22} \tilde{\mathbf{R}}_{23} \right) \cdot \left( \beta_{21} \tilde{\mathbf{R}}_{21} \right) \right]^{-1} \cdot \left( \alpha_{22} \beta_{32} \tilde{T}_{32} \right)
\]

\[
= \beta_{32}^2 \left\{ \tilde{\mathbf{R}}_{32} + \beta_{21}^2 \alpha_{22} \tilde{T}_{23} \cdot \tilde{\mathbf{R}}_{21} \cdot \left[ \mathbf{I} - \beta_{21}^2 \alpha_{22} \tilde{\mathbf{R}}_{23} \cdot \tilde{\mathbf{R}}_{21} \right]^{-1} \cdot \tilde{T}_{32} \right\}
\]

\[
= \beta_{32}^2 \tilde{\mathbf{R}}_{32}. \quad (2.51)
\]
Therefore,

\[ \hat{\mathbf{R}}_{32} = \mathbf{R}_{32} + \beta_{21}^2 \alpha_{22}^2 \mathbf{T}_{23} \cdot \hat{\mathbf{R}}_{21} \cdot \left[ \mathbf{I} - \beta_{21}^2 \alpha_{22}^2 \mathbf{R}_{23} \cdot \hat{\mathbf{R}}_{21} \right]^{-1} \cdot \mathbf{T}_{32}. \]  

(2.52)

Again, the property of \(|\beta_{21} \alpha_{22}| \leq 1\) stabilizes the numerical evaluation of (2.52).

It should be emphasized that the associated multiplicative factors to the redefined *generalized* reflection coefficients are the same as those to the redefined reflection coefficients (See (2.43a), (2.43b) and (2.48), (2.51)). This characteristic makes it easier to redefine all generalized reflection coefficients in cylindrically stratified media. In other words, the associated multiplicative factors remain the same irrespective of the number of cylindrical layers. In summary, redefined generalized reflection coefficients between two adjacent layers are expressed as

\[ \tilde{\mathbf{R}}_{i,i+1} = \alpha_{ii}^2 \hat{\mathbf{R}}_{i,i+1}, \]  

(2.53a)

\[ \tilde{\mathbf{R}}_{i+1,i} = \beta_{i+1,i}^2 \hat{\mathbf{R}}_{i+1,i}. \]  

(2.53b)

### 2.3.3 Generalized Transmission Coefficients

Before redefining generalized transmission coefficients, we first need to look at the so-called \(S\)-coefficients, which are used to define generalized transmission coefficients. The \(S\)-coefficients can be regarded as transmission coefficients between two adjacent layers incorporating multiple reflections and transmissions in relevant interfaces. Therefore, the \(S\)-coefficients are defined in the medium with at least three cylindrical layers, otherwise they become (local) transmission coefficients as shown in (2.43c) and (2.43d). Two types of the \(S\)-coefficients are described in Figure 2.5a and 2.5b.
For the outgoing-wave case, the $S$-coefficient is defined as

$$\bar{S}_{12} = [I - \bar{R}_{21} \cdot \bar{R}_{23}]^{-1} \cdot \bar{T}_{12}. \quad (2.54)$$

Using the range-conditioned cylindrical functions, (2.54) is redefined as

$$\bar{S}_{12} = [I - (\beta_{21}^2 \hat{R}_{21}) \cdot (\alpha_{22}^2 \hat{R}_{23})]^{-1} \cdot (\alpha_{11} \beta_{21} \hat{T}_{12})$$

$$= \alpha_{11} \beta_{21} [I - \beta_{21}^2 \alpha_{22}^2 \hat{R}_{21} \cdot \hat{R}_{23}]^{-1} \cdot \hat{T}_{12}$$

$$= \alpha_{11} \beta_{21} N_{2+} \cdot \hat{T}_{12}$$

$$= \alpha_{11} \beta_{21} \hat{S}_{12}. \quad (2.55)$$

For the definition of $N_{2+}$, refer to Section 2.3.4. Therefore,

$$\hat{S}_{12} = N_{2+} \cdot \hat{T}_{12}. \quad (2.56)$$

For the standing-wave case, the $S$-coefficient is defined as

$$\bar{S}_{32} = [I - \bar{R}_{23} \cdot \bar{R}_{21}]^{-1} \cdot \bar{T}_{32}. \quad (2.57)$$
Similarly, (2.57) is redefined using the range-conditioned cylindrical functions as

\[
\mathbf{S}_{32} = \left[ \mathbf{I} - \left( \alpha_{22}^2 \hat{\mathbf{R}}_{23} \cdot \beta_{21}^2 \hat{\mathbf{R}}_{21} \right) \right]^{-1} \cdot \left( \alpha_{22}\beta_{32} \hat{T}_{32} \right)
\]

\[
= \alpha_{22}\beta_{32} \left[ \mathbf{I} - \beta_{21}^2 \alpha_{22}^2 \hat{\mathbf{R}}_{23} \cdot \hat{\mathbf{R}}_{21} \right]^{-1} \cdot \hat{T}_{32}
\]

\[
= \alpha_{22}\beta_{32} \mathbf{N}_{2-} \cdot \hat{T}_{32}
\]

\[
= \alpha_{22}\beta_{32} \mathbf{S}_{32}.
\] (2.58)

Again, refer to Section 2.3.4 for the definition of \( \mathbf{N}_{2-} \). Therefore,

\[
\hat{\mathbf{S}}_{32} = \mathbf{N}_{2-} \cdot \hat{T}_{32}.
\] (2.59)

When more than three layers are concerned, the reflection coefficients \( \mathbf{R}_{23} \) for the outgoing-wave case and \( \mathbf{R}_{21} \) for the standing-wave case should be replaced with generalized ones such as \( \hat{\mathbf{R}}_{23} \) and \( \hat{\mathbf{R}}_{21} \). However, the redefinition of the \( S \)-coefficients is consistent with such generalization. In summary, redefined \( S \)-coefficients can be written as

\[
\hat{\mathbf{S}}_{i,i+1} = \mathbf{N}_{(i+1)+} \cdot \hat{T}_{i,i+1},
\] (2.60a)

\[
\hat{\mathbf{S}}_{i+1,i} = \mathbf{N}_{i-} \cdot \hat{T}_{i+1,1}.
\] (2.60b)

Using (2.60a) and (2.60b), generalized transmission coefficients can be redefined as well. Figure 2.6 describes the generalized transmission coefficient for the outgoing-wave case in cylindrically stratified media and its definition is given in (2.61).

When \( i > j \),

\[
\hat{T}_{ji} = \hat{T}_{i-1,i} \cdot \hat{\mathbf{S}}_{i-2,i-1} \cdot \ldots \cdot \hat{\mathbf{S}}_{j,j+1}.
\] (2.61)
Figure 2.6: Generalized transmission coefficient \( \tilde{T}_{ji} \) for the outgoing-wave case in cylindrically stratified media.

(2.61) can be redefined using the range-conditioned cylindrical functions as

\[
\tilde{T}_{ji} = T_{i-1,i} \cdot S_{j-2,i-1} \cdot \cdots \cdot S_{j,j+1}
\]

\[
= T_{i-1,i} \cdot X_{j,i-1} \cdot I
\]

\[
= \left( \beta_{i,i-1} \hat{T}_{i-1,i} \right) \cdot \left( \prod_{k=j+1}^{i-1} \beta_{k,k-1} \alpha_{kk} \hat{N}_{k+} \cdot \hat{T}_{k-1,k} \right) \cdot (\alpha_{jj} I)
\]

\[
= \beta_{i,i-1} \hat{T}_{ji} \cdot (\alpha_{jj} I),
\]

where

\[
\hat{N}_{k+} = \left[ I - \beta_{k,k-1}^2 \alpha_{kk}^2 \hat{R}_{k,k-1} \cdot \hat{R}_{k,k+1} \right]^{-1}.
\]

Therefore,

\[
\hat{T}_{ji} = \hat{T}_{i-1,i} \cdot X_{j,i-1}
\]

\[
= \hat{T}_{i-1,i} \cdot \left( \prod_{k=j+1}^{i-1} \beta_{k,k-1} \alpha_{kk} \hat{N}_{k+} \cdot \hat{T}_{k-1,k} \right).
\]

It should be noted that the product in (2.64) is indeed the product of a number of 2×2 matrices, so the order of the product should be clarified. The 2×2 matrix for \( k = j+1 \) and 2×2 matrix for \( k = i-1 \) should be placed in the rightmost and leftmost
in the matrix product, respectively. Furthermore, when \( i = j + 1 \), the matrix product is not defined. In this case, \( \mathbf{X}_{j,i-1} = \mathbf{I} \).

For the standing-wave case, Figure 2.7 describes the generalized transmission coefficient in cylindrically stratified media and its definition is given in (2.65).

When \( i < j \),

\[
\tilde{T}_{ji} = T_{i+1,i} \cdot S_{i+2,i+1} \cdots S_{j,j-1}.
\]  

(2.65)

Similarly, using the range-conditioned cylindrical functions, (2.65) can be redefined as

\[
\tilde{T}_{ji} = \tilde{T}_{i+1,i} \cdot S_{i+2,i+1} \cdots S_{j,j-1}
\]

\[
= \tilde{T}_{i+1,i} \cdot \mathbf{X}_{j,i+1} \cdot \mathbf{I}
\]

\[
= (\alpha_{ii} \tilde{T}_{i+1,i}) \cdot \left( \prod_{k=i+1}^{j-1} \beta_{k,k-1} \alpha_{kk} \mathbf{N}_k \cdot \tilde{T}_{k+1,k} \right) \cdot (\beta_{j,j-1} \mathbf{I})
\]

\[
= \alpha_{ii} \tilde{T}_{ji} \cdot (\beta_{j,j-1} \mathbf{I}),
\]  

(2.66)
where
\[ \hat{N}_{k-} = \left[ \mathbf{I} - \beta_{k,k-1}^2 \mathbf{\hat{A}}_{kk}^2 \mathbf{\hat{R}}_{k,k+1} \cdot \mathbf{\hat{R}}_{k,k-1} \right]^{-1}. \] (2.67)

Therefore,
\[
\mathbf{\hat{T}}_{ji} = \mathbf{\hat{T}}_{i+1,i} \cdot \mathbf{X}_{j,i+1} = \mathbf{\hat{T}}_{i+1,i} \cdot \left( \prod_{k=i+1}^{j-1} \beta_{k,k-1} \mathbf{\hat{A}}_{kk} \hat{N}_{k-} \cdot \mathbf{\hat{T}}_{k+1,k} \right). \] (2.68)

Again, the product in (2.68) is indeed the product of 2×2 matrices. In contrast to (2.64), the order of the matrix product is just the opposite. The 2×2 matrix for \( k = i+1 \) and 2×2 matrix for \( k = j-1 \) should be placed in the leftmost and rightmost in the matrix product, respectively. Furthermore, when \( i = j-1 \), the matrix product is again not defined, so \( \mathbf{X}_{j,i+1} = \mathbf{I} \).

### 2.3.4 Auxiliary Coefficients

In addition to generalized reflection and transmission coefficients, \( \mathbf{\tilde{M}}_{j\pm} \) and \( \mathbf{N}_{i\pm} \) are required in the computation of \( \mathbf{F}_n(\rho,\rho') \) in (2.11a) – (2.11d). The difference between the two coefficients is that \( \mathbf{\tilde{M}}_{j\pm} \) need two generalized reflection coefficients whereas \( \mathbf{N}_{i\pm} \) need one generalized reflection coefficient. These auxiliary coefficients can be also redefined using the range-conditioned reflection coefficients such that

\[
\mathbf{\tilde{M}}_{j+} = \left[ \mathbf{I} - \mathbf{\tilde{R}}_{j,j-1} \cdot \mathbf{\tilde{R}}_{j,j+1} \right]^{-1} = \left[ \mathbf{I} - \beta_{j,j-1}^2 \mathbf{\hat{A}}_{jj}^2 \mathbf{\hat{R}}_{j,j-1} \cdot \mathbf{\hat{R}}_{j,j+1} \right]^{-1}, \] (2.69a)

\[
\mathbf{\tilde{M}}_{j-} = \left[ \mathbf{I} - \mathbf{\tilde{R}}_{j,j+1} \cdot \mathbf{\tilde{R}}_{j,j-1} \right]^{-1} = \left[ \mathbf{I} - \beta_{j,j+1}^2 \mathbf{\hat{A}}_{jj}^2 \mathbf{\hat{R}}_{j,j+1} \cdot \mathbf{\hat{R}}_{j,j-1} \right]^{-1}, \] (2.69b)

\[
\mathbf{N}_{i+} = \left[ \mathbf{I} - \mathbf{R}_{i,i-1} \cdot \mathbf{\tilde{R}}_{i,i+1} \right]^{-1} = \left[ \mathbf{I} - \beta_{i,i-1}^2 \mathbf{\hat{A}}_{ii}^2 \mathbf{\hat{R}}_{i,i-1} \cdot \mathbf{\hat{R}}_{i,i+1} \right]^{-1}, \] (2.69c)

\[
\mathbf{N}_{i-} = \left[ \mathbf{I} - \mathbf{R}_{i,i+1} \cdot \mathbf{\tilde{R}}_{i,i-1} \right]^{-1} = \left[ \mathbf{I} - \beta_{i,i+1}^2 \mathbf{\hat{A}}_{ii}^2 \mathbf{\hat{R}}_{i,i+1} \cdot \mathbf{\hat{R}}_{i,i-1} \right]^{-1}. \] (2.69d)
2.3.5 Modified Integrand

Using the redefined coefficients for all types of arguments obtained in Section 2.3.1 through Section 2.3.4, the integrand in (2.10) can be modified accordingly. Recall that there are four cases for the integrand in terms of the relative position of $\rho$ and $\rho'$ (see (2.11a) – (2.11d)). For Case 1, there are four arguments of interest: $k_{jp}a_{j-1}$, $k_{jp}\rho'$, $k_{jp}\rho$, and $k_{jp}a_j$. For convenience, we let $a_{j-1} = a_1$, $\rho' = a_2$, $\rho = a_3$, and $a_j = a_4$ so that $a_1 < a_2 < a_3 < a_4$. The integrand is modified to

$$
\Phi_n(\rho, \rho') = \left[ H_n^{(1)}(k_{jp}\rho) \mathbf{I} + J_n(k_{jp}\rho) \mathbf{R}_{j,j+1} \right] \cdot \mathbf{M}_{j+} \cdot \left[ J_n(k_{jp}\rho') \mathbf{I} + H_n^{(1)}(k_{jp}\rho') \mathbf{R}_{j,j-1} \right]
$$

$$
= \left[ \alpha_{j3} \hat{H}_n^{(1)}(k_{jp}\rho) \mathbf{I} + \beta_{j3} \hat{J}_n(k_{jp}\rho) \alpha_{j4}^2 \hat{\mathbf{R}}_{j,j+1} \right] \cdot \mathbf{M}_{j+} \cdot \left[ \beta_{j2} \hat{J}_n^{(1)}(k_{jp}\rho') \mathbf{I} + \alpha_{j2} \hat{H}_n^{(1)}(k_{jp}\rho') \beta_{j4}^2 \hat{\mathbf{R}}_{j,j-1} \right]
$$

$$
= \left[ \beta_{j2} \alpha_{j3} \hat{H}_n^{(1)}(k_{jp}\rho) \mathbf{I} + (\beta_{j2} \alpha_{j4}) (\beta_{j3} \alpha_{j4}) \hat{J}_n(k_{jp}\rho) \hat{\mathbf{R}}_{j,j+1} \right] \cdot \mathbf{M}_{j+}
$$

$$
= \left[ A_1 \hat{H}_n^{(1)}(k_{jp}\rho) \mathbf{I} + A_2 \hat{J}_n(k_{jp}\rho) \hat{\mathbf{R}}_{j,j+1} \right] \cdot \mathbf{M}_{j+}
$$

$$
\cdot \left[ A_3 \hat{J}_n^{(1)}(k_{jp}\rho') \mathbf{I} + A_4 \hat{H}_n^{(1)}(k_{jp}\rho') \hat{\mathbf{R}}_{j,j-1} \right].
$$

(2.70)

The reciprocity property $\beta_{j2}^{-1} = \alpha_{j2}$ is used above. Also, all multiplicative factors ($A_1$, $A_2$, $A_3$, $A_4$) ensure that their absolute values are never greater than unity due to the boundness property. The full list of the multiplicative factors for all possible scenarios of Case 1 is shown in Table A.1 in Appendix A.

For Case 2, four arguments are of interest: $k_{jp}a_{j-1}$, $k_{jp}\rho$, $k_{jp}\rho'$, and $k_{jp}a_j$. Again, we let $a_{j-1} = a_1$, $\rho = a_2$, $\rho' = a_3$, and $a_j = a_4$ so that $a_1 < a_2 < a_3 < a_4$. Similarly,
the integrand is modified to

$$F_n(\rho, \rho') = \left[ J_n(k_j \rho) \mathbf{I} + H_n^{(1)}(k_j \rho) \hat{\mathbf{R}}_{j,j-1} \right] \cdot \mathbf{M}_{j-} \cdot \left[ H_n^{(1)}(k_j \rho') \mathbf{I} + J_n(k_j \rho') \hat{\mathbf{R}}_{j,j+1} \right]$$

$$= \left[ \beta_{j2} \tilde{J}_n(k_j \rho) \mathbf{I} + \alpha_{j2} \tilde{H}_n^{(1)}(k_j \rho) \beta_{j1}^2 \hat{\mathbf{R}}_{j,j-1} \right] \cdot \mathbf{M}_{j-} \cdot \left[ \alpha_{j3} \tilde{H}_n^{(1)}(k_j \rho') \mathbf{I} + \beta_{j3} \tilde{J}_n(k_j \rho') \alpha_{j4}^2 \hat{\mathbf{R}}_{j,j+1} \right]$$

$$= \left[ \beta_{j2} \alpha_{j3} \tilde{J}_n(k_j \rho) \mathbf{I} + (\beta_{j1} \alpha_{j2})(\beta_{j1} \alpha_{j3}) \tilde{H}_n^{(1)}(k_j \rho) \beta_{j4} \hat{\mathbf{R}}_{j,j-1} \right] \cdot \mathbf{M}_{j-} \cdot \left[ \tilde{H}_n^{(1)}(k_j \rho') \mathbf{I} + (\beta_{j3} \alpha_{j4})^2 \tilde{J}_n(k_j \rho') \hat{\mathbf{R}}_{j,j+1} \right]$$

$$= \left[ B_1 \tilde{J}_n(k_j \rho) \mathbf{I} + B_2 \tilde{H}_n^{(1)}(k_j \rho) \hat{\mathbf{R}}_{j,j-1} \right] \cdot \mathbf{M}_{j-} \cdot \left[ B_3 \tilde{H}_n^{(1)}(k_j \rho') \mathbf{I} + B_4 \tilde{J}_n(k_j \rho') \hat{\mathbf{R}}_{j,j+1} \right].$$

(2.71)

The reciprocity property $\alpha_{j3}^{-1} = \beta_{j3}$ is used above. Also, all multiplicative factors $(B_1, B_2, B_3, B_4)$ ensure that their absolute values are never greater than unity due to the boundness property. The full list of the multiplicative factors for all possible scenarios of Case 2 is given in Table A.2 in Appendix A.

For Case 3, there are 6 arguments of interest: $k_{i\rho} a_{i-1}, k_{i\rho} \rho, k_{i\rho} a_i, k_{j\rho} a_{j-1}, k_{j\rho} \rho'$, and $k_{j\rho} a_j$. We let $a_{i-1} = a_1, \rho = a_2, a_i = a_3, a_{j-1} = b_1, \rho' = b_2$, and $a_j = b_3$ so that
\[ F_n(\rho, \rho') = \left[ H_n^{(1)}(k_{ip}\rho)\tilde{I} + J_n(k_{ip}\rho)\tilde{R}_{i,i+1} \right] \cdot \tilde{N}_{i+} \cdot \tilde{T}_{ji} \cdot \tilde{M}_{j+} \cdot \left[ J_n(k_{j\rho}\rho')\tilde{I} + H_n^{(1)}(k_{j\rho}\rho')\tilde{R}_{j,j-1} \right] \]

\[ = \left[ H_n^{(1)}(k_{ip}\rho)\tilde{I} + J_n(k_{ip}\rho)\tilde{R}_{i,i+1} \right] \cdot \tilde{N}_{i+} \cdot \tilde{T}_{i-1,i} \cdot \tilde{X}_{j,i-1} \cdot \tilde{I} \cdot \tilde{M}_{j+} \cdot \left[ J_n(k_{j\rho}\rho')\tilde{I} + H_n^{(1)}(k_{j\rho}\rho')\tilde{R}_{j,j-1} \right] \]

\[ = \left[ \beta_{i1}\alpha_{i2}H_n^{(1)}(k_{ip}\rho)\tilde{I} + (\beta_{i1}\alpha_{i3})(\beta_{i2}\alpha_{i3})\hat{J}_n(k_{ip}\rho)\tilde{R}_{i,i+1} \right] \cdot \tilde{N}_{i+} \cdot \tilde{T}_{ji} \cdot \tilde{M}_{j+} \cdot \left[ \beta_{j1}\alpha_{j2}\hat{J}_n(k_{j\rho}\rho')\tilde{I} + (\beta_{j1}\alpha_{j3})\hat{H}_n^{(1)}(k_{j\rho}\rho')\tilde{R}_{j,j-1} \right] \]

\[ = \left[ C_1\hat{H}_n^{(1)}(k_{ip}\rho)\tilde{I} + C_2\hat{J}_n(k_{ip}\rho)\tilde{R}_{i,i+1} \right] \cdot \tilde{N}_{i+} \cdot \tilde{T}_{ji} \cdot \tilde{M}_{j+} \cdot \left[ C_3\hat{J}_n(k_{j\rho}\rho')\tilde{I} + C_4\hat{H}_n^{(1)}(k_{j\rho}\rho')\tilde{R}_{j,j-1} \right]. \quad (2.72) \]

All multiplicative factors \((C_1, C_2, C_3, C_4)\) ensure that their absolute values are never greater than unity due to the boundness property. The full list of the multiplicative factors for all possible scenarios of Case 3 is given in Table A.3 in Appendix A.
For Case 4, the arguments of interest are the same as those for Case 3. The integrand is modified to

$$\mathbf{F}_n(\rho, \rho') = \left[ J_n(k_i \rho) \mathbf{I} + H_n^{(1)}(k_i \rho) \tilde{\mathbf{R}}_{i,i-1} \right] \cdot \tilde{\mathbf{N}}_{i-} \cdot \tilde{\mathbf{T}}_{ji} \cdot \tilde{\mathbf{M}}_{j-}$$

$$\cdot \left[ H_n^{(1)}(k_j \rho') \mathbf{I} + J_n(k_j \rho') \tilde{\mathbf{R}}_{j,j+1} \right] = \left[ J_n(k_i \rho) \mathbf{I} + H_n^{(1)}(k_i \rho) \tilde{\mathbf{R}}_{i,i-1} \right] \cdot \tilde{\mathbf{N}}_{i-} \cdot \tilde{\mathbf{T}}_{i+1,i} \cdot \tilde{\mathbf{X}}_{j,i+1} \cdot \mathbf{I} \cdot \tilde{\mathbf{M}}_{j-}$$

$$\cdot \left[ H_n^{(1)}(k_j \rho') \mathbf{I} + J_n(k_j \rho') \tilde{\mathbf{R}}_{j,j+1} \right] = \left[ \beta_2 \alpha_3 \tilde{J}_n(k_i \rho) \mathbf{I} + (\beta_{i1} \alpha_{i2}) (\beta_{i1} \alpha_{i3}) \tilde{H}_n^{(1)}(k_i \rho) \tilde{\mathbf{R}}_{i,i-1} \right] \cdot \tilde{\mathbf{N}}_{i-} \cdot \tilde{\mathbf{T}}_{ji} \cdot \tilde{\mathbf{M}}_{j-}$$

$$\cdot \left[ \beta_1 \alpha_2 \tilde{H}_n^{(1)}(k_j \rho') \mathbf{I} + (\beta_{j1} \alpha_{j2}) (\beta_{j2} \alpha_{j3}) \tilde{J}_n(k_j \rho') \tilde{\mathbf{R}}_{j,j+1} \right] = \left[ D_1 \tilde{J}_n(k_i \rho) \mathbf{I} + D_2 \tilde{H}_n^{(1)}(k_i \rho) \tilde{\mathbf{R}}_{i,i-1} \right] \cdot \tilde{\mathbf{N}}_{i-} \cdot \tilde{\mathbf{T}}_{ji} \cdot \tilde{\mathbf{M}}_{j-}$$

$$\cdot \left[ D_3 \tilde{H}_n^{(1)}(k_j \rho') \mathbf{I} + D_4 \tilde{J}_n(k_j \rho') \tilde{\mathbf{R}}_{j,j+1} \right]. \quad (2.73)$$

All multiplicative factors ($D_1, D_2, D_3, D_4$) ensure that their absolute values are never greater than unity due to the boundness property. It should be noted that $D_1$ and $D_2$ for Case 4 have the same form as $C_3$ and $C_4$ for Case 3, respectively. Also, $D_3$ and $D_4$ for Case 4 have the same form as $C_1$ and $C_2$ for Case 3, respectively. The full list of the multiplicative factors for all possible scenarios of Case 4 is given in Table A.4 in Appendix A.

2.3.6 Representation of an Arbitrary Electric Dipole Source and Its Effect

In addition to $\mathbf{F}_n(\rho, \rho')$, the source factor $\tilde{\mathbf{D}}'_j$ should be determined in order to compute the numerical integration (2.10). Since the source is considered to be arbitrarily oriented, the source factor is first expressed in Cartesian coordinates and then
converted to its counterpart in cylindrical coordinates.

**Source Represented by Cartesian Coordinates**

A Hertzian electric dipole along the \( \hat{\alpha} \) direction can be represented as \( J(r) = Il\hat{\alpha}\delta(r - r') \). In Cartesian coordinates, \( \hat{\alpha} \) can be described as

\[
\hat{\alpha} = \hat{x}\alpha_x + \hat{y}\alpha_y + \hat{z}\alpha_z.
\]

(2.74)

Note that the dipole is located at \( r' \) hence the primed variables in the above. When the \( z \)-components of the electromagnetic fields are of interest, the source factor can be represented by

\[
\vec{D}_j' = \frac{i}{2} \left[ (\hat{z}k^2 - ik_z \nabla') \cdot \hat{\alpha}' \right],
\]

(2.75)

where

\[
\nabla' = \rho' \frac{\partial}{\partial \rho'} - \phi' \frac{\partial}{\partial \phi'} - \hat{z}k_z.
\]

(2.76)

Each component of \( \hat{\alpha}' \) is next examined separately. When \( \hat{\alpha}' = \hat{x}' \),

\[
\nabla' \cdot \hat{x}' = \cos \phi' \frac{\partial}{\partial \rho'} + \sin \phi' \frac{\partial}{\partial \phi'},
\]

(2.77a)

\[
\hat{z} \cdot \nabla' \times \hat{x}' = -\sin \phi' \frac{\partial}{\partial \rho'} + \cos \phi' \frac{\partial}{\partial \phi'}
\]

(2.77b)

Using (2.77a) and (2.77b), the source factor is

\[
\vec{D}_j' = \frac{i}{2} \left[ \frac{\partial}{\partial z} \left( \cos \phi' \frac{\partial}{\partial \rho'} + \sin \phi' \frac{\partial}{\partial \phi'} \right) \right] = \frac{i}{2} \left[ (-ik_z) \left( \cos \phi' \frac{\partial}{\partial \rho'} + \sin \phi' \frac{\partial}{\partial \phi'} \right) \right].
\]

(2.78)

When \( \hat{\alpha}' = \hat{y}' \),

\[
\nabla' \cdot \hat{y}' = \sin \phi' \frac{\partial}{\partial \rho'} - \cos \phi' \frac{\partial}{\partial \phi'},
\]

(2.79a)

\[
\hat{z} \cdot \nabla' \times \hat{y}' = \cos \phi' \frac{\partial}{\partial \rho'} + \sin \phi' \frac{\partial}{\partial \phi'}
\]

(2.79b)
Using (2.79a) and (2.79b), the source factor is

\[
\mathbf{D}'_j = \frac{i}{2} \left[ \frac{\partial}{\partial z'} \left( \sin \phi' \frac{\partial}{\partial \rho'} - \cos \phi' \frac{\partial}{\partial \rho'} \right) \right] = \frac{i}{2} \left[ \frac{\partial}{\partial \rho'} \left( \sin \phi' \frac{\partial}{\partial \rho'} - \cos \phi' \frac{\partial}{\partial \rho'} \right) \right]. \tag{2.80}
\]

Finally, when \( \hat{\alpha}' = \hat{z}' \),

\[
\nabla' \cdot \hat{z}' = -ik_z, \tag{2.81a}
\]

\[
\hat{z} \cdot \nabla' \times \hat{z}' = 0. \tag{2.81b}
\]

Using (2.81a) and (2.81b), the source factor is

\[
\mathbf{D}' = \frac{i}{2} \left[ \begin{array}{c}
-ik_z \\
0
\end{array} \right] = \frac{i}{2} \left[ \begin{array}{c}
k_z \\
0
\end{array} \right]. \tag{2.82}
\]

In general, with an arbitrary source direction, the source factor can be represented by

\[
\mathbf{D}'_j = \frac{i}{2} \left[ \begin{array}{c}
\cos \phi' \frac{\partial}{\partial \rho'} + \sin \phi' \frac{\partial}{\partial z'} \\
-\sin \phi' \frac{\partial}{\partial \rho'} + \cos \phi' \frac{\partial}{\partial z'}
\end{array} \right] \frac{\partial}{\partial \rho'} \left( \sin \phi' \frac{\partial}{\partial \rho'} - \cos \phi' \frac{\partial}{\partial \rho'} \right) \tag{2.83}
\]

Alternatively, (2.83) can be described as

\[
\mathbf{D}'_j = \frac{i}{2} \left[ \begin{array}{c}
(k^2_j)_{\alpha_x'} \\
0
\end{array} \right] + \frac{i}{2} \left[ \begin{array}{c}
\frac{k_z}{\rho'} \left( \alpha_x' \sin \phi' - \alpha_y' \cos \phi' \right) \\
\frac{k_z}{\rho'} \left( \alpha_x' \cos \phi' + \alpha_y' \sin \phi' \right)
\end{array} \right] \frac{\partial}{\partial \rho'}
\]

\[
= \frac{i}{2} \left( \mathbf{D}'_{j1} + \mathbf{D}'_{j2} + \mathbf{D}'_{j3} \frac{\partial}{\partial \rho'} \right). \tag{2.84}
\]

In (2.84) for a fixed \( k_z \), \( \mathbf{D}'_{j1} \) is constant, \( \mathbf{D}'_{j2} \) depends on the modal index \( n \), and \( \mathbf{D}'_{j3} \) is associated with the partial derivative with respect to \( \rho' \).
Source Represented by Cylindrical Coordinates

The source factor can be represented in cylindrical coordinates via a simple coordinate transformation. The source polarization vector is described in cylindrical coordinates as

\[ \hat{\alpha}' = \hat{\rho}' \alpha_{\rho}' + \hat{\phi}' \alpha_{\phi}' + \hat{z}' \alpha_z' \]

\[ = \hat{x}' \alpha_{x'} + \hat{y}' \alpha_{y'} + \hat{z}' \alpha_z', \]

where

\[ \alpha_{x'} = \alpha_{\rho}' \cos \phi' - \alpha_{\phi}' \sin \phi', \]  
\[ \alpha_{y'} = \alpha_{\rho}' \sin \phi' + \alpha_{\phi}' \cos \phi'. \]

The two trigonometric functions appearing on \( \hat{D}_{j2}' \) in (2.84) are converted to their cylindrical counterparts using (2.86a) and (2.86b).

\[ \alpha_{x'} \sin \phi' - \alpha_{y'} \cos \phi' \]

\[ = (\alpha_{\rho}' \cos \phi' - \alpha_{\phi}' \sin \phi') \sin \phi' - (\alpha_{\rho}' \sin \phi' + \alpha_{\phi}' \cos \phi') \cos \phi' \]

\[ = -\alpha_{\phi}'. \]  
\[ \alpha_{x'} \cos \phi' + \alpha_{y'} \sin \phi' \]

\[ = (\alpha_{\rho}' \cos \phi' - \alpha_{\phi}' \sin \phi') \cos \phi' + (\alpha_{\rho}' \sin \phi' + \alpha_{\phi}' \cos \phi') \sin \phi' \]

\[ = \alpha_{\rho}'. \]
Similarly, the two trigonometric functions appearing on $\vec{D}_{j_3}'$ in (2.84) are converted to their cylindrical counterparts using (2.86a) and (2.86b).

\[
\alpha_{x'} \cos \phi' + \alpha_{y'} \sin \phi' = (\alpha_{x'} \cos \phi' - \alpha_{y'} \sin \phi') \cos \phi' + (\alpha_{x'} \sin \phi' + \alpha_{y'} \cos \phi') \sin \phi' = \alpha_{\rho'}.
\]

(2.87c)

\[-\alpha_{x'} \sin \phi' + \alpha_{y'} \cos \phi' = - (\alpha_{x'} \cos \phi' - \alpha_{y'} \sin \phi') \sin \phi' + (\alpha_{x'} \sin \phi' + \alpha_{y'} \cos \phi') \cos \phi' = \alpha_{\phi'}.
\]

(2.87d)

Using (2.87a) – (2.87d), $\vec{D}_{j_2}'$ and $\vec{D}_{j_3}'$ can be simplified to

\[
\vec{D}_{j_2}' = \begin{bmatrix} -\frac{n k z \alpha_{\phi'}}{\rho'} \\ -\frac{n \omega \epsilon_j}{\rho} \alpha_{\rho'} \end{bmatrix}, \\
\vec{D}_{j_3}' = \begin{bmatrix} -i k z \alpha_{\phi'} \\ i \omega \epsilon_j \alpha_{\phi'} \end{bmatrix}.
\]

(2.88a)

(2.88b)

Therefore, with a source representation in cylindrical coordinates, the source factor becomes

\[
\vec{D}_j' = \frac{i}{2} \left( \vec{D}_{j_1}' + \vec{D}_{j_2}' + \vec{D}_{j_3}' \frac{\partial}{\partial \rho'} \right)
\]

\[
= \frac{i}{2} \left( \begin{bmatrix} (k_{j_{\rho'}}^2) \alpha_{\phi'} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{n k z \alpha_{\phi'}}{\rho'} \\ -\frac{n \omega \epsilon_j}{\rho} \alpha_{\rho'} \end{bmatrix} \right)
\]

(2.89)

2.3.7 Azimuth Series Folding

As $J_n(z)$ and $H_n^{(1)}(z)$ for a positive and negative integer order $n$ are related, the summation over the azimuthal mode number $n$ can be folded. When an order is a
negative integer, the cylindrical functions and their derivatives can be expressed as

\[ J_{-n}(z) = (-1)^n J_n(z), \quad (2.90a) \]
\[ J'_{-n}(z) = (-1)^n J'_n(z), \quad (2.90b) \]
\[ H^{(1)}_{-n}(z) = (-1)^n H^{(1)}_n(z), \quad (2.90c) \]
\[ H'^{(1)}_{-n}(z) = (-1)^n H'^{(1)}_n(z), \quad (2.90d) \]

where \( n \) is a positive integer.

**z-components**

First of all, let us consider the \( z \)-components of electromagnetic fields.

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix}
= i \frac{Il}{4 \pi \omega \epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \cdot \mathbf{D}'_j \right].
\]
(2.91)

Recall that reflection and transmission coefficients are \( 2 \times 2 \) matrices in cylindrical coordinates and they include the matrices in (2.30a) – (2.31b). Therefore, only the off-diagonal elements of reflection and transmission coefficients as well as the auxiliary coefficients (\( \tilde{M}_{j+} \) and \( \tilde{N}_{i\pm} \)) in \( F_n(\rho, \rho') \) change sign. As a result, only the off-diagonal elements of \( F_n(\rho, \rho') \) change sign. Since the order of summation and integration can be interchanged, (2.91) becomes

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix}
= i \frac{Il}{4 \pi \omega \epsilon_j} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \cdot \mathbf{D}'_j \right].
\]
\]
(2.92)
Using (2.84) or (2.89), the integrand in (2.92) excluding the exponential part can be expanded as
\[
\left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \cdot \vec{D}_j' \right] = \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \right] \cdot \vec{D}_j' \\
+ \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \cdot \vec{D}_j'' \right] \\
+ \frac{i}{2} \frac{\partial}{\partial \rho'} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho, \rho') \right] \cdot \vec{D}_j''].
\]
\[(2.93)\]

Note that \(\vec{D}_j'\) and \(\vec{D}_j''\) do not depend on the azimuth mode number \(n\), so it can be pulled out of the squared bracket. With additional coefficients defined as \(\vec{F}_n(\rho, \rho') = X_{z,n}, Y_{z,n}, Z_{z,n}\), (2.93) can be simplified as
\[
\frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} X_{z,n} \right] \cdot \vec{D}_j' \\
+ \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} Y_{z,n} \right] + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} Z_{z,n} \right] \cdot \vec{D}_j''.
\]
\[(2.94)\]

For the definitions of \(\vec{X}_{z,n}, \vec{Y}_{z,n}, \vec{Z}_{z,n}\) in (2.94), refer to Appendix B. As a result, (2.92) can be modified to
\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix}
= \frac{iI}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \vec{F}_n(\rho, \rho') \cdot \vec{D}_j' \right]
\]
\[
= -\frac{II}{8\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left\{ \left[ \vec{X}_{z,0} + \sum_{n=1}^{\infty} \vec{X}_{z,n} \right] \cdot \vec{D}_j' + \left[ \sum_{n=1}^{\infty} \vec{Y}_{z,n} \right] \cdot \vec{D}_j'' + \left[ \vec{Z}_{z,0} + \sum_{n=1}^{\infty} \vec{Z}_{z,n} \right] \cdot \vec{D}_j'' \right\},
\]
\[(2.95)\]
so that only non-negative integer orders are involved.
The \( \rho \)-components of electromagnetic fields can be obtained by substituting (2.91) into (2.14).

\[
\begin{bmatrix}
E_\rho \\
H_\rho
\end{bmatrix} = \frac{iI_l}{4\pi\omega \epsilon_0} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{1}{k_{z'}^2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{B}_n \cdot \mathbf{F}_n(\rho, \rho') \cdot \mathbf{D}'_j \right],
\]

(2.96)

In order to perform azimuth series folding, it is necessary to decompose \( \mathbf{F}_n(\rho, \rho') \) into three independent matrices, i.e.,

\[
\mathbf{F}_n(\rho, \rho') = \mathbf{L}_n(\rho) \cdot \mathbf{M}_n \cdot \mathbf{R}_n(\rho'),
\]

(2.97)

where \( \mathbf{L}_n(\rho) \) is a function of \( \rho \), \( \mathbf{M}_n \) is a function of cylindrical layers’ interfaces, and \( \mathbf{R}_n(\rho') \) is a function of \( \rho' \). It should be noted that \( \mathbf{B}_n \) only acts on \( \mathbf{L}_n(\rho) \) and \( \mathbf{D}'_j \) only acts on \( \mathbf{R}_n \). Therefore, \( \mathbf{B}_n \cdot \mathbf{L}_n(\rho) \) and \( \mathbf{R}_n(\rho') \cdot \mathbf{D}'_j \) can be calculated separately. Furthermore, \( \mathbf{R}_n(\rho') \cdot \mathbf{D}'_j \) can be decomposed into three parts as

\[
\mathbf{R}_n(\rho') \cdot \mathbf{D}'_j = \frac{i}{2} \mathbf{R}_n(\rho') \cdot \left( \mathbf{D}'_{j1} + \mathbf{D}'_{j2} + \frac{\partial}{\partial \rho'} \mathbf{D}'_{j3} \right).
\]

(2.98)

With a new coefficient defined as \( \mathbf{B}_n \cdot \mathbf{L}_n(\rho) \cdot \mathbf{M}_n = \mathbf{W}_{\rho,n} \), the squared bracket shown in (2.96) becomes

\[
\left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') \cdot \mathbf{D}'_j \right] = \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') \right] \cdot \mathbf{D}'_{j1}
\]

\[
\quad + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') \right] \cdot \mathbf{D}'_{j2}
\]

\[
\quad + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{W}_{\rho,n} \cdot \frac{\partial}{\partial \rho'} \left[ \mathbf{R}_n(\rho') \right] \right] \cdot \mathbf{D}'_{j3}.
\]

(2.99)

Again, for negative integer orders, the off-diagonal elements of \( \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') \) and \( \mathbf{W}_{\rho,n} \cdot \frac{\partial}{\partial \rho'} \left[ \mathbf{R}_n(\rho') \right] \) in (2.99) change sign. With additional coefficients defined as

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\[ \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') = \mathbf{X}_{\rho,n}, \quad \mathbf{W}_{\rho,n} \cdot \mathbf{R}_n(\rho') \cdot \hat{\mathbf{D}}_{j2} = \mathbf{Y}_{\rho,n}, \quad \text{and} \quad \mathbf{W}_{\rho,n} \cdot \frac{\partial}{\partial \rho} [\mathbf{R}_n(\rho')] = \mathbf{Z}_{\rho,n}, \]

(2.99) can be simplified as

\[
\frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{X}_{\rho,n} \right] \cdot \hat{\mathbf{D}}'_{j1} + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{Y}_{\rho,n} \right] + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{Z}_{\rho,n} \right] \cdot \hat{\mathbf{D}}'_{j3} = \frac{i}{2} \left[ \sum_{n=1}^{\infty} e^{in(\phi-\phi')} \mathbf{X}_{\rho,0} \right] \cdot \hat{\mathbf{D}}'_{j1} + \frac{i}{2} \left[ \sum_{n=1}^{\infty} e^{in(\phi-\phi')} \mathbf{Y}_{\rho,0} \right] + \frac{i}{2} \left[ \sum_{n=1}^{\infty} e^{in(\phi-\phi')} \mathbf{Z}_{\rho,0} \right] \cdot \hat{\mathbf{D}}'_{j3}. \quad (2.100)
\]

For the definitions of \( \dddot{\mathbf{X}}_{\rho,n}, \dddot{\mathbf{Y}}_{\rho,n}, \dddot{\mathbf{Z}}_{\rho,n} \) in (2.100), refer to Appendix B. As a result, (2.96) is modified to

\[
\begin{bmatrix} E_{\rho} \\ H_{\rho} \end{bmatrix} = \frac{iL}{4\pi\omega_1} \int_{-\infty}^{\infty} dk_z e^{i k_z (z-z')} \frac{1}{k_{ip}^2} \left[ \sum_{n=-\infty}^{\infty} e^{i n(\phi-\phi')} \mathbf{B}_n \cdot \mathbf{F}_n(\rho, \rho') \cdot \hat{\mathbf{D}}_j \right],
\]

\[
\begin{bmatrix} E_{\phi} \\ H_{\phi} \end{bmatrix} = \frac{-iL}{8\pi\omega_1} \int_{-\infty}^{\infty} dk_z e^{i k_z (z-z')} \frac{1}{k_{ip}^2} \left\{ \left[ \mathbf{X}_{\rho,0} + \sum_{n=1}^{\infty} \dddot{\mathbf{X}}_{\rho,n} \right] \cdot \hat{\mathbf{D}}'_j + \left[ \sum_{n=1}^{\infty} \dddot{\mathbf{Y}}_{\rho,n} \right] \right\}, \quad (2.101)
\]

so that only non-negative integer orders are involved.

**\( \phi \)-components**

Similarly, the \( \phi \)-components can be obtained by substituting (2.91) into (2.15).

\[
\begin{bmatrix} E_{\phi} \\ H_{\phi} \end{bmatrix} = \frac{iL}{4\pi\omega_1} \int_{-\infty}^{\infty} dk_z e^{i k_z (z-z')} \frac{1}{k_{ip}^2} \left[ \sum_{n=-\infty}^{\infty} e^{i n(\phi-\phi')} \mathbf{C}_n \cdot \mathbf{F}_n(\rho, \rho') \cdot \hat{\mathbf{D}}_j \right]. \quad (2.102)
\]
With a new coefficient defined as $\overline{C}_n \cdot L_n(\rho) \cdot \overline{M}_n = \overline{W}_{\phi,n}$, the squared bracket shown in (2.102) becomes

$$
\left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} W_{\phi,n} \cdot R_n(\rho') \cdot \hat{D}_j \right] = \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} W_{\phi,n} \cdot R_n(\rho') \right] \cdot \hat{D}_{j1} + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} W_{\phi,n} \cdot R_n(\rho') \right] \cdot \hat{D}_{j2} + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} W_{\phi,n} \cdot \frac{\partial}{\partial \rho'} [R_n(\rho')] \right] \cdot \hat{D}_{j3}.
$$

(2.103)

In contrast to the $\rho$-components, the diagonal elements of $W_{\phi,n} \cdot R_n(\rho')$ and $W_{\phi,n} \cdot \frac{\partial}{\partial \rho'} [R_n(\rho')]$ in (2.103) change sign. With additional coefficients defined as $\overline{W}_{\phi,n} \cdot R_n(\rho') = X_{\phi,n}, W_{\phi,n} \cdot R_n(\rho') \cdot \hat{D}_{j2} = Y_{\phi,n},$ and $W_{\phi,n} \cdot \frac{\partial}{\partial \rho'} [R_n(\rho')] = Z_{\phi,n}$ (2.103) can be simplified as

$$
\frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} X_{\phi,n} \right] \cdot \hat{D}_{j1} + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} Y_{\phi,n} \right] + \frac{i}{2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} Z_{\phi,n} \right] \cdot \hat{D}_{j3} = \frac{i}{2} \left[ X_{\phi,0} + \sum_{n=1}^{\infty} \ddot{X}_{\phi,n} \right] \cdot \hat{D}_{j1} + \frac{i}{2} \left[ Y_{\phi,0} + \sum_{n=1}^{\infty} \ddot{Y}_{\phi,n} \right] + \frac{i}{2} \left[ Z_{\phi,0} + \sum_{n=1}^{\infty} \ddot{Z}_{\phi,n} \right] \cdot \hat{D}_{j3}.
$$

(2.104)

For the definitions of $\ddot{X}_{\phi,n}, \ddot{Y}_{\phi,n}, \ddot{Z}_{\phi,n}$ in (2.104), refer to Appendix B. As a result, (2.102) is modified to

$$
\begin{bmatrix}
E_{\phi} \\
H_{\phi}
\end{bmatrix} = \frac{iI}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk z e^{ikz(z-z')} \frac{1}{k^2} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \overline{C}_n \cdot \overline{F}_n(\rho, \rho') \cdot \hat{D}_j \right] = -\frac{I}{8\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk z e^{ikz(z-z')} \frac{1}{k^2} \left\{ \left[ X_{\phi,0} + \sum_{n=1}^{\infty} \ddot{X}_{\phi,n} \right] \cdot \hat{D}_{j1} + \left[ \sum_{n=1}^{\infty} \ddot{Y}_{\phi,n} \right] \right. \\
\left. + \left[ Z_{\phi,0} + \sum_{n=1}^{\infty} \ddot{Z}_{\phi,n} \right] \cdot \hat{D}_{j3} \right\}.
$$

(2.105)

Again, only non-negative integer orders are involved in (2.105).
2.4 Robust Integration Paths

In this section, numerically robust integration paths in the complex $k_z$ plane for evaluation of the spectral integrals are introduced. It will be observed that optimal choices leading to fast convergence depend on the observation point $(\rho, \phi, z)$ and source point $(\rho', \phi', z')$. Even though the integration paths considered here yield accurate results and fast computing time, they fail under two particular scenarios. These two scenarios will be identified and strategies to compute the fields in those scenarios will be presented as well.

2.4.1 Sommerfeld Integration Path

A robust Sommerfeld Integration Path (SIP) depicted in Figure 2.8 can be used to avoid integrand singularities in the complex $k_z$ plane due to poles and various branch points/cuts associated with $k_\rho = \sqrt{k^2 - k_z^2}$ and the logarithmic singularity of $H_1^{(1)}(z)$ at $z = 0$. As shown in Figure 2.8, there are three basic free parameters for the numerical integration along this path: $\delta_1$, $\delta_2$, and $\delta_{3,SIP}$. Here, the parameters $\delta_1$ and $\delta_2$ are chosen as a function of the branch point nearest to the origin, such that

\[
\delta_1 = \frac{1}{5} \Re[k_{\min}], \quad \delta_2 = \frac{1}{5} \Im[k_{\min}].
\]

The chosen scaling factor $1/5$ was verified to produce accurate results, but it can be modified as long as the path does not cross branch points. If the smallest branch point is purely real because the associated layer is lossless, $\delta_2$ is set equal to $\delta_1$. In this case, branch cut crossing is inevitable and a proper tracking of the correct $k_\rho$ value is required. The SIP is truncated according to $\delta_{3,SIP}$. When the SIP is appropriate, the
The integrand will decrease exponentially away from the origin and $\delta_{3,\text{SIP}}$ can be taken to be a point at which the ratio of the magnitudes of the integrand evaluated at $k_z = p_2$ and $k_z = p_1$, as indicated in Figure 2.8, is below some threshold. Here, we choose this threshold ratio equal to $10^{-20}$. In some instances, the integrand along the SIP does not decay in magnitude but simply oscillates, which is an indication that the integral along such SIP does not converge. In this case, a modified integration path should be used instead, as discussed next.

### 2.4.2 Deformed Sommerfeld Integration Path

The SIP considered above is simple to implement, but it produces inaccurate results under certain conditions. For example, when the integrand is slowly decaying as the real part of $k_z$, $\Re[k_z]$, goes to $-\infty$ or $\infty$, the exponential part in the integrand oscillates faster. Therefore, it is difficult to obtain very accurate numerical
integration values. In this case, the SIP can be deformed such that the two horizontal paths are bent upwards while making sure that all branch cuts and singularities are enclosed [70]. This type of an integration path is called a Deformed Sommerfeld Integration Path (DSIP) and is depicted in Figure 2.9.

There are four parameters for the numerical integration along the DSIP: \( \delta_1, \delta_2, \delta_{3,DSIP}, \) and \( \delta_4 \). The parameters \( \delta_1 \) and \( \delta_2 \) are defined in the same manner as for the SIP (see (2.106) and (2.107)). For the DSIP, the truncation is set by \( \delta_4 \). If the imaginary part of \( k_z, \Im[k_z] \), is positive and \( z - z' > 0 \), the exponential part of the integrand decays exponentially as shown in (2.108). For the case, when \( z - z' < 0 \), refer to Section 2.4.3.

\[
e^{i k_z (z-z')} = e^{i (k'_z + ik''_z)(z-z')} = e^{-k''_z(z-z')} e^{i k'_z(z-z')}, \text{ when } z - z' > 0.
\] (2.108)
Figure 2.10: Determination of $\delta_{3, DSIP}$: (a) Case 1 and (b) Case 2.

Hence, $\delta_4$ is determined by $z - z'$ so that the factor $e^{-\delta_4 |z - z'|}$ is below some tolerance $\gamma$. In other words,

$$\delta_4 = -\frac{\log_e(\gamma)}{|z - z'|}. \quad (2.109)$$

The parameter $\delta_{3, DSIP}$ is related to $\delta_4$ but defined in a different way compared to $\delta_{3, SIP}$. Figure 2.10a and 2.10b show two different cases to help clarify the determination of $\delta_{3, DSIP}$. When the imaginary parts of some branch points are greater than $\delta_4$, those branch points are excluded in the determination of $\delta_{3, DSIP}$ because the contribution of the integrand path near those points are negligible and the computing time along such integration path (enclosing all branch points) will be excessive. This is indeed the case when some cylindrical layers exhibit very large conductivity. Consequently, $\delta_{3, DSIP}$ is defined as some factor $f$ times the value of the largest real part of the remaining branch points.

$$\delta_{3, DSIP} = f \text{Re}[\tilde{k}_{\text{max}}], \quad (2.110)$$
where \( \tilde{k} \) represents the group of the remaining branch points. We have verified that \( f = 2 \) gives good results. This number can be increased or decreased under certain limits but it should not be made too close to unity because the integration path would then approach a singularity point.

For most cases, the DSIP yields more accurate results than the SIP, so the primary integration path is set here to be the DSIP. However, the DSIP is not always the most efficient. Under some conditions as noted below, the SIP shows the better convergence.

In two particular scenarios, the SIP or DSIP does not yield convergent integrals. In those cases, alternative strategies need to be implemented. This will be discussed in Section 2.4.4.

2.4.3 Reversed Deformed Sommerfeld Integration Path

The DSIP introduced in Section 2.4.2 assumes \( z > z' \). For \( z < z' \), the exponential factor in the integrand increases for \( k_z \) with a positive imaginary part. To have the desired exponential decay, the DSIP is simply reversed about the origin, as depicted in Figure 2.11. All numerical parameters \( \delta_1, \delta_2, \delta_{3, DSIP}, \delta_4 \) remain the same as before.

2.4.4 Treatment of Non-Convergent Cases

Both the SIP and DSIP are good on their own but under two scenarios the numerical integration does not converge. In this section, these two scenarios will be identified and strategies to the scenarios will be discussed.
Two Scenarios of Non-Convergent Numerical Integration

The first non-convergent scenario occurs for the DSIP when \( z \approx z' \), so that \( \delta_4 \) would be very large. The exponential part in the integrand, the role of which is to attenuate the magnitude of the associated integrand, becomes unity. Since sufficient attenuation of the integrand is not present, convergence is very slow or absent.

The second non-convergent scenario occurs when \( \rho \approx \rho' \). In this case, the numerical integration using the DSIP converges but the summation over \( n \) does not. In fact, the direct field terms contribute to this non-convergence behavior. The magnitudes of the azimuth series terms representing the direct fields remain the same or decrease very slowly as the mode index \( n \) increases. It should be emphasized that the contribution of the direct fields is only present when the field point and source point are in the same layer. In other words, this scenario only happens when (2.11a)
Table 2.8: Non- or slow-convergent behavior of direct field terms.

<table>
<thead>
<tr>
<th>Order (n)</th>
<th>( \rho = \rho' )</th>
<th>( \rho = 1.1\rho' )</th>
<th>( \rho = 2.0\rho' )</th>
<th>( \rho = 5.0\rho' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.9670</td>
<td>5.4113</td>
<td>2.8948</td>
<td>1.0064</td>
</tr>
<tr>
<td>10</td>
<td>3.0189</td>
<td>1.0581</td>
<td>1.4732×10(^{-3})</td>
<td>6.1529×10(^{-8})</td>
</tr>
<tr>
<td>20</td>
<td>3.0189</td>
<td>0.4079</td>
<td>1.4390×10(^{-6})</td>
<td>6.3148×10(^{-15})</td>
</tr>
<tr>
<td>30</td>
<td>3.0189</td>
<td>0.1573</td>
<td>1.4054×10(^{-9})</td>
<td>6.4717×10(^{-22})</td>
</tr>
<tr>
<td>50</td>
<td>3.0189</td>
<td>0.0234</td>
<td>1.3404×10(^{-15})</td>
<td>6.7907×10(^{-36})</td>
</tr>
</tbody>
</table>

and (2.11b) are concerned. Table 2.8 illustrates this fact. The numbers in Table 2.8 are \( |H_n^{(1)}(k_j\rho)J_n(k_j\rho')| \), which corresponds to the \( \rho \)-component of the direct field terms when the source is \( z \)-oriented. It is assumed that \( k_j\rho = 0.25147 + i0.79122 \) and \( \rho' = 0.1270 \).

When \( \rho = \rho' \), the magnitude does not decrease at all as the mode number increases. For \( \rho = 1.1\rho' \), the magnitude decreases but only very slowly. Mathematically, the non-convergent behavior of this series can be explained using the small argument approximations for (2.19a) – (2.19d). The possible direct field terms for Case 1 (refer
to (2.11a)) then become

$$H_n^{(1)}(k_{j\rho},p)J_n(k_{j\rho}p') = G_i^{-1}\rho^{-n}\left(-\frac{i}{n\pi}\right)\cdot G_i(\rho')^n = -\frac{i}{n\pi}\left(\frac{\rho'}{\rho}\right)^n,$$

(2.111a)

$$H'_n^{(1)}(k_{j\rho},p)J_n(k_{j\rho}p') = G_i^{-1}\rho^{-n}\left(\frac{i}{\pi k_{j\rho}p}\right)\cdot G_i(\rho')^n = \frac{i}{\pi k_{j\rho}p}\left(\frac{\rho'}{\rho}\right)^n,$$

(2.111b)

$$H''_n^{(1)}(k_{j\rho},p)J'_n(k_{j\rho}p') = G_i^{-1}\rho^{-n}\left(-\frac{i}{n\pi}\right)\cdot G_i(\rho')^n \left(\frac{n}{k_{j\rho}p'}\right) = -\frac{i}{\pi k_{j\rho}p'}\left(\frac{\rho'}{\rho}\right)^n,$$

(2.111c)

$$H''_n^{(1)}(k_{j\rho},p)J'_n(k_{j\rho}p') = G_i^{-1}\rho^{-n}\left(\frac{i}{\pi k_{j\rho}p}\right)\cdot G_i(\rho')^n \left(\frac{n}{k_{j\rho}p'}\right) = \frac{in}{\pi k_{j\rho}^2 p p'}\left(\frac{\rho'}{\rho}\right)^n.$$

(2.111d)

Except for (2.111a), it is seen that all three cases show non-convergent or even divergent behavior as a function of $n$. Note that (2.111a) refers to the $z$-component produced by a $z$-directed source but similar conclusions can be made in other cases. Physically, the integrand excluding the exponential factor represents cylindrical waves along the $\rho$-direction. When $\rho = \rho'$, all cylindrical order modes should be present in the wave representation.

In order to examine this non-convergent behavior in more detail, the square region of 25 cm × 25 cm in the $\rho z$-plane as shown in Figure 2.12 is considered. The medium is homogeneous and its electromagnetic properties are $\epsilon_r = 1$, $\mu_r = 1$, and $\sigma = 1$. The source is a $\phi$-directed magnetic dipole and the $\phi$-components of the magnetic field in the squared region are computed. The operating frequency is 36 kHz.

Figure 2.13a – 2.13d show the relative error distribution using the DSIP for various maximum orders $n_{max}$ and the number of integration points $n_{int}$ along the integration path. The relative error is computed as

$$\text{relative error}_{dB} = 10\log_{10}\left|\frac{H_{\phi,a} - H_{\phi,n}}{H_{\phi,a}}\right|,$$

(2.112)
where \(H_{\phi,a}\) is an analytical (exact) value and \(H_{\phi,n}\) is the numerical integration value. As the maximum order and integration points increase, smaller relative error is obtained except at the two regions \(z \approx z'\) and \(\rho \approx \rho'\).

Figure 2.14a – 2.14d show a similar distribution of relative errors now using the SIP. The SIP yields a very small relative error in the region beyond \(\rho - \rho' = 18\) cm (where the integrand has sufficient decay along the SIP). In this case, an accurate numerical integration is obtained with smaller \(\delta_{3,\text{SIP}}\). Nevertheless, the DSIP is more efficient than the SIP in all other cases. That is the reason why the DSIP is set to the primary integration path.

When \(z \approx z'\), the relative errors using the DSIP and SIP are shown in Figure 2.15a and 2.15b. Note that the sampling region span is changed. In the limit as \(z \rightarrow z'\), \(\delta_{4}\) becomes infinite and the numerical integration is not feasible. This is illustrated by the large error visible near the horizontal axis in Figure 2.15a. However, as shown in
Figure 2.13: Relative errors of the DSIP: (a) \( n_{\text{max}} = 10, n_{\text{int}} = 2000 \), (b) \( n_{\text{max}} = 20, n_{\text{int}} = 2000 \), (c) \( n_{\text{max}} = 10, n_{\text{int}} = 4000 \), and (d) \( n_{\text{max}} = 20, n_{\text{int}} = 4000 \).

Figure 2.15b, the SIP is capable of calculating the field value even when \( z = z' \). It can be concluded that the SIP becomes more efficient than the DSIP when \( z \approx z' \). The choice of the SIP as an integration path is discussed below. Also, when \( \rho \approx \rho' \), neither the DSIP nor SIP is convergent, and a strategy to address this case is introduced below.
If a field point does not lie on any of these two non-convergent zones, the DSIP works well and its accuracy gradually increases as the maximum order and the number of integration points increase. Figure 2.16 shows the improvement of the numerical integration results using the DSIP for various maximum orders and number of integration points. In this case, we have $\rho - \rho' = 10$ cm, $\phi - \phi' = 0^\circ$, and $z - z' = 10$ cm.
Figure 2.15: Comparison of relative errors between the DSIP and SIP with $n_{\text{max}} = 30$ and $n_{\text{int}} = 4000$ when $z \approx z'$: (a) DSIP and (b) SIP.

Figure 2.16: Relative errors using the DSIP in terms of various maximum orders $n_{\text{max}}$ and integration points $n_{\text{int}}$ when $\rho - \rho' = 10 \text{ cm}$, $\phi - \phi' = 0^\circ$, and $z - z' = 10 \text{ cm}$.

as an example. For different values of $\phi$ and $\phi'$, Figure 2.17a and Figure 2.17b show the relative error distribution when $\phi - \phi' = 30^\circ$ and $\phi - \phi' = 105^\circ$, respectively.
Figure 2.17: Relative errors using the DSIP in terms of various maximum orders $n_{max}$ and integration points $n_{int}$ with $\rho - \rho' = 10$ cm and $z - z' = 10$ cm: (a) $\phi - \phi' = 30^\circ$ and (b) $\phi - \phi' = 105^\circ$. 
Strategy for $z \approx z'$ Scenario

When $z \approx z'$, comparison between the SIP and DSIP is performed. From Section 2.4.1 and 2.4.2, the longest and dominant path segment is associated with $\delta_{3,\text{SIP}}$ for the SIP and $\delta_4$ for the DSIP. With the same number of integration points, the smaller value between $\delta_{3,\text{SIP}}$ and $\delta_4$ should be used. Namely, when $z \approx z'$,

- use the SIP for $\delta_{3,\text{SIP}} < \delta_4$. \hspace{1cm} (2.113a)
- use the DSIP for $\delta_{3,\text{SIP}} \geq \delta_4$. \hspace{1cm} (2.113b)

To verify (2.113a) and (2.113b), one example is provided in Figure 2.18. The operating frequency is 36 kHz. The modal index, $n$, ranges from $-30$ to $30$. The number of numerical integration points along $k_z$ is the same. The source is assumed to be a $\phi$-oriented magnetic dipole and $H_\phi$ is computed. The two cylindrical layers have the same electromagnetic properties and hence there are no reflections at the
Table 2.9: Comparison between the SIP and DSIP for \( z \approx z' \).

<table>
<thead>
<tr>
<th>( z - z' )</th>
<th>( \delta_{3,SIP} )</th>
<th>Relative Error of ( H_\phi ) (SIP)</th>
<th>( \delta_4 )</th>
<th>Relative Error of ( H_\phi ) (DSIP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 m</td>
<td>617.6628</td>
<td>0.1393</td>
<td>50.6568</td>
<td>5.3080 \times 10^{-3}</td>
</tr>
<tr>
<td>0.1 m</td>
<td>617.6628</td>
<td>3.8746 \times 10^{-5}</td>
<td>506.568</td>
<td>1.3774 \times 10^{-4}</td>
</tr>
<tr>
<td>0.01 m</td>
<td>617.6628</td>
<td>1.6291 \times 10^{-5}</td>
<td>5065.68</td>
<td>4.4262 \times 10^{-3}</td>
</tr>
<tr>
<td>0.001 m</td>
<td>617.6628</td>
<td>1.6113 \times 10^{-5}</td>
<td>50656.8</td>
<td>212.8221</td>
</tr>
</tbody>
</table>

interface and only direct field terms are present. Table 2.9 shows the comparison between the SIP and DSIP results. The relative error defined in (2.33) is used for the above results where the analytical values are obtained as discussed in Appendix C. The integration path with the smaller delta parameter shows the smaller relative error. For \( z - z' = 0.01 \) m, the delta parameters are of a similar order and the relative errors from the two paths are small enough. Therefore, it can be concluded that the choice of (2.113a) and (2.113b) is a good strategy.

**Strategy for \( \rho \approx \rho' \) Scenario**

As explained above, direct field terms are the cause of the non-convergent behavior for \( \rho \approx \rho' \). In this case, the direct field terms, which can be evaluated analytically in a closed-form, can be subtracted from the integrand. In other words,
since the numerical expressions for all three components of electromagnetic fields are

\[
\begin{align*}
\begin{bmatrix} E^\rho \\ H^\rho \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{1}{k^2_{ip}} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} B_n \cdot \mathbf{F}_n(\rho,\rho') \cdot \hat{D}_j \right], \\
\begin{bmatrix} E^\phi \\ H^\phi \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{1}{k^2_{ip}} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} C_n \cdot \mathbf{F}_n(\rho,\rho') \cdot \hat{D}_j \right], \\
\begin{bmatrix} E^z \\ H^z \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} F_n(\rho,\rho') \cdot \hat{D}_j \right].
\end{align*}
\] (2.114)

Then, by subtracting and compensating for the direct field terms, (2.114), (2.115), and (2.116) can be modified to

\[
\begin{align*}
\begin{bmatrix} E^\rho \\ H^\rho \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{1}{k^2_{ip}} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} B_n \right] \\
&\quad \cdot \left\{ \mathbf{F}_n(\rho,\rho') - \mathbf{F}_n(\rho,\rho') \right\} \cdot \hat{D}_j \right] + \begin{bmatrix} E^\rho_o \\ H^\rho_o \end{bmatrix}, \\
\begin{bmatrix} E^\phi \\ H^\phi \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{1}{k^2_{ip}} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} C_n \right] \\
&\quad \cdot \left\{ \mathbf{F}_n(\rho,\rho') - \mathbf{F}_n(\rho,\rho') \right\} \cdot \hat{D}_j \right] + \begin{bmatrix} E^\phi_o \\ H^\phi_o \end{bmatrix}, \\
\begin{bmatrix} E^z \\ H^z \end{bmatrix} &= \frac{iiL}{4\pi\omega\epsilon_j} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \right] \\
&\quad \cdot \left\{ \mathbf{F}_n(\rho,\rho') - \mathbf{F}_n(\rho,\rho') \right\} \cdot \hat{D}_j \right] + \begin{bmatrix} E^z_o \\ H^z_o \end{bmatrix},
\end{align*}
\] (2.117)

(2.118)

(2.119)

where the superscript o in $\mathbf{F}_n^o(\rho,\rho')$ and the last terms in the right hand sides of (2.117), (2.118), and (2.119) indicate the direct field terms. $\mathbf{F}_n^o(\rho,\rho')$ is defined as $\mathbf{F}_n(\rho,\rho')$ with all generalized reflection coefficients being zero. The last terms are analytical expressions of the direct field terms as detailed in Appendix C.

When $\rho \approx \rho'$, for a given maximum order ($n_{\text{max}} = NM$), we compare the magnitudes of the direct field of $n = NM$ with $n = 0$. If the ratio of the two magnitudes are
Figure 2.19: Summary of numerical integration strategies according to field points in the $\rho z$-plane.

less than $10^{-20}$, then a direct field subtraction is not needed, otherwise the subtraction should be performed.

$$\frac{\text{integrand}|_{n=NM}}{\text{integrand}|_{n=0}} < 10^{-20}, \text{ direct field subtraction is not necessary.} \quad (2.120a)$$

$$\frac{\text{integrand}|_{n=NM}}{\text{integrand}|_{n=0}} \geq 10^{-20}, \text{ direct field subtraction is necessary.} \quad (2.120b)$$

Figure 2.19 schematically summarizes the various numerical integration strategies according to a variety of field point locations.
2.5 Numerical Results

This section provides some validation results for the new algorithm, (2.95), (2.101), and (2.105), using the robust integration paths introduced in Section 2.4. Several cases of practical significance in borehole problems are considered. In all the cases, it is implicitly assumed that layers are circularly cylindrical and concentric, and both the relative permittivity $\epsilon_r$ and relative permeability $\mu_r$ are equal to one. The numerical integration results from the new algorithm are compared to the Finite Element Method (FEM) results. All results below were produced using a double-precision Fortran 77 code running on a PC with specifications of 2.6 GHz Opterons, 8 cores each, 32 GB memory, and 225 GB local disk space. In order to obtain accurate results, the Fortran code is designed to automatically increase the number of integration points and azimuth summation terms until the relative error for two successive iterations is below some given threshold. The relative error is defined as

$$\text{relative error} = \frac{|F_i - F_{i-1}|}{|F_i|},$$

where $F_i$ is the requested electric or magnetic field component, and the subscript $i$ indicates the iteration number. In the following, an error threshold of $10^{-4}$ is used for the numerical integration results. Note that field values are expressed in a phasor form with $e^{j\omega t}$ convention. In Section 2.5.1, all results and comparison for the practical cases are provided. In Section 2.5.2, the convergence behavior of the new algorithm is discussed.

2.5.1 Results Validation

For all cases shown below, both the transmitter and receiver are small coil antennas represented as a magnetic dipole with unity magnetic dipole moment.
Figure 2.20: Practical cases in the $\rho z$-plane: (a) Case 1, (b) Case 2, and (c) Case 3.

Case 1 is depicted in Figure 2.20a. The magnetic dipole is $\phi$-directed. It should be noted that since the two layers have the same medium properties, the analytical solution is also available (refer to Appendix C). Table 2.10 compares the three different results. Case 2 is depicted in Figure 2.20b. Now, the magnetic dipole is $z$-directed. Again, the analytical solution is also available for Case 2 (refer to Appendix C). Comparison of three different results is shown in Table 2.11. Case 3 illustrated in Figure 2.20c is similar to Case 1 but the innermost layer has now a larger resistivity of $1000 \, \Omega \cdot m$. Table 2.12 shows the comparison of the results.

Case 4 is depicted in Figure 2.21a, where the innermost layer is changed to a metal having very low resistivity. The transmitter and receiver are $\phi$-directed magnetic dipoles. Case 5 illustrated in Figure 2.21b has the same medium properties as Case 4. Difference is that the transmitter and receiver are $z$-directed magnetic dipoles.
Table 2.10: Comparison of the magnetic fields for Case 1.

<table>
<thead>
<tr>
<th>Magnetic Field [A/m]</th>
<th>Analytical</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.1884</td>
<td>4.0476</td>
<td>4.1884</td>
</tr>
<tr>
<td>∠91.0681°</td>
<td>∠91.1087°</td>
<td>∠91.0681°</td>
<td></td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>4 sec.</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.11: Comparison of the magnetic fields for Case 2.

<table>
<thead>
<tr>
<th>Magnetic Field [A/m]</th>
<th>Analytical</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8.3259</td>
<td>8.2662</td>
<td>8.3259</td>
</tr>
<tr>
<td>∠-91.2105°</td>
<td>∠-91.2178°</td>
<td>∠-91.2105°</td>
<td></td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>4 sec.</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.12: Comparison of the magnetic fields for Case 3.

<table>
<thead>
<tr>
<th>Magnetic Field [A/m]</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.0473</td>
<td>4.1881</td>
</tr>
<tr>
<td>∠91.2619°</td>
<td>∠91.2172°</td>
<td></td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>4 sec.</td>
</tr>
</tbody>
</table>

Case 6 shown in Figure 2.21c has three layers and both the transmitter and receiver are placed in the middle layer. The added formation layer (outermost layer) has resistivity of $5 \, \Omega \cdot m$. Case 7 shown in Figure 2.21d is the same as Case 6 except that the outermost layer represents casing with resistivity of $10^{-5} \, \Omega \cdot m$. Tables 2.13, 2.14, 2.15, and 2.16 show the comparison of the results, respectively.
Results with Casing

In this case, antennas are outside a metallic casing.

Results obtained with the FEM code are below:

FEM
Magnetic Field 12.174515∠100.98107149 A/m
with Coil Antenna Scaling 0.98363989∠-79.0189285° V
Rf = 1 Ωm
Rc = 2.7e-8 Ωm
f = 36 kHz
εr = 1
μr = 1
Transmitter
Receiver

3 Layer Results

In this simulation, a second formation layer with 5 Ω-m resistivity was added. (Note that casing resistivity is changed to 1e-5 Ω-m.)

Results obtained with the FEM code are below:

FEM
Magnetic Field 10.31164∠98.189855 A/m
with Coil Antenna Scaling 0.8331289∠-81.81015° V
Rc = 1e-5 Ωm
f = 36 kHz
εr = 1
μr = 1
Transmitter
Receiver

Results When Antennas are Between 2 Conductive Layers

Second formation layer is replaced with a layer that has the same resistivity with the casing.

Results obtained with the FEM code are below:

FEM
Magnetic Field 46.42578012∠114.24284158 A/m
with Coil Antenna Scaling 3.7509707∠-65.757158° V
Rc = 1e-5 Ωm
Rf = 1 Ωm
f = 36 kHz
εr = 1
μr = 1
Transmitter
Receiver

Figure 2.21: Practical cases in the ρz-plane: (a) Case 4, (b) Case 5, (c) Case 6, and (d) Case 7.
Table 2.13: Comparison of the magnetic fields for Case 4.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>12.1745 100.9810°</td>
<td>12.4300 100.7265°</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>17 sec.</td>
</tr>
</tbody>
</table>

Table 2.14: Comparison of the magnetic fields for Case 5.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>11.2415 -91.0181°</td>
<td>11.3623 -90.9977°</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>4 sec.</td>
</tr>
</tbody>
</table>

Table 2.15: Comparison of the magnetic fields for Case 6.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>10.3116 98.1898°</td>
<td>10.5471 98.1354°</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>31 sec.</td>
</tr>
</tbody>
</table>

Table 2.16: Comparison of the magnetic fields for Case 7.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>46.4257 114.2428°</td>
<td>46.4265 114.2420°</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>30 sec.</td>
</tr>
</tbody>
</table>
In this simulation, the outer conductive layer is reduced to a width of 0.125". A formation layer with 5 Ω-m resistivity fills the rest of the space. Results obtained with the FEM code are below:

FEM Magnetic Field 46.63098758 ∠118.43290636 A/m with Coil Antenna Scaling 3.76755048 ∠-61.56709° V

Rc = 1e-5 Ωm

4 Layer Results

This time frequency was increased to 125 kHz.

Results obtained with the FEM code are below:

FEM Magnetic Field 18.79570781 ∠110.9753189 A/m with Coil Antenna Scaling 18.308728128 ∠-69.02468110° V

Rc = 1e-5 Ωm

4 Layer Results at High Frequency

Figure 2.22: Practical cases in the ρz-plane: (a) Case 8, (b) Case 9, and (c) Case 10.

Case 8 depicted in Figure 2.22a has four layers in order to consider the situation when casing and formation coexist. Table 2.17 provides the corresponding results for comparison. Case 9 depicted in Figure 2.22b is the same as Case 8 except that the operating frequency is decreased to 1 kHz. The comparison of the results is shown in Table 2.18. Case 10 depicted in Figure 2.22c is the same as Case 8 and Case 9 but operating frequency is now 125 kHz. Table 2.19 provides the comparison of the results.

Case 11 depicted in Figure 2.23a has the same as Case 8 but with a z-directed transmitter and ρ-directed receiver. Table 2.20 provides the comparative results. Case 12 is illustrated in Figure 2.23b. There are three layers and the borehole (second layer) radius is extended to 16". Table 2.21 shows the comparison of the results. Case 13 is similar to Case 8 except for the longitudinal distance between the transmitter and
Table 2.17: Comparison of the magnetic fields for Case 8.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>46.6309</td>
<td>46.6307</td>
</tr>
<tr>
<td></td>
<td>$\angle 118.4329^\circ$</td>
<td>$\angle 118.4320^\circ$</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>41 sec.</td>
</tr>
</tbody>
</table>

Table 2.18: Comparison of the magnetic fields for Case 9.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>290.2144</td>
<td>290.2652</td>
</tr>
<tr>
<td></td>
<td>$\angle 127.4332^\circ$</td>
<td>$\angle 127.4151^\circ$</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>9 sec.</td>
</tr>
</tbody>
</table>

Table 2.19: Comparison of the magnetic fields for Case 10.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>18.7957</td>
<td>18.8069</td>
</tr>
<tr>
<td></td>
<td>$\angle 110.9753^\circ$</td>
<td>$\angle 110.9191^\circ$</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>36 sec.</td>
</tr>
</tbody>
</table>

receiver, which is changed to 4”. The comparison of the results is shown in Table 2.22.
Case 14 depicted in Figure 2.23d is similar to Case 13. The longitudinal distance is now increased to 64”. Table 2.23 shows the comparison.
4 Layer Results for Cross Components

- In this simulation, transmitter was z-directed while the receiver was ρ-directed.
- Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>FEM</th>
<th>Magnetic Field</th>
<th>1.2589421518 A/m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with Coil Antenna Scaling</td>
<td>0.1017162267 V</td>
</tr>
<tr>
<td>Rc</td>
<td>= 1e-5 Ωm</td>
<td></td>
</tr>
</tbody>
</table>

3 Layer with Large Borehole Results

- In this simulation, borehole was extended to 16".
- Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>FEM</th>
<th>Magnetic Field</th>
<th>10.78550306 A/m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with Coil Antenna Scaling</td>
<td>-0.8714146818 V</td>
</tr>
<tr>
<td>Rc</td>
<td>= 1e-5 Ωm</td>
<td></td>
</tr>
</tbody>
</table>

4 Layer Results with a Short Tool

- In this case, transmitter-receiver spacing is 4 inches.
- Results obtained below:

<table>
<thead>
<tr>
<th>FEM</th>
<th>Magnetic Field</th>
<th>1007.5854699 A/m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with Coil Antenna Scaling</td>
<td>-81.407864481 V</td>
</tr>
<tr>
<td>Rc</td>
<td>= 1e-5 Ωm</td>
<td></td>
</tr>
</tbody>
</table>

4 Layer Results with a Longer Tool

- In this simulation, transmitter-receiver spacing is increased to 64 inches.
- Results obtained below:

<table>
<thead>
<tr>
<th>FEM</th>
<th>Magnetic Field</th>
<th>12.12710397 A/m</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with Coil Antenna Scaling</td>
<td>-0.9798093226 V</td>
</tr>
<tr>
<td>Rc</td>
<td>= 1e-5 Ωm</td>
<td></td>
</tr>
</tbody>
</table>

Figure 2.23: Practical cases in the ρz-plane: (a) Case 11, (b) Case 12, (c) Case 13, and (d) Case 14.
Table 2.20: Comparison of the magnetic fields for Case 11.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>1.2589 (\angle 67.6502^\circ)</td>
<td>1.2589 (\angle 67.5950^\circ)</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>41 sec.</td>
</tr>
</tbody>
</table>

Table 2.21: Comparison of the magnetic fields for Case 12.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>10.7855 (\angle 100.0572^\circ)</td>
<td>10.7856 (\angle 100.0573^\circ)</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>29 sec.</td>
</tr>
</tbody>
</table>

Table 2.22: Comparison of the magnetic fields for Case 13.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>1007.5854 (\angle 107.9553^\circ)</td>
<td>1045.5344 (\angle 107.8155^\circ)</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>20 sec.</td>
</tr>
</tbody>
</table>

Table 2.23: Comparison of the magnetic fields for Case 14.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field [A/m]</td>
<td>12.1271 (\angle 111.9689^\circ)</td>
<td>12.1266 (\angle 111.9684^\circ)</td>
</tr>
<tr>
<td>Computing Time</td>
<td>N.A.</td>
<td>40 sec.</td>
</tr>
</tbody>
</table>
In this case, tool was moved to the third layer at a radial distance of 6" from the center of the borehole.

Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Magnetic Field</td>
<td>8.1258 (\angle 97.0379)°</td>
<td>8.1326 (\angle 97.0322)°</td>
</tr>
<tr>
<td>with Coil Antenna Scaling</td>
<td>0.65652839 (\angle -82.9620)°</td>
<td></td>
</tr>
</tbody>
</table>

Finally, in Case 15 both the transmitter and receiver are moved to the third layer, as shown in Figure 2.24. Table 2.24 shows the comparison of the results. All fifteen cases above show good agreement between the FEM results and new algorithm results.

Among the fifteen cases, field distributions of Case 6 and Case 8 are compared. Magnitude of magnetic fields, not its \(\phi\)-component, is plotted in the following figures.
Figure 2.25: Magnitude of the magnetic fields around the magnetic dipole in 3-D view at 36 kHz: (a) Case 6 and (b) Case 8.

Due to a large variation of the magnitude, it is illustrated using a log-scale; i.e., $10 \log_{10} |\mathbf{H}|$. In all figures below, thicker black lines represent interfaces between cylindrical layers and thinner black lines represent contours of the plot.

Figure 2.25 shows the comparison the magnetic fields in 3-D view for Case 6 and Case 8. It should be noted that the third layer of Case 8 is too thin to identify. For Case 8, the magnetic field tends to be trapped in the source layer (second layer) due to the outer conductive layer (third layer). Figures 2.26, 2.27, and 2.28 show the magnetic fields on $z = 16''$, $z = 0''$, and $y = 0''$ cut-planes, respectively.

The field distributions of Case 6 and Case 8 are also compared at 1 kHz operating frequency. Note that Case 8 at 1 kHz is Case 9. Figure 2.29 shows the comparison of the magnetic fields in 3-D view. Figures 2.30, 2.31, and 2.32 show the magnetic fields on $z = 16''$, $z = 0''$, and $y = 0''$ cut-planes, respectively.
Figure 2.26: Magnitude of the magnetic fields at $z = 16''$ plane around the magnetic dipole at 36 kHz: (a) Case 6 and (b) Case 8.

Figure 2.27: Magnitude of the magnetic fields at $z = 0''$ plane around the magnetic dipole at 36 kHz: (a) Case 6 and (b) Case 8.

Furthermore, the magnetic field distributions of Case 6 and Case 8 are compared at 125 kHz operating frequency. Note that Case 8 at 125 kHz is Case 10. Figure
Figure 2.28: Magnitude of the magnetic fields at $y = 0^\prime\prime$ plane around the magnetic dipole at 36 kHz: (a) Case 6 and (b) Case 8.

Figure 2.29: Magnitude of the magnetic fields around the magnetic dipole in 3-D view at 1 kHz: (a) Case 6 and (b) Case 8.

2.33 shows the comparison of the magnetic fields in 3-D view. The magnetic fields
Figure 2.30: Magnitude of the magnetic fields at $z = 16''$ plane around the magnetic dipole at 1 kHz: (a) Case 6 and (b) Case 8.

Figure 2.31: Magnitude of the magnetic fields at $z = 0''$ plane around the magnetic dipole at 1 kHz: (a) Case 6 and (b) Case 8.

on $z = 16''$, $z = 0''$, and $y = 0''$ cut-planes are shown in Figures 2.34, 2.35, and 2.36, respectively.
Figure 2.32: Magnitude of the magnetic fields at $y = 0''$ plane around the magnetic dipole at 1 kHz: (a) Case 6 and (b) Case 8.

Figure 2.33: Magnitude of the magnetic fields around the magnetic dipole in 3-D view at 125 kHz: (a) Case 6 and (b) Case 8.
Figure 2.34: Magnitude of the magnetic fields at $z = 16''$ plane around the magnetic dipole at 125 kHz: (a) Case 6 and (b) Case 8.

Figure 2.35: Magnitude of the magnetic fields at $z = 0''$ plane around the magnetic dipole at 125 kHz: (a) Case 6 and (b) Case 8.
Figure 2.36: Magnitude of the magnetic fields at $y = 0''$ plane around the magnetic dipole at 125 kHz: (a) Case 6 and (b) Case 8.

2.5.2 Convergence Behavior

In Figures 2.37 – 2.39, the convergence behavior of the magnetic field for the practical cases is presented as the number of integration points $n_{\text{int}}$ or maximum order $n_{\text{max}}$ vary. Since analytical solutions exist for Case 1 and Case 2, they are not considered here. The number of integration points for the first iteration in Figure 2.37a, Figure 2.38a, and Figure 2.39a is 4,400. From the second iteration on, the number of integration points is doubled for each successive iteration. After the iterations for integration points are done, the determination of an optimal maximum order for a given error threshold is carried out. The maximum order is set to 5 in the first iteration and increased by 5 from the second iteration on, as shown in Figure 2.37b, Figure 2.38b, and Figure 2.39b. The horizontal dashed lines in these figures represent
the given error threshold of $10^{-4}$. Note that relative errors are available from the second iteration.

As shown below, some cases show smaller relative errors than the error threshold in the second iteration, so that very quick convergence is obtained. The other cases need only four iterations at most before the magnetic field values converge under this chosen threshold. This fast convergence behavior is reflected in fast computing times obtained in Section 2.5.1.
Figure 2.38: Convergence behavior of the magnetic fields from Case 8 to Case 11 in terms of (a) integration points $n_{\text{int}}$ and (b) maximum orders $n_{\text{max}}$.

Figure 2.39: Convergence behavior of the magnetic fields from Case 12 to Case 15 in terms of (a) integration points $n_{\text{int}}$ and (b) maximum orders $n_{\text{max}}$. 
Chapter 3

Electromagnetic Fields from Dipole Sources in Cylindrically Stratified and Anisotropic Media

This chapter discusses the computation of electromagnetic fields due to arbitrarily-oriented Hertzian dipoles in cylindrically stratified and anisotropic media. Similar discussion can be found in [50, 71]. This chapter can be considered as an extension of the work in Chapter 2. Among some relevant applications, the community of geological exploration has paid much attention to this type of problems because earth formations often feature anisotropy resulting from geological structures of grains (microscopic anisotropy) and/or the sequences of thinly laminated sand and shale (macroscopic anisotropy). Since the effect of anisotropy is significant for highly deviated and even horizontal drilling for geophysical and petrophysical prospecting, the effect of formation anisotropy on electromagnetic responses has been one of top subjects in this application.

Anisotropy has various forms. As the simplest form, uniaxial anisotropy has been often assumed in many analytical studies because earth formations with practical interest are mostly composed of sedimentary rocks. Kuns and Moran [72] examined a variety of cases with two half-space anisotropic media. Moran and Gianzero [73] extended these cases by considering dipping beds with various dip angles. Kong [68]
discussed the radiation from dipoles when both permittivity and permeability have
uniaxial anisotropy. Bittar and Rodney [74] developed analytical modeling without
the effect of borehole and invasion zones to investigate the effects of multiple-frequency
and multiple-spacing in deviated wells. Anderson et al. [75] developed the numerical
modeling of responses of induction logging tools in crossbedded layers with electrical
anisotropy through a triple Fourier integral. Howard, Jr. [14] demonstrated that for-
mation anisotropy affect resistivity log data to various degrees. Wang [76] introduced
the technique of weak-anisotropy approximation for up to anisotropy ratio of 2 for
coplanar components for crossbedding formations. Zhong et al. [9] developed numer-
ical simulations based on the analytical approach of Hankel transform integrals and
the coefficient propagator method without borehole effect and invasion zones. Lüling
[77] stressed the paradox of formation anisotropy with simple examples. Zhang et al.
[8] and Hagiwara [78, 79] focused on determination of formation properties such as
anisotropy ratios and relative dip to bedding interfaces from measured logging data
assuming uniaxial anisotropy. Biaxial anisotropy was also assumed in some litera-
tures [80, 81], where the analytical derivations of the effect of biaxial anisotropy on
the response of induction tools are discussed but the extension to multi-layered media
is not available. Furthermore, arbitrary anisotropy represented by a full $3 \times 3$ tensor
has been taken into account. Teitler and Henvis [82] and Morgan et al. [83] derived
electromagnetic scattering by stratified, arbitrary anisotropic media using numerical
solution of the state-vector equation. Yin and Maurer [84] computed electromagnetic
fields from two scalar potentials and Fourier transform. Sainath and Teixeira
[85, 86, 87, 88] developed efficient algorithms for the evaluation of two-dimensional
Fourier integrals. These analytical efforts rely on various forms of inverse Fourier-Hankel transformations, equivalently Green’s functions, under the assumption of planarly stratified media in order to interpret changes in electromagnetic responses while logging tools move over many formation layers with different characteristics.

Apart from planarly stratified media, there have been numerous efforts to compute and analyze electromagnetic responses through modeling the surrounding medium as a number of cylindrical layers because such a medium can easily represent the borehole and invaded zones with negligible errors. Advances in the analytical expressions of electromagnetic fields in cylindrically stratified and anisotropic media were made against the complexity of derivations and involvement of intractable cylindrical eigenfunctions. To name a few, Hue and Teixeira [57] presented the exposition of reflection and transmission coefficients among multiple cylindrical layers, which is then combined with the numerical mode matching for full 3-D data. Wang et al. [58] and Liu et al. [59] relied on the spectral representation of cylindrical eigenfunctions but they both assumed that permeability is isotropic.

In this chapter, further advances in the analytical expression of electromagnetic fields from arbitrarily oriented Hertzian dipoles in cylindrically stratified and anisotropic media are provided. It is assumed that each layer is doubly uniaxial; i.e., both complex permittivity $\bar{\varepsilon}$ and permeability $\bar{\mu}$ are independently uniaxial. Note that the assumption of uniaxial permeability allows to use the concept of electromagnetic duality when magnetic sources such as small coil antennas are employed. After that, stable numerical computation of the analytical expressions is focused using the concept of the range-conditioning, the judicious manipulation of canonical cylindrical
eigenfunctions. Two distinct integration paths introduced in Section 2.4 are again employed to ensure fast convergence in any scenarios.

This chapter is organized as follows. In Section 3.1, the analytical formulations are presented. In Section 3.2, the modification of the analytical formulations for stable numerical computation is discussed. To do so, the range-conditioned cylindrical functions are extended to accommodate uniaxial properties of cylindrically stratified media. Several numerical examples will be provided and compared to the Finite Element Method (FEM) data for validation in Section 3.3.

3.1 Analytical Formulations

3.1.1 Vector Wave Equation

For uniaxial, homogeneous, and source-free media in a phasor form with time-harmonic dependence of $e^{-i\omega t}$, two Maxwell’s curl equations are expressed as

$$\nabla \times \mathbf{E} = i\omega \overline{\mu} \mathbf{H}, \quad (3.1)$$

$$\nabla \times \mathbf{H} = -i\omega \overline{\varepsilon} \mathbf{E} + \mathbf{J}, \quad (3.2)$$

where $\mathbf{J}$ does not represent a causative impressed source but rather an induced conduction current. With the use of permittivity tensor $\overline{\varepsilon}$ and conductivity tensor $\overline{\sigma}$, (3.2) is rearranged to

$$\nabla \times \mathbf{H} = -i\omega \overline{\varepsilon} \mathbf{E} + \mathbf{J} = -i\omega \overline{\varepsilon} \mathbf{E} + \overline{\sigma} \mathbf{E} = -i\omega \left( \overline{\varepsilon} - \frac{\overline{\sigma}}{i\omega} \right) \mathbf{E} = -i\omega \overline{\varepsilon}_e \mathbf{E}, \quad (3.3)$$

where $\overline{\varepsilon}_e$ represents the complex permittivity. When uniaxial anisotropy is assumed, $\overline{\varepsilon}_e$ is written as

$$\overline{\varepsilon}_e = \begin{bmatrix} \epsilon_h + i\sigma_h/\omega & 0 & 0 \\ 0 & \epsilon_h + i\sigma_h/\omega & 0 \\ 0 & 0 & \epsilon_v + i\sigma_v/\omega \end{bmatrix} = \begin{bmatrix} \epsilon_{e,h} & 0 & 0 \\ 0 & \epsilon_{e,h} & 0 \\ 0 & 0 & \epsilon_{e,v} \end{bmatrix}, \quad (3.4)$$
where $\epsilon_h$ and $\epsilon_v$ are horizontal and vertical permittivity, and $\sigma_h$ and $\sigma_v$ are horizontal and vertical conductivity. Hereinafter, the subscript $e$ is omitted for simplifying the notation, so all permittivities below are assumed in general to be complex-valued.

Also, permeability tensor $\bar{\mu}$ is written as

$$
\bar{\mu} = \begin{bmatrix}
\mu_h & 0 & 0 \\
0 & \mu_h & 0 \\
0 & 0 & \mu_v
\end{bmatrix},
$$

(3.5)

where $\mu_h$ and $\mu_v$ are horizontal and vertical permeability, respectively.

The two other Maxwell’s equations can be written by

$$
\nabla \cdot \bar{\epsilon} E = 0, \quad (3.6)
$$

$$
\nabla \cdot \bar{\mu} H = 0. \quad (3.7)
$$

Note that in general $\nabla \cdot E$ and $\nabla \cdot H$ in uniaxial and source-free media are nonzero.

Indeed, the left hand side of (3.6) in cylindrical coordinates is written as

$$
\nabla \cdot \bar{\epsilon} E = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \epsilon_h E_{\rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left( \epsilon_h E_{\phi} \right) + \frac{\partial}{\partial z} \left( \epsilon_v E_z \right)
$$

$$
= \epsilon_h \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho E_{\rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} E_{\phi} + \frac{\partial}{\partial z} \frac{\partial E_z}{\partial z} - \left( 1 - \frac{\epsilon_v}{\epsilon_h} \right) \frac{\partial E_z}{\partial z} \right\}
$$

$$
= \epsilon_h \left\{ \nabla \cdot E - \left( 1 - \frac{\epsilon_v}{\epsilon_h} \right) \frac{\partial E_z}{\partial z} \right\}. \quad (3.8)
$$

From (3.6) and (3.8), we have

$$
\nabla \cdot E = \left( 1 - \frac{\epsilon_v}{\epsilon_h} \right) \frac{\partial E_z}{\partial z}. \quad (3.9)
$$

Similarly, we can obtain

$$
\nabla \cdot H = \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial H_z}{\partial z}. \quad (3.10)
$$
To obtain the vector wave equation for $E$, taking the curl of (3.1) gives

$$\nabla \times \nabla \times E = i\omega \nabla \times \bar{p}H,$$

$$\nabla \left( \nabla \cdot E \right) - \nabla^2 E = i\omega \left[ \left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\mu_h \mathbf{H}_s + \mu_v \mathbf{H}_z) \right],$$

$$\nabla^2 E - \left( 1 - \frac{\varepsilon_v}{\varepsilon_h} \right) \nabla \frac{\partial E_z}{\partial z} = -i\omega \left[ \left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\mu_h \mathbf{H}_s + \mu_v \mathbf{H}_z) \right], \quad (3.11)$$

where $\nabla = \nabla_s + \hat{z} \frac{\partial}{\partial z}$ is used and subscript $s$ indicates the transverse components to the $z$-component. When the $z$-component is extracted from (3.11), the equation for $E_z$ is

$$\nabla^2 E_z - \left( 1 - \frac{\varepsilon_v}{\varepsilon_h} \right) \frac{\partial^2 E_z}{\partial z^2} + i\omega \left( \nabla_s \times \mu_h \mathbf{H}_s \right) = 0,$$

$$\nabla^2 E_z - \left( 1 - \frac{\varepsilon_v}{\varepsilon_h} \right) \frac{\partial^2 E_z}{\partial z^2} + i\omega \mu_h \left( -i\omega \varepsilon_v E_z \right) = 0,$$

$$\nabla^2 E_z - \left( 1 - \frac{\varepsilon_v}{\varepsilon_h} \right) \frac{\partial^2 E_z}{\partial z^2} + \omega^2 \mu_h \varepsilon_v E_z = 0. \quad (3.12)$$

The vector wave equation for $H$ can be obtained by taking the curl of (3.2) such that

$$\nabla \times \nabla \times H = -i\omega \nabla \times \bar{p}E,$$

$$\nabla \left( \nabla \cdot H \right) - \nabla^2 H = -i\omega \left[ \left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\varepsilon_h \mathbf{E}_s + \varepsilon_v \mathbf{E}_z) \right],$$

$$\nabla^2 H - \left( 1 - \frac{\mu_v}{\mu_h} \right) \nabla \frac{\partial H_z}{\partial z} = i\omega \left[ \left( \nabla_s + \hat{z} \frac{\partial}{\partial z} \right) \times (\varepsilon_h \mathbf{E}_s + \varepsilon_v \mathbf{E}_z) \right]. \quad (3.13)$$

When the $z$-component is extracted from (3.13), the equation for $H_z$ is

$$\nabla^2 H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} - i\omega \left( \nabla_s \times \varepsilon_h \mathbf{E}_s \right) = 0,$$

$$\nabla^2 H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} - i\omega \mu_h \left( i\omega \mu_v H_z \right) = 0,$$

$$\nabla^2 H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} + \omega^2 \mu_v \varepsilon_h H_z = 0. \quad (3.14)$$
To solve (3.12), we apply the separation of variables technique and assume $E_z = R_e(\rho)\Phi(\phi)Z(z)$ so that (3.12) is modified to

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_e}{d\rho} \right) \Phi Z + \frac{1}{\rho^2} \frac{d^2\Phi}{d\phi^2} R_e Z + \frac{\epsilon_v d^2Z}{\epsilon_h dz^2} R_e \Phi + \omega^2 \mu_h \epsilon_v R_e \Phi Z = 0. \tag{3.15}
$$

Dividing the both sides of (3.15) by $R_e \Phi Z$ yields

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_e}{d\rho} \right) \Phi + \frac{1}{\rho^2} \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} + \frac{\epsilon_v}{\epsilon_h} \frac{d^2Z}{dz^2} + \omega^2 \mu_h \epsilon_v = 0. \tag{3.16}
$$

Equation (3.16) can be decomposed into three independent equations:

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_e}{d\rho} \right) + \left[ \frac{\epsilon_v}{\epsilon_h} (\omega^2 \mu_h \epsilon_h - k_z^2) - \frac{n^2}{\rho^2} \right] R_e = 0, \tag{3.17a}
$$

$$
\frac{d^2\Phi}{d\phi^2} + n^2 \Phi = 0, \tag{3.17b}
$$

$$
\frac{d^2Z}{dz^2} + k_z^2 Z = 0. \tag{3.17c}
$$

Similarly, to solve (3.14), it is assumed that $H_z = R_h(\rho)\Phi(\phi)Z(z)$ so that (3.14) becomes

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_h}{d\rho} \right) \Phi Z + \frac{1}{\rho^2} \frac{d^2\Phi}{d\phi^2} R_h Z + \frac{\mu_v d^2Z}{\mu_h dz^2} R_h \Phi + \omega^2 \mu_v \epsilon_h R_h \Phi Z = 0. \tag{3.18}
$$

Again, dividing the both sides of (3.18) by $R_h \Phi Z$ yields

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_h}{d\rho} \right) \Phi + \frac{1}{\rho^2} \frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} + \frac{\mu_v}{\mu_h} \frac{d^2Z}{dz^2} + \omega^2 \mu_v \epsilon_h = 0. \tag{3.19}
$$

Equation (3.19) can be decomposed into three independent equations:

$$
\frac{1}{\rho} \frac{d}{d\rho} \left( \frac{dR_h}{d\rho} \right) + \left[ \frac{\mu_v}{\mu_h} (\omega^2 \mu_h \epsilon_h - k_z^2) - \frac{n^2}{\rho^2} \right] R_h = 0, \tag{3.20a}
$$

$$
\frac{d^2\Phi}{d\phi^2} + n^2 \Phi = 0, \tag{3.20b}
$$

$$
\frac{d^2Z}{dz^2} + k_z^2 Z = 0. \tag{3.20c}
$$
As usual, we define the propagation constant as

$$k = \omega \sqrt{\mu_h \epsilon_h}. \quad (3.21)$$

The dispersion relation becomes

$$\omega^2 \mu_h \epsilon_h - k_z^2 = k_\rho^2. \quad (3.22)$$

There are two different anisotropic ratios of the medium: (i) anisotropy ratio of the complex permittivity

$$\kappa_\epsilon = \sqrt{\frac{\epsilon_h}{\epsilon_v}}, \quad (3.23)$$

and (ii) anisotropy ratio of permeability

$$\kappa_\mu = \sqrt{\frac{\mu_h}{\mu_v}}. \quad (3.24)$$

Using (3.23) and (3.24), two stretched radial wavenumbers are defined such that

$$\tilde{k}_\rho = \frac{k_\rho}{\kappa_\epsilon}, \quad (3.25a)$$

$$\ddot{k}_\rho = \frac{k_\rho}{\kappa_\mu}. \quad (3.25b)$$

With the above definition, the general solutions can be written as

$$E_z = \left[ A_n J_n \left( \tilde{k}_\rho \rho \right) + B_n H_n^{(1)} \left( \tilde{k}_\rho \rho \right) \right] e^{i n \phi} e^{i k_z z}, \quad (3.26a)$$

$$H_z = \left[ C_n J_n \left( \ddot{k}_\rho \rho \right) + D_n H_n^{(1)} \left( \ddot{k}_\rho \rho \right) \right] e^{i n \phi} e^{i k_z z}, \quad (3.26b)$$

where $A_n$, $B_n$, $C_n$, and $D_n$ are coefficients determined by boundary conditions. The dispersion relations for $\tilde{k}_\rho$ and $\ddot{k}_\rho$ are

$$\frac{\epsilon_v}{\epsilon_h} \left( \omega^2 \mu_h \epsilon_h - k_z^2 \right) = \tilde{k}_\rho^2, \quad (3.27a)$$

$$\frac{\mu_v}{\mu_h} \left( \omega^2 \mu_h \epsilon_h - k_z^2 \right) = \ddot{k}_\rho^2. \quad (3.27b)$$
The transverse $\rho$- and $\phi$-components can be derived from the $z$-components. Let us begin with two Maxwell’s curl equations (3.1) and (3.2). By extracting the transverse components, we obtain

\[
\nabla_s \times \hat{z}E_s + \frac{\partial}{\partial z} \hat{z} \times E_s = i\omega\mu_h H_s,
\]

\[
\nabla_s \times \hat{z}H_s + \frac{\partial}{\partial z} \hat{z} \times H_s = -i\omega\epsilon_h E_s.
\]

(3.28)

After some algebra [17], we obtain

\[
\nabla_s \frac{\partial E_z}{\partial z} + k_2^2 E_s = -i\omega\mu_h \nabla_s \times \hat{z}H_s + \omega^2 \mu_h \epsilon_h E_s.
\]

(3.29)

(3.30)

Rearranging (3.30) for $E_s$ gives

\[
E_s = \frac{1}{\omega^2 \mu_h \epsilon_h - k_2^2} \left[ \nabla_s \frac{\partial E_z}{\partial z} - i\omega\mu_h \hat{z} \times \nabla_s H_s \right] = \frac{1}{k_\rho^2} \left[ ik_z \nabla_s E_z - i\omega\mu_h \hat{z} \times \nabla_s H_s \right].
\]

(3.31)

Similarly, we can obtain

\[
H_s = \frac{1}{\omega^2 \mu_h \epsilon_h - k_2^2} \left[ \nabla_s \frac{\partial H_z}{\partial z} + i\omega\epsilon_h \hat{z} \times \nabla_s E_z \right] = \frac{1}{k_\rho^2} \left[ ik_z \nabla_s H_z + i\omega\epsilon_h \hat{z} \times \nabla_s E_z \right].
\]

(3.32)

Matrix representations of the transverse components are

\[
\begin{bmatrix}
E_\rho \\
H_\rho
\end{bmatrix} = \frac{1}{k^2} \begin{bmatrix}
\frac{ik_z}{\rho} & -\frac{i\omega\mu_h}{\rho} \\
\frac{i\omega\epsilon_h}{\rho} & \frac{ik_z}{\rho}
\end{bmatrix}
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{1}{k^2} \mathbf{B}_{hn} \begin{bmatrix}
E_z \\
H_z
\end{bmatrix},
\]

(3.33)

\[
\begin{bmatrix}
E_\phi \\
H_\phi
\end{bmatrix} = \frac{1}{k^2} \begin{bmatrix}
-\frac{nk_z}{\rho} & -\frac{i\omega\mu_h}{\rho} \\
\frac{i\omega\epsilon_h}{\rho} & -\frac{nk_z}{\rho}
\end{bmatrix}
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{1}{k^2} \mathbf{C}_{hn} \begin{bmatrix}
E_z \\
H_z
\end{bmatrix}.
\]

(3.34)

It should be noted that $\mathbf{B}_{hn}$ and $\mathbf{C}_{hn}$ are slightly different from $\mathbf{B}_n$ and $\mathbf{C}_n$ in (2.16) and (2.17) as $\mathbf{B}_{hn}$ and $\mathbf{C}_{hn}$ only depend on horizontal medium properties $\epsilon_h$ and $\mu_h$.

3.1.2 Reflection and Transmission Coefficients

As the general solution of $E_z$ and $H_z$ for uniaxial media are slightly different from those for isotropic media, local reflection and transmission coefficients are different
Outgoing-Wave Case

Based on (3.26a) and (3.26b), the outgoing waves in a uniaxial medium can be expressed as

\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \begin{bmatrix}
H_n^{(1)}(\tilde{k}_\rho \rho) & 0 \\
0 & H_n^{(1)}(\tilde{k}_\rho \rho)
\end{bmatrix} \begin{bmatrix}
e_z \\
h_z
\end{bmatrix} = \mathbf{P}^{(1)}_{zn}(k_\rho \rho) \cdot \mathbf{a},
\]

where the column vector \( \mathbf{a} \) includes \( e^{in\phi+ik_z z} \) dependence.

When the medium consists of only two cylindrical layers and the source is embedded in layer 1, reflection and transmission exist at the interface \( a_1 \), which is illustrated in Figure 3.1. Therefore, the total \( z \)-components of the fields in layer 1 and layer 2
are expressed as
\[
\begin{bmatrix}
E_{z1} \\
H_{z1}
\end{bmatrix} = \mathbf{H}^{(1)}(k_1 \rho) \cdot \mathbf{a}_1 + \mathbf{J}_{zn}(k_1 \rho) \cdot \mathbf{R}_{12} \cdot \mathbf{a}_1, \quad (3.36a)
\]
\[
\begin{bmatrix}
E_{z2} \\
H_{z2}
\end{bmatrix} = \mathbf{H}^{(1)}(k_2 \rho) \cdot \mathbf{T}_{12} \cdot \mathbf{a}_1. \quad (3.36b)
\]

Using (3.34), the \(\phi\)-components in layer 1 and layer 2 are expressed as
\[
\begin{bmatrix}
H_{\phi1} \\
E_{\phi1}
\end{bmatrix} = \mathbf{H}^{(1)}(k_1 \rho) \cdot \mathbf{a}_1 + \mathbf{J}_{\phi n}(k_1 \rho) \cdot \mathbf{R}_{12} \cdot \mathbf{a}_1, \quad (3.37a)
\]
\[
\begin{bmatrix}
H_{\phi2} \\
E_{\phi2}
\end{bmatrix} = \mathbf{H}^{(1)}(k_2 \rho) \cdot \mathbf{T}_{12} \cdot \mathbf{a}_1. \quad (3.37b)
\]

Two types of matrices only depending on \(\rho\) rather than \(\phi\) and \(z\) are defined such that
\[
\mathbf{B}_{zn}(k_i \rho) = \begin{bmatrix}
B_n(\tilde{k}_i \rho) & 0 \\
0 & B_n(\tilde{k}_i \rho)
\end{bmatrix}, \quad (3.38a)
\]
\[
\mathbf{B}_{\phi n}(k_i \rho) = \frac{1}{k_i^2 \rho} \begin{bmatrix}
i \omega \epsilon_{hi} \tilde{k}_i \rho B'_n(\tilde{k}_i \rho) & -n k_z B_n(\tilde{k}_i \rho) \\
-n k_z B_n(\tilde{k}_i \rho) & -i \omega \mu_{hi} \tilde{k}_i \rho B'_n(\tilde{k}_i \rho)
\end{bmatrix}, \quad (3.38b)
\]

where \(B_n\) is either \(H_n^{(1)}\) or \(J_n\), \(k_i \rho = \omega^2 \mu_{hi} \epsilon_{hi} - k_z^2\), and \(\epsilon_{hi}\) and \(\mu_{hi}\) are the horizontal permittivity and permeability in layer \(i\), respectively. Applying the boundary conditions, viz. continuity of \(z\)- and \(\phi\)-components at \(\rho = a_1\), to (3.36a)–(3.37b) yields
\[
\begin{bmatrix}
\mathbf{H}^{(1)}(k_1 \rho a_1) + \mathbf{J}_{zn}(k_1 \rho a_1) \cdot \mathbf{R}_{12}
\end{bmatrix} \cdot \mathbf{a}_1 = \mathbf{H}^{(1)}(k_2 \rho a_1) \cdot \mathbf{T}_{12} \cdot \mathbf{a}_1, \quad (3.39a)
\]
\[
\begin{bmatrix}
\mathbf{H}^{(1)}(k_1 \rho a_1) + \mathbf{J}_{\phi n}(k_1 \rho a_1) \cdot \mathbf{R}_{12}
\end{bmatrix} \cdot \mathbf{a}_1 = \mathbf{H}^{(1)}(k_2 \rho a_1) \cdot \mathbf{T}_{12} \cdot \mathbf{a}_1. \quad (3.39b)
\]

For notational simplicity, a shorthand notation is defined such that
\[
\mathbf{B}_{\alpha n}(k_i \rho a_j) = \mathbf{B}_{\alpha ij}. \quad (3.40)
\]
In the right hand side of (3.40), the first, second, and third subscripts indicate the relevant components of the fields, radial wavenumbers, and radial distances, respectively. Also, the kind of the Hankel function (1) and modal number \( n \) are suppressed in the followings. Using the shorthand notation (3.40), several matrices can be expressed in a simple way:

\[
\begin{align*}
\mathbf{H}_{zn}^{(1)}(k_1 \rho_1 a_1) &= \mathbf{H}_{z11}, \\
\mathbf{H}_{zn}^{(1)}(k_2 \rho_1 a_1) &= \mathbf{H}_{z21}, \\
\mathbf{J}_{zn}(k_1 \rho_1 a_1) &= \mathbf{J}_{z11}, \\
\mathbf{H}_{\phi n}^{(1)}(k_1 \rho_1 a_1) &= \mathbf{H}_{\phi11}, \\
\mathbf{H}_{\phi n}^{(1)}(k_2 \rho_1 a_1) &= \mathbf{H}_{\phi21}, \\
\mathbf{J}_{\phi n}(k_1 \rho_1 a_1) &= \mathbf{J}_{\phi11}.
\end{align*}
\]

(3.41) a-f

To obtain \( \mathbf{R}_{12} \), \( a_1 \) in the both sides are cancelled and (3.39a) and (3.39b) are rearranged as

\[
\begin{align*}
\mathbf{H}_{z21}^{-1} \cdot [\mathbf{H}_{z11} + \mathbf{J}_{z11} \cdot \mathbf{R}_{12}] &= \mathbf{T}_{12}, \\
\mathbf{H}_{\phi21}^{-1} \cdot [\mathbf{H}_{\phi11} + \mathbf{J}_{\phi11} \cdot \mathbf{R}_{12}] &= \mathbf{T}_{12}.
\end{align*}
\]

(3.42) a-b

From (3.42a) and (3.42b),

\[
\mathbf{H}_{z21}^{-1} \cdot \mathbf{H}_{z11} + \mathbf{H}_{z21}^{-1} \cdot \mathbf{J}_{z11} \cdot \mathbf{R}_{12} = \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} + \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \cdot \mathbf{R}_{12},
\]

(3.43)

which is rearranged as

\[
\left[ \mathbf{H}_{z21}^{-1} \cdot \mathbf{J}_{z11} - \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right] \cdot \mathbf{R}_{12} = \left[ \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z21}^{-1} \cdot \mathbf{H}_{z11} \right].
\]

(3.44)

Multiplying the both sides of (3.44) by \( \mathbf{H}_{z21} \) gives

\[
\left[ \mathbf{J}_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right] \cdot \mathbf{R}_{12} = \left[ \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \right].
\]

(3.45)
Therefore,
\[
\mathbf{R}_{12} = \left[ \mathbf{J}_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right]^{-1} \cdot \left[ \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \right].
\]  \hspace{1cm} (3.46)

To obtain \( \mathbf{T}_{12} \), \( a_1 \) in the both sides are cancelled again and (3.39a) and (3.39b) are rearranged as
\[
\mathbf{R}_{12} = \mathbf{J}_{z11}^{-1} \cdot \left[ \mathbf{H}_{z21} \cdot \mathbf{T}_{12} - \mathbf{H}_{z11} \right],
\]  \hspace{1cm} (3.47a)
\[
\mathbf{R}_{12} = \mathbf{J}_{\phi11}^{-1} \cdot \left[ \mathbf{H}_{\phi21} \cdot \mathbf{T}_{12} - \mathbf{H}_{\phi11} \right].
\]  \hspace{1cm} (3.47b)

From (3.47a) and (3.47b),
\[
\mathbf{J}_{z11}^{-1} \cdot \mathbf{H}_{z21} \cdot \mathbf{T}_{12} - \mathbf{J}_{z11}^{-1} \cdot \mathbf{H}_{z11} = \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \cdot \mathbf{T}_{12} - \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi11},
\]  \hspace{1cm} (3.48)
which is rearranged as
\[
\left[ \mathbf{J}_{z11}^{-1} \cdot \mathbf{H}_{z21} - \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right] \cdot \mathbf{T}_{12} = \left[ \mathbf{J}_{z11}^{-1} \cdot \mathbf{H}_{z11} - \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi11} \right].
\]  \hspace{1cm} (3.49)

Multiplying the both sides of (3.49) by \( \mathbf{J}_{z11} \) gives
\[
\left[ \mathbf{H}_{z21} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right] \cdot \mathbf{T}_{12} = \left[ \mathbf{H}_{z11} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi11} \right].
\]  \hspace{1cm} (3.50)

Therefore,
\[
\mathbf{T}_{12} = \left[ \mathbf{H}_{z21} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right]^{-1} \cdot \left[ \mathbf{H}_{z11} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi11} \right].
\]  \hspace{1cm} (3.51)

**Standing-Wave Case**

For the standing-wave case illustrated in Figure 3.2, the source is embedded in layer 2 instead of layer 1. The \( z \)-components in layer 1 and layer 2 are expressed as
Figure 3.2: Standing-wave case with two uniaxial cylindrical layers in the $\rho z$-plane.

\[
\begin{bmatrix}
E_{z1} \\
H_{z1}
\end{bmatrix} = \mathbf{J}_{zn}(k_1 \rho) \cdot \mathbf{T}_{21} \cdot \mathbf{a}_2, \tag{3.52a}
\]

\[
\begin{bmatrix}
E_{z2} \\
H_{z2}
\end{bmatrix} = \mathbf{H}^{(1)}_{zn}(k_2 \rho) \cdot \mathbf{R}_{21} \cdot \mathbf{a}_2 + \mathbf{J}_{zn}(k_2 \rho) \cdot \mathbf{a}_2. \tag{3.52b}
\]

Using (3.34), the $\phi$-components in layer 1 and layer 2 are expressed as

\[
\begin{bmatrix}
H_{\phi1} \\
E_{\phi1}
\end{bmatrix} = \mathbf{J}_{\phi n}(k_1 \rho) \cdot \mathbf{T}_{21} \cdot \mathbf{a}_2, \tag{3.53a}
\]

\[
\begin{bmatrix}
H_{\phi2} \\
E_{\phi2}
\end{bmatrix} = \mathbf{H}^{(1)}_{\phi n}(k_2 \rho) \cdot \mathbf{R}_{21} \cdot \mathbf{a}_2 + \mathbf{J}_{\phi n}(k_2 \rho) \cdot \mathbf{a}_2. \tag{3.53b}
\]

Applying the boundary conditions at $\rho = a_1$ to (3.52a)–(3.53b) yields

\[
\mathbf{J}_{zn}(k_{1\rho}a_1) \cdot \mathbf{T}_{21} \cdot \mathbf{a}_2 = \left[ \mathbf{H}^{(1)}_{zn}(k_{2\rho}a_1) \cdot \mathbf{R}_{21} + \mathbf{J}_{zn}(k_{2\rho}a_1) \right] \cdot \mathbf{a}_2, \tag{3.54a}
\]

\[
\mathbf{J}_{\phi n}(k_{1\rho}a_1) \cdot \mathbf{T}_{21} \cdot \mathbf{a}_2 = \left[ \mathbf{H}^{(1)}_{\phi n}(k_{2\rho}a_1) \cdot \mathbf{R}_{21} + \mathbf{J}_{\phi n}(k_{2\rho}a_1) \right] \cdot \mathbf{a}_2. \tag{3.54b}
\]
To obtain $\mathbf{R}_{21}$, $a_2$ in the both sides are cancelled and (3.54a) and (3.54b) are rearranged as

$$
\mathbf{T}_{21} = J_{z11}^{-1} \cdot [\mathbf{H}_{z21} \cdot \mathbf{R}_{21} + J_{z21}] ,
$$
(3.55a)

$$
\mathbf{T}_{21} = J_{\phi11}^{-1} \cdot [\mathbf{H}_{\phi21} \cdot \mathbf{R}_{21} + J_{\phi21}] .
$$
(3.55b)

Equating (3.55a) and (3.55b) gives

$$
\left[ J_{z11}^{-1} \cdot \mathbf{H}_{z21} - J_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right] \cdot \mathbf{R}_{21} = \left[ J_{z11}^{-1} \cdot J_{\phi21} - J_{z11}^{-1} \cdot J_{z21} \right] .
$$
(3.56)

By multiplying the both sides by $J_{z11}$, we obtain

$$
\mathbf{R}_{21} = \left[ \mathbf{H}_{z21} - J_{z11} \cdot J_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right]^{-1} \cdot \left[ J_{z11}^{-1} \cdot J_{\phi21} - J_{z11} \cdot J_{z21} \right] .
$$
(3.57)

For obtaining $\mathbf{T}_{21}$, $a_2$ in the both sides are cancelled again and (3.54a) and (3.54b) are rearranged as

$$
\mathbf{H}_{z21}^{-1} \cdot [\mathbf{J}_{z11} \cdot \mathbf{T}_{21} - J_{z21}] = \mathbf{R}_{21} ,
$$
(3.58a)

$$
\mathbf{H}_{\phi21}^{-1} \cdot [\mathbf{J}_{\phi11} \cdot \mathbf{T}_{21} - J_{\phi21}] = \mathbf{R}_{21} .
$$
(3.58b)

Equating (3.58a) and (3.58b) gives

$$
\left[ \mathbf{H}_{z21}^{-1} \cdot J_{z11} - \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi11} \right] \cdot \mathbf{T}_{21} = \left[ \mathbf{H}_{z21}^{-1} \cdot J_{z21} - \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi21} \right] .
$$
(3.59)

By multiplying the both sides by $\mathbf{H}_{z21}$, we obtain

$$
\mathbf{T}_{21} = \left[ J_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi11} \right]^{-1} \cdot \left[ \mathbf{H}_{z21}^{-1} \cdot J_{z21} - \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi21} \right] .
$$
(3.60)

In summary, the local reflection and transmission coefficients are

$$
\mathbf{R}_{12} = \left[ J_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi11} \right]^{-1} \cdot \left[ \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \right] ,
$$
(3.61a)

$$
\mathbf{R}_{21} = \left[ \mathbf{H}_{z21} - J_{z11} \cdot J_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right]^{-1} \cdot \left[ J_{z11} \cdot J_{\phi11}^{-1} \cdot J_{\phi21} - J_{z21} \right] ,
$$
(3.61b)

$$
\mathbf{T}_{12} = \left[ \mathbf{H}_{z21} - J_{z11} \cdot J_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right]^{-1} \cdot \left[ \mathbf{H}_{z11} - J_{z11} \cdot J_{\phi11}^{-1} \cdot \mathbf{H}_{\phi11} \right] ,
$$
(3.61c)

$$
\mathbf{T}_{21} = \left[ J_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi11} \right]^{-1} \cdot \left[ \mathbf{J}_{z21} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot J_{\phi21} \right] .
$$
(3.61d)
We can succinctly rewrite (3.61a)–(3.61d) as

\[
\begin{align*}
\mathbf{R}_{12} &= \mathbf{D}_{\mathcal{A}}^{-1} \cdot \left[ \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \right], \\
\mathbf{R}_{21} &= \mathbf{D}_{\mathcal{B}}^{-1} \cdot \left[ \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{J}_{\phi21} - \mathbf{J}_{z21} \right], \\
\mathbf{T}_{12} &= \mathbf{D}_{\mathcal{B}}^{-1} \cdot \left[ \mathbf{H}_{z21} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right], \\
\mathbf{T}_{21} &= \mathbf{D}_{\mathcal{A}}^{-1} \cdot \left[ \mathbf{J}_{z21} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi21} \right],
\end{align*}
\] (3.62a–d)

where

\[
\begin{align*}
\mathbf{D}_{\mathcal{A}} &= \left[ \mathbf{J}_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right], \\
\mathbf{D}_{\mathcal{B}} &= \left[ \mathbf{H}_{z21} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right].
\end{align*}
\] (3.63a–b)

It should be noted that (3.62a)–(3.62d) recover the coefficients in isotropic media. In this case, \(\mathbf{H}_{z11}, \mathbf{H}_{z21}, \mathbf{J}_{z11},\) and \(\mathbf{J}_{z21}\) are no longer matrices, but scalars. As a result, two \(\mathbf{D}\) matrices are modified to

\[
\begin{align*}
\mathbf{D}_{\mathcal{A}} &= \left[ \mathbf{J}_{z11} - \mathbf{H}_{z21} \cdot \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right] \\
&= \left[ \mathbf{J}_{z11} \mathbf{I} - \mathbf{H}_{z21} \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{J}_{\phi11} \right] \\
&= \mathbf{H}_{\phi21}^{-1} \cdot \left[ \mathbf{J}_{z11} \mathbf{H}_{\phi21} - \mathbf{H}_{z21} \mathbf{J}_{\phi11} \right], \\
\mathbf{D}_{\mathcal{B}} &= \left[ \mathbf{H}_{z21} - \mathbf{J}_{z11} \cdot \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right] \\
&= \left[ \mathbf{H}_{z21} \mathbf{I} - \mathbf{J}_{z11} \mathbf{J}_{\phi11}^{-1} \cdot \mathbf{H}_{\phi21} \right] \\
&= \mathbf{J}_{\phi11}^{-1} \cdot \left[ \mathbf{H}_{z21} \mathbf{J}_{\phi11} - \mathbf{J}_{z11} \mathbf{H}_{\phi21} \right],
\end{align*}
\] (3.64a–b)

where \(\mathbf{I}\) is the \(2 \times 2\) identity matrix. Therefore, reflection coefficient \(\overline{\mathbf{R}}_{12}\) is modified to

\[
\begin{align*}
\overline{\mathbf{R}}_{12} &= \left[ \mathbf{J}_{z11} \mathbf{H}_{\phi21} - \mathbf{H}_{z21} \mathbf{J}_{\phi11} \right]^{-1} \cdot \mathbf{H}_{\phi21} \cdot \left[ \mathbf{H}_{z21} \mathbf{H}_{\phi21}^{-1} \cdot \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \mathbf{I} \right] \\
&= \left[ \mathbf{J}_{z11} \mathbf{H}_{\phi21} - \mathbf{H}_{z21} \mathbf{J}_{\phi11} \right]^{-1} \cdot \left[ \mathbf{H}_{z21} \mathbf{H}_{\phi11} - \mathbf{H}_{z11} \mathbf{H}_{\phi21} \right],
\end{align*}
\] (3.65)

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reflection coefficient $\overline{R}_{21}$ is modified to

$$\overline{R}_{21} = \left[ H_{z21} \overline{J}_{\phi11} - J_{z11} \overline{H}_{\phi21} \right]^{-1} \cdot \overline{J}_{\phi11} \cdot \left[ J_{z11} \overline{J}_{\phi11}^{-1} \cdot \overline{J}_{\phi21} - J_{z21} \overline{I} \right]$$

(3.66)

transmission coefficient $\overline{T}_{12}$ is modified to

$$\overline{T}_{12} = \left[ H_{z21} \overline{J}_{\phi11} - J_{z11} \overline{H}_{\phi21} \right]^{-1} \cdot \overline{J}_{\phi11} \cdot \left[ H_{z11} \overline{I} - J_{z11} \overline{J}_{\phi11}^{-1} \cdot \overline{H}_{\phi11} \right]$$

(3.67)

and transmission coefficient $\overline{T}_{21}$ is modified to

$$\overline{T}_{21} = \left[ J_{z11} \overline{H}_{\phi21} - H_{z21} \overline{J}_{\phi11} \right]^{-1} \cdot \overline{H}_{\phi21} \cdot \left[ J_{z21} \overline{I} - H_{z21} \overline{H}_{\phi21}^{-1} \cdot \overline{J}_{\phi21} \right]$$

(3.68)

It is observed that (3.65)–(3.68) are the same as those shown in [17, Ch. 3].

3.1.3 Hertzian Source and Its Spectral Representation

Let us consider uniaxial and homogeneous media where a source is the Hertzian electric dipole, i.e.,

$$J(r) = IIl \hat{\alpha} \delta(r - r'),$$

(3.69)

where $Il$ is the dipole moment. The $z$-component of the electric field is obtained as follows. When a source is present, the curl of (3.1) becomes

$$\nabla \times \nabla \times \mathbf{E} = i\omega \nabla \times \overline{\mu} \mathbf{H},$$

$$\nabla^2 \mathbf{E} - \nabla \left( \nabla \cdot \mathbf{E} \right) = -i\omega \nabla \times \overline{\mu} \mathbf{H},$$

$$\nabla^2 \mathbf{E} - \nabla \left[ \frac{\rho_v}{\epsilon_h} + \left( 1 - \frac{\epsilon_v}{\epsilon_h} \right) \frac{\partial E_z}{\partial z} \right] = -i\omega \nabla \times \overline{\mu} \mathbf{H},$$

(3.70)

$$\nabla^2 \mathbf{E} - \left( 1 - \frac{\epsilon_v}{\epsilon_h} \right) \nabla \frac{\partial E_z}{\partial z} = -i\omega \nabla \times \overline{\mu} \mathbf{H} + \nabla \left( \frac{\nabla \cdot J}{i\omega \epsilon_h} \right).$$

(3.71)
In (3.70), $\rho_v$ is the volume charge density and the divergence of $\mathbf{E}$ is from (3.8).

In (3.71), the continuity equation $\nabla \cdot \mathbf{J} - i \omega \rho_v = 0$ is applied. By extracting the $z$-components, the equation for $E_z$ becomes

$$\nabla^2 E_z - \left(1 - \frac{\epsilon_v}{\epsilon_h}\right) \frac{\partial^2 E_z}{\partial z^2} = -i\omega \nabla_s \times \mu_h \mathbf{H}_s + \frac{\partial}{\partial z} \left(\frac{\nabla \cdot \mathbf{J}}{i\omega\epsilon_h}\right),$$

$$\nabla^2 E_z - \left(1 - \frac{\epsilon_v}{\epsilon_h}\right) \frac{\partial^2 E_z}{\partial z^2} = -i\omega \mu_h \left(-i\omega \epsilon_v E_z + J_z\right) + \frac{\partial}{\partial z} \left(\frac{\nabla \cdot \mathbf{J}}{i\omega\epsilon_h}\right),$$

$$\nabla^2 E_z + \omega^2 \mu_h \epsilon_v E_z - \left(1 - \frac{\epsilon_v}{\epsilon_h}\right) \frac{\partial^2 E_z}{\partial z^2} = -i\omega \mu_h \hat{z} \cdot \mathbf{J} + \frac{\partial}{\partial z} \left(\frac{\nabla \cdot \mathbf{J}}{i\omega\epsilon_h}\right). \quad (3.72)$$

Using (3.69), we obtain

$$\nabla^2 E_z + \omega^2 \mu_h \epsilon_v E_z - \left(1 - \frac{\epsilon_v}{\epsilon_h}\right) \frac{\partial^2 E_z}{\partial z^2} =$$

$$-i\omega \mu_h Il \left[\hat{z} \cdot \hat{\alpha}' + \frac{1}{\omega^2 \mu_h \epsilon_h} \frac{\partial}{\partial z} \nabla \cdot \hat{\alpha}'\right] \delta(r - r'). \quad (3.73)$$

Since $E_z$ has $e^{ik_zz}$ dependence as shown in (3.26a), (3.73) is reduced to

$$\nabla^2 E_z + \tilde{k}^2 E_z = -\frac{Il}{\omega \epsilon_h} \left[k^2 (\hat{z} \cdot \hat{\alpha}') + \frac{\partial}{\partial z} \nabla \cdot \hat{\alpha}'\right] \delta(r - r'), \quad (3.74)$$

where

$$\tilde{k}^2 = \omega^2 \mu_h \epsilon_v + \left(1 - \frac{\epsilon_v}{\epsilon_h}\right) k_z^2 = \frac{\epsilon_v}{\epsilon_h} \left(\omega^2 \mu_h \epsilon_v - k_z^2\right) + k_z^2 = \tilde{k}_p^2 + k_z^2, \quad (3.75)$$

$$k^2 = \omega^2 \mu_h \epsilon_h. \quad (3.76)$$

Therefore, $E_z$ can be easily obtained via the scalar Green’s function,

$$E_z = \frac{Il}{\omega \epsilon_h} \left[k^2 (\hat{z} \cdot \hat{\alpha}') + \frac{\partial}{\partial z} \nabla \cdot \hat{\alpha}'\right] \tilde{g}(r - r'), \quad (3.77)$$

where

$$\tilde{g}(r - r') = \frac{e^{i\tilde{k}|r-r'|}}{4\pi|r-r'|}. \quad (3.78)$$
For the $z$-component of the magnetic field, taking the curl of (3.2) yields
\[
\nabla \times \nabla \times H = -i \omega \nabla \times \tilde{\mathbf{r}} \mathbf{E} + \nabla \times \mathbf{J},
\]
\[
\nabla^2 \mathbf{H} - \nabla (\nabla \cdot \mathbf{H}) = i \omega \nabla \times \tilde{\mathbf{r}} \mathbf{E} - \nabla \times \mathbf{J}, \tag{3.79}
\]
\[
\nabla^2 \mathbf{H} - \nabla \left[ \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial H_z}{\partial z} \right] = i \omega \nabla \times \tilde{\mathbf{r}} \mathbf{E} - \nabla \times \mathbf{J}. \tag{3.80}
\]

In (3.79), the divergence of $\mathbf{H}$ is obtained from (3.10). By extracting the $z$-components, the equation for $H_z$ becomes
\[
\nabla^2 H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} = i \omega \nabla \times \tilde{\mathbf{r}} \mathbf{E}_s - \tilde{z} \cdot \nabla \times \mathbf{J},
\]
\[
\nabla^2 H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} = i \omega \epsilon_h (i \omega \mu_v H_z) - \tilde{z} \cdot \nabla \times \mathbf{J},
\]
\[
\nabla^2 H_z + \omega^2 \mu_v \epsilon_h H_z - \left( 1 - \frac{\mu_v}{\mu_h} \right) \frac{\partial^2 H_z}{\partial z^2} = -\tilde{z} \cdot \nabla \times \mathbf{J}. \tag{3.81}
\]

Since $H_z$ also has $e^{ik_z z}$ dependence shown in (3.26b), (3.81) is reduced to
\[
\nabla^2 H_z + \tilde{k}^2 H_z = -\tilde{z} \cdot \nabla \times \mathbf{J}, \tag{3.82}
\]
where
\[
\tilde{k}^2 = \omega^2 \mu_v \epsilon_h + \left( 1 - \frac{\mu_v}{\mu_h} \right) k_z^2 = \frac{\mu_v}{\mu_h} (\omega^2 \mu_h \epsilon_h - k_z^2) + k_z^2 = \tilde{k}^2 + k_z^2. \tag{3.83}
\]

Therefore, we again obtain $H_z$ via the scalar Green’s function,
\[
H_z = -i \tilde{z} \cdot \nabla' \times \tilde{\alpha}' \tilde{g}(\mathbf{r} - \mathbf{r}'), \tag{3.84}
\]
where
\[
\tilde{g}(\mathbf{r} - \mathbf{r}') = \frac{e^{i k |\mathbf{r}\!-\!\mathbf{r}'|}}{4\pi |\mathbf{r}\!-\!\mathbf{r}'|}. \tag{3.85}
\]

Using the spectral representation of the scalar Green’s function
\[
e^{i k |\mathbf{r}\!-\!\mathbf{r}'|} = \sum_{n=-\infty}^{\infty} \frac{i e^{in(\phi - \phi')}}{2} \int_{-\infty}^{\infty} dk_z e^{i k_z (z - z')} J_n(k_{\rho} \rho_<) H_n^{(1)}(k_{\rho} \rho_>) \tag{3.86},
\]
the $z$-components of electromagnetic fields in (3.77) and (3.84) can be written as

$$\begin{bmatrix} E_z \\ H_z \end{bmatrix} = \frac{iI}{4\pi\omega\varepsilon_h} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \begin{bmatrix} J_n(\tilde{k}_\rho\rho_<)H_n^{(1)}(\tilde{k}_\rho\rho_>) \\ 0 \\ 0 \\ J_n(\tilde{k}_\rho\rho_<)H_n^{(1)}(\tilde{k}_\rho\rho_>) \end{bmatrix} \cdot \tilde{D}', \quad (3.87)$$

where

$$\tilde{D}' = \frac{i}{2} \left[ (\tilde{z}k_j^2 + \frac{\partial}{\partial z'} \nabla') \cdot \tilde{\alpha}' \right] . \quad (3.88)$$

When the medium consists of a number of cylindrical layers and the point source is present in layer $j$, (3.87) is modified to account for reflections from the boundaries at $\rho = a_j$ and $\rho = a_{j-1}$.

$$\begin{bmatrix} E_{zj} \\ H_{zj} \end{bmatrix} = \frac{iI}{4\pi\omega\varepsilon_{hj}} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left\{ \tilde{J}_{zj\rho_<} \cdot \tilde{H}_{zj\rho_>} + \tilde{H}_{zj\rho} \cdot \tilde{a}_{jn}(\rho') + \tilde{J}_{zj\rho} \cdot \tilde{b}_{jn}(\rho') \right\} \cdot \tilde{D}_j', \quad (3.89)$$

where

$$\tilde{J}_{zj\rho_<} \cdot \tilde{H}_{zj\rho_>} = \begin{bmatrix} J_n(\tilde{k}_\rho\rho_<)H_n^{(1)}(\tilde{k}_\rho\rho_>) \\ 0 \\ 0 \\ J_n(\tilde{k}_\rho\rho_<)H_n^{(1)}(\tilde{k}_\rho\rho_>) \end{bmatrix} , \quad (3.90a)$$

$$\tilde{H}_{zj\rho} = \begin{bmatrix} H_n^{(1)}(\tilde{k}_\rho\rho) \\ 0 \\ 0 \\ H_n^{(1)}(\tilde{k}_\rho\rho) \end{bmatrix} , \quad (3.90b)$$

$$\tilde{J}_{zj\rho} = \begin{bmatrix} J_n(\tilde{k}_\rho\rho) \\ 0 \\ 0 \\ J_n(\tilde{k}_\rho\rho) \end{bmatrix} , \quad (3.90c)$$

$$\tilde{D}_j' = \frac{i}{2} \left[ (\tilde{z}k_j^2 - i\tilde{k}_z \nabla') \cdot \tilde{\alpha}' \right] . \quad (3.90d)$$

Recall that notations of (3.90a), (3.90b), and (3.90c) are based on (3.38a) and (3.40).

In (3.90d), $k_j = \omega\mu_{hj}\varepsilon_{hj}$, and $\mu_{hj}$ and $\varepsilon_{hj}$ represent horizontal permeability and
permittivity in layer \( j \), respectively. When \( \hat{\alpha}' \) is represented in cylindrical coordinates,

\[
\hat{\alpha}' = \hat{\rho}' \alpha_{\rho}' + \hat{\phi}' \alpha_{\phi}' + \hat{z}' \alpha_z',
\]

\[
\mathbf{D}_j' = \frac{i}{2} \left( \mathbf{D}_{j1}' + \mathbf{D}_{j2}' + \frac{\partial}{\partial \rho'} \mathbf{D}_{j3}' \right)
\]

\[
= \frac{i}{2} \left( \begin{bmatrix}
(k_{j\rho}')^2 \alpha_{z}' \\
0
\end{bmatrix} + \begin{bmatrix}
-\frac{n k_{\rho}'}{\rho} \alpha_{\phi}' \\
-\frac{n c_{k} h_{i}'}{\rho} \alpha_{\rho}'
\end{bmatrix} + \frac{\partial}{\partial \rho'} \begin{bmatrix}
-ik_z \alpha_{\rho}' \\
i \omega \epsilon_{\rho} \alpha_{\phi}'
\end{bmatrix} \right).
\]

(3.91)

Alternatively, when \( \hat{\alpha}' \) is represented in Cartesian coordinates, \( \hat{\alpha}' = \hat{x}' \alpha_{x}' + \hat{y}' \alpha_{y}' + \hat{z}' \alpha_z' \),

\[
\mathbf{D}_j' = \frac{i}{2} \left( \mathbf{D}_{j1}' + \mathbf{D}_{j2}' + \frac{\partial}{\partial \rho'} \mathbf{D}_{j3}' \right)
\]

\[
= \frac{i}{2} \left[ \begin{bmatrix}
(k_{j\rho}')^2 \alpha_{x}' \\
0
\end{bmatrix} + \frac{nk_{\rho}'}{\rho} \alpha_{x}' \sin \phi' - \alpha_{y}' \cos \phi' \\
-\frac{n c_{k} h_{i}'}{\rho} \alpha_{x}' \cos \phi' + \alpha_{y}' \sin \phi'
\end{bmatrix} + \frac{\partial}{\partial \rho'} \begin{bmatrix}
-ik_z \alpha_{x}' \cos \phi' + \alpha_{y}' \sin \phi' \\
i \omega \epsilon_{\rho} \alpha_{x}' \cos \phi'
\end{bmatrix} \right].
\]

(3.92)

To solve for \( \mathbf{a}_j (\rho') \) and \( \mathbf{b}_j (\rho') \) appeared in (3.89), two constraint conditions at \( \rho = a_j \) and \( \rho = a_{j-1} \) are necessary, which relates the outgoing waves to the standing waves such that

\[
\mathbf{D}_j \cdot \mathbf{D}_j' = \mathbf{R}_{j,j+1} \cdot \begin{bmatrix}
\mathbf{J}_{zj\rho}' + \mathbf{a}_j \\
\mathbf{H}_{zj\rho}' + \mathbf{b}_j
\end{bmatrix} \cdot \mathbf{D}_j',
\]

(3.93a)

\[
\mathbf{a}_j \cdot \mathbf{D}_j' = \mathbf{R}_{j,j-1} \cdot \begin{bmatrix}
\mathbf{J}_{zj\rho}' + \mathbf{a}_j \\
\mathbf{H}_{zj\rho}' + \mathbf{b}_j
\end{bmatrix} \cdot \mathbf{D}_j'.
\]

(3.93b)

From (3.93a) and (3.93b), we obtain

\[
\mathbf{a}_j = \left[ \mathbf{I} - \mathbf{R}_{j,j-1} \cdot \mathbf{R}_{j,j+1} \right] \cdot \mathbf{R}_{j,j-1} \cdot \begin{bmatrix}
\mathbf{J}_{zj\rho}' + \mathbf{R}_{j,j+1} \cdot \mathbf{J}_{zj\rho}' \\
\mathbf{H}_{zj\rho}' + \mathbf{R}_{j,j+1} \cdot \mathbf{H}_{zj\rho}'
\end{bmatrix},
\]

(3.94a)

\[
\mathbf{b}_j = \left[ \mathbf{I} - \mathbf{R}_{j,j+1} \cdot \mathbf{R}_{j,j-1} \right] \cdot \mathbf{R}_{j,j+1} \cdot \begin{bmatrix}
\mathbf{J}_{zj\rho}' + \mathbf{R}_{j,j-1} \cdot \mathbf{J}_{zj\rho}' \\
\mathbf{H}_{zj\rho}' + \mathbf{R}_{j,j-1} \cdot \mathbf{H}_{zj\rho}'
\end{bmatrix}.
\]

(3.94b)

Note that the second brackets in the right hand sides of (3.94a) and (3.94b) are slightly different from those for isotropic media. When \( \rho > \rho' \), \( \rho < = \rho' \) and \( \rho > = \rho. \)
The curly bracket in (3.89) is expressed as
\[
\bar{J}_{zj\rho^<} \cdot \bar{H}_{zj\rho^>} + \bar{H}_{zj\rho^<} \cdot \bar{a}_{jn}(\rho') + \bar{J}_{zj\rho^<} \cdot \bar{b}_{jn}(\rho')
= \left[ \bar{H}_{zj\rho^<} + \bar{J}_{zj\rho^<} \cdot \bar{\tilde{R}}_{j,j+1} \right] \cdot \bar{M}_{j^+} \cdot \left[ \bar{J}_{zj\rho'} + \bar{\tilde{R}}_{j,j-1} \cdot \bar{H}_{zj\rho'} \right]. \tag{3.95}
\]
On the other hand, when \(\rho < \rho'\), \(\rho^< = \rho\) and \(\rho^> = \rho'\). The curly bracket in (3.89) is now expressed as
\[
\bar{J}_{zj\rho^<} \cdot \bar{H}_{zj\rho^>} + \bar{H}_{zj\rho^<} \cdot \bar{a}_{jn}(\rho') + \bar{J}_{zj\rho^<} \cdot \bar{b}_{jn}(\rho')
= \left[ \bar{J}_{zj\rho^<} + \bar{H}_{zj\rho^<} \cdot \bar{\tilde{R}}_{j,j-1} \right] \cdot \bar{M}_{j^-} \cdot \left[ \bar{H}_{zj\rho'} + \bar{\tilde{R}}_{j,j+1} \cdot \bar{J}_{zj\rho'} \right]. \tag{3.96}
\]
Again, (3.95) and (3.96) are slightly different from those for isotropic media. When the field layer is not the same as the source layer, the approach is the same as that in [17]. In summary,
\[
\begin{bmatrix}
E_z \\
H_z
\end{bmatrix} = \frac{iI}{4\pi \omega \epsilon_{hj}} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')} \int_{-\infty}^{\infty} dk_z e^{ik_z(z'-z)} F_n(\rho, \rho') \cdot \bar{D}_j', \tag{3.97}
\]
where
for Case 1: \(\rho\) and \(\rho'\) are in the same layer. \((\rho \geq \rho')\)
\[
F_n(\rho, \rho') = \left[ \bar{H}_{zj\rho^+} + \bar{J}_{zj\rho^+} \cdot \bar{\tilde{R}}_{j,j+1} \right] \cdot \bar{M}_{j^+} \cdot \left[ \bar{J}_{zj\rho'} + \bar{\tilde{R}}_{j,j-1} \cdot \bar{H}_{zj\rho'} \right], \tag{3.98a}
\]
for Case 2: \(\rho\) and \(\rho'\) are in the same layer. \((\rho < \rho')\)
\[
F_n(\rho, \rho') = \left[ \bar{J}_{zj\rho} + \bar{H}_{zj\rho} \cdot \bar{\tilde{R}}_{j,j-1} \right] \cdot \bar{M}_{j^-} \cdot \left[ \bar{H}_{zj\rho'} + \bar{\tilde{R}}_{j,j+1} \cdot \bar{J}_{zj\rho'} \right], \tag{3.98b}
\]
for Case 3: \(\rho\) and \(\rho'\) are in different layers. \((\rho > \rho')\)
\[
F_n(\rho, \rho') = \left[ \bar{H}_{zj\rho^+} + \bar{J}_{zj\rho^+} \cdot \bar{\tilde{R}}_{i,i+1} \right] \cdot \bar{N}_{i^+} \cdot \bar{T}_{j'i} \cdot \bar{M}_{j^+} \cdot \left[ \bar{J}_{zj\rho'} + \bar{\tilde{R}}_{j,j-1} \cdot \bar{H}_{zj\rho'} \right], \tag{3.98c}
\]
for Case 4: \( \rho \) and \( \rho' \) are in different layers. \((\rho < \rho')\)

\[
\mathbf{F}_n(\rho, \rho') = \left[ \mathbf{J}_{zi\rho} + \mathbf{H}_{zi\rho} \cdot \mathbf{R}_{i,i-1} \right] \cdot \mathbf{N}_{i-} \cdot \mathbf{T}_{ji} \cdot \mathbf{M}_{j-} \cdot \left[ \mathbf{H}_{zj\rho'} + \mathbf{R}_{j,j+1} \cdot \mathbf{J}_{zj\rho'} \right]. 
\]

(3.98d)

### 3.2 Range-Conditioning

This section discusses how to stabilize the numerical computation of electromagnetic fields in uniaxial media and provides the detailed mathematical derivations. To do so, the range-conditioning technique in Section 2.3 is employed. It will be shown that the range-conditioning for isotropic media is indeed a special case for uniaxial media in a way that all types of associated multiplicative factors for uniaxial media below reduce to diagonal matrices (and can be represented by scalar factors instead).

#### 3.2.1 Range-Conditioned Cylindrical Matrices

The range-conditioned cylindrical functions are slightly modified for uniaxial media because the two stretched radial wavenumbers (see (3.25a) and (3.25b)) are a part of arguments for the functions. To recall, the stretched radial wavenumbers indicated by a tilde or two dots are defined as

\[
\tilde{k}_{i\rho} = \frac{k_{i\rho}}{\kappa_{ie}}, \quad (3.99a)
\]

\[
\hat{k}_{i\rho} = \frac{k_{i\rho}}{\kappa_{i\mu}}, \quad (3.99b)
\]

where the subscript \( i \) indicates layer \( i \), and \( \kappa_{ie} = \sqrt{\epsilon_{hi}/\epsilon_{vi}} \) and \( \kappa_{i\mu} = \sqrt{\mu_{hi}/\mu_{vi}} \) are the anisotropy ratios of permittivity and permeability in layer \( i \), respectively. Table 3.1 shows the definitions of the range-conditioned cylindrical functions for permittivity-uniaxial media. Note that subscripts \( i \) and \( j \) are arbitrary.
Table 3.1: Definition of the range-conditioned cylindrical functions for permittivity-uniaxial media for all types of arguments.

<table>
<thead>
<tr>
<th>Function</th>
<th>Small Arguments</th>
<th>Moderate Arguments</th>
<th>Large Arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_n(\tilde{k}_i a_j)$</td>
<td>$\tilde{G}_i a_j^* J_n(\tilde{k}_i a_j)$</td>
<td>$\tilde{P}_{ij} J_n(\tilde{k}_i a_j)$</td>
<td>$e^{\tilde{k}_i a_j a_j} J_n(\tilde{k}_i a_j)$</td>
</tr>
<tr>
<td>$J_n'(\tilde{k}_i a_j)$</td>
<td>$\tilde{G}_i a_j^* J_n'(\tilde{k}_i a_j)$</td>
<td>$\tilde{P}_{ij} J_n'(\tilde{k}_i a_j)$</td>
<td>$e^{\tilde{k}_i a_j a_j} J_n'(\tilde{k}_i a_j)$</td>
</tr>
<tr>
<td>$H_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$\tilde{G}_i^{-1} a_j^{-n} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$\tilde{P}_{ij}^{-1} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$e^{-\tilde{k}_i a_j a_j} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
</tr>
<tr>
<td>$H_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$\tilde{G}_i^{-1} a_j^{-n} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$\tilde{P}_{ij}^{-1} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
<td>$e^{-\tilde{k}_i a_j a_j} \tilde{H}_n^{(1)}(\tilde{k}_i a_j)$</td>
</tr>
</tbody>
</table>

Similarly, the range-conditioned cylindrical functions for permeability-uniaxial media can be constructed, which are not discussed here. In Table 3.1,

$$\tilde{G}_i = \frac{1}{n!} \left( \frac{\tilde{k}_i}{2} \right)^n,$$

(3.100)

$$\tilde{P}_{ij} = \begin{cases} 1, & \text{if } |J_n(\tilde{k}_i a_j)|^{-1} < T_{\text{moderate}}, \\ |J_n(\tilde{k}_i a_j)|, & \text{if } |J_n(\tilde{k}_i a_j)|^{-1} \geq T_{\text{moderate}}, \end{cases}$$

(3.101)

$$\tilde{k}_i'' = \Im \left[ \frac{k_i' \rho}{\kappa_i' \epsilon} \right] = \Im \left[ \frac{k_i' + i k_i''}{\kappa_i' + i \kappa_i''} \right] = \Im \left[ \frac{k_i' + i k_i''}{k_i' \kappa_i' + i \kappa_i''} \right] = \frac{k_i' k_i'' - \kappa_i'' k_i'}{|\kappa_i'|^2},$$

(3.102)

where $T_{\text{moderate}}$ is the magnitude threshold for moderate arguments given in (2.39b).

The multiplicative factors associated with the new functions shown in Table 3.1 can be again classified into two types: $\alpha$-type and $\beta$-type. Therefore, the cylindrical
Table 3.2: Definition of the multiplicative factors, $\tilde{\alpha}_{ij}$ and $\tilde{\beta}_{ij}$, for electromagnetic fields in permittivity-uniaxial media ($\bar{\alpha}_{ij}$ and $\bar{\beta}_{ij}$ in permeability-uniaxial media can be constructed in a similar way).

<table>
<thead>
<tr>
<th>Argument Type</th>
<th>$\tilde{\alpha}_{ij}$</th>
<th>$\tilde{\beta}_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>$\tilde{G}_i^{-1} a_j^i$</td>
<td>$\tilde{G}_i a_j^i$</td>
</tr>
<tr>
<td>Moderate</td>
<td>$\tilde{P}_{ij}$</td>
<td>$\tilde{P}_{ij}$</td>
</tr>
<tr>
<td>Large</td>
<td>$e^{-\tilde{k}_{ij}} a_j^i$</td>
<td>$e^{\tilde{k}_{ij}} a_j^i$</td>
</tr>
</tbody>
</table>

functions for permittivity-uniaxial media can be succinctly expressed as

$$J_n(\tilde{k}_{i\rho}a_j) = \tilde{\beta}_{ij} \hat{J}_n(\tilde{k}_{i\rho}a_j), \quad (3.103a)$$

$$J'_n(\tilde{k}_{i\rho}a_j) = \tilde{\beta}_{ij} \hat{J}'_n(\tilde{k}_{i\rho}a_j), \quad (3.103b)$$

$$H^{(1)}_n(\tilde{k}_{i\rho}a_j) = \tilde{\alpha}_{ij} \hat{H}^{(1)}_n(\tilde{k}_{i\rho}a_j), \quad (3.103c)$$

$$H'_{n}^{(1)}(\tilde{k}_{i\rho}a_j) = \tilde{\alpha}_{ij} \hat{H}'_{n}^{(1)}(\tilde{k}_{i\rho}a_j). \quad (3.103d)$$

The definitions of $\tilde{\alpha}_{ij}$ and $\tilde{\beta}_{ij}$ are provided in Table 3.2. There are two important characteristics of $\tilde{\alpha}_{ij}$ and $\tilde{\beta}_{ij}$, which are identical to those for isotropic media given in (2.44) and (2.45) of Section 2.3.1.

Furthermore, range-conditioned cylindrical matrices can be defined for cylindrically stratified and doubly-uniaxial media. From (3.38a) and (3.38b), we obtain

$$\mathbf{J}_{zn}(\tilde{k}_{i\rho}a_j) = \begin{bmatrix} J_n(\tilde{k}_{i\rho}a_j) & 0 \\ 0 & J_n(\tilde{k}_{i\rho}a_j) \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{J}_n(\tilde{k}_{i\rho}a_j) & 0 \\ 0 & \tilde{J}_n(\tilde{k}_{i\rho}a_j) \end{bmatrix} \cdot \begin{bmatrix} \tilde{\beta}_{ij} & 0 \\ 0 & \tilde{\beta}_{ij} \end{bmatrix}$$

$$= \tilde{\beta}_{ij} \cdot \mathbf{J}_{zn} = \tilde{\mathbf{J}}_{zn} \cdot \tilde{\beta}_{ij}, \quad (3.104)$$
\[
\mathbf{H}^{(1)}_{zn}(k_{ip}a_j) = \begin{bmatrix}
H_n^{(1)}(\tilde{k}_{ip}a_j) & 0 \\
0 & H_n^{(1)}(\bar{k}_{ip}a_j)
\end{bmatrix}
= \begin{bmatrix}
\tilde{H}_n^{(1)}(\tilde{k}_{ip}a_j) & 0 \\
0 & \bar{H}_n^{(1)}(\bar{k}_{ip}a_j)
\end{bmatrix} \cdot \begin{bmatrix}
\tilde{\alpha}_{ij} & 0 \\
0 & \bar{\alpha}_{ij}
\end{bmatrix}
= \tilde{\alpha}_{ij} \cdot \hat{\mathbf{H}}^{(1)}_{zn} = \hat{\mathbf{H}}^{(1)}_{zn} \cdot \tilde{\alpha}_{ij},
\tag{3.105}
\]

\[
\mathbf{J}_{\phi n}(k_{ip}a_j) = \frac{1}{k_{ip}^2 a_j} \begin{bmatrix}
i\omega \epsilon_h \bar{k}_{ip}a_j J'_n(\bar{k}_{ip}a_j) & -nk_z J_n(\bar{k}_{ip}a_j) \\
-nk_z J_n(\bar{k}_{ip}a_j) & -i\omega \mu_h \bar{k}_{ip}a_j J'_n(\bar{k}_{ip}a_j)
\end{bmatrix}
= \frac{1}{k_{ip}^2 a_j} \begin{bmatrix}
i\omega \epsilon_h \bar{k}_{ip}a_j \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) & -nk_z \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) \\
-nk_z \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) & -i\omega \mu_h \bar{k}_{ip}a_j \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j)
\end{bmatrix} \cdot \begin{bmatrix}
\tilde{\beta}_{ij} & 0 \\
0 & \bar{\beta}_{ij}
\end{bmatrix}
= \hat{\mathbf{J}}_{\phi n} \cdot \tilde{\beta}_{ij},
\tag{3.106}
\]

\[
\mathbf{\bar{H}}^{(1)}_{\phi n}(k_{ip}a_j) = \frac{1}{k_{ip}^2 a_j} \begin{bmatrix}
i\omega \epsilon_h \bar{k}_{ip}a_j H_n^{(1)}(\bar{k}_{ip}a_j) & -nk_z H_n^{(1)}(\bar{k}_{ip}a_j) \\
-nk_z H_n^{(1)}(\bar{k}_{ip}a_j) & -i\omega \mu_h \bar{k}_{ip}a_j H_n^{(1)}(\bar{k}_{ip}a_j)
\end{bmatrix}
= \frac{1}{k_{ip}^2 a_j} \begin{bmatrix}
i\omega \epsilon_h \bar{k}_{ip}a_j \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) & -nk_z \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) \\
-nk_z \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j) & -i\omega \mu_h \bar{k}_{ip}a_j \tilde{H}_n^{(1)}(\bar{k}_{ip}a_j)
\end{bmatrix} \cdot \begin{bmatrix}
\hat{\alpha}_{ij} & 0 \\
0 & \tilde{\alpha}_{ij}
\end{bmatrix}
= \hat{\mathbf{H}}^{(1)}_{\phi n} \cdot \alpha_{ij}.
\tag{3.107}
\]

The associated matrices in (3.104) and (3.105) are all diagonal, so they commute. Moreover, \( \mathbf{J}_{zn} \) and \( \mathbf{J}_{\phi n} \) exhibit the same multiplicative matrix factor \( \bar{\beta}_{ij} \), and \( \mathbf{\bar{H}}^{(1)}_{zn} \) and \( \mathbf{\bar{H}}^{(1)}_{\phi n} \) exhibit the same multiplicative matrix factor \( \alpha_{ij} \).

### 3.2.2 Reflection and Transmission Coefficients

Using the redefined matrices in (3.104) – (3.107), local reflection and transmission coefficients depicted in Figure 3.3 can be also redefined. Two intermediate matrices,
Figure 3.3: Reflection and transmission coefficients for two cylindrical layers in the \( \rho z \)-plane.

\( \overline{D}_A \) and \( \overline{D}_B \), are redefined as

\[
\overline{D}_A = \left[ \hat{\mathbf{J}}_{z11} - \hat{\mathbf{H}}_{z21} \cdot \hat{\mathbf{H}}_{\phi21}^{-1} \cdot \hat{\mathbf{J}}_{\phi11} \right] = \left[ \hat{\mathbf{J}}_{z11} \cdot \beta_{11} - \hat{\mathbf{H}}_{z21} \cdot \alpha_{21} \cdot \alpha_{21}^{-1} \cdot \hat{\mathbf{H}}_{\phi21}^{-1} \cdot \hat{\mathbf{J}}_{\phi11} \cdot \beta_{11} \right] = \left[ \hat{\mathbf{J}}_{z11} - \hat{\mathbf{H}}_{z21} \cdot \hat{\mathbf{H}}_{\phi21}^{-1} \cdot \hat{\mathbf{J}}_{\phi11} \right] \cdot \beta_{11} = \overline{D}_A \cdot \beta_{11} \tag{3.108a}
\]

\[
\overline{D}_B = \left[ \hat{\mathbf{H}}_{z21} - \hat{\mathbf{J}}_{z11} \cdot \hat{\mathbf{J}}_{\phi11}^{-1} \cdot \hat{\mathbf{H}}_{\phi21} \right] = \left[ \hat{\mathbf{H}}_{z21} \cdot \alpha_{21} - \hat{\mathbf{J}}_{z11} \cdot \beta_{11} \cdot \beta_{11}^{-1} \cdot \hat{\mathbf{J}}_{\phi11} \cdot \hat{\mathbf{H}}_{\phi21} \cdot \alpha_{21} \right] = \left[ \hat{\mathbf{H}}_{z21} - \hat{\mathbf{J}}_{z11} \cdot \hat{\mathbf{J}}_{\phi11}^{-1} \cdot \hat{\mathbf{H}}_{\phi21} \right] \cdot \alpha_{21} = \overline{D}_B \cdot \alpha_{21} \tag{3.108b}
\]
Therefore, the reflection coefficient \( R_{12} \) is redefined as

\[
R_{12} = \hat{D}_A^{-1} \cdot \left[ \hat{H}_{z21} \cdot \hat{H}_{\phi 21}^{-1} \cdot \hat{H}_{\phi 11} \cdot \hat{H}_{z11} \right] \\
= \hat{\beta}_{11}^{-1} \cdot \hat{D}_A^{-1} \cdot \left[ \hat{H}_{z21} \cdot \hat{\alpha}_{21} \cdot \alpha_{21}^{-1} \cdot \hat{H}_{\phi 21}^{-1} \cdot \hat{H}_{\phi 11} \cdot \alpha_{11} \cdot \hat{H}_{z11} \right] \\
= \alpha_{11} \cdot \hat{D}_A^{-1} \cdot \left[ \hat{H}_{z21} \cdot \hat{H}_{\phi 21}^{-1} \cdot \hat{H}_{\phi 11} \cdot \hat{H}_{z11} \right] \cdot \alpha_{11} \\
= \alpha_{11} \cdot \hat{R}_{12} \cdot \alpha_{11}, \tag{3.109a}
\]

the reflection coefficient \( R_{21} \) is redefined as

\[
R_{21} = \hat{D}_B^{-1} \cdot \left[ \hat{J}_{z11} \cdot \hat{J}_{\phi 11} \cdot \hat{J}_{\phi 21} \cdot \hat{J}_{z21} \right] \\
= \alpha_{21}^{-1} \cdot \hat{D}_B^{-1} \cdot \left[ \hat{J}_{z11} \cdot \hat{\beta}_{11} \cdot \beta_{11}^{-1} \cdot \hat{J}_{\phi 11} \cdot \beta_{21} \cdot \hat{J}_{21} \cdot \beta_{21} \right] \\
= \beta_{21} \cdot \hat{D}_B^{-1} \cdot \left[ \hat{J}_{z11} \cdot \hat{J}_{\phi 11} \cdot \hat{J}_{\phi 21} \cdot \hat{J}_{z21} \right] \cdot \beta_{21} \\
= \beta_{21} \cdot \hat{R}_{21} \cdot \beta_{21}, \tag{3.109b}
\]

the transmission coefficient \( T_{12} \) is redefined as

\[
T_{12} = \hat{D}_B^{-1} \cdot \left[ \hat{H}_{z11} \cdot \hat{J}_{z11} \cdot \hat{J}_{\phi 11} \cdot \hat{H}_{\phi 11} \right] \\
= \alpha_{21}^{-1} \cdot \hat{D}_B^{-1} \cdot \left[ \hat{H}_{z11} \cdot \alpha_{11} \cdot \hat{J}_{z11} \cdot \alpha_{11} \cdot \hat{J}_{\phi 11} \cdot \alpha_{11} \cdot \hat{H}_{\phi 11} \cdot \alpha_{11} \right] \\
= \beta_{21} \cdot \hat{D}_B^{-1} \cdot \left[ \hat{H}_{z11} \cdot \hat{J}_{z11} \cdot \hat{J}_{\phi 11} \cdot \hat{H}_{\phi 11} \right] \cdot \alpha_{11} \\
= \beta_{21} \cdot \hat{T}_{12} \cdot \alpha_{11}, \tag{3.109c}
\]

and the transmission coefficient \( T_{21} \) is redefined as

\[
T_{21} = \hat{D}_A^{-1} \cdot \left[ \hat{J}_{z21} \cdot \hat{H}_{z21} \cdot \hat{H}_{\phi 21}^{-1} \cdot \hat{J}_{\phi 21} \right] \\
= \beta_{11}^{-1} \cdot \hat{D}_A^{-1} \cdot \left[ \hat{J}_{z21} \cdot \beta_{21} \cdot \alpha_{21} \cdot \alpha_{21}^{-1} \cdot \hat{H}_{\phi 21} \cdot \hat{J}_{\phi 21} \cdot \beta_{21} \right] \\
= \alpha_{11} \cdot \hat{D}_A^{-1} \cdot \left[ \hat{J}_{z21} \cdot \hat{H}_{z21} \cdot \hat{H}_{\phi 21} \cdot \hat{J}_{\phi 21} \right] \cdot \beta_{21} \\
= \alpha_{11} \cdot \hat{T}_{21} \cdot \beta_{21}. \tag{3.109d}
\]
Figure 3.4: Generalized reflection coefficients for three cylindrical layers: (a) \( \tilde{R}_{12} \) for the outgoing-wave case and (b) \( \tilde{R}_{32} \) for the standing-wave case.

We can proceed to redefine generalized reflection coefficients, which are the functions of local reflection and transmission coefficients. The generalized reflection coefficient for the outgoing-wave case for three cylindrical layers depicted in Figure 3.4a is modified to

\[
\tilde{R}_{12} = R_{12} + T_{21} \cdot R_{23} \cdot \left[ I - R_{21} \cdot R_{23} \right]^{-1} \cdot T_{12}
\]

\[
= \alpha_{11} \cdot \tilde{R}_{12} \cdot \alpha_{11} + \tilde{T}_{21} \cdot \beta_{21} \cdot \alpha_{22} \cdot \tilde{R}_{23} \cdot \alpha_{22} \cdot (\beta_{21} \cdot \alpha_{21})
\]

\[
\cdot \left[ I - \beta_{21} \cdot \tilde{R}_{21} \cdot \beta_{21} \cdot \alpha_{22} \cdot \tilde{R}_{23} \cdot \alpha_{22} \right]^{-1}
\]

\[
\cdot (\beta_{21} \cdot \alpha_{21}) \cdot \beta_{21} \cdot \tilde{T}_{12} \cdot \alpha_{11}
\]

(3.110)

\[
= \alpha_{11} \cdot \left\{ \tilde{R}_{12} + \tilde{T}_{21} \cdot \beta_{21} \cdot \alpha_{22} \cdot \tilde{R}_{23} \cdot \beta_{21} \cdot \alpha_{22}
\right.
\]

\[
\cdot \left[ I - \tilde{R}_{21} \cdot \beta_{21} \cdot \alpha_{22} \cdot \tilde{R}_{23} \cdot \beta_{21} \cdot \alpha_{22} \right]^{-1} \cdot \tilde{T}_{12} \left\} \cdot \alpha_{11}
\]

(3.111)

\[
= \alpha_{11} \cdot \tilde{R}_{12} \cdot \alpha_{11}.
\]

(3.112)

In (3.110), \( \beta_{21} \cdot \alpha_{21} = I \) is inserted twice for mathematical manipulation. Recall that all \( \beta \)'s and \( \alpha \)'s are diagonal matrices, so they commute. The magnitude of the...
product of multiplicative factors \( \beta_{21} \cdot \alpha_{22} \) in the curly bracket in (3.111) is never greater than one due to the boundness property, which guarantees a moderate magnitude of \( \hat{\mathbf{R}}_{12} \) in any case. Since the associated multiplicative factors \( \alpha_{11} \) shown in (3.112) are the same as those for \( \mathbf{R}_{12} \) (see (3.109a)), they do not change when more than three layers are present. Therefore, the redefined generalized reflection coefficient between two arbitrarily-indexed adjacent layers for the outgoing-wave case can be expressed in general as

\[
\hat{\mathbf{R}}_{i,i+1} = \alpha_{ii} \cdot \hat{\mathbf{R}}_{i,i+1} \cdot \alpha_{ii}, \tag{3.113}
\]

where

\[
\hat{\mathbf{R}}_{i,i+1} = \hat{\mathbf{R}}_{i,i+1} + \hat{\mathbf{T}}_{i+1,i} \cdot \beta_{i+1,i} \cdot \alpha_{i+1,i+1} \cdot \hat{\mathbf{R}}_{i+1,i+2} \cdot \beta_{i+1,i} \cdot \alpha_{i+1,i+1} \cdot \left[ \mathbf{I} - \hat{\mathbf{R}}_{i+1,i} \cdot \beta_{i+1,i} \cdot \alpha_{i+1,i+1} \cdot \hat{\mathbf{R}}_{i+1,i+2} \cdot \beta_{i+1,i} \cdot \alpha_{i+1,i+1} \right]^{-1} \cdot \hat{\mathbf{T}}_{i,i+1}. \tag{3.114}
\]

The generalized reflection coefficient for the standing-wave case for three cylindrical layers depicted in Figure 3.4b is modified to

\[
\hat{\mathbf{R}}_{32} = \mathbf{R}_{32} + \mathbf{T}_{23} \cdot \mathbf{R}_{21} \cdot \left[ \mathbf{I} - \mathbf{R}_{23} \cdot \mathbf{R}_{21} \right]^{-1} \cdot \mathbf{T}_{32} \\
= \beta_{32} \cdot \hat{\mathbf{R}}_{32} \cdot \beta_{32} + \beta_{32} \cdot \hat{\mathbf{T}}_{23} \cdot \alpha_{22} \cdot \beta_{21} \cdot \hat{\mathbf{R}}_{21} \cdot \beta_{21} \cdot \left( \alpha_{22} \cdot \beta_{22} \right) \\
\cdot \left[ \mathbf{I} - \hat{\mathbf{R}}_{23} \cdot \alpha_{22} \cdot \beta_{21} \cdot \hat{\mathbf{R}}_{21} \cdot \beta_{21} \right]^{-1} \\
\cdot \left( \alpha_{22} \cdot \beta_{22} \right) \cdot \alpha_{22} \cdot \hat{\mathbf{T}}_{32} \cdot \beta_{32} \tag{3.115}
\]

\[
\hat{\mathbf{R}}_{32} = \beta_{32} \cdot \left\{ \hat{\mathbf{R}}_{32} + \hat{\mathbf{T}}_{23} \cdot \alpha_{22} \cdot \hat{\mathbf{R}}_{21} \cdot \beta_{21} \cdot \alpha_{22} \\
\cdot \left[ \mathbf{I} - \hat{\mathbf{R}}_{23} \cdot \alpha_{22} \cdot \beta_{21} \cdot \hat{\mathbf{R}}_{21} \cdot \beta_{21} \cdot \alpha_{22} \right]^{-1} \cdot \hat{\mathbf{T}}_{32} \right\} \cdot \beta_{32} \tag{3.116}
\]

\[
= \beta_{32} \cdot \hat{\mathbf{R}}_{32} \cdot \beta_{32}. \tag{3.117}
\]

Similarly, \( \alpha_{22} \beta_{22} = \mathbf{I} \) is inserted twice in (3.115). Again, the product of multiplicative factors \( \beta_{21} \cdot \alpha_{22} \) in the curly bracket in (3.116) is never greater than one in magnitude.
due to the boundness property, which also guarantees a moderate magnitude of $\hat{\tilde{R}}_{32}$ in any case. Note that when more than three layers are present, the multiplicative factors in (3.117) are not altered. Therefore, the redefined generalized reflection coefficient between two arbitrarily-indexed adjacent layers for the standing-wave case can be in general expressed as

$$
\tilde{\tilde{R}}_{i+1,i} = \hat{\beta}_{i+1,i} \cdot \hat{\tilde{R}}_{i+1,i} \cdot \hat{\beta}_{i+1,i},
$$

(3.118)

where

$$
\hat{\tilde{R}}_{i+1,i} = \hat{R}_{i+1,i} + \hat{T}_{i+1,i} \cdot \hat{\beta}_{i,i-1} \cdot \hat{\alpha}_{ii} \cdot \hat{R}_{i,i-1} \cdot \hat{\beta}_{i,i-1} \cdot \hat{\alpha}_{ii}

\cdot \left[ I - \hat{R}_{i,i-1} \cdot \hat{\beta}_{i,i-1} \cdot \hat{\alpha}_{ii} \cdot \hat{R}_{i,i-1} \cdot \hat{\beta}_{i,i-1} \cdot \hat{\alpha}_{ii} \right]^{-1} \cdot \hat{T}_{i+1,i}.
$$

(3.119)

Before we proceed to obtain generalized transmission coefficients, the redefinition of the $S$-coefficients [17, Ch. 3] is necessary, as they represent local transmission coefficients in the presence of multi-layers. The $S$-coefficient for the outgoing-wave case
case for three cylindrical layers depicted in Figure 3.5a is modified to

\[ S_{12} = \left[ I - \hat{R}_{21} \cdot \hat{R}_{23} \right]^{-1} \cdot \hat{T}_{12} \]

\[ = (\hat{\beta}_{21} \cdot \hat{\alpha}_{21}) \cdot \left[ I - \hat{R}_{21} \cdot \hat{\beta}_{21} \cdot \hat{\alpha}_{22} \cdot \hat{R}_{23} \cdot \hat{\alpha}_{22} \right]^{-1} \]

\[ \cdot \left( \hat{\beta}_{21} \cdot \hat{\alpha}_{21} \right) \cdot \hat{T}_{12} \cdot \hat{\alpha}_{11} \]

\[ = \hat{\beta}_{21} \cdot \left[ I - \hat{R}_{21} \cdot \hat{\beta}_{21} \cdot \hat{\alpha}_{22} \cdot \hat{R}_{23} \cdot \hat{\beta}_{21} \cdot \hat{\alpha}_{22} \right]^{-1} \cdot \hat{T}_{12} \cdot \hat{\alpha}_{11} \]  (3.120)

\[ = \hat{\beta}_{21} \cdot \hat{S}_{12} \cdot \hat{\alpha}_{11}. \]  (3.121)

Note that \( \hat{\beta}_{21} \cdot \hat{\alpha}_{21} = I \) is inserted twice in (3.120). Therefore, the redefined arbitrarily-indexed \( S \)-coefficient for the outgoing-wave case is written as

\[ S_{i,i+1} = \hat{\beta}_{i+1,i} \cdot \hat{S}_{i,i+1} \cdot \hat{\alpha}_{ii}, \]  (3.122)

where

\[ \hat{S}_{i,i+1} = \left[ I - \hat{R}_{i+1,i} \cdot \hat{\beta}_{i+1,i} \cdot \hat{\alpha}_{i+1,i+1} \cdot \hat{R}_{i+1,i+2} \cdot \hat{\beta}_{i+1,i} \cdot \hat{\alpha}_{i+1,i+1} \right]^{-1} \cdot \hat{T}_{i,i+1}. \]  (3.123)

The \( S \)-coefficient for the standing-wave case for three cylindrical layers depicted in Figure 3.5b is modified to

\[ S_{32} = \left[ I - \hat{R}_{23} \cdot \hat{R}_{21} \right]^{-1} \cdot \hat{T}_{32} \]

\[ = (\hat{\alpha}_{22} \cdot \hat{\beta}_{22}) \cdot \left[ I - \hat{R}_{23} \cdot \hat{\alpha}_{22} \cdot \hat{R}_{21} \cdot \hat{\beta}_{21} \right]^{-1} \]

\[ \cdot (\hat{\alpha}_{22} \cdot \hat{\beta}_{22}) \cdot \hat{T}_{32} \cdot \hat{\beta}_{32} \]  (3.124)

\[ = \hat{\alpha}_{22} \cdot \left[ I - \hat{R}_{23} \cdot \hat{\beta}_{21} \cdot \hat{\alpha}_{22} \cdot \hat{R}_{21} \cdot \hat{\beta}_{21} \cdot \hat{\alpha}_{22} \right]^{-1} \cdot \hat{T}_{32} \cdot \hat{\beta}_{32} \]

\[ = \hat{\alpha}_{22} \cdot \hat{S}_{32} \cdot \hat{\beta}_{32}. \]  (3.125)

Note that \( \hat{\alpha}_{22} \cdot \hat{\beta}_{22} = \hat{I} \) is inserted twice in (3.124). As a result, the redefined arbitrarily-indexed \( S \)-coefficient for the standing-wave case is written as

\[ \bar{S}_{i+1,i} = \hat{\alpha}_{ii} \cdot \hat{S}_{i+1,i} \cdot \hat{\beta}_{i+1,i}. \]  (3.126)
where

\[
\tilde{S}_{i+1,i} = \left[ I - \tilde{R}_{i,i+1} \cdot \beta_{i,i-1} \cdot \alpha_{ii} \cdot \tilde{R}_{i,i-1} \cdot \beta_{i,i-1} \cdot \alpha_{ii} \right]^{-1} \cdot \tilde{T}_{i+1,i}.
\] (3.127)

Let us now consider the generalized transmission coefficient for the outgoing-wave case \((i > j)\) in cylindrically stratified media, which is expressed as

\[
\tilde{T}_{ji} = T_{i-1,i} \cdot S_{i-2,i-1} \cdots S_{j,j+1},
\]

which can be modified in a way that

\[
\tilde{T}_{ji} = T_{i-1,i} \cdot S_{i-2,i-1} \cdots S_{j,j+1} = \left( \beta_{i,i-1} \cdot \tilde{T}_{i-1,i} \cdot \alpha_{i-1,i-1} \right) \cdot \left( \prod_{k=j}^{i-2} \beta_{k+1,k} \cdot \tilde{S}_{k,k+1} \cdot \alpha_{kk} \right)
\]

\[
= \beta_{i,i-1} \cdot \tilde{T}_{i-1,i} \cdot \left( \prod_{k=j}^{i-2} \beta_{k+1,k} \cdot \alpha_{k+1,k+1} \cdot \tilde{S}_{k,k+1} \right) \cdot \alpha_{jj}
\] (3.129)

\[
= \beta_{i,i-1} \cdot \tilde{T}_{ji} \cdot \alpha_{jj}.
\] (3.130)

The magnitudes of the products of multiplicative factors \(\beta_{k+1,k} \cdot \alpha_{k+1,k+1}\) in (3.129) are never greater than one, which stabilizes the computation of \(\tilde{T}_{ji}\). The product in (3.129) is the product of a number of \(2 \times 2\) matrices, so the order should be specified. The \(2 \times 2\) matrix for \(k = j\) and \(2 \times 2\) matrix for \(k = i - 2\) should be placed in the rightmost and leftmost in the matrix product, respectively. Furthermore, when \(i = j + 1\), the matrix product reduces to an identity matrix. It should be also noted that the associated multiplicative factors shown in (3.130) are the generalized version of those shown in (3.109c). As an example, let us consider \(\tilde{T}_{25}\) depicted in Figure 3.6,
which is expressed as

\[
\tilde{T}_{25} = T_{45} \cdot S_{34} \cdot S_{23} \\
= (\beta_{54} \cdot \hat{T}_{45} \cdot \alpha_{44}) \cdot \left( \prod_{k=2}^{3} (\beta_{k+1,k} \cdot \hat{S}_{k,k+1} \cdot \alpha_{kk}) \right) \\
= (\beta_{54} \cdot \hat{T}_{45} \cdot \alpha_{44}) \cdot (\beta_{43} \cdot \hat{S}_{34} \cdot \alpha_{33}) \cdot (\beta_{32} \cdot \hat{S}_{23} \cdot \alpha_{22}) \\
= \beta_{54} \cdot \left[ \hat{T}_{45} \cdot (\beta_{43} \cdot \alpha_{44}) \cdot \hat{S}_{34} \cdot (\beta_{32} \cdot \alpha_{33}) \cdot \hat{S}_{23} \right] \cdot \alpha_{22} \\
= \beta_{54} \cdot \hat{T}_{25} \cdot \alpha_{22}. \quad (3.131)
\]

Furthermore, the generalized transmission coefficient for the standing-wave case \((i < j)\) in cylindrically stratified media is expressed as

\[
\tilde{T}_{ji} = T_{i+1,i} \cdot S_{i+2,i+1} \cdot \ldots \cdot S_{j,j-1}. \quad (3.132)
\]
Similarly, (3.132) is modified to

\[
\tilde{T}_{ji} = \mathbf{T}_{i+1,i} \cdot \tilde{S}_{i+2,i+1} \cdots \tilde{S}_{j,j-1} = \left( \frac{1}{\alpha_{ii}} \cdot \tilde{T}_{i+1,i} \cdot \overline{\beta}_{i+1,i} \right) \cdot \left( \prod_{k=i+1}^{j-1} \alpha_{kk} \cdot \hat{S}_{k+1,k} \cdot \overline{\beta}_{k+1,k} \right) = \alpha_{ii} \cdot \tilde{T}_{ji} \cdot \overline{\beta}_{j,j-1}\]

(3.133)

Similarly, the magnitudes of the product of multiplicative factors \( \overline{\beta}_{k,k-1} \cdot \alpha_{kk} \) in (3.133) are never greater than one in any case. For the matrix product in (3.133), the \( 2 \times 2 \) matrix for \( k = i + 1 \) and \( 2 \times 2 \) matrix for \( k = j - 1 \) should be placed in the leftmost and rightmost, which is opposite to the outgoing-wave case. Furthermore, when \( i = j - 1 \), the matrix product reduces to an identity matrix. The associated multiplicative factors shown in (3.134) are the generalized version of those shown in (3.109d). As an example, let us consider \( \tilde{T}_{52} \) depicted in Figure 3.7, which is expressed as

\[
\tilde{T}_{52} = \mathbf{T}_{32} \cdot \tilde{S}_{43} \cdot \tilde{S}_{54} = \left( \frac{1}{\alpha_{22}} \cdot \tilde{T}_{32} \cdot \overline{\beta}_{32} \right) \cdot \left( \prod_{k=i+1}^{j-1} \alpha_{kk} \cdot \hat{S}_{k+1,k} \cdot \overline{\beta}_{k+1,k} \right) = \alpha_{22} \cdot \tilde{T}_{52} \cdot \overline{\beta}_{54}\]

(3.135)

Several auxiliary coefficients appeared in (3.98a) – (3.98d) should be redefined properly as well. For the first type of the integrand shown in (3.98a), \( \tilde{M}_{j+} \) is redefined
as shown below.

\[
\tilde{M}_{j+} = \left[ I - \tilde{R}_{j,j-1} \cdot \tilde{R}_{j,j+1} \right]^{-1} \\
= \overline{\beta}_{j,j-1} \cdot \overline{\alpha}_{j,j-1} \cdot \left[ I - \tilde{R}_{j,j-1} \cdot \tilde{R}_{j,j-1} \cdot \overline{\beta}_{j,j-1} \cdot \overline{\alpha}_{j,j} \cdot \tilde{R}_{j,j+1} \cdot \overline{\alpha}_{j,j} \right]^{-1} \\
= \overline{\beta}_{j,j-1} \cdot \tilde{M}_{j+} \cdot \overline{\alpha}_{j,j-1},
\]

where \( \overline{\beta}_{j,j-1} \cdot \overline{\alpha}_{j,j-1} = I \) is inserted twice for mathematical manipulation and the radial distance corresponding to subscript \([j - 1, j]\) is \( a_{[j - 1, j]} = ca_{j-1} + (1 - c)a_{j} \), \( 0 \leq c \leq 1 \). Two extreme choices of \( a_{[j - 1, j]} \) \( (a_{[j - 1, j]} = a_{j-1} \) and \( a_{[j - 1, j]} = a_{j} \)) would be used for notational convenience but these are not useful in the redefinition of the integrand, as clarified further below in Section 3.2.3. For the second type of the
inteogrand shown in (3.98b), \( \tilde{M}_{j-} \) is similarly redefined as

\[
\tilde{M}_{j-} = \left[ I - \tilde{R}_{j, j+1} \cdot \tilde{R}_{j, j-1} \right]^{-1}
\]

\[
= \alpha_{j, [j-1, j]} \cdot \tilde{\beta}_{j, [j-1, j]} \cdot \left[ I - \alpha_{jj} \cdot \tilde{R}_{j, j+1} \cdot \alpha_{jj} \cdot \tilde{\beta}_{j, j-1} \cdot \tilde{R}_{j, j-1} \cdot \tilde{\beta}_{j, j-1} \right]^{-1}
\]

\[
= \alpha_{j, [j-1, j]} \cdot \tilde{\beta}_{j, [j-1, j]} \cdot \left[ I - \tilde{\beta}_{j, [j-1, j]} \cdot \alpha_{jj} \cdot \tilde{R}_{j, j+1} \cdot \tilde{\beta}_{j, j-1} \cdot \alpha_{jj} \cdot \tilde{R}_{j, j-1} \cdot \tilde{\beta}_{j, j-1} \cdot \tilde{\beta}_{j, j-1} \right]^{-1}
\]

\[
= \alpha_{j, [j-1, j]} \cdot \tilde{M}_{j-} \cdot \tilde{\beta}_{j, [j-1, j]},
\]

(3.137)

where \( \alpha_{j, [j-1, j]} \cdot \tilde{\beta}_{j, [j-1, j]} = I \) is inserted twice and the same subscript \([j - 1, j]\) is used as well. Again, the two extreme cases of \( a_{[j-1, j]} \) are undesired for the proper redefinition of the integrand as shown in Section 3.2.3. For the third type of the integrand shown in (3.98c), \( N_{i+} \) is redefined as

\[
N_{i+} = \left[ I - \tilde{R}_{i, i-1} \cdot \tilde{R}_{i, i+1} \right]^{-1}
\]

\[
= \tilde{\beta}_{i, i-1} \cdot \alpha_{i, i-1} \cdot \left[ I - \tilde{\beta}_{i, i-1} \cdot \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{i, i-1} \cdot \alpha_{ii} \cdot \tilde{R}_{i, i+1} \cdot \alpha_{ii} \right]^{-1} \cdot \tilde{\beta}_{i, i-1} \cdot \alpha_{i, i-1}
\]

\[
= \tilde{\beta}_{i, i-1} \cdot \left[ I - \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{i, i-1} \cdot \alpha_{ii} \cdot \tilde{R}_{i, i+1} \cdot \tilde{\beta}_{i, i-1} \cdot \alpha_{ii} \right]^{-1} \cdot \alpha_{i, i-1}
\]

\[
= \tilde{\beta}_{i, i-1} \cdot \tilde{N}_{i+} \cdot \alpha_{i, i-1},
\]

(3.138)

where \( \tilde{\beta}_{i, i-1} \cdot \alpha_{i, i-1} = I \) is inserted twice for mathematical manipulation. For the fourth type of the integrand shown in (3.98d), \( N_{i-} \) is redefined as

\[
N_{i-} = \left[ I - \tilde{R}_{i, i+1} \cdot \tilde{R}_{i, i-1} \right]^{-1}
\]

\[
= \tilde{\beta}_{ii} \cdot \alpha_{ii} \cdot \left[ I - \tilde{\beta}_{ii} \cdot \tilde{R}_{i, i+1} \cdot \tilde{\beta}_{ii} \cdot \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{ii} \cdot \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{ii} \right]^{-1} \cdot \alpha_{ii} \cdot \tilde{\beta}_{ii}
\]

\[
= \tilde{\beta}_{ii} \cdot \left[ I - \tilde{R}_{i, i+1} \cdot \tilde{\beta}_{ii} \cdot \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{ii} \cdot \tilde{R}_{i, i-1} \cdot \tilde{\beta}_{ii} \right]^{-1} \cdot \alpha_{ii}
\]

\[
= \tilde{\beta}_{ii} \cdot \tilde{N}_{i-} \cdot \alpha_{ii},
\]

(3.139)
where \( \mathbf{\alpha}_i \cdot \mathbf{\beta}_i = \mathbf{I} \) is inserted twice for mathematical manipulation as well.

### 3.2.3 Modified Integrand

The modification of the integrand for cylindrically stratified and uniaxial media is performed in a similar fashion to Section 2.3.5. As can be seen below, the modified integrand looks a bit more complicated due to uniaxial anisotropy. For Case 1 in (3.98a), there are four arguments of interest: \( k_j \rho a_j^{-1}, k_j \rho \rho', k_j \rho, \) and \( k_j \rho a_j \). For convenience, we let \( a_j^{-1} = a_1, \rho' = a_2, \rho = a_3, \) and \( a_j = a_4 \) so that \( a_1 < a_2 < a_3 < a_4 \). The integrand is redefined as

\[
\mathbf{\Phi}_n(\rho, \rho') = \left[ \mathbf{H}_{zj\rho} + \mathbf{J}_{zj\rho} \cdot \mathbf{R}_{z,j+1} \right] \cdot \mathbf{M}_{j+} \cdot \left[ \mathbf{J}_{zj\rho'} + \mathbf{R}_{z,j-1} \cdot \mathbf{H}_{zj\rho'} \right]
\]

\[
= \left[ \mathbf{H}_{zj\rho} \cdot \mathbf{\alpha}_3 + \mathbf{J}_{zj\rho} \cdot \mathbf{\beta}_3 \cdot \mathbf{\alpha}_{j4} \cdot \mathbf{R}_{z,j+1} \cdot \mathbf{\alpha}_{j4} \right] \cdot \mathbf{\beta}_{j2} \cdot \mathbf{\alpha}_{j2} \cdot \mathbf{\alpha}_{j2}
\]

\[
= \left[ \mathbf{J}_{zj\rho'} \cdot \mathbf{\beta}_{j2} + \mathbf{R}_{z,j-1} \cdot \mathbf{\beta}_{j1} \cdot \mathbf{\alpha}_j \cdot \mathbf{H}_{zj \rho'} \cdot \mathbf{\alpha}_{j2} \right]
\]

\[
= \left[ \mathbf{J}_{zj\rho'} \cdot \mathbf{\beta}_{j2} \cdot \mathbf{\alpha}_3 \cdot \mathbf{H}_{zj\rho} + \mathbf{J}_{zj\rho} \cdot \mathbf{R}_{z,j+1} \cdot \mathbf{\beta}_{j2} \cdot \mathbf{\alpha}_{j4} \right] \cdot \mathbf{\alpha}_{j2} \cdot \mathbf{\alpha}_{j2} \cdot \mathbf{\alpha}_{j2}
\]

Note that, as (3.140) shows, the corresponding radial distance to the subscript \([j-1, j]\) in (3.136) is chosen to be \( a_2 \), neither \( a_1 \) nor \( a_4 \). To be more specific, the radial distance of the source \( \rho' \) is selected. The choice enables the left and right squared brackets in (3.141) to be balanced in magnitude and to yield stable computation. For Case 2 in (3.98b), four arguments are of interest: \( k_j \rho a_j^{-1}, k_j \rho \rho', k_j \rho, \) and \( k_j \rho a_j \). Similarly, we let \( a_{j-1} = a_1, \rho = a_2, \rho' = a_3, \) and \( a_j = a_4 \) so that \( a_1 < a_2 < a_3 < a_4 \). The integrand
is redefined as

$$
\mathbf{F}_\nu(\rho, \rho') = \left[ \mathbf{J}_{zj\rho} + \mathbf{H}_{zj\rho} \cdot \tilde{\mathbf{R}}_{j,j-1} \right] \cdot \tilde{\mathbf{M}}_{j-} \cdot \left[ \mathbf{H}_{zj\rho'} + \tilde{\mathbf{R}}_{j,j+1} \cdot \mathbf{J}_{zj\rho'} \right]
$$

$$
= \left[ \mathbf{J}_{zj\rho} \cdot \tilde{\mathbf{H}}_{zj\rho} + \tilde{\mathbf{R}}_{j,j-1} \cdot \tilde{\mathbf{J}}_{zj\rho} \right] \cdot \alpha_{j3} \cdot \tilde{\mathbf{M}}_{j-} \cdot \tilde{\mathbf{B}}_{j3} \quad (3.142)
$$

$$
= \left[ \tilde{\mathbf{H}}_{zj\rho} \cdot \tilde{\mathbf{J}}_{zj\rho} + \tilde{\mathbf{R}}_{j,j-1} \cdot \tilde{\mathbf{J}}_{zj\rho} \cdot \tilde{\mathbf{B}}_{j3} \right]
$$

$$
= \left[ \tilde{\mathbf{H}}_{zj\rho} + \tilde{\mathbf{R}}_{j,j+1} \cdot \tilde{\mathbf{J}}_{zj\rho} \right] \cdot \tilde{\mathbf{M}}_{j-} \cdot \tilde{\mathbf{B}}_{j3} \quad (3.143)
$$

It should be noted that, as (3.142) shows, the corresponding radial distance to the subscript \([j - 1, j]\) in (3.137) is chosen to be \(a_3\), the radial distance of the source (neither \(a_1\) nor \(a_4\)). Again, this choice makes the left and right squared brackets in (3.143) balanced in magnitude and yields stable computation. For Case 3 in (3.98c), there are 6 arguments of interest: \(k_{i\rho}a_{i-1}, k_{i\rho}a_i, k_{i\rho}a_{j-1}, k_{j\rho}a_j\). We let \(a_{i-1} = a_1, \rho = a_2, a_i = a_3, a_{j-1} = b_1, \rho' = b_2, \) and \(a_j = b_3\) so that \(a_1 < a_2 < a_3\) and \(b_1 < b_2 < b_3\). The integrand is redefined as

$$
\mathbf{F}_\nu(\rho, \rho') = \left[ \mathbf{H}_{zi\rho} + \mathbf{J}_{zi\rho} \cdot \tilde{\mathbf{R}}_{i,i+1} \right] \cdot \tilde{\mathbf{N}}_{i+} \cdot \tilde{\mathbf{T}}_{j i} \cdot \tilde{\mathbf{M}}_{j+} \cdot \left[ \mathbf{J}_{zj\rho'} + \tilde{\mathbf{R}}_{j,j-1} \cdot \mathbf{H}_{zj\rho'} \right]
$$

$$
= \left[ \tilde{\mathbf{H}}_{zi\rho} + \tilde{\mathbf{R}}_{i,i+1} \cdot \tilde{\mathbf{J}}_{zi\rho} \right] \cdot \left( \tilde{\mathbf{B}}_{i1} \cdot \tilde{\mathbf{T}}_{j1} \cdot \tilde{\mathbf{N}}_{i+} \right) \cdot \left( \tilde{\mathbf{B}}_{j2} \cdot \tilde{\mathbf{M}}_{j+} \cdot \tilde{\mathbf{B}}_{j3} \right) \quad (3.144)
$$

$$
= \left[ \tilde{\mathbf{H}}_{zi\rho} + \tilde{\mathbf{R}}_{i,i+1} \cdot \tilde{\mathbf{J}}_{zi\rho} \right] \cdot \left( \tilde{\mathbf{B}}_{i1} \cdot \tilde{\mathbf{T}}_{j1} \cdot \tilde{\mathbf{N}}_{i+} \right) \cdot \left( \tilde{\mathbf{B}}_{j2} \cdot \tilde{\mathbf{M}}_{j+} \cdot \tilde{\mathbf{B}}_{j3} \right) \quad (3.145)
$$

In should be stressed that the corresponding radial distance for \(\tilde{\mathbf{M}}_{j+}\) in (3.144) is now \(b_2\), which is the radial distance of the source. For Case 4 in (3.98d), the arguments

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of interest are the same as those for Case 3. The integrand is redefined as

\[
\mathbf{F}_n(\rho, \rho') = \left[ \mathbf{J}_{zi\rho} + \mathbf{H}_{zi\rho} \cdot \mathbf{\hat{R}}_{i,i-1} \right] \cdot \mathbf{N}_{i-1} \cdot \mathbf{\hat{T}}_{ji} \cdot \mathbf{\hat{M}}_{j-1} \cdot \left[ \mathbf{H}_{zj\rho'} + \mathbf{\hat{R}}_{j,j+1} \cdot \mathbf{J}_{zj\rho'} \right] \\
= \left[ \mathbf{J}_{zi\rho} \cdot \mathbf{\beta}_{i2} + \mathbf{H}_{zi\rho} \cdot \mathbf{\alpha}_{i2} \cdot \mathbf{\beta}_{i1} \cdot \mathbf{\hat{R}}_{i,i-1} \cdot \mathbf{\beta}_{i1} \right] \\
\cdot \left( \mathbf{\alpha}_{i3} \cdot \mathbf{\hat{N}}_{i-1} \cdot \mathbf{\beta}_{i3} \right) \cdot \left( \mathbf{\alpha}_{j3} \cdot \mathbf{\hat{T}}_{ji} \cdot \mathbf{\beta}_{j1} \right) \cdot \left( \mathbf{\alpha}_{j2} \cdot \mathbf{\hat{M}}_{j-1} \cdot \mathbf{\beta}_{j2} \right) \\
\cdot \left[ \mathbf{\hat{H}}_{zj\rho'} \cdot \mathbf{\alpha}_{j2} + \mathbf{\alpha}_{j3} \cdot \mathbf{\hat{R}}_{j,j+1} \cdot \mathbf{\alpha}_{j3} \cdot \mathbf{\hat{J}}_{zj\rho'} \cdot \mathbf{\beta}_{j2} \right] \\
(3.146)
\]

\[
= \mathbf{\beta}_{i2} \cdot \mathbf{\alpha}_{i3} \cdot \mathbf{\hat{J}}_{zi\rho} + \mathbf{\beta}_{i1} \cdot \mathbf{\alpha}_{i2} \cdot \mathbf{\hat{H}}_{zi\rho} \cdot \mathbf{\hat{R}}_{i,i-1} \cdot \mathbf{\beta}_{i1} \cdot \mathbf{\alpha}_{i3} \\
\cdot \mathbf{\hat{N}}_{i-1} \cdot \mathbf{\hat{T}}_{ji} \cdot \mathbf{\beta}_{j1} \cdot \mathbf{\alpha}_{j2} \\
\cdot \mathbf{\hat{M}}_{j-1} \cdot \left[ \mathbf{\hat{H}}_{zj\rho'} + \mathbf{\beta}_{j2} \cdot \mathbf{\alpha}_{j3} \cdot \mathbf{\hat{R}}_{j,j+1} \cdot \mathbf{\hat{J}}_{zj\rho'} \cdot \mathbf{\beta}_{j2} \cdot \mathbf{\alpha}_{j3} \right] . \\
(3.147)
\]

Again, the radial distance of the source \( b_2 \) is chosen to be the corresponding radial distance for \( \mathbf{\hat{M}}_{j-} \) in (3.146).

3.2.4 Azimuth Series Folding

The folded summation over azimuth modes shown in Section 2.3.7 can also be applied to the integrands shown in (3.141), (3.143), (3.145), and (3.147) because the symmetrical properties of involved cylindrical eigenfunctions can still be utilized. In this section, the final expressions of all three components of electromagnetic fields that are slightly different from those in Section 2.3.7 are provided.

\[
\begin{align*}
\left[ \begin{array}{c}
E_z \\
H_z
\end{array} \right] &= \frac{iI}{4\pi\omega\varepsilon_{hj}} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left[ \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \mathbf{F}_n(\rho, \rho') \cdot \mathbf{\hat{D}}_j \right] \\
&= \frac{-II}{8\pi\omega\varepsilon_{hj}} \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \left\{ \left[ \mathbf{X}_{z,0} + \sum_{n=1}^{\infty} \mathbf{\ddot{X}}_{z,n} \right] \cdot \mathbf{\hat{D}}_{j1} + \left[ \sum_{n=1}^{\infty} \mathbf{\dddot{X}}_{z,n} \right] \\
&\quad + \left[ \mathbf{Z}_{z,0} + \sum_{n=1}^{\infty} \mathbf{\ddot{Z}}_{z,n} \right] \cdot \mathbf{\hat{D}}_{j3} \right\} , \\
(3.148)
\end{align*}
\]
\[ \begin{bmatrix} E_\rho \\ H_\rho \end{bmatrix} = \frac{i I_l}{4 \pi \omega \epsilon_h j} \int_{-\infty}^{\infty} dk_z e^{i k_z (z - z')} \frac{1}{k_{i\rho}^2} \left[ \sum_{n=-\infty}^{\infty} e^{i n(\phi - \phi')} B_{hn} \cdot F_n(\rho, \rho') \cdot \hat{D}'_j \right] , \]

\[ = - \frac{I_l}{8 \pi \omega \epsilon_h j} \int_{-\infty}^{\infty} dk_z e^{i k_z (z - z')} \frac{1}{k_{i\rho}^2} \left\{ \begin{bmatrix} X_{\rho,0} + \sum_{n=1}^{\infty} \ddot{X}_{\rho,n} \\ \dddot{Z}_{\rho,0} + \sum_{n=1}^{\infty} \dddot{Z}_{\rho,n} \end{bmatrix} \cdot \hat{D}'_j + \sum_{n=1}^{\infty} \dot{Y}_{\rho,n} \right\} , \]  

(3.149)

\[ \begin{bmatrix} E_\phi \\ H_\phi \end{bmatrix} = \frac{i I_l}{4 \pi \omega \epsilon_h j} \int_{-\infty}^{\infty} dk_z e^{i k_z (z - z')} \frac{1}{k_{i\rho}^2} \left[ \sum_{n=-\infty}^{\infty} e^{i n(\phi - \phi')} C_{hn} \cdot F_n(\rho, \rho') \cdot \hat{D}'_j \right] , \]

\[ = - \frac{I_l}{8 \pi \omega \epsilon_h j} \int_{-\infty}^{\infty} dk_z e^{i k_z (z - z')} \frac{1}{k_{i\rho}^2} \left\{ \begin{bmatrix} X_{\phi,0} + \sum_{n=1}^{\infty} \ddot{X}_{\phi,n} \\ \dddot{Z}_{\phi,0} + \sum_{n=1}^{\infty} \dddot{Z}_{\phi,n} \end{bmatrix} \cdot \hat{D}'_j + \sum_{n=1}^{\infty} \dot{Y}_{\phi,n} \right\} , \]  

(3.150)

### 3.3 Numerical Results

This section provides some validation results from the new algorithm detailed above. In Section 3.3.1, the results are compared to analytical solutions available in homogeneous and anisotropic media. In Section 3.3.2, the results are compared to the Finite Element Method (FEM) results for several selected cases of practical interest. These results also examine the effect of anisotropy ratios in surrounding earth formations. Throughout this section, the cross-sections of all layers are assumed to be circular and concentric. Field values are expressed in a phasor form under \( e^{i\omega t} \) convention.

#### 3.3.1 Homogeneous and Uniaxial Media

To validate the new algorithm, the results from the new algorithm are first compared to the analytical expressions of electromagnetic fields in homogeneous uniaxial
media. For detailed derivations of such analytical solutions, refer to the Appendix D. For comparison, a square region of 10 cm \( \times \) 10 cm in the \( \rho z \)-plane, where receiver points are located, is considered. The source is a \( z \)-directed Hertzian electric dipole with unit dipole moment. The operating frequency is 36 kHz. The medium has \( \epsilon_h = 16 \epsilon_0 \) [F/m], \( \mu_h = 16 \mu_0 \) [H/m], \( \sigma_h = 16 \) [S/m], where \( \epsilon_0 \) and \( \mu_0 \) represent permittivity and permeability in free space. These horizontal values are fixed whereas different \( \epsilon_v, \mu_v, \) and \( \sigma_v \) values are considered to yield various anisotropy ratios. Figures 3.8a – 3.8d show the relative error between the new algorithm and the analytical solution for different maximum orders \( n_{\text{max}} \) used for the azimuth summation, and for various numbers of quadrature points \( n_{\text{int}} \) in the numerical integral. It is assumed that \( \epsilon_v = \epsilon_0 \) [F/m], \( \mu_v = \mu_0 \) [H/m], and \( \sigma_v = 1 \) [S/m], with \( \kappa_\epsilon = \kappa_\mu = \kappa = 4 \). The relative error is defined as

\[
\text{relative error}_{\text{dB}} = 10 \log_{10} \frac{|E_{z,a} - E_{z,n}|}{|E_{z,a}|},
\]

where \( E_{z,a} \) and \( E_{z,n} \) indicate analytical and numerical values, respectively. As expected, smaller relative errors are obtained for larger number of quadrature points or summation terms.

Figures 3.9a – 3.9d show the relative error distribution for \( \kappa_\epsilon = \kappa_\mu = \kappa = 2 \), under the assumption of \( \epsilon_v = 4 \epsilon_0 \) [F/m], \( \mu_v = 4 \mu_0 \) [H/m], and \( \sigma_v = 4 \) [S/m]. As expected, higher \( n_{\text{max}} \) and \( n_{\text{int}} \) produce smaller relative errors. Compared to the case with \( \kappa = 4 \), the error distribution in this case has a slower decay along the vertical direction.

Figures 3.10a – 3.10d show the relative error distribution for \( \kappa_\epsilon = \kappa_\mu = \kappa = \sqrt{2} \), under the assumption of \( \epsilon_v = 8 \epsilon_0 \) [F/m], \( \mu_v = 8 \mu_0 \) [H/m], and \( \sigma_v = 8 \) [S/m]. Compared to the previous cases shown, the error distribution in this case has a
Figure 3.8: Relative error distribution with $\kappa = 4$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$.

slower decay along the vertical direction. Similar observation is obtained and the error distribution is more expanded vertically.

Finally, Figures 3.11a – 3.11d show the relative error distribution for $\kappa_\epsilon = \kappa_\mu = \kappa = 1$, under the assumption of $\epsilon_v = 16\epsilon_0$ [F/m], $\mu_v = 16\mu_0$ [H/m], and $\sigma_v = 16$ [S/m], which indeed recovers the isotropic case. Even though the relative error is bigger than the other cases, the relative error is reduced by higher $n_{\text{max}}$ and $n_{\text{int}}$. 

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Figure 3.9: Relative error distribution with $\kappa = 2$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$.

Compared to the previous cases, the error distribution in this case has the slowest decay along the vertical direction. In order to further scrutinize the effect of $n_{\text{max}}$ and $n_{\text{int}}$, the receiver point is next fixed at $\rho - \rho' = 10$ cm, $\phi - \phi' = 0^\circ$, and $z - z' = 10$ cm. Figures 3.12a – 3.12d show the error as $n_{\text{max}}$ and $n_{\text{int}}$ vary. In all cases, an improvement in the accuracy achieved by increasing the number of quadrature points as long as a sufficient large $n_{\text{max}}$ is chosen ($\gtrsim 30$).
Figure 3.10: Relative error distribution with $\kappa = \sqrt{2}$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$. 
Figure 3.11: Relative error distribution with $\kappa = 1$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$. 
Figure 3.12: Relative error distribution in terms of various maximum orders $n_{\text{max}}$ and integration points $n_{\text{int}}$ with the receiver point at $\rho - \rho' = 10$ cm, $\phi - \phi' = 0^\circ$, and $z - z' = 10$ cm: (a) $\kappa = 4$, (b) $\kappa = 2$, (c) $\kappa = \sqrt{2}$, and (d) $\kappa = 1$. 

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3.3.2 Cylindrically Layered Media

In this section, a number of practical cases are considered to validate the new algorithm. In all the cases, both the relative permittivity $\varepsilon_r$ and relative permeability $\mu_r$ are set to one, whereas the conductivity (or complex permittivity $\tilde{\varepsilon}$, see (3.4)) only exhibits uniaxial anisotropy. Furthermore, the horizontal resistivity, reciprocal of horizontal conductivity, is fixed to $5 \, \Omega \cdot m$, whereas the vertical resistivity varies for different anisotropy ratios $\kappa_\varepsilon$.

Case 1 is depicted in Figure 3.13a. There are three layers representing a metallic mandrel of high conductivity, the borehole filled with isotropic fluid, and the outermost layer (earth formation) with uniaxial anisotropy. Case 2 is depicted in Figure 3.13b. A metallic casing (third layer) is inserted between the borehole and anisotropic formation. Table 3.3 provides the comparison of corresponding results for Case 1 in terms of the square of anisotropy ratios. The discrepancy in the magnitude of magnetic fields can be traced to limitations in the FEM modeling such as inaccurate computation of strong gradient field near to the source (due to the finite cell size) and mesh truncation artifacts. In particular, the fields obtained by FEM have smaller magnitudes because the Dirichlet boundary condition enforced at the mesh boundary moves the zero reference potential (originally at infinity) closer to the source location. This causes a small offset in the results as illustrated in Table 3.4, which shows the relative difference in the computed field magnitudes with excellent agreement. Table 3.5 and 3.6 provide corresponding results for Case 2.

Case 3 and 4 are depicted in Figures 3.13c and 3.13d, which are the same as Case 2 except for the operating frequencies, which for Case 3 is 1 kHz and for Case 4 is 125 kHz. Tables 3.7, 3.8, 3.9, and 3.10 provide the corresponding results.
In this simulation, a second formation layer with a horizontal resistivity of 5 Ω-m was added. (Note that casing resistivity is changed to 1e-5 Ω-m.)

Results obtained with the FEM code for different anisotropy ratios (AR = Rv2/Rh2) for the second medium are below:

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field (A/m)</th>
<th>Angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.3116</td>
<td>98.1899</td>
</tr>
<tr>
<td>2</td>
<td>10.2365</td>
<td>97.5910</td>
</tr>
<tr>
<td>5</td>
<td>10.1565</td>
<td>96.9883</td>
</tr>
<tr>
<td>10</td>
<td>10.1070</td>
<td>96.6327</td>
</tr>
</tbody>
</table>

\[ Rc = 1e-5 \, \text{Ωm} \]

In this simulation, the outer conductive layer is reduced to a width of 0.125". A formation layer with 5 Ω-m horizontal resistivity fills the rest of the space.

Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field (A/m)</th>
<th>Angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>46.6091</td>
<td>118.4181</td>
</tr>
<tr>
<td>2</td>
<td>46.6099</td>
<td>118.4234</td>
</tr>
<tr>
<td>5</td>
<td>46.6110</td>
<td>118.4283</td>
</tr>
<tr>
<td>10</td>
<td>46.6118</td>
<td>118.4310</td>
</tr>
</tbody>
</table>

\[ Rc = 1e-5 \, \text{Ωm} \]

In this simulation, the outer conductive layer is reduced to a width of 0.125". A formation layer with 5 Ω-m horizontal resistivity fills the rest of the space.

Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field (A/m)</th>
<th>Angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>299.2</td>
<td>110.9753</td>
</tr>
<tr>
<td>2</td>
<td>299.5</td>
<td>110.9762</td>
</tr>
<tr>
<td>5</td>
<td>299.7</td>
<td>110.9770</td>
</tr>
<tr>
<td>10</td>
<td>299.8</td>
<td>110.9774</td>
</tr>
</tbody>
</table>

\[ Rc = 1e-5 \, \text{Ωm} \]

This time frequency was increased to 125 kHz.

Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field (A/m)</th>
<th>Angle (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18.7967</td>
<td>110.9753</td>
</tr>
<tr>
<td>2</td>
<td>18.7959</td>
<td>110.9762</td>
</tr>
<tr>
<td>5</td>
<td>18.7962</td>
<td>110.9770</td>
</tr>
<tr>
<td>10</td>
<td>18.7963</td>
<td>110.9774</td>
</tr>
</tbody>
</table>

\[ Rc = 1e-5 \, \text{Ωm} \]

Figure 3.13: Practical cases in the \( \rho z \)-plane (a) Case 1, (b) Case 2, (c) Case 3, and (d) Case 4.
Table 3.3: Comparison of magnetic fields in terms of various anisotropy ratios for Case 1.

<table>
<thead>
<tr>
<th>Anisotropy Ratio $\kappa^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10.3116 \angle 98.1899^\circ$</td>
<td>$10.5475 \angle 98.1390^\circ$</td>
<td>10 sec.</td>
</tr>
<tr>
<td>2</td>
<td>$10.2365 \angle 97.5910^\circ$</td>
<td>$10.4723 \angle 97.5486^\circ$</td>
<td>32 sec.</td>
</tr>
<tr>
<td>5</td>
<td>$10.1565 \angle 96.9883^\circ$</td>
<td>$10.3924 \angle 96.9612^\circ$</td>
<td>32 sec.</td>
</tr>
<tr>
<td>10</td>
<td>$10.1070 \angle 96.6327^\circ$</td>
<td>$10.3428 \angle 96.6152^\circ$</td>
<td>31 sec.</td>
</tr>
</tbody>
</table>

Table 3.4: Comparison of magnitude difference in magnetic fields for Case 1.

<table>
<thead>
<tr>
<th>Magnitude Difference</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2 = 1$ and $\kappa^2 = 2$</td>
<td>0.0751</td>
<td>0.0752</td>
</tr>
<tr>
<td>Between $\kappa^2 = 2$ and $\kappa^2 = 5$</td>
<td>0.0800</td>
<td>0.0799</td>
</tr>
<tr>
<td>Between $\kappa^2 = 5$ and $\kappa^2 = 10$</td>
<td>0.0495</td>
<td>0.0496</td>
</tr>
</tbody>
</table>

Table 3.5: Comparison of magnetic fields in terms of various anisotropy ratios for Case 2.

<table>
<thead>
<tr>
<th>Anisotropy Ratio $\kappa^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$46.6091 \angle 118.4181^\circ$</td>
<td>$46.6303 \angle 118.4324^\circ$</td>
<td>15 sec.</td>
</tr>
<tr>
<td>2</td>
<td>$46.6099 \angle 118.4234^\circ$</td>
<td>$46.6311 \angle 118.4381^\circ$</td>
<td>44 sec.</td>
</tr>
<tr>
<td>5</td>
<td>$46.6110 \angle 118.4283^\circ$</td>
<td>$46.6321 \angle 118.4432^\circ$</td>
<td>44 sec.</td>
</tr>
<tr>
<td>10</td>
<td>$46.6118 \angle 118.4310^\circ$</td>
<td>$46.6329 \angle 118.4459^\circ$</td>
<td>44 sec.</td>
</tr>
</tbody>
</table>

Table 3.6: Comparison of magnitude difference in magnetic fields for Case 2.

<table>
<thead>
<tr>
<th>Magnitude Difference</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2 = 1$ and $\kappa^2 = 2$</td>
<td>-0.0008</td>
<td>-0.0008</td>
</tr>
<tr>
<td>Between $\kappa^2 = 2$ and $\kappa^2 = 5$</td>
<td>-0.0011</td>
<td>-0.0010</td>
</tr>
<tr>
<td>Between $\kappa^2 = 5$ and $\kappa^2 = 10$</td>
<td>-0.0008</td>
<td>-0.0008</td>
</tr>
</tbody>
</table>
Table 3.7: Comparison of magnetic fields in terms of various anisotropy ratios for Case 3.

<table>
<thead>
<tr>
<th>Square of Anisotropy Ratio $\kappa^2_\varepsilon$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>290.2144 $\angle 127.4332^\circ$</td>
<td>290.2711 $\angle 127.4185^\circ$</td>
<td>7 sec.</td>
</tr>
<tr>
<td>2</td>
<td>289.9780 $\angle 127.4252^\circ$</td>
<td>290.0564 $\angle 127.4170^\circ$</td>
<td>22 sec.</td>
</tr>
<tr>
<td>5</td>
<td>289.7678 $\angle 127.4089^\circ$</td>
<td>289.8157 $\angle 127.4175^\circ$</td>
<td>22 sec.</td>
</tr>
<tr>
<td>10</td>
<td>289.6754 $\angle 127.3988^\circ$</td>
<td>289.6589 $\angle 127.4189^\circ$</td>
<td>22 sec.</td>
</tr>
</tbody>
</table>

Table 3.8: Comparison of magnitude difference in magnetic fields for Case 3.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2_\varepsilon = 1$ and $\kappa^2_\varepsilon = 2$</td>
<td>0.2364</td>
<td>0.2147</td>
</tr>
<tr>
<td>Between $\kappa^2_\varepsilon = 2$ and $\kappa^2_\varepsilon = 5$</td>
<td>0.2102</td>
<td>0.2407</td>
</tr>
<tr>
<td>Between $\kappa^2_\varepsilon = 5$ and $\kappa^2_\varepsilon = 10$</td>
<td>0.0924</td>
<td>0.1568</td>
</tr>
</tbody>
</table>

Table 3.9: Comparison of magnetic fields in terms of various anisotropy ratios for Case 4.

<table>
<thead>
<tr>
<th>Square of Anisotropy Ratio $\kappa^2_\varepsilon$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18.7957 $\angle 110.9753^\circ$</td>
<td>18.8074 $\angle 110.9191^\circ$</td>
<td>7 sec.</td>
</tr>
<tr>
<td>2</td>
<td>18.7959 $\angle 110.9762^\circ$</td>
<td>18.8076 $\angle 110.9200^\circ$</td>
<td>19 sec.</td>
</tr>
<tr>
<td>5</td>
<td>18.7962 $\angle 110.9770^\circ$</td>
<td>18.8079 $\angle 110.9209^\circ$</td>
<td>19 sec.</td>
</tr>
<tr>
<td>10</td>
<td>18.7963 $\angle 110.9774^\circ$</td>
<td>18.8080 $\angle 110.9213^\circ$</td>
<td>19 sec.</td>
</tr>
</tbody>
</table>

Table 3.10: Comparison of magnitude difference in magnetic fields for Case 4.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2_\varepsilon = 1$ and $\kappa^2_\varepsilon = 2$</td>
<td>-0.0002</td>
<td>-0.0002</td>
</tr>
<tr>
<td>Between $\kappa^2_\varepsilon = 2$ and $\kappa^2_\varepsilon = 5$</td>
<td>-0.0003</td>
<td>-0.0003</td>
</tr>
<tr>
<td>Between $\kappa^2_\varepsilon = 5$ and $\kappa^2_\varepsilon = 10$</td>
<td>-0.0001</td>
<td>-0.0001</td>
</tr>
</tbody>
</table>
Case 5 and 6 are depicted in Figures 3.14a and 3.14b. For Case 5, the orientations of the transmitter (z-directed) and receiver (ρ-directed) are changed but the layer information is the same as Case 2. For Case 6, the borehole is extended to 16” without casing. Tables 3.11 and 3.13 provide the comparison of corresponding results for Case 5 and Case 6 in terms of the anisotropy ratios squared. Tables 3.12 and 3.14 show the relative difference in the field magnitude for each case.

Case 7 and Case 8 are depicted in Figures 3.14c and 3.14d. For these cases, the layer distribution is the same as in Case 2 but the separation between the transmitter and receiver is changed. The separations for Case 7 and Case 8 are 4” and 64”, respectively. Tables 3.15 and 3.17 provide the results for Case 7 and Case 8 in terms of the anisotropy ratios squared. Tables 3.16 and 3.18 show the relative difference in the field magnitudes for each case.

Case 9 is depicted in Figure 3.15. In this case, both the transmitter and receiver are positioned inside the formation, which again has uniaxial anisotropy. Table 3.19 provides the comparison of corresponding results in terms of the anisotropy ratios squared. Table 3.20 shows the relative difference in the field magnitudes.
4 Layer Results for Cross-Components (Including Anisotropy)

- In this simulation, transmitter was z-directed while the receiver was ρ-directed.
- Results obtained with the FEM code are below:

\[
R_c = 1 \times 10^{-5} \, \Omega m
\]

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.7855±100,0572 , A/m</td>
</tr>
<tr>
<td>2</td>
<td>10.6723±99,4009 , A/m</td>
</tr>
<tr>
<td>5</td>
<td>10.5553±98,7180 , A/m</td>
</tr>
<tr>
<td>10</td>
<td>10.4847±98,3108 , A/m</td>
</tr>
</tbody>
</table>

AR Magnetic Field Coefficient Scaling

1 10.7855∠100.0572° A/m 0.8714∠-79.9428° V
2 10.6723∠99.4009° A/m 0.8623∠-80.5991° V
5 10.5553∠98.7180° A/m 0.8528∠-81.2820° V
10 10.4847∠98.3108° A/m 0.8471∠-81.6892° V

Figure 3.14: Practical cases in the ρz-plane (a) Case 5, (b) Case 6, (c) Case 7, and (d) Case 8.
Table 3.11: Comparison of magnetic fields in terms of various anisotropy ratios for Case 5.

<table>
<thead>
<tr>
<th>Anisotropy Ratio $\kappa^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.2589 \angle 67.6503^\circ$</td>
<td>$1.2589 \angle 67.5966^\circ$</td>
<td>8 sec.</td>
</tr>
<tr>
<td>2</td>
<td>$1.2588 \angle 67.6532^\circ$</td>
<td>$1.2588 \angle 67.6008^\circ$</td>
<td>22 sec.</td>
</tr>
<tr>
<td>5</td>
<td>$1.2588 \angle 67.6555^\circ$</td>
<td>$1.2587 \angle 67.6038^\circ$</td>
<td>22 sec.</td>
</tr>
<tr>
<td>10</td>
<td>$1.2588 \angle 67.6565^\circ$</td>
<td>$1.2587 \angle 67.6050^\circ$</td>
<td>22 sec.</td>
</tr>
</tbody>
</table>

Table 3.12: Comparison of magnitude difference in magnetic fields for Case 5.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2 = 1$ and $\kappa^2 = 2$</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>Between $\kappa^2 = 2$ and $\kappa^2 = 5$</td>
<td>0.0000</td>
<td>0.0001</td>
</tr>
<tr>
<td>Between $\kappa^2 = 5$ and $\kappa^2 = 10$</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table 3.13: Comparison of magnetic fields in terms of various anisotropy ratios for Case 6.

<table>
<thead>
<tr>
<th>Anisotropy Ratio $\kappa^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$10.7855 \angle 100.0572^\circ$</td>
<td>$10.7857 \angle 100.0586^\circ$</td>
<td>10 sec.</td>
</tr>
<tr>
<td>2</td>
<td>$10.6723 \angle 99.4009^\circ$</td>
<td>$10.6721 \angle 99.4024^\circ$</td>
<td>29 sec.</td>
</tr>
<tr>
<td>5</td>
<td>$10.5553 \angle 98.7180^\circ$</td>
<td>$10.5546 \angle 98.7190^\circ$</td>
<td>29 sec.</td>
</tr>
<tr>
<td>10</td>
<td>$10.4847 \angle 98.3108^\circ$</td>
<td>$10.4839 \angle 98.3113^\circ$</td>
<td>29 sec.</td>
</tr>
</tbody>
</table>

Table 3.14: Comparison of magnitude difference in magnetic fields for Case 6.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2 = 1$ and $\kappa^2 = 2$</td>
<td>0.1132</td>
<td>0.1136</td>
</tr>
<tr>
<td>Between $\kappa^2 = 2$ and $\kappa^2 = 5$</td>
<td>0.1170</td>
<td>0.1175</td>
</tr>
<tr>
<td>Between $\kappa^2 = 5$ and $\kappa^2 = 10$</td>
<td>0.0706</td>
<td>0.0707</td>
</tr>
</tbody>
</table>
Table 3.15: Comparison of magnetic fields in terms of various anisotropy ratios for Case 7.

<table>
<thead>
<tr>
<th>Ani. Ratio $\kappa_2^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1007.5855 \angle 107.9553^\circ$</td>
<td>$1045.5278 \angle 107.8159^\circ$</td>
<td>16 sec.</td>
</tr>
<tr>
<td></td>
<td>$1007.5847 \angle 107.9556^\circ$</td>
<td>$1045.5251 \angle 107.8164^\circ$</td>
<td>48 sec.</td>
</tr>
<tr>
<td></td>
<td>$1007.5843 \angle 107.9559^\circ$</td>
<td>$1045.5237 \angle 107.8167^\circ$</td>
<td>48 sec.</td>
</tr>
<tr>
<td></td>
<td>$1007.5844 \angle 107.9561^\circ$</td>
<td>$1045.5234 \angle 107.8169^\circ$</td>
<td>48 sec.</td>
</tr>
</tbody>
</table>

Table 3.16: Comparison of magnitude difference in magnetic fields for Case 7.

<table>
<thead>
<tr>
<th>Between $\kappa_2^2 = 1$ and $\kappa_2^2 = 2$</th>
<th>Between $\kappa_2^2 = 2$ and $\kappa_2^2 = 5$</th>
<th>Between $\kappa_2^2 = 5$ and $\kappa_2^2 = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM 0.0008</td>
<td>FEM 0.004</td>
<td>FEM 0.0001</td>
</tr>
<tr>
<td>New Algorithm 0.0027</td>
<td>New Algorithm 0.0014</td>
<td>New Algorithm 0.0003</td>
</tr>
</tbody>
</table>

Table 3.17: Comparison of magnetic fields in terms of various anisotropy ratios for Case 8.

<table>
<thead>
<tr>
<th>Anisotropy Ratio $\kappa_2^2$</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$12.1215 \angle 111.9454^\circ$</td>
<td>$12.1271 \angle 111.9689^\circ$</td>
<td>7 sec.</td>
</tr>
<tr>
<td>2</td>
<td>$12.1213 \angle 111.9601^\circ$</td>
<td>$12.1281 \angle 111.9831^\circ$</td>
<td>20 sec.</td>
</tr>
<tr>
<td>5</td>
<td>$12.1224 \angle 111.9742^\circ$</td>
<td>$12.1292 \angle 111.9975^\circ$</td>
<td>20 sec.</td>
</tr>
<tr>
<td>10</td>
<td>$12.1233 \angle 111.9829^\circ$</td>
<td>$12.1299 \angle 112.0060^\circ$</td>
<td>20 sec.</td>
</tr>
</tbody>
</table>

Table 3.18: Comparison of magnitude difference in magnetic fields for Case 8.

<table>
<thead>
<tr>
<th>Between $\kappa_2^2 = 1$ and $\kappa_2^2 = 2$</th>
<th>Between $\kappa_2^2 = 2$ and $\kappa_2^2 = 5$</th>
<th>Between $\kappa_2^2 = 5$ and $\kappa_2^2 = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM 0.0002</td>
<td>FEM -0.0001</td>
<td>FEM -0.0009</td>
</tr>
<tr>
<td>New Algorithm -0.0010</td>
<td>New Algorithm -0.0011</td>
<td>New Algorithm -0.0007</td>
</tr>
</tbody>
</table>
3 Layer Results with Tool in Third Layer (Including Anisotropy)

- In this case, tool was moved to the third layer at a radial distance of 6” from the center of the borehole.

- Results obtained with the FEM code are below:

<table>
<thead>
<tr>
<th>AR</th>
<th>Magnetic Field (FEM)</th>
<th>Magnetic Field (New Algorithm)</th>
<th>Computing Time (New Algorithm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.1259 °79.0379\degree A/m</td>
<td>8.1326 °79.0341\degree</td>
<td>11 sec.</td>
</tr>
<tr>
<td>2</td>
<td>8.0817 °96.5262\degree A/m</td>
<td>8.0814 °96.4841\degree</td>
<td>32 sec.</td>
</tr>
<tr>
<td>5</td>
<td>8.0276 °95.9964\degree A/m</td>
<td>8.0271 °95.9416\degree</td>
<td>32 sec.</td>
</tr>
<tr>
<td>10</td>
<td>7.9939 °95.6786\degree A/m</td>
<td>7.9933 °95.6240\degree</td>
<td>32 sec.</td>
</tr>
</tbody>
</table>

Table 3.19: Comparison of magnetic fields in terms of various anisotropy ratios for Case 9.

Table 3.20: Comparison of magnitude difference in magnetic fields for Case 9.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between $\kappa^2_{\epsilon} = 1$ and $\kappa^2_{\epsilon} = 2$</td>
<td>0.0442</td>
<td>0.0512</td>
</tr>
<tr>
<td>Between $\kappa^2_{\epsilon} = 2$ and $\kappa^2_{\epsilon} = 5$</td>
<td>0.0541</td>
<td>0.0543</td>
</tr>
<tr>
<td>Between $\kappa^2_{\epsilon} = 5$ and $\kappa^2_{\epsilon} = 10$</td>
<td>0.0337</td>
<td>0.0338</td>
</tr>
</tbody>
</table>
Chapter 4

Electric Potentials from Current Electrodes in Cylindrically Stratified and Anisotropic Media

This chapter discusses the computation of electric potentials from current electrodes in cylindrically stratified and anisotropic media. Similar discussion can be found in [89]. As mentioned in Section 1.1, this type of computation is necessary for electrode logging [1, 2, 10, 11, 13, 15, 36, 54, 73, 90, 91, 92]. As the formations often feature uniaxial anisotropy, it is required that the effects of uniaxial properties be properly detected for a variety of environmental situations. Since operating frequencies for this specific type of tools are either extremely low or zero (DC), quasi-static approximations can be valid and the main physical mechanism can be described by Poisson’s equation [3, 10, 12, 93].

In this chapter, a robust algorithm to perform numerical computation of electric potentials due to arbitrarily situated point electrodes is discussed. The algorithm is based on semi-analytical formulations, which are motivated by handling the poor scaling of the modified cylindrical functions for extreme arguments and higher azimuthal modes. The semi-analytical formulations are constructed by two strategies. The first one is to rescale the modified cylindrical functions to obviate underflow and overflow,
and to minimize round-off errors. The second one is the use of extrapolation methods to accelerate the convergence of the numerical evaluation of a Sommerfeld-type integral.

This chapter is organized as follows. In Section 4.1, analytical formulations for various types of media are provided. In Section 4.2, the modification of those analytical formulations using the newly proposed set of functions is discussed. After that, several extrapolation methods for fast numerical integration are discussed in Section 4.3. Finally, several numerical examples of practical interest are provided in Section 4.4 for the validation of the proposed algorithm.

4.1 Analytical Formulations

In this section, analytical formulations of electric potentials in media with three different levels of complexity are discussed.

4.1.1 Homogeneous and Isotropic Media

For a homogeneous and isotropic medium, which is the lowest level of complexity, the source-free Poisson’s equation (Laplace equation) is written as

\[ \nabla^2 \psi = 0, \]

where \( \psi \) is the electric potential. In cylindrical coordinates, \( (4.1) \) can be expanded as

\[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = 0. \]  

(4.2)

With the assumptions of a solution of the form \( \psi = R(\rho)\Phi(\phi)Z(z) \) and a source at the origin, three conditions can be imposed for \( \psi \) from physical considerations: (i) \( R(\rho) = 0 \) when \( \rho \to \infty \); (ii) \( d\Phi/d\phi = 0 \); and (iii) \( Z(-z) = Z(z) \). The use of the
separation of variables leads to three independent equations,

\[
\rho \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) - \left[ (\lambda \rho)^2 + n^2 \right] R = 0,
\]

(4.3a)

\[
\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0,
\]

(4.3b)

\[
\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0,
\]

(4.3c)

where \(\lambda\) is the separation variable. Possible solutions to (4.3a) are the two modified Bessel functions, \(I_n(\lambda \rho)\) and \(K_n(\lambda \rho)\), but only \(K_n(\lambda \rho)\) satisfies condition (i). Since \(\Phi\) is a constant from condition (ii), \(n = 0\) from (4.3b). Finally, from condition (iii), it is seen that \(Z(z)\) should be a cosine function. Consequently, \(\psi\) can be generally expressed as

\[
\psi = \int_0^\infty f(\lambda) K_0(\lambda \rho) \cos(\lambda z) d\lambda,
\]

(4.4)

where \(f(\lambda)\) represents a coefficient for each \(\lambda\). Note that one might assert that it is also possible to have

\[
\frac{d^2 Z}{dz^2} - \lambda^2 Z = 0
\]

(4.5)

instead of (4.3c). Then, (4.3a) becomes the classical Bessel equation. However, the only solution to (4.5) with the third condition is a hyperbolic cosine function, \(Z(z) = \cosh(z)\), which diverges as \(z\) goes to infinity. When the point source condition at the origin is imposed, it can be shown that \(f(\lambda) = I/(2\pi^2 \sigma)\) [3, p. 46]. Therefore,

\[
\psi = \frac{I}{2\pi^2 \sigma} \int_0^\infty K_0(\lambda \rho) \cos(\lambda z) d\lambda,
\]

(4.6)

where \(I\) is the electric current flowing into the medium from the electrode and \(\sigma\) is the conductivity of the medium. Alternatively, (4.6) can be obtained using one of integral representations of the modified Bessel functions. It is well known that the
electric potential $\psi$ due to a point source at the origin of a homogeneous medium can be expressed as
\[
\psi = \frac{I}{4\pi\sigma R} = \frac{I}{4\pi\sigma \sqrt{\rho^2 + z^2}}. \tag{4.7}
\]

Using the complete Lipschitz-Hankel integral [94, p. 388],
\[
\frac{1}{\sqrt{\rho^2 + z^2}} = \frac{2}{\pi} \int_0^\infty K_0(\lambda \rho) \cos(\lambda z) d\lambda, \tag{4.8}
\]
we can directly obtain (4.6). Note that the zeroth order of the modified Bessel function of the second kind suffices to describe the electric potential because the source is assumed to be located at the origin. This is not true when the source is off the origin. In this case, we need to incorporate all orders to properly describe the electric potential. Let primed coordinates $(\rho', \phi', z')$ represent the source location and unprimed coordinates $(\rho, \phi, z)$ represent the observation point. Then, the distance between the source and observation point is
\[
R = \sqrt{\rho^2 + \rho'^2 - 2\rho \rho' \cos(\phi - \phi') + (z - z')^2},
\]
and the two functions in the integrand of (4.6) are modified to $K_0(\lambda |\rho - \rho'|)$ and $\cos(\lambda (z - z'))$, respectively. Note that $\rho$ and $\rho'$ are vectors such that $\rho = \rho \hat{\rho}$ and $\rho' = \rho' \hat{\rho}$. The addition theorem of the modified Bessel function of the second kind [94, p. 361],
\[
K_0(\lambda |\rho - \rho'|) = \sum_{n=-\infty}^{\infty} I_n(\lambda \rho_<) K_n(\lambda \rho_> e^{i n(\phi - \phi')}), \tag{4.9}
\]
can be used, where $\rho_< = \min(\rho, \rho')$ and $\rho_> = \max(\rho, \rho')$. Therefore, in a homogeneous medium, (4.6) is modified to
\[
\psi = \frac{I}{2\pi^2\sigma} \int_0^\infty K_0(\lambda |\rho - \rho'|) \cos(\lambda (z - z')) d\lambda
\]
\[
= \frac{I}{2\pi^2\sigma} \sum_{n=-\infty}^{\infty} e^{i n(\phi - \phi')} \int_0^\infty I_n(\lambda \rho_<) K_n(\lambda \rho_> \cos(\lambda (z - z')) d\lambda. \tag{4.10}
\]
4.1.2 Homogeneous and Uniaxial Media

For the next level, let us consider a homogeneous and uniaxial medium, where the conductivity tensor with uniaxial anisotropy is generally expressed as

$$\begin{bmatrix}
\sigma_h & 0 & 0 \\
0 & \sigma_h & 0 \\
0 & 0 & \sigma_v
\end{bmatrix}, \quad (4.11)$$

where \(\sigma_h\) and \(\sigma_v\) are horizontal and vertical conductivities, respectively. Note that the vertical direction coincides with the \(z\)-direction. Laplace equation is then slightly modified from \((4.1)\) and expressed as

$$\nabla \cdot (\overline{\sigma} \nabla \psi) = 0. \quad (4.12)$$

Note that \((4.12)\) is constructed from the continuity equation using \(J = \overline{\sigma} E\) and \(E = -\nabla \psi\). In cylindrical coordinates, the left hand side of \((4.12)\) can be expanded as

$$\nabla \cdot (\overline{\sigma} \nabla \psi) = \nabla \cdot \left( \sigma_h \frac{\partial \psi}{\partial \rho} + \sigma_h \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} + \sigma_v \frac{\partial \psi}{\partial z} \right)$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \sigma_h \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left( \sigma_h \frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \right) + \frac{\partial}{\partial z} \left( \sigma_v \frac{\partial \psi}{\partial z} \right)$$

$$= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\sigma_v}{\sigma_h} \frac{\partial^2 \psi}{\partial z^2}. \quad (4.13)$$

Using the separation of variables with a solution of the form \(\psi = R(\rho) \Phi(\phi) Z(z)\), the three characteristic equations become

$$\frac{d}{d\rho} \left( \frac{dR}{d\rho} \right) - \left[ \left( \frac{\lambda \rho}{\kappa} \right)^2 + n^2 \right] R = 0, \quad (4.14a)$$

$$\frac{d^2 \Phi}{d\phi^2} + n^2 \Phi = 0, \quad (4.14b)$$

$$\frac{d^2 Z}{dz^2} + \lambda^2 Z = 0. \quad (4.14c)$$
In (4.14a), the anisotropy ratio $\kappa$ is defined as

$$
\kappa = \sqrt{\frac{\sigma_h}{\sigma_v}}. 
$$

(4.15)

Similar to (4.4), the general solution can be represented as an integral over the separation variable $\lambda$, i.e.,

$$
\psi = \int_0^\infty f(\lambda) K_0 \left( \frac{\lambda \rho}{\kappa} \right) \cos(\lambda z) d\lambda.
$$

(4.16)

Note that $\kappa$ appears within the argument of the modified Bessel function of the second kind, $K_0$. Assuming a point source at the origin, the electric potential in a homogeneous and uniaxial medium is

$$
\psi = \frac{I}{2\pi^2 \kappa \sigma_h} \int_0^\infty K_0 \left( \frac{\lambda \rho}{\kappa} \right) \cos(\lambda z) d\lambda.
$$

(4.17)

An alternative expression can be obtained by associating $\kappa$ with the equation for $Z$ (4.14c) instead of the equation for $R$ (4.14a) above. Then, these two characteristic equations are slightly modified to

$$
\rho \frac{d}{d\rho} \left( \rho \frac{dR}{d\rho} \right) - \left[ (\lambda \rho)^2 + n^2 \right] R = 0,
$$

(4.18a)

$$
\frac{d^2 Z}{dz^2} + (\kappa \lambda)^2 Z = 0.
$$

(4.18b)

In this way, the electric potential is written as

$$
\psi = \frac{I}{2\pi^2 \sigma_h} \int_0^\infty K_0(\lambda \rho) \cos(\kappa \lambda z) d\lambda.
$$

(4.19)

These two expressions, (4.17) and (4.19), can be obtained from each other by applying the change of variables $\lambda / \kappa = \tilde{\lambda}$. For problems with geometry invariant along $z$, the first expression (4.17) is more convenient than the second one (4.19) because phase matching along the $z$-direction can be easily enforced. Therefore, discussion about
cylindrically layered media in the next section is based on (4.17). The closed-form expression of the electric potential in a homogeneous and uniaxial medium can be obtained from (4.17) as

\[ \psi = \frac{I}{2\pi^2 \kappa \sigma_h} \int_0^\infty K_0 \left( \frac{\lambda \rho}{\kappa} \right) \cos(\lambda z) d\lambda = \frac{I}{2\pi^2 \kappa \sigma_h} \left[ \frac{\pi}{2} \frac{1}{\sqrt{(\rho/\kappa)^2 + z^2}} \right] \]

\[ = \frac{I}{4\pi \sigma_h \sqrt{\rho^2 + (\kappa z)^2}}, \]  \hspace{1cm} (4.20)

where the complete Lipschitz-Hankel integral [94, p. 388],

\[ \frac{1}{\sqrt{\rho^2 + z^2}} = \frac{2}{\pi} \int_0^\infty K_0(\lambda \rho) \cos(\lambda z) d\lambda, \]  \hspace{1cm} (4.21)

has been used. When a source is off the origin, (4.17) is modified to

\[ \psi = \frac{I}{2\pi^2 \kappa \sigma_h} \int_0^\infty K_0 \left( \frac{\lambda |\rho - \rho'|}{\kappa} \right) \cos(\lambda(z - z')) d\lambda \]

\[ = \frac{I}{2\pi^2 \kappa \sigma_h} \sum_{n=-\infty}^\infty e^{i(n \phi - \phi')} \int_0^\infty I_n \left( \frac{\lambda}{\kappa \rho_<} \right) K_n \left( \frac{\lambda}{\kappa \rho_>} \right) \cos(\lambda(z - z')) d\lambda. \]  \hspace{1cm} (4.22)

### 4.1.3 Cylindrically Stratified and Uniaxial Media

When a medium is cylindrically stratified and uniaxial, the terms accounted for multiple reflections and transmissions need to be incorporated in (4.22). Let us first assume two uniaxial cylindrical layers only. When the source is embedded in layer 1 (innermost), we denote it the outgoing-potential case illustrated in Figure 4.1a. In this case, the electric potentials in the two layers are expressed as

\[ \psi_1 = [K_n(\lambda_1 \rho) + R_{12} I_n(\lambda_1 \rho)] A_0, \]  \hspace{1cm} (4.23a)

\[ \psi_2 = T_{12} K_n(\lambda_2 \rho) A_0, \]  \hspace{1cm} (4.23b)
Figure 4.1: Two different cases of two uniaxial cylindrical layers in the \( \rho z \)-plane: (a) Outgoing-potential case and (b) Standing-potential case.

where two stretched variables are defined as

\[
\lambda_1 = \frac{\lambda}{\kappa_1} = \lambda \sqrt{\frac{\sigma_{v1}}{\sigma_{h1}}}, \tag{4.24a}
\]

\[
\lambda_2 = \frac{\lambda}{\kappa_2} = \lambda \sqrt{\frac{\sigma_{v2}}{\sigma_{h2}}}, \tag{4.24b}
\]

with horizontal conductivity \( \sigma_{hi} \) and vertical conductivity \( \sigma_{vi} \), \( i = 1, 2 \), in the respective layers. From the boundary conditions (phase matching), \( A_0 \) should exhibit the same \( z \) dependence in all layers. Note that in each layer, different \( \lambda \)'s occur (\( \lambda_1 \) or \( \lambda_2 \)). The derivation of the \( R_{12} \) and \( T_{12} \) coefficients come from the continuity of the electric potential \( \psi_1 = \psi_2 \) and normal current density \( J_{\rho 1} = J_{\rho 2} \) at the boundary \( \rho = a_1 \). From the first boundary condition,

\[
K_n(\lambda_1 a_1) + R_{12} I_n(\lambda_1 a_1) = T_{12} K_n(\lambda_2 a_1). \tag{4.25}
\]

From the second boundary condition, \( \sigma_{h1} (\partial \psi_1 / \partial \rho) = \sigma_{h2} (\partial \psi_2 / \partial \rho) \) at \( \rho = a_1 \),

\[
\lambda_1 \sigma_{h1} [K'_n(\lambda_1 a_1) + R_{12} I'_n(\lambda_1 a_1)] = \lambda_2 \sigma_{h2} T_{12} K'_n(\lambda_2 a_1). \tag{4.26}
\]
Using (4.25) and (4.26), we obtain

\[ R_{12} = \frac{\lambda_2 \sigma h_2 K_n(\lambda_1 a_1) K_n'(\lambda_2 a_1) - \lambda_1 \sigma h_1 K_n(\lambda_2 a_1) K_n'(\lambda_1 a_1)}{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}, \] (4.27a)

\[ T_{12} = \frac{\sigma h_1}{a_1} \frac{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}, \] (4.27b)

When the source is embedded in layer 2, we denote it the standing-potential case illustrated in Figure 4.1b, where the electric potentials in the two layers are written as

\[ \psi_1 = T_{21} I_n(\lambda_1 \rho) B_0, \] (4.28a)

\[ \psi_2 = [R_{21} K_n(\lambda_2 \rho) + I_n(\lambda_2 \rho)] B_0, \] (4.28b)

where again \( \lambda_1 = \lambda/\kappa_1 = \lambda \sqrt{\sigma v_1/\sigma h_1}, \) \( \lambda_2 = \lambda/\kappa_2 = \lambda \sqrt{\sigma v_2/\sigma h_2}, \) and \( B_0 \) has the same \( z \) dependence in the two layers. From two boundary conditions,

\[ T_{21} I_n(\lambda_1 a_1) = R_{21} K_n(\lambda_2 a_1) + I_n(\lambda_2 a_1). \] (4.29)

\[ \lambda_1 \sigma h_1 T_{21} I_n'(\lambda_1 a_1) = \lambda_2 \sigma h_2 [R_{21} K_n'(\lambda_2 a_1) + I_n'(\lambda_2 a_1)]. \] (4.30)

Using (4.29) and (4.30), we obtain

\[ R_{21} = \frac{\lambda_2 \sigma h_2 I_n(\lambda_1 a_1) I_n'(\lambda_2 a_1) - \lambda_1 \sigma h_1 I_n(\lambda_2 a_1) I_n'(\lambda_1 a_1)}{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}, \] (4.31a)

\[ T_{21} = \frac{\sigma h_2}{a_1} \frac{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}{\lambda_1 \sigma h_1 I_n'(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma h_2 I_n(\lambda_1 a_1) K_n'(\lambda_2 a_1)}, \] (4.31b)

When more than two layers are present, the expressions of the generalized reflection and transmission coefficients for uniaxial media are the same as those in Chapter
2 and Chapter 3 as summarized below. For more details, refer to [17].

\[
\tilde{R}_{i,i+1} = R_{i,i+1} + T_{i+1,i} \tilde{R}_{i+1,i+2}(1 - R_{i+1,i} \tilde{R}_{i+1,i+2})^{-1} T_{i,i+1}, \quad (4.32a)
\]

\[
\tilde{R}_{i,i-1} = R_{i,i-1} + T_{i-1,i} \tilde{R}_{i-1,i-2}(1 - R_{i-1,i} \tilde{R}_{i-1,i-2})^{-1} T_{i,i-1}, \quad (4.32b)
\]

\[
\tilde{T}_{ji} = T_{i-1,i} X_{j,i-1}, \text{ for the outgoing-potential case}, \quad (4.32c)
\]

\[
\tilde{T}_{ji} = T_{i+1,i} X_{j,i+1}, \text{ for the standing-potential case}. \quad (4.32d)
\]

In summary, for a cylindrically stratified and uniaxial medium, the electric potential in layer \(i\) is expressed as

\[
\psi_i = \frac{I}{2\pi^2 \kappa_j \sigma_{hj}} \sum_{n=-\infty}^{\infty} e^{in(\phi - \phi')} \int_0^\infty F_n(\rho, \rho') \cos(\lambda(z - z')) d\lambda, \quad (4.33)
\]

where

for Case 1 : \(\rho\) and \(\rho'\) are in the same layer. \((\rho \geq \rho')\)

\[
F_n = \left[ K_n(\lambda_j \rho) + I_n(\lambda_j \rho) \tilde{R}_{j,j+1} \right] \left[ I_n(\lambda_j \rho') + K_n(\lambda_j \rho') \tilde{R}_{j,j-1} \right] M_{j}, \quad (4.34a)
\]

for Case 2 : \(\rho\) and \(\rho'\) are in the same layer. \(\rho < \rho'\)

\[
F_n = \left[ I_n(\lambda_j \rho) + K_n(\lambda_j \rho) \tilde{R}_{j,j-1} \right] \left[ K_n(\lambda_j \rho') + I_n(\lambda_j \rho') \tilde{R}_{j,j+1} \right] M_{j}, \quad (4.34b)
\]

for Case 3 : \(\rho\) and \(\rho'\) are in different layers. \(\rho > \rho'\)

\[
F_n = \left[ K_n(\lambda_j \rho) + I_n(\lambda_j \rho) \tilde{R}_{i,i+1} \right] \left[ I_n(\lambda_j \rho') + K_n(\lambda_j \rho') \tilde{R}_{j,j-1} \right] N_{i+\tilde{T}_{ji}} M_{j}, \quad (4.34c)
\]

for Case 4 : \(\rho\) and \(\rho'\) are in different layers. \(\rho < \rho'\)

\[
F_n = \left[ I_n(\lambda_i \rho) + K_n(\lambda_i \rho) \tilde{R}_{i,i-1} \right] \left[ K_n(\lambda_j \rho') + I_n(\lambda_j \rho') \tilde{R}_{j,j+1} \right] N_{i-\tilde{T}_{ji}} M_{j}. \quad (4.34d)
\]

In (4.33), \(\kappa_j = \sqrt{\sigma_{hj}/\sigma_{vj}}\), \(\sigma_{hj}\) and \(\sigma_{vj}\) are the horizontal and vertical conductivities in layer \(j\), respectively. Also, \(\lambda_j = \lambda \sqrt{\sigma_{vj}/\sigma_{hj}}\) and \(\lambda_i = \lambda \sqrt{\sigma_{vi}/\sigma_{hi}}\) from (4.34a) to
It should be noted that (4.34a) through (4.34d) can be succinctly expressed as

\[
F_n = \left[ I_n(\lambda_s \rho_m) + K_n(\lambda_s \rho_m)\tilde{R}_{s,s-1} \right] \left[ K_n(\lambda_l \rho_M) + I_n(\lambda_l \rho_M)\tilde{R}_{l,l+1} \right] T_l M_j, \quad (4.35)
\]

where

\[
T_l = \begin{cases} 
1, & \text{for Case 1 and Case 2,} \\
N_{i+\tilde{T}_{ji}}, & \text{for Case 3,} \\
N_{i-\tilde{T}_{ji}}, & \text{for Case 4.}
\end{cases} \quad (4.36)
\]

In (4.35), the definition of \( M_j \) is given in Section 4.2.2, \( \rho_m = \min(\rho, \rho') \), \( \rho_M = \max(\rho, \rho') \), the subscript \( s \) is the smaller of \( i \) and \( j \), and the subscript \( l \) is the larger of \( i \) and \( j \). Figure 4.2 describes the relevant parameters in the integrand (4.35).

### 4.2 Range-Conditioning

As can be seen from (4.33), the computation of the electric potential in cylindrically stratified and uniaxial media requires the evaluation of products of the modified Bessel function of the first and second kind, viz. \( I_n \) and \( K_n \). Those products sometimes involve very small or very large factors due to the exponential behavior of the functions. For example, when \(|z| << 1\), \( K_n(z) \) has very large values whereas \( I_n(z) \) has very small values. This disparity becomes progressively greater for higher order modes. On the other hand, when \( \Re z >> 1 \), \( K_n(z) \) has very small values while \( I_n(z) \) has very large values. In numerical computations, such disparate values can lead to numerical overflow, underflow, round-off errors, and eventually unreliable results. To tackle this challenge, a new set of functions called range-conditioned modified cylindrical functions are defined in a similar fashion to what has been done in Chapter 2 and Chapter 3 for other cylindrical functions, \( J_n \) and \( H_n^{(1)} \).
Figure 4.2: Schematic description of the relevant parameters for the integrand: (a) Case 1 and 2, and (b) Case 3 and 4.

4.2.1 Range-Conditioned Modified Cylindrical Functions

Since the general characteristics of $J_n$ and $H_n^{(1)}$ are similar to those of $I_n$ and $K_n$, we can establish definitions for the range-conditioned modified cylindrical functions. When $|z| \ll 1$, $I_n(z)$ and $K_n(z)$ can be expressed via the small argument approximations for $n > 0$. Noting that, for $-\pi < \text{arg}(z) < \pi/2$, the relationship between
cylindrical functions and modified cylindrical functions is

\[ I_n(z) = i^{-n} J_n(iz), \quad (4.37a) \]

\[ K_n(z) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(iz). \quad (4.37b) \]

Therefore, if \( z = \lambda_i a_i \), the following small argument approximations for \( I_n \) and \( K_n \) can be easily constructed.

\[ I_n(\lambda_i a_i) \approx \frac{1}{n!} \left( \frac{\lambda_i a_i}{2} \right)^n = \frac{1}{n!} \left( \frac{\lambda_i}{2} \right)^n \cdot a_i^n \cdot 1 = G_i a_i^n \hat{I}_n(\lambda_i a_i), \quad (4.38a) \]

\[ I_n'(\lambda_i a_i) \approx \frac{1}{2(n-1)!} \left( \frac{\lambda_i a_i}{2} \right)^{n-1} = \frac{1}{n!} \left( \frac{\lambda_i}{2} \right)^n \cdot a_i^n \cdot \frac{n}{\lambda_i a_i} = G_i a_i^n \hat{I}_n'(\lambda_i a_i), \quad (4.38b) \]

\[ K_n(\lambda_i a_i) \approx \frac{(n-1)!}{2} \left( \frac{2}{\lambda_i a_i} \right)^n = n! \left( \frac{2}{\lambda_i} \right)^n \cdot a_i^{-n} \cdot \left( \frac{1}{2n} \right) = G_i^{-1} a_i^{-n} \hat{K}_n(\lambda_i a_i), \quad (4.38c) \]

\[ K_n'(\lambda_i a_i) \approx -\frac{n!}{4} \left( \frac{2}{\lambda_i a_i} \right)^{n+1} = n! \left( \frac{2}{\lambda_i} \right)^n \cdot a_i^{-n} \cdot \left( -\frac{1}{2\lambda_i a_i} \right) = G_i^{-1} a_i^{-n} \hat{K}_n'(\lambda_i a_i). \quad (4.38d) \]

It should be noted that the multiplicative factor \( G_i \) above is defined so as to not depend on the radial distance \( a_i \). Also, the same \( G_i \) is associated to one function and its derivative, and the multiplicative factors appearing in \( I_n \) and \( K_n \) are reciprocal to each other. This will facilitate some computations later on.

When \( \Re e[z] \gg 1 \), the large argument approximations for the modified Bessel functions [69, p. 207] are written as

\[ I_n(z) = \frac{e^z}{\sqrt{2\pi z}} \left[ 1 - \frac{(\mu - 1)}{1!(8z)} + \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \cdots \right], \quad (4.39a) \]

\[ K_n(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{(\mu - 1)}{1!(8z)} + \frac{(\mu - 1)(\mu - 9)}{2!(8z)^2} + \cdots \right]. \quad (4.39b) \]
If we let $\lambda_i$ be complex, i.e., $\lambda_i = \lambda_i' + i\lambda_i''$, (4.39a) and (4.39b) can be rewritten as

\[
I_n(\lambda_i a_i) = \frac{e^{\lambda_i a_i}}{\sqrt{2\pi \lambda_i a_i}} \left[ 1 - \frac{(\mu - 1)}{1!(8\lambda_i a_i)} + \frac{(\mu - 1)(\mu - 9)}{2!(8\lambda_i a_i)^2} + \ldots \right] 
\]

\[
= e^{\chi_i a_i} \frac{e^{i\chi_i'' a_i}}{\sqrt{2\pi \lambda_i a_i}} \left[ 1 - \frac{(\mu - 1)}{1!(8\lambda_i a_i)} + \frac{(\mu - 1)(\mu - 9)}{2!(8\lambda_i a_i)^2} + \ldots \right] 
\]

\[
= e^{\chi_i a_i} \hat{I}_n(\lambda_i a_i), \quad (4.40a)
\]

\[
K_n(\lambda_i a_i) = \sqrt{\frac{\pi}{2\lambda_i a_i}} e^{-\lambda_i a_i} \left[ 1 + \frac{(\mu - 1)}{1!(8\lambda_i a_i)} + \frac{(\mu - 1)(\mu - 9)}{2!(8\lambda_i a_i)^2} + \ldots \right] 
\]

\[
= \sqrt{\frac{\pi}{2\lambda_i a_i}} e^{-\chi_i a_i} e^{i\chi_i'' a_i} \left[ 1 + \frac{(\mu - 1)}{1!(8\lambda_i a_i)} + \frac{(\mu - 1)(\mu - 9)}{2!(8\lambda_i a_i)^2} + \ldots \right] 
\]

\[
= e^{-\chi_i a_i} \hat{K}_n(\lambda_i a_i), \quad (4.40b)
\]

where $\mu = 4n^2$. Again, the associated multiplicative factors are reciprocal to each other. The derivatives of the range-conditioned modified cylindrical functions for large arguments can be derived through the recursive formulas [69, p. 206],

\[
I_n'(\lambda_i a_i) = I_{n-1}(\lambda_i a_i) - \frac{n}{\lambda_i a_i} I_n(\lambda_i a_i) 
\]

\[
= e^{\chi_i a_i} \hat{I}_{n-1}(\lambda_i a_i) - e^{\chi_i a_i} \frac{n}{\lambda_i a_i} \hat{I}_n(\lambda_i a_i) 
\]

\[
= e^{\chi_i a_i} \hat{I}_n'(\lambda_i a_i), \quad (4.41)
\]

\[
K_n'(\lambda_i a_i) = -K_{n-1}(\lambda_i a_i) - \frac{n}{\lambda_i a_i} K_n(\lambda_i a_i) 
\]

\[
= -e^{\chi_i a_i} \hat{K}_{n-1}(\lambda_i a_i) - e^{\chi_i a_i} \frac{n}{\lambda_i a_i} \hat{K}_n(\lambda_i a_i) 
\]

\[
= e^{\chi_i a_i} \hat{K}_n'(\lambda_i a_i). \quad (4.42)
\]
Table 4.1: Definition of the range-conditioned modified cylindrical functions for uniaxial media for all types of arguments.

<table>
<thead>
<tr>
<th></th>
<th>Small Arguments</th>
<th>Moderate Arguments</th>
<th>Large Arguments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n(\lambda_i a_i)$</td>
<td>$G_i a_i^n \hat{I}_n(\lambda_i a_i)$</td>
<td>$P_{ii} \hat{I}_n(\lambda_i a_i)$</td>
<td>$e^{\lambda_i a_i} \hat{I}_n(\lambda_i a_i)$</td>
</tr>
<tr>
<td>$I_n'(\lambda_i a_i)$</td>
<td>$G_i a_i^n \hat{I}_n'(\lambda_i a_i)$</td>
<td>$P_{ii} \hat{I}_n'(\lambda_i a_i)$</td>
<td>$e^{\lambda_i a_i} \hat{I}_n'(\lambda_i a_i)$</td>
</tr>
<tr>
<td>$K_n(\lambda_i a_i)$</td>
<td>$G_i^{-1} a_i^{-n} \hat{K}_n(\lambda_i a_i)$</td>
<td>$P_{ii}^{-1} \hat{K}_n(\lambda_i a_i)$</td>
<td>$e^{-\lambda_i a_i} \hat{K}_n(\lambda_i a_i)$</td>
</tr>
<tr>
<td>$K_n'(\lambda_i a_i)$</td>
<td>$G_i^{-1} a_i^{-n} \hat{K}_n'(\lambda_i a_i)$</td>
<td>$P_{ii}^{-1} \hat{K}_n'(\lambda_i a_i)$</td>
<td>$e^{-\lambda_i a_i} \hat{K}_n'(\lambda_i a_i)$</td>
</tr>
</tbody>
</table>

If the argument is neither small nor large, the range-conditioned modified cylindrical functions are defined, in analogy to small and large arguments, as

$$I_n(\lambda_i a_i) = P_{ii} \hat{I}_n(\lambda_i a_i), \quad (4.43a)$$
$$I_n'(\lambda_i a_i) = P_{ii} \hat{I}_n'(\lambda_i a_i), \quad (4.43b)$$
$$K_n(\lambda_i a_i) = P_{ii}^{-1} \hat{K}_n(\lambda_i a_i), \quad (4.43c)$$
$$K_n'(\lambda_i a_i) = P_{ii}^{-1} \hat{K}_n'(\lambda_i a_i), \quad (4.43d)$$

where another multiplicative factor $P_{ii}$ defined down below in (4.45) appears, and its first and second subscripts are linked to the stretched variable $\lambda_i$ and radial distance $a_i$, respectively. In summary, arguments for the range-conditioned modified cylindrical functions can be categorized into small, moderate, and large, with different and appropriate multiplicative factors defined accordingly. This is summarized in Table 4.1.
Table 4.2: Definition of the multiplicative factors, $\alpha_{ij}$ and $\beta_{ij}$, for the electric potential in uniaxial media.

<table>
<thead>
<tr>
<th>Argument Type</th>
<th>$\alpha_{ij}$</th>
<th>$\beta_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>$G_i^{-1}a_j^{-n}$</td>
<td>$G_i a_j^n$</td>
</tr>
<tr>
<td>Moderate</td>
<td>$P_{ij}^{-1}$</td>
<td>$P_{ij}$</td>
</tr>
<tr>
<td>Large</td>
<td>$e^{-\lambda_i a_j}$</td>
<td>$e^{\lambda_i a_j}$</td>
</tr>
</tbody>
</table>

Several multiplicative factors seen in Table 4.1 are given by

$$G_i = \frac{1}{n!} \left( \frac{\lambda_i}{2} \right)^n = \frac{1}{n!} \left( \frac{\lambda}{2} \sqrt{\frac{\sigma_{vi}}{\sigma_{hi}}} \right)^n,$$

and

$$P_{ii} = \begin{cases} 
1, & \text{if } |I_n(\lambda_i a_i)|^{-1} < T_{moderate}, \\
|I_n(\lambda_i a_i)|, & \text{if } |I_n(\lambda_i a_i)|^{-1} \geq T_{moderate},
\end{cases}$$

where $\sigma_{hi}$ and $\sigma_{vi}$ are horizontal and vertical conductivities in layer $i$, and $T_{moderate}$ is the magnitude threshold for moderate arguments given in (2.39b). For other threshold values for small and large arguments, see Section 2.2.2.

4.2.2 Reflection and Transmission Coefficients

The multiplicative factors shown in Table 4.1 can be classified into two types, denoted as $\alpha$ and $\beta$, shown in Table 4.2. Similarly to the isotropic media case for electromagnetic fields in Chapter 2, $\alpha$ and $\beta$ for uniaxial media have now two subscripts because the argument $\lambda$ of the modified Bessel functions varies according to associated layers: the first subscript represents stretched variable $\lambda_i = \lambda \sqrt{\sigma_{vi}/\sigma_{hi}}$ and the second subscript represents the radial distance. It should be noted that $\alpha$ is
associated with $K_n(\cdot)$ whereas $\beta$ is associated with $I_n(\cdot)$. Recall that $\alpha$ and $\beta$ have two properties, which are discussed in (2.44) and (2.45).

Using such $\alpha$ and $\beta$, the reflection coefficient $R_{12}$ for the outgoing-potential case is modified to

$$R_{12} = \frac{\lambda_2 \sigma_{h2} K_n^\prime(\lambda_1 a_1) K_n^\prime(\lambda_2 a_1) - \lambda_1 \sigma_{h1} K_n(\lambda_2 a_1) K_n^\prime(\lambda_1 a_1)}{\lambda_1 \sigma_{h1} I_n^\prime(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} I_n(\lambda_1 a_1) K_n^\prime(\lambda_2 a_1)}$$

$$= \alpha_{11} \frac{\lambda_2 \sigma_{h2} \hat{K}_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1) - \lambda_1 \sigma_{h1} \hat{K}_n(\lambda_2 a_1) \hat{K}_n^\prime(\lambda_1 a_1)}{\lambda_1 \sigma_{h1} \hat{I}_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} \hat{I}_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1)}$$

$$= \alpha_{11}^2 \hat{R}_{12}, \quad (4.46)$$

the reflection coefficient $R_{21}$ for the standing-potential case is modified to

$$R_{21} = \frac{\lambda_2 \sigma_{h2} I_n(\lambda_1 a_1) I_n^\prime(\lambda_2 a_1) - \lambda_1 \sigma_{h1} I_n(\lambda_2 a_1) I_n^\prime(\lambda_1 a_1)}{\lambda_1 \sigma_{h1} I_n^\prime(\lambda_1 a_1) K_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} I_n(\lambda_1 a_1) K_n^\prime(\lambda_2 a_1)}$$

$$= \beta_{21} \frac{\lambda_2 \sigma_{h2} \hat{I}_n(\lambda_1 a_1) \hat{I}_n^\prime(\lambda_2 a_1) - \lambda_1 \sigma_{h1} \hat{I}_n(\lambda_2 a_1) \hat{I}_n^\prime(\lambda_1 a_1)}{\lambda_1 \sigma_{h1} \hat{I}_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} \hat{I}_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1)}$$

$$= \beta_{21}^2 \hat{R}_{21}, \quad (4.47)$$

the transmission coefficient $T_{12}$ for the outgoing-potential case is modified to

$$T_{12} = \frac{\sigma_{h1}}{a_1 \left[ \lambda_1 \sigma_{h1} I_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} I_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1) \right]}$$

$$= \alpha_{11} \beta_{21} \frac{\sigma_{h1}}{a_1 \left[ \lambda_1 \sigma_{h1} \hat{I}_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} \hat{I}_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1) \right]}$$

$$= \alpha_{11} \beta_{21} \hat{T}_{12}, \quad (4.48)$$

and the transmission coefficient $T_{21}$ for the standing-potential case is modified to

$$T_{21} = \frac{\sigma_{h2}}{a_1 \left[ \lambda_1 \sigma_{h1} I_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} I_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1) \right]}$$

$$= \alpha_{11} \beta_{21} \frac{\sigma_{h2}}{a_1 \left[ \lambda_1 \sigma_{h1} \hat{I}_n^\prime(\lambda_1 a_1) \hat{K}_n(\lambda_2 a_1) - \lambda_2 \sigma_{h2} \hat{I}_n(\lambda_1 a_1) \hat{K}_n^\prime(\lambda_2 a_1) \right]}$$

$$= \alpha_{11} \beta_{21} \hat{T}_{21}. \quad (4.49)$$
Note that the multiplicative factors associated with \( \hat{T}_{12} \) and \( \hat{T}_{21} \) are not necessarily unity while \( \hat{T}_{12} = T_{12} \) and \( \hat{T}_{21} = T_{21} \) in isotropic media.

Generalized reflection coefficients can be conditioned using the (local) reflection and transmission coefficients above and rewritten as

\[
\tilde{R}_{i,i+1} = \alpha_{ii}^2 \hat{R}_{i,i+1}, \quad (4.50a)
\]
\[
\tilde{R}_{i+1,i} = \beta_{i+1,i}^2 \hat{R}_{i+1,i}, \quad (4.50b)
\]

where

\[
\hat{R}_{i,i+1} = \hat{R}_{i+1,i} + \beta_{i,i+1}^2 \alpha_{i+1,i+1}^2 \hat{T}_{i+1,i} \hat{R}_{i+1,i+2}(1 - \beta_{i+1,i}^2 \alpha_{i+1,i+1}^2 \hat{R}_{i+1,i} \hat{R}_{i+1,i+2})^{-1} \hat{T}_{i,i+1}, \quad (4.51a)
\]
\[
\hat{R}_{i+1,i} = \hat{R}_{i+1,i} + \beta_{i+1,i}^2 \alpha_{i+1,i}^2 \hat{T}_{i+1,i} \hat{R}_{i+1,i+1}(1 - \beta_{i+1,i}^2 \alpha_{i+1,i}^2 \hat{R}_{i+1,i} \hat{R}_{i+1,i-1})^{-1} \hat{T}_{i+1,i}. \quad (4.51b)
\]

Generalized transmission coefficients can be written as follows. For the outgoing-potential case, \( i > j \),

\[
\tilde{T}_{ji} = T_{i-1,j} i^{-2} \prod_{k=j}^{i-2} S_{k,k+1} = \left( \beta_{i-1,i} \hat{T}_{i-1,i} \right) \left( \prod_{k=j+1}^{i-1} \beta_{k,k-1} \alpha_{k,k} N_{k-1,k} \hat{T}_{k-1,k} \right) (\alpha_{jj})
\]
\[
= \beta_{i,j-1} \alpha_{jj} \hat{T}_{ji}, \quad (4.52)
\]

where the \( S \)-coefficient for the outgoing-potential case is

\[
S_{i,i+1} = (1 - R_{i+1,i} \hat{R}_{i+1,i+2})^{-1} T_{i,i+1} = (1 - \beta_{i+1,i}^2 \alpha_{i+1,i+1} \hat{R}_{i+1,i} \hat{R}_{i+1,i+2})^{-1} \alpha_{ii} \beta_{i+1,i} \hat{T}_{i,i+1}
\]
\[
= \alpha_{ii} \beta_{i+1,i} \hat{S}_{i,i+1}. \quad (4.53)
\]

For the standing-potential case, \( j > i \), the generalized transmission coefficient is written as

\[
\tilde{T}_{ji} = T_{i+1,j} j^{-2} \prod_{k=j}^{i+2} S_{k,k+1} = \left( \alpha_{ii} \hat{T}_{i+1,j} \right) \left( \prod_{k=i+1}^{j-1} \beta_{k,k-1} \alpha_{k,k} N_{k-1,k} \hat{T}_{k-1,k} \right) (\beta_{j,j-1})
\]
\[
= \alpha_{ii} \beta_{j,j-1} \hat{T}_{ji}, \quad (4.54)
\]
where the $S$-coefficient for the standing-potential case is

$$S_{i+1,j} = (1 - R_{i,i+1} R_{i,i-1})^{-1} T_{i+1,i} = (1 - R_{i,i-1} \alpha_{ii}^2 \hat{R}_{i,i+1} \hat{R}_{i,i-1})^{-1} \alpha_{ii} \beta_{j+1,i} \tilde{T}_{i+1,i},$$

(4.55)

Furthermore, three auxiliary coefficients in (4.34a) through (4.34d) can be conditioned as well. The auxiliary coefficients $N_k$ are rewritten as

$$N_{k+} = \left[1 - R_{k,k-1} \hat{R}_{k,k+1}\right]^{-1} = \left[1 - \beta_{k,k-1}^2 \alpha_{kk} \hat{R}_{k,k-1} \hat{R}_{k,k+1}\right]^{-1},$$

(4.56)

$$N_{k-} = \left[1 - R_{k,k+1} \hat{R}_{k,k-1}\right]^{-1} = \left[1 - \beta_{k,k-1}^2 \alpha_{kk} \hat{R}_{k,k+1} \hat{R}_{k,k-1}\right]^{-1}.$$

(4.57)

The auxiliary coefficients $M_j$ is rewritten as

$$M_j = \left[1 - \hat{R}_{j,j-1} \hat{R}_{j,j+1}\right]^{-1} = \left[1 - \beta_{j,j-1}^2 \alpha_{jj} \hat{R}_{j,j-1} \hat{R}_{j,j+1}\right]^{-1}.$$

(4.58)

### 4.2.3 Modified Integrand

Using the reflection, transmission, and auxiliary coefficients defined in the previous section, the integrand of the integral for uniaxial media shown in (4.34a) – (4.34d) can be modified accordingly. It is emphasized that, as (4.35) and (4.36) show, the integrands $F_n(\rho, \rho')$ for Case 1 and Case 2 are identical due to the reciprocity theorem, and that the integrands for Case 3 and Case 4 are also identical except for the factor associated with the generalized transmission coefficients $T_i$. Throughout this section, the subscript $s$ and $l$ represent $\min(i, j)$ and $\max(i, j)$, respectively.

For Case 1 and Case 2, there are four radial distances of interest: $a_{s-1}$, $\rho_m$, $\rho_M$, and $a_s$. Note that $s = l$ for these two cases. For convenience, we let $a_{s-1} = a_1$, $\rho_m = a_2$, $\rho_M = a_3$, and $a_s = a_4$ so that $a_1 < a_2 < a_3 < a_4$, so that the integrand is
rewritten as

\[ F_n(\rho, \rho') = \left[ I_n(\lambda s \rho_m) + K_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \left[ K_n(\lambda l \rho_M) + I_n(\lambda l \rho_M) \tilde{R}_{l,l+1} \right] M_j \]

\[ = \left[ \beta_{s2} \alpha_{l3} \hat{I}_n(\lambda s \rho_m) + \beta_{s1}^2 \alpha_{s2} \alpha_{l3} \hat{K}_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \]

\[ \left[ \hat{K}_n(\lambda l \rho_M) + \beta_{l3}^2 \alpha_{l4} \hat{I}_n(\lambda l \rho_M) \tilde{R}_{l,l+1} \right] M_j \]

\[ = \left[ \beta_{s2} \alpha_{s3} \hat{I}_n(\lambda s \rho_m) + (\beta_{s1} \alpha_{s2}) (\beta_{s1} \alpha_{s3}) \hat{K}_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \cdot \]

\[ \left[ \hat{K}_n(\lambda l \rho_M) + (\beta_{s3} \alpha_{s4})^2 \hat{I}_n(\lambda l \rho_M) \tilde{R}_{s,s-1} \right] M_j \]

\[ = \left[ A_1 \hat{I}_n(\lambda s \rho_m) + A_2 \hat{K}_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \]

\[ \left[ A_3 \hat{K}_n(\lambda l \rho_M) + A_4 \hat{I}_n(\lambda l \rho_M) \tilde{R}_{s,s-1} \right] M_j. \]  \hspace{1cm} (4.59)

All multiplicative factors \((A_1, A_2, A_3, A_4)\) are never greater than one in magnitude due to the boundness property. Refer to Figure 4.2a for the schematic description of the parameters in (4.59).

For Case 3 and Case 4, there are six radial distances of interest: \(a_{s-1}, \rho_m, a_s, a_{l-1}, \rho_M,\) and \(a_l\). For convenience, we let \(a_{s-1} = a_1, \rho_m = a_2, a_s = a_3, a_{l-1} = a_4, \rho_M = a_5,\) and \(a_l = a_6\) so that \(a_1 < a_2 < a_3 \leq a_4 < a_5 < a_6\), so that the integrand is rewritten as

\[ F_n(\rho, \rho') = \left[ I_n(\lambda s \rho_m) + K_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \left[ K_n(\lambda l \rho_M) + I_n(\lambda l \rho_M) \tilde{R}_{l,l+1} \right] T_t M_j \]

\[ = \left[ \beta_{s2} \hat{I}_n(\lambda s \rho_m) + \alpha_{s2} \hat{K}_n(\lambda s \rho_m) \beta_{s1} \tilde{R}_{s,s-1} \right] \cdot \]

\[ \left[ \alpha_{l5} \hat{K}_n(\lambda l \rho_M) + \beta_{l5} \hat{I}_n(\lambda l \rho_M) \alpha_{l0} \tilde{R}_{l,l+1} \right] \alpha_{s3} \beta_{l4} \hat{T}_t M_j \]  \hspace{1cm} (4.60)

\[ = \left[ \beta_{s2} \alpha_{s3} \hat{I}_n(\lambda s \rho_m) + (\beta_{s1} \alpha_{s2}) (\beta_{s1} \alpha_{s3}) \hat{K}_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \cdot \]

\[ \left[ \beta_{l4} \alpha_{l5} \hat{K}_n(\lambda l \rho_M) + (\beta_{l4} \alpha_{l6}) (\beta_{l5} \alpha_{l6}) \hat{I}_n(\lambda l \rho_M) \tilde{R}_{l,l+1} \right] \hat{T}_t M_j \]

\[ = \left[ D_1 \hat{I}_n(\lambda s \rho_m) + D_2 \hat{K}_n(\lambda s \rho_m) \tilde{R}_{s,s-1} \right] \]

\[ \left[ D_3 \hat{K}_n(\lambda l \rho_M) + D_4 \hat{I}_n(\lambda l \rho_M) \tilde{R}_{l,l+1} \right] \hat{T}_t M_j. \]  \hspace{1cm} (4.61)
In (4.60), $T_t = \alpha_{s3}\beta_{l4}\hat{T}_t$ can be obtained from (4.52), (4.54), (4.56), and (4.57). In (4.61), the left squared bracket is the function of the subscript $s$ whereas the right squared bracket is the function of the subscript $l$. $T_t$ and its conditioned version $\hat{T}_t$ are different for Case 3 and Case 4 as shown in (4.36). Again, all multiplicative factors $(B_1, B_2, B_3, B_4)$ are never greater than one in magnitude. Again, refer to Figure 4.2b for the schematic description of the parameters in (4.61).

### 4.2.4 Azimuth Series Folding

The computation of the electric potential in (4.33) involves an infinite series over the azimuth mode $n$. As two modified Bessel functions are even functions with respect to $n$, $I_{-n}(z) = I_n(z)$ and $K_{-n}(z) = K_n(z)$, the series can be folded in order to exclude negative orders as shown below.

$$
\psi_i = \frac{I}{2\pi^2\kappa_j\sigma_{nj}} \sum_{n=-\infty}^{\infty} e^{in(\phi-\phi')} \int_0^{\infty} F_n(\rho,\rho') \cos(\lambda(z-z'))d\lambda
$$

$$
= \frac{I}{2\pi^2\kappa_j\sigma_{nj}} \int_0^{\infty} \left[ F_0(\rho,\rho') + 2 \sum_{n=1}^{\infty} F_n(\rho,\rho') \cos(n(\phi-\phi')) \right] \cos(\lambda(z-z'))d\lambda. \quad (4.62)
$$

### 4.3 Numerical Integration

The expression of the electric potential in cylindrically stratified media includes a semi-infinite integral and an infinite series, as shown in (4.33). Therefore, truncation errors in numerical computation are inevitable and an error analysis should be made to ensure computational reliability. In addition, the Sommerfeld-type integral can be notoriously slowly convergent. A variety of extrapolation methods are reviewed and compared here in order to find the most efficient method for fast convergence.
4.3.1 Extrapolation Methods

Among numerous extrapolation methods [95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109], some popular ones for Sommerfeld-type integrals are very briefly reviewed here. For more details, readers are referred to [108, 109]. Before a given sequence is extrapolated, a Sommerfeld-type integral can be divided into a number of integrals as shown below.

\[
S = \int_{\lambda}^{\infty} g(\lambda) p(\lambda) d\lambda = \sum_{i=0}^{\infty} \int_{\lambda_{i-1}}^{\lambda_i} g(\lambda) p(\lambda) d\lambda = \sum_{i=0}^{\infty} u_i, \tag{4.63}
\]

where \(g(\lambda)\) is an exponentially decaying part and \(p(\lambda)\) is an oscillatory part. This approach is called partition-extrapolation [110]. In general, the remainders are defined as

\[
r_n = S_n - S = -\int_{\lambda_n}^{\infty} g(\lambda) p(\lambda) d\lambda. \tag{4.64}
\]

Furthermore, the remainders are assumed to feature Poincaré-type asymptotic expansions [109] of the form

\[
r_n \sim \omega_n \sum_{i=0}^{\infty} a_i \lambda_n^{-i}, \quad n \to \infty, \tag{4.65}
\]

where \(\omega_n\) is the remainder estimate and \(a_i\) are associated coefficients. The estimates \(\omega_n\) play an important role in the extrapolation and can be obtained analytically or numerically. The coefficients \(a_i\) are unknowns but they are not necessary for the extrapolation itself. In our case, \(g(\lambda)\) in (4.63) can be asymptotically expressed as

\[
g(\lambda) \sim \frac{e^{-\lambda|\rho-\rho'|}}{\lambda} \sum_{i=0}^{\infty} \frac{c_i}{\lambda^i}, \tag{4.66}
\]

where \(c_i\) are arbitrary constants. Furthermore, \(p(\lambda) = \cos(\lambda(z - z'))\) with half-period equal to \(\pi/|\rho - \rho'|\). After some algebra, it can be shown that the remainder estimates
write as

\[ \omega_n = (-1)^{n+1} e^\frac{n|\rho - \rho'|}{|z - z'|} \lambda_n. \]  \quad (4.67)

We list next three popular extrapolation methods for a given sequence, \( \{S_n\} = S_0, S_1, S_2, \ldots, S_n \), where we consider \( S_n = S_n^{(0)} \).

(i) Euler transformation. For every sequence,

\[ S_n^{(k+1)} = \frac{1}{2} \left( S_n^{(k)} + S_{n+1}^{(k)} \right), \quad n, k \geq 0. \]  \quad (4.68)

The best approximation in this case is \( S_0^{(k)} \big|_{k=n} \), and this choice is the most effective for logarithmic alternating sequences.

(ii) Iterative Aitken transformation

\[ S_n^{(k+1)} = S_n^{(k)} - \frac{[\Delta S_n^{(k)}]^2}{\Delta^2 S_n^{(k)}}, \quad n, k \geq 0. \]  \quad (4.69)

Note that \( \Delta S_n^{(k)} = S_{n+1}^{(k)} - S_n^{(k)} \) and \( \Delta^2 S_n^{(k)} = S_{n+2}^{(k)} - 2S_{n+1}^{(k)} + S_{n+1}^{(k)} \). Obviously, this is a nonlinear transformation and it can be applied to both linear monotone and alternating sequences. The best approximation depends on the number of sequences. When an odd number of sequences are given, the best approximation is \( S_0^{(k)} \big|_{k=n} \). On the other hand, for an even number of given sequences, the best approximation is \( S_1^{(k)} \big|_{k=n} \).

(iii) Weighted-averages method

\[ S_n^{(k+1)} = \frac{S_n^{(k)} + \eta_n^{(k)} S_{n+1}^{(k)}}{1 + \eta_n^{(k)}}, \quad n, k \geq 0, \]  \quad (4.70)
where \( \eta^{(k)}_n \) is the weight and defined as

\[
\eta^{(k)}_n = -\frac{\omega_n}{\omega_{n+1}} = \frac{\lambda_{n+1}}{\lambda_n} e^{\frac{\pi|\rho - \rho'|}{|z - z'|}}.
\]  

(4.71)

As (4.70) implies, the weighted-averages method [103, 111] can be regarded as generalized Euler transformation since \( \eta = 1 \) recovers the Euler transformation. Note that the power of this method comes from using remainder estimates compared to the two other methods. As pointed out in [109], the weighted-averages method is more suitable for extrapolating the Sommerfeld-type integrals. We will examine the strength of this method for our problem later.

### 4.3.2 Numerical Analysis

This section details how to use extrapolation methods and provides some relevant numerical parameters. To evaluate the integral (4.63), the subinterval length \( q \) should be first determined. As suggested in [109], the half-period of the oscillating part of the integrand is a good choice for \( q \), i.e.,

\[
q = \frac{\pi}{|z - z'|}.
\]  

(4.72)

This is because it makes the sequences alternating. However, this is not appropriate in other circumstances. Let us consider the two scenarios described in Figure 4.3. The two figures show the behavior of \( g(\lambda) \) and \( p(\lambda) \) for different combinations of \( |\rho - \rho'| \) and \( |z - z'| \) as \( \lambda \) increases. For better understanding, the functions are normalized by their respective maximum. The Type 1 integrand occurs when \( |\rho - \rho'| > |z - z'| \) and shows linear monotone convergence. On the other hand, the Type 2 integrand occurs when \( |\rho - \rho'| < |z - z'| \) and shows logarithmic alternating convergence. Therefore, (4.72) is not appropriate for the Type 1 because the integrand is near zero before the
Figure 4.3: Two scenarios of the integrand: (a) Type 1 with $|\rho - \rho'| = 0.05$, $|z - z'| = 0.01$ and (b) Type 2 with $|\rho - \rho'| = 0.05$, $|z - z'| = 0.5$.

first half-period. For the Type 1 integrand, a different subinterval length should be defined so that

$$q = \frac{C}{|\rho - \rho'|} = \frac{\pi}{|\rho - \rho'|}, \quad \text{for the Type 1,}$$

(4.73)

where we let $C = \pi$ to be consistent with (4.72). Consequently, the subinterval length can be written in general as

$$q = \frac{\pi}{\max(|\rho - \rho'|, |z - z'|)}, \quad \text{for all cases.}$$

(4.74)

Once the subinterval length is determined, it is necessary to determine how many subintervals are required to achieve sufficient convergence. To do so, the relative error below is defined

$$e_i = \frac{|T(S_{i+1}) - T(S_i)|}{|T(S_{i+1})|},$$

(4.75)

where $T(S_i)$ is the extrapolated (transformed) value for given sequences, $\{S_i\} = S_0, S_1, \cdots, S_i$. If $e_i$ is less than the given error tolerance $e_{tol}$, the sequence is stopped.
and the number of subintervals is determined. Next, the number of quadrature points per the subinterval is increased until the relative error between two adjacent iterations meets the desired criterion. To distinguish from \( e_{\text{tol}} \), the criterion in this step is called the error threshold \( e_{\text{thr}} \). The same step is repeated for determining the number of orders. The overall procedure is schematically described in Figure 4.4.

4.3.3 Convergence Tests

A number of simple convergence tests are performed to verify the procedure described in the previous section. The medium is assumed homogeneous because exact (analytical) solutions are available as reference. Three cases of source/observation distances are considered.

Case 1: \( |\rho - \rho'| = 0.001 \text{ m}, \ |z - z'| = 0.1 \text{ m} \)

Case 2: \( |\rho - \rho'| = 0.1 \text{ m}, \ |z - z'| = 0.1 \text{ m} \)

Case 3: \( |\rho - \rho'| = 10 \text{ m}, \ |z - z'| = 0.1 \text{ m} \)

In each case, three extrapolation methods are compared. Both current magnitude and the medium resistivity are assumed to be equal to one. The relevant error parameters are \( e_{\text{tol}} = 10^{-6} \) and \( e_{\text{thr}} = 10^{-4} \). The smaller \( e_{\text{tol}} \) is chosen to examine the effect of the methods on the number of subintervals. Tables 4.3, 4.4, and 4.5 compare the results, where the first rows represent the number of subintervals needed to achieve convergence in terms of \( e_{\text{tol}} \) and the second rows represent the relative error against the analytical solution. As Table 4.3 shows, the iterative Aitken method does not work for Case 1 because it corresponds to the Type 2 integrand with logarithmic alternating convergence. The Euler transformation works well for all cases, but it can be seen that the weighted-averages method provides the best results. This corroborates the
Figure 4.4: Flowchart of the procedure for computation of the electric potential using extrapolation methods.

collection [109] that the weighted-averages method is very efficient for Sommerfeld-type integrals.
Table 4.3: Comparison of the three extrapolation methods for Case 1.

<table>
<thead>
<tr>
<th></th>
<th>Euler</th>
<th>Aitken</th>
<th>Weighted-Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Subintervals</td>
<td>16</td>
<td>&gt;100</td>
<td>10</td>
</tr>
<tr>
<td>Relative Error</td>
<td>$1.9990 \times 10^{-6}$</td>
<td>N. A.</td>
<td>$3.5863 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 4.4: Comparison of the three extrapolation methods for Case 2.

<table>
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<th>Aitken</th>
<th>Weighted-Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Subintervals</td>
<td>13</td>
<td>13</td>
<td>4</td>
</tr>
<tr>
<td>Relative Error</td>
<td>$1.4131 \times 10^{-6}$</td>
<td>$1.1921 \times 10^{-6}$</td>
<td>$1.4058 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

Table 4.5: Comparison of the three extrapolation methods for Case 3.

<table>
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<tr>
<th></th>
<th>Euler</th>
<th>Aitken</th>
<th>Weighted-Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Subintervals</td>
<td>13</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>Relative Error</td>
<td>$4.9227 \times 10^{-7}$</td>
<td>$4.9227 \times 10^{-7}$</td>
<td>$4.9799 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

4.4 Numerical Results

In this section, validation results for the new algorithm are provided. For all the cases shown below, the relative permittivity $\epsilon_r$ and relative permeability $\mu_r$ are set to one. In Section 4.4.1, the computed electric potentials due to point DC (static) current electrodes by the new algorithm are compared to the analytical solutions in
homogeneous and uniaxial media given in (4.20). In Section 4.4.2, the results from the new algorithm are compared to the Finite Element Method (FEM) results under double-precision arithmetics. A number of cases of practical interest in geophysical exploration are considered. All results from the new algorithm are produced using error tolerance of $10^{-4}$ and error threshold of $10^{-4}$ discussed in Section 4.3.2. The units of the electric potential and resistivity here are $V$ and $\Omega \cdot m$, respectively.

4.4.1 Homogeneous and Uniaxial Media

To recall, the electric potential in a homogeneous and uniaxial medium due to a point current electrode is expressed as

$$
\psi(\rho, z) = \frac{I}{4\pi \sigma_h \sqrt{\rho^2 + (\kappa z)^2}},
$$

(4.76)

where $\kappa$ is the anisotropy ratio defined as $\kappa = \sqrt{\sigma_h/\sigma_v}$. Computations of the electric potential on a square region of dimensions $2.5\,\text{cm} \times 2.5\,\text{cm}$ lying on the $\rho z$-plane are examined in this section. It is assumed that the medium has $\epsilon_r = 1$, $\mu_r = 1$, and $\sigma_h = 1$. For different anisotropy ratios, vertical conductivity $\sigma_v$ varies.

Figures 4.5a – 4.5d show the relative error between the results from the new algorithm and analytical expression results for various maximum orders $n_{\text{max}}$ used in the azimuth summation and various numbers of integration points per subinterval $n_{\text{int}}$, assuming the anisotropy ratio equal to one ($\kappa = 1$). The relative error is defined as

$$
\text{relative error}_{dB} = 10 \log_{10} \frac{|\psi_a - \psi_n|}{|\psi_a|},
$$

(4.77)
where $\psi_a$ is the analytical (exact) value and $\psi_n$ is the numerical value. A smaller relative error is observed when the maximum order or the number of integration points are increased, as expected.

Figures 4.6a – 4.6d show a similar comparison now assuming the anisotropy ratio equal to $\sqrt{2}$, so that $\sigma_h/\sigma_v = 2$. As seen, the error distribution pattern is “compressed” vertically due to smaller vertical conductivity.
Figure 4.6: Relative error distribution with $\kappa = \sqrt{2}$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$.

Similarly, Figures 4.7a – 4.7d show the relative error distribution with the anisotropy ratio of $\sqrt{5}$. The error pattern is again compressed vertically.

Finally, Figures 4.8a – 4.8d show the relative error distribution with the anisotropy ratio of $\sqrt{10}$. Further compression of the error distribution is once more observed.
Figure 4.7: Relative error distribution with $\kappa = \sqrt{5}$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$.

In order to examine the effect of $n_{\text{max}}$ and $n_{\text{int}}$, the receiver point is next fixed at $\rho - \rho' = 1$ cm, $\phi - \phi' = 0^\circ$, and $z - z' = 1$ cm. Figures 4.9a – 4.9d show the error distribution as $n_{\text{max}}$ and $n_{\text{int}}$ vary. In all cases above, the improvement of accuracy achieved by increasing the number of integration points is observed only if sufficient large $n_{\text{max}}$ is chosen ($\gtrsim 20$).
Figure 4.8: Relative error distribution with $\kappa = \sqrt{10}$: (a) $n_{\text{max}} = 10$, $n_{\text{int}} = 1000$, (b) $n_{\text{max}} = 20$, $n_{\text{int}} = 1000$, (c) $n_{\text{max}} = 10$, $n_{\text{int}} = 2000$, and (d) $n_{\text{max}} = 20$, $n_{\text{int}} = 2000$. 
Figure 4.9: Relative error distribution in terms of various maximum orders $n_{\text{max}}$ and integration points $n_{\text{int}}$ with the receiver point at $\rho - \rho' = 1$ cm, $\phi - \phi' = 0^\circ$, and $z - z' = 1$ cm: (a) $\kappa = 1$, (b) $\kappa = \sqrt{2}$, (c) $\kappa = \sqrt{5}$, and (d) $\kappa = \sqrt{10}$. 
4.4.2 Cylindrically Layered Media

Two types of practical cases are presented for validation in cylindrically layered media. For the first type, transmitting and receiving electrodes are point electrodes as illustrated in Figure 4.10a. Case 1 to Case 9 are of this type. A transmitting electrode emitting a DC current of 1 A is located at the Survey Electrode position. The electric potential is measured at two receiving points: the Measurement Electrode 1 ($V_{16''}$) positioned 16" vertically away from the source and the Measurement Electrode 2 ($V_{32''}$) positioned 32" vertically away from the source. Both the transmitting and receiving electrodes are spaced 5" away from the $z$-axis and their azimuthal positions are the same ($\phi = \phi' = 0$). Furthermore, the voltage reference (ground potential) is located 1000" vertically away from the transmitter. The innermost layer has 6" radius and represents the borehole filled with mud, so mud resistivity is also of interest. The outer layers represent the adjacent earth formations, invasion zones, and/or casing layers.

For the second type, ring electrodes are used as transmitting and receiving electrodes as illustrated in Figure 4.13a. Case 10 and Case 11 are of this type. As Figure 4.13a illustrates, the ring electrode has 1" radius and 0.5" thickness. Again, a transmitting electrode emits a DC current of 1 A radially. The longitudinal separation of the transmitting and receiving electrodes is 32". The location of voltage reference is the same as the first type. The innermost layer represents an insulator with very high resistivity and the middle layer represents the borehole characterized by mud resistivity. The outermost layer represents earth formations, which could feature uniaxial anisotropy.
Homogeneous Formation Results
(Rm = 1 Ω-m, Rf = 1 Ω-m)

- Cross-section of the formation in ρ-z plane is shown.
- Survey electrode transmits a DC current of 1 A.
- Both mud resistivity and formation resistivity are 1 Ω-m.
- Voltages at measure electrodes and voltage difference between measure electrodes for this case are:
  - Rm = 1 Ωm
  - εr = 1
  - μr = 1
  - V_ME1 [V] = 0.17878839
  - V_ME2 [V] = 0.094416797
  - ΔV [V] = 0.084371591

Case 1 depicted in Figure 4.10a corresponds to a homogeneous problem, where analytical solutions are available. Both mud resistivity of the inner layer and formation resistivity of the outer layer are the same as 1 Ω·m. As Table 4.6 shows, the new algorithm produces very accurate results with very fast computing time. Note that fast computing times are very important for being able to solve the inverse problem (requiring many repeated forward solutions) in a reasonable time window.

Case 2 is depicted in Figure 4.10b. Now, formation resistivity is increased from 1 Ω·m to 2 Ω·m. Table 4.7 provides the comparison of the electric potentials.

Case 3 depicted in figure 4.10c has mud resistivity of 2 Ω·m and formation resistivity of 1 Ω·m. Table 4.8 provides the comparison of the corresponding electric potentials.
Table 4.6: Comparison of the electric potentials for Case 1.

<table>
<thead>
<tr>
<th></th>
<th>Analytical</th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$1.9581 \times 10^{-1}$</td>
<td>$1.7878 \times 10^{-1}$</td>
<td>$1.9580 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$9.7905 \times 10^{-2}$</td>
<td>$9.4416 \times 10^{-2}$</td>
<td>$9.7900 \times 10^{-2}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$9.7905 \times 10^{-2}$</td>
<td>$8.4371 \times 10^{-2}$</td>
<td>$9.7902 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 4.7: Comparison of the electric potentials for Case 2.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$3.7136 \times 10^{-1}$</td>
<td>$3.9092 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$1.9798 \times 10^{-1}$</td>
<td>$2.0246 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$1.7338 \times 10^{-1}$</td>
<td>$1.8846 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table 4.8: Comparison of the electric potentials for Case 3.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$1.7334 \times 10^{-1}$</td>
<td>$1.9914 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$9.1880 \times 10^{-2}$</td>
<td>$9.7078 \times 10^{-2}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$8.1464 \times 10^{-2}$</td>
<td>$1.0206 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
Figure 4.11: Practical cases with point electrodes in the ρz-plane: (a) Case 4, (b) Case 5, and (c) Case 6.

Case 4 depicted in Figure 4.11a is similar to Case 2 but the formation resistivity becomes 5 Ω · m. Table 4.9 provides the comparison of the results.

On the other hand, Case 5 depicted in Figure 4.11b is similar to Case 3 but the mud resistivity becomes 5 Ω · m. Table 4.10 provides the comparison of the results.

Case 6 illustrated in Figure 4.11c includes a highly conductive casing with $10^{-8}$ Ω · m. In Table 4.11, the comparison of the electric potentials are provided.

In Case 7 depicted in Figure 4.12a, there are three layers. A highly conductive casing is placed between the borehole and earth formation with resistivity of 2 Ω · m. Table 4.12 provides the comparison of the electric potentials. Note that there is disagreement between the absolute potentials, but the difference in the electric potential is similar. This disagreement comes from the fact that this FEM result is affected by the necessary mesh truncation. The current that flows along the thin,
Table 4.9: Comparison of the electric potentials for Case 4.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$9.5544 \times 10^{-1}$</td>
<td>$9.7802 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$5.4241 \times 10^{-1}$</td>
<td>$5.4981 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$4.1303 \times 10^{-1}$</td>
<td>$4.2822 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table 4.10: Comparison of the electric potentials for Case 5.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$1.6945 \times 10^{-1}$</td>
<td>$2.0533 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$9.0310 \times 10^{-2}$</td>
<td>$9.7677 \times 10^{-2}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$7.9145 \times 10^{-2}$</td>
<td>$1.0766 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table 4.11: Comparison of the electric potentials for Case 6.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$1.2980 \times 10^{-4}$</td>
<td>$1.3873 \times 10^{-4}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$2.1372 \times 10^{-7}$</td>
<td>$2.1415 \times 10^{-7}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$1.2959 \times 10^{-4}$</td>
<td>$1.3852 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

highly conductive case does not produce enough potential decay before it reaches the mesh boundary so the FEM result shows a spurious potential offset. FEM has difficulty is simulating this problem unless an extremely long mesh is used along the $z$-direction.
In this simulation, casing width is shrinked to an eighth of an inch and a formation layer that fills the rest of the computational domain is added.

Mud resistivity is $R_m = 1 \ \Omega \cdot m$. Casing resistivity is $R_c = 10^{-8} \ \Omega \cdot m$. Formation resistivity is $R_f = 2 \ \Omega \cdot m$.

Voltages at measure electrodes and voltage difference between measure electrodes for this case are:

- $V_{ME1} [V] = 0.22658744 \times 10^{-3}$
- $V_{ME2} [V] = 0.96331022 \times 10^{-4}$
- $\Delta V [V] = 0.13025642 \times 10^{-3}$

In this simulation, casing in the previous slide is replaced with a resistive layer of resistivity $R_f_1 = 1.5 \ \Omega \cdot m$.

Resistivity of second layer is $R_f_1 = 1.5 \ \Omega \cdot m$. Resistivity of second layer is $R_f_2 = 2 \ \Omega \cdot m$.

Voltages at measure electrodes and voltage difference between measure electrodes for this case are:

- $V_{ME1} [V] = 0.37097601$
- $V_{ME2} [V] = 0.19796549$
- $\Delta V [V] = 0.17301053$

In Case 8 depicted in Figure 4.12b, the middle layer is replaced with another formation layer with resistivity of $1.5 \ \Omega \cdot m$. Table 4.13 provides the comparison of the electric potentials.

Case 9 depicted in Figure 4.12b has thicker middle layer of $3''$ than that in Case 8. Table 4.14 provides the comparison of the electric potentials.

Figure 4.12: Practical cases with point electrodes in the $\rho z$-plane: (a) Case 7, (b) Case 8, and (c) Case 9.
Table 4.12: Comparison of the electric potentials for Case 7.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$2.2658 \times 10^{-4}$</td>
<td>$1.7241 \times 10^{-3}$ (3 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$9.6331 \times 10^{-5}$</td>
<td>$1.5885 \times 10^{-3}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$1.3025 \times 10^{-4}$</td>
<td>$1.3562 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 4.13: Comparison of the electric potentials for Case 8.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$3.7097 \times 10^{-1}$</td>
<td>$3.9061 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$1.9796 \times 10^{-1}$</td>
<td>$2.0245 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$1.7301 \times 10^{-1}$</td>
<td>$1.8816 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Table 4.14: Comparison of the electric potentials for Case 9.

<table>
<thead>
<tr>
<th></th>
<th>FEM</th>
<th>New Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_{16''}$</td>
<td>$3.6077 \times 10^{-1}$</td>
<td>$3.8160 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$V_{32''}$</td>
<td>$1.9694 \times 10^{-1}$</td>
<td>$2.0156 \times 10^{-1}$ (2 sec.)</td>
</tr>
<tr>
<td>$\Delta V$</td>
<td>$1.6383 \times 10^{-1}$</td>
<td>$1.8003 \times 10^{-1}$</td>
</tr>
</tbody>
</table>
Figure 4.13: Practical cases with ring electrodes in the $\rho z$-plane: (a) Case 10 and (b) Case 11.

Apart from the previous ten cases, Cases 10 and 11 illustrated in Figures 4.13a and 4.13b have ring electrodes for measurement of the electric potential. The innermost insulator is assumed to have resistivity of $10^{20} \, \Omega \cdot m$. In the middle layer, mud resistivity varies from $10^{-3}$ up to $10^{3} \, \Omega \cdot m$. The outermost formation is isotropic for Case 10 with resistivity of $1 \, \Omega \cdot m$ whereas the formation for Case 11 features uniaxial anisotropy with the anisotropy ratio of $\sqrt{2}$. Comparison of the electric potentials for Case 10 and Case 11 are provided in Tables 4.15 and 4.16, respectively.
Table 4.15: Comparison of the electric potentials for Case 10.

<table>
<thead>
<tr>
<th>Mud Resistivity</th>
<th>FEM</th>
<th>New Algorithm</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_m = 10^{-3}$</td>
<td>-</td>
<td>0.030552523</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^{-2}$</td>
<td>-</td>
<td>0.076391719</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^{-1}$</td>
<td>0.106663694</td>
<td>0.106600110</td>
<td>$5.9612 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_m = 10^0$</td>
<td>0.092693130</td>
<td>0.092695986</td>
<td>$3.0821 \times 10^{-5}$</td>
</tr>
<tr>
<td>$R_m = 10^1$</td>
<td>0.087568565</td>
<td>0.087620773</td>
<td>$5.9620 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_m = 10^2$</td>
<td>-</td>
<td>0.091243455</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^3$</td>
<td>-</td>
<td>0.133535615</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.16: Comparison of the electric potentials for Case 11.

<table>
<thead>
<tr>
<th>Mud Resistivity</th>
<th>FEM</th>
<th>New Algorithm</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_m = 10^{-3}$</td>
<td>-</td>
<td>0.031723158</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^{-2}$</td>
<td>-</td>
<td>0.081692362</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^{-1}$</td>
<td>0.118557013</td>
<td>0.118485808</td>
<td>$6.0060 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_m = 10^0$</td>
<td>0.100404494</td>
<td>0.100416669</td>
<td>$1.2126 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_m = 10^1$</td>
<td>0.092388761</td>
<td>0.092452525</td>
<td>$6.9017 \times 10^{-4}$</td>
</tr>
<tr>
<td>$R_m = 10^2$</td>
<td>-</td>
<td>0.095730898</td>
<td>-</td>
</tr>
<tr>
<td>$R_m = 10^3$</td>
<td>-</td>
<td>0.137988427</td>
<td>-</td>
</tr>
</tbody>
</table>
Moreover, the electric potential distributions from Case 2 to Case 7 are illustrated. Since the electric potential varies by many orders of magnitude near the electrodes, a log-scale is used for the following plots; i.e., we plot the quantity $10 \log_{10} |\psi|$. Figure 4.14 depicts potential distributions in 3-D view. Figure 4.15 and Figure 4.16 illustrate cross-section of the electric potential distributions at the $z = 16''$ and $z = 32''$ planes, respectively, while Figure 4.17 depicts the cross-section at the $y = 0''$ plane. In all figures below, thicker black lines represent interfaces between cylindrical layers and thinner black lines represent potential contours. Note that the third layer of Case 7 is too thin for visualization in these plots.
Figure 4.14: Electric potential distribution around the transmitting electrode in 3-D view: (a) Case 2, (b) Case 3, (c) Case 4, (d) Case 5, (e) Case 6, and (f) Case 7.
Figure 4.15: Electric potential distribution at $z = 16''$ plane around the transmitting electrode: (a) Case 2, (b) Case 3, (c) Case 4, (d) Case 5, (e) Case 6, and (f) Case 7.
Figure 4.16: Electric potential distribution at $z = 32''$ plane around the transmitting electrode: (a) Case 2, (b) Case 3, (c) Case 4, (d) Case 5, (e) Case 6, and (f) Case 7.
Figure 4.17: Electric potential distribution at $y = 0''$ plane around the transmitting electrode: (a) Case 2, (b) Case 3, (c) Case 4, (d) Case 5, (e) Case 6, and (f) Case 7.
Chapter 5

Particle-in-cell Algorithm on Unstructured Grids for Kinetic Charge Transport

The importance of particle-in-cell (PIC) algorithms for simulating the behavior of kinetic charges has continuously grown over last several decades [16, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122]. They are utilized in many plasma-related applications such as particle accelerators [123, 124, 125], laser-plasma interactions [126, 127, 128, 129, 130, 131, 132], astrophysics [133, 134], high-power microwave generation [135], and semiconductor devices [136, 137, 138, 139]. The PIC algorithms are based on a self-consistent coupling of the many-particle kinetics with the electromagnetic field evolution. In particular, it is possible to integrate electromagnetic field solvers based on finite elements or finite-differences, with little or no modification, with particles updates on the same grid. One of the reasons for the success of the PIC simulations is that they also allow for an accurate representation of the underlying physics by means of “superparticles,” in which case each (computational) particle can represent millions of electrons or ions. This is often the strategy adopted for collisionless plasma simulations [16]. Nevertheless, a key challenge for the PIC algorithms on unstructured grids is whether charge conservation is fulfilled or not. When charge is not conserved, the undesired charge accumulation at grid nodes occurs, which violates
Gauss’ law and eventually results in spurious solutions. The traditional approach was to apply some correction terms to updated electric fields at every time step to enforce Gauss’ law [140, 141, 142]. The use of correction terms is usually translated into computational burden, which is dramatically increased for a system with a large number of particles. Therefore, the need for a highly efficient PIC algorithm arose and, as an alternative, the use of the (discrete) continuity equation at every grid cell was proposed by many researchers [143, 144, 145, 146, 147, 148, 149, 150, 151], who initially focused on rectangular grids. To overcome the well-known limitations of the rectangular grids, unstructured grids are then employed with the various choices of basis functions [143, 152, 153, 154, 155, 156].

Among the PIC algorithms developed for unstructured grids, the algorithm proposed by Candel et al. [153, 154] was formulated from the vector-wave equation with time-integrated electric field as an unknown. In general, solution space from the vector-wave equation is larger than that from the mixed E-B Finite-Element Time-Domain (FETD) method [157, 158, 159, 160, 161]. For example, gradient-like solutions of the form $t \nabla \phi$ supported in the vector-wave equation is likely to dominate the solution space in later time [158, 160, 162] unless specialized strategies are adopted [163, 164, 165]. Also, the accuracy order for the electric field is one less than the proposed unknown because numerical integration should be performed. The Newmark-beta scheme [166] was used in the algorithm for unconditionally stable time integration with arbitrary time step. However, the overlooked burden of the scheme is not fully examined and the cost will be significant for highly refined and multiscaled problems. For more discussions, see Appendix F.
Another PIC algorithm on unstructured grids developed by Squire et al. [155] was based on variational principles and the use of gauge symmetry to achieve charge conservation. Also, they assumed the diagonal representation of discrete Hodge operators called mass matrices, which encode the information of spatial metrics. This diagonal representation can only be applied to the specific type of unstructured grids such as Delaunay triangulation with the Voronoi diagram, where orthogonality between primal and dual grid elements must be attained. For general unstructured grids without such orthogonality (See [167, 168, 169]), the validity of the algorithm is not, therefore, guaranteed.

More recently, a comprehensive PIC algorithm was developed by Campos-Pinto et al. [156]. This algorithm featured many desirable aspects such as curl-conforming elements of arbitrary orders, arbitrary shape factors, and particle trajectories being piecewise polynomial.

In this chapter, a novel charge-conserving algorithm for the PIC simulations on unstructured grids is discussed. Similar discussion can be found in [170]. The scatter-gather part of the present algorithm is identical to the algorithms in [155, 156] at a fundamental level but is constructed in a more streamlined manner from geometrical perspectives inspired by differential forms. Also, the present algorithm can be applied to general grids from irregular triangulation compared to the algorithm in [155]. The formulation of the present algorithm starts with the two coupled Maxwell’s curl equations with Whitney forms of various degrees. Leap-frog time integration is selected for updating both electromagnetic fields and particle attributes to ensure symmetric-positive-definite linear system without numerical costs due to high condition numbers depending on different mesh refinement levels. Some aspects of the present algorithm
are the use of geometrical interpretation of differential forms, which links all physical variables to specific geometrical objects, and unique localization rules for all grid elements. Also, the consistent use of Whitney forms [161, 171, 172] for scatter-gather scheme not only results in optimized interpolation between field values and particle attributes, but also leads to a closed-form of the scattered charges and currents, which excludes the need for numerical quadratures. It can be understood later that the elegant closed-form of the scattered values are closely related to the geometrical characteristics of the chosen interpolation functions.

This chapter is organized as follows. In Section 5.1, the formulation of the present particle-in-cell algorithm is discussed. Four substeps comprising the entire PIC simulations for arbitrarily unstructured grids are presented. Also, charge conservation and Gauss’ law are verified analytically, geometrically, and numerically. In Section 5.2, a variety of the PIC examples are provided for demonstrating the validity of the present algorithm.

5.1 Formulation

5.1.1 Field Update

For field update, finite elements based on two Maxwell’s curl equations instead of the vector wave equation in time domain are chosen. Two unknowns, the electric field intensity \( \mathbf{E}(\mathbf{r}, t) \) and the magnetic flux density \( \mathbf{B}(\mathbf{r}, t) \) are expanded using Whitney functions of different degrees as [161, 171, 172]

\[
\mathbf{E}(\mathbf{r}, t) = \sum_{i=1}^{N_e} e_i(t) \mathbf{W}_i^1(\mathbf{r}),
\]

\[
\mathbf{B}(\mathbf{r}, t) = \sum_{i=1}^{N_f} b_i(t) \mathbf{W}_i^2(\mathbf{r}),
\]
where $W_1^i(r)$ and $W_2^i(r)$ are Whitney edge basis functions and Whitney face basis functions, respectively. $W_1^i(r)$ is defined on each edge, so the summation in (5.1) is over $N_e$, the total number of edges in the grid. On the other hand, $W_2^i(r)$ is defined on each face, so the summation in (5.2) is over $N_f$, the total number of faces in the grid. Note that the metric information is stored in the Whitney functions, i.e., $W_1^i(r)$ in m$^{-1}$ and $W_2^i(r)$ in m$^{-2}$, and time-dependency is associated with two different types of degrees of freedom, $e_i(t)$ and $h_i(t)$, which have units of Volts [V] and Webers [Wb], respectively. From the perspective of differential forms [173, 174, 175, 176], these Whitney basis functions are the vectorial expression of Whitney forms. As the electric field is inherently a 1-form in the language of differential forms, it is expanded via Whitney 1-forms or, equivalently, Whitney edge basis functions. Similarly, as the magnetic flux density is inherently a 2-form, it is expanded via Whitney 2-forms or, equivalently, Whitney face basis functions. For further discussions about Whitney basis functions and their relation to Whitney forms are discussed in Appendix E. Moreover, the electric current density is expanded using Whitney edge basis functions like the electric field.

$$J_*(r, t) = \sum_{i=1}^{N_e} i_i(t) W_1^i(r),$$

(5.3)

where the subscript $*$ is used to indicate that $J$, in fact, resides in the second grid that is dual to the primal grid where $E$ resides. The second grid is called a dual grid. The distinction between the primal and dual grids stems from the conceptual description of various physical quantities and manifests itself in the language of differential forms. As $J$ is a 2-form in the dual grid and $E$ is a 1-form in the primal grid in three-dimensional space [175], there exists one-to-one correspondence between $J$ and $E$. Therefore, the same Whitney basis functions are used and the summation is over $N_e$. 201
The spatial discretization is implemented through Galerkin testing and Maxwell’s equations are modified to [161]

\[ C \cdot e = -\frac{d}{dt} b, \]  
\[ \tag{5.4} \]

\[ C^T \cdot [\star_{\mu^{-1}}] \cdot b = \frac{d}{dt} [\star_{\epsilon}] \cdot e + i, \]  
\[ \tag{5.5} \]

where \( C \) is called an incidence matrix, which is the discrete representation of curl operator [177, 178]. It is noteworthy that the elements of \( C \) only have three integer values: -1, 0, or 1. The superscript \( T \) in (5.5) means the transpose. The arrays of degrees of freedom are identified and they are defined as

\[ e = [e_1(t), e_2(t), \cdots, e_{N_e}(t)]^T, \]  
\[ \tag{5.6} \]

\[ b = [b_1(t), b_2(t), \cdots, b_{N_f}(t)]^T, \]  
\[ \tag{5.7} \]

\[ i = [i_1(t), i_2(t), \cdots, i_{N_e}(t)]^T. \]  
\[ \tag{5.8} \]

In (5.5), \([\star_{\mu^{-1}}]\) and \([\star_{\epsilon}]\) are mass matrices or called discrete Hodge star operators defined as [161, 179],

\[ [\star_{\epsilon}]_{ij} = \int_{\Omega} \epsilon W_i^1(r) \cdot W_j^1(r) dV, \]  
\[ \tag{5.9} \]

\[ [\star_{\mu^{-1}}]_{ij} = \int_{\Omega} \frac{1}{\mu} W_i^2(r) \cdot W_j^2(r) dV. \]  
\[ \tag{5.10} \]

It should be pointed out that the mass matrix associated with \( i \) in (5.5) is assumed to be an identity matrix for simplicity.

The temporal discretization of (5.4) and (5.5) is performed using the leap-frog time update.

\[ b^{n+\frac{1}{2}} = b^{n-\frac{1}{2}} - \Delta t C \cdot e^n, \]  
\[ \tag{5.11} \]

\[ [\star_{\epsilon}] \cdot e^{n+1} = [\star_{\epsilon}] \cdot e^n + \Delta t \left( C^T \cdot [\star_{\mu^{-1}}] \cdot b^{n+\frac{1}{2}} - i^{n+\frac{1}{2}} \right), \]  
\[ \tag{5.12} \]
where wholly discrete values are defined as

\[ b^{n+\frac{1}{2}} = \begin{bmatrix} b_1 \left( (n + \frac{1}{2})\Delta t \right), b_2 \left( (n + \frac{1}{2})\Delta t \right), \ldots, b_{N_f} \left( (n + \frac{1}{2})\Delta t \right) \end{bmatrix}^T, \quad (5.13) \]
\[ i^{n+\frac{1}{2}} = \begin{bmatrix} i_1 \left( (n + \frac{1}{2})\Delta t \right), i_2 \left( (n + \frac{1}{2})\Delta t \right), \ldots, i_{N_e} \left( (n + \frac{1}{2})\Delta t \right) \end{bmatrix}^T, \quad (5.14) \]
\[ e^n = [e_1(n\Delta t), e_2(n\Delta t), \ldots, e_{N_e}(n\Delta t)]^T. \quad (5.15) \]

The above time update scheme is conditionally stable because of the leap-frog time update as well as positive-definiteness of (5.9) and (5.10) (See [161],[178]). Each element of \( e^{n+1} \) and \( b^{n+\frac{1}{2}} \) obtained from (5.11) and (5.12) can be expanded in time using two different basis functions as shown below [180].

\[ e_i(t) = \sum_n e^n_i \Pi^n(t), \quad (5.16) \]
\[ b_i(t) = \sum_n b^{n+\frac{1}{2}}_i \Lambda^{n+\frac{1}{2}}(t), \quad (5.17) \]

where \( \Pi^n(t) \) are piecewise constant functions while \( \Lambda^{n+\frac{1}{2}}(t) \) are piecewise linear functions as shown in Figure 5.1. Furthermore, they are centered at different discrete times. \( \Pi^n(t) \) are centered on integer discrete times while \( \Lambda^{n+\frac{1}{2}}(t) \) are centered on half-integer discrete times. The choice of (5.16) and (5.17) is motivated by examining (5.4) that both \( \frac{d}{dt}b \) and \( e \) should be described by the same temporal basis functions for equality. In a similar sense, from (5.5), \( b \) and \( i \) should have the same form in time. Therefore, it is determined that \( i_i(t) \) are expanded using the same piecewise linear functions.

### 5.1.2 Gather

In order to update particle attributes, field values at particle positions should be gathered a priori. In this step, Whitney basis functions of two different orders are
chosen for the interpolation of the field values, which are expressed as

\[
E(\vec{r}_p, n\Delta t) = E^n(\vec{r}_p) = \sum_{i=1}^{N_p} e_i^n W_i^1(\vec{r}_p), \tag{5.18}
\]

\[
B(\vec{r}_p, (n+\frac{1}{2})\Delta t) = B^{n+\frac{1}{2}}(\vec{r}_p) = \sum_{i=1}^{N_p} b_i^{n+\frac{1}{2}} W_i^2(\vec{r}_p), \tag{5.19}
\]

where \(\vec{r}_p\) represents the position vector of the \(p\)-th particle. These expressions are easily obtained by substituting (5.16) and (5.17) into (5.1) and (5.2). Note that the summations in (5.16) and (5.17) are collapsed to single terms due to the respective discrete times of the field values as shown in Figure 5.1.

In addition, it is necessary to determine the initial positions of particles on the unstructured grids. Figure 5.2a and Figure 5.2b depict two possible scenarios. If \(x_{in}\) is inside the triangle, \((\vec{e}_1 \times \vec{a}) \cdot (\vec{e}_2 \times \vec{a}) < 0\). On the other hand, if \(x_{out}\) is outside the triangle, \((\vec{e}_1 \times \vec{b}) \cdot (\vec{e}_2 \times \vec{b}) > 0\). As a special case, when \(\vec{e}_1\) and \(\vec{a}\) are parallel to each other, their cross product is zero. In this case, the position of the particle should be between the two nodes. In order to place the particle exactly inside a certain triangle, this testing should be done repeatedly for all three nodes of each triangle.
Figure 5.2: Illustration of the particle positioning about \( \nu_1 \) (vertex 1) in the triangular mesh: (a) Typical scenario and (b) Special scenario.

### 5.1.3 Particle Update

In order to update particle positions \( \mathbf{r}_p(t) \) and velocities \( \mathbf{v}_p(t) \), the equation of motion and Lorentz-Newton equation are combined. First, a non-relativistic case is discussed and relativistic case will follow.

**Non-Relativistic Case**

The particle update in the non-relativistic regime consists of

\[
\frac{d\mathbf{r}_p}{dt} = \mathbf{v}_p, \quad (5.20)
\]

\[
\frac{d\mathbf{v}_p}{dt} = \frac{q}{m} \left( \mathbf{E} + \mathbf{v}_p \times \mathbf{B} \right), \quad (5.21)
\]
where \( q \) and \( m \) are the charge and mass of the particle, respectively. Implementing temporal discretization of (5.20) and (5.21) using the leap-frog time update gives

\[
\begin{align*}
\mathbf{r}_{p,n+1} - \mathbf{r}_{p} &= \Delta t \mathbf{v}_{p}^{n+\frac{1}{2}}, \\
\mathbf{v}_{p}^{n+\frac{1}{2}} - \mathbf{v}_{p}^{n-\frac{1}{2}} &= \frac{q\Delta t}{m} (\mathbf{E}^{n} + \mathbf{v}_{p}^{n} \times \mathbf{B}^{n}).
\end{align*}
\]  

(5.22)  
(5.23)

Note that the update of (5.22) is straightforward whereas the update of (5.23) is somewhat complicated because \( \mathbf{v}_{p}^{n} \) and \( \mathbf{B}^{n} \) should be interpolated a priori. Among many ways for the interpolation such as Newton, Runge-Kutta, and Boris method [16], the simplest form is used in this scheme, i.e.,

\[
\begin{align*}
\mathbf{v}_{p}^{n} &= \frac{1}{2} \left( \mathbf{v}_{p}^{n+\frac{1}{2}} + \mathbf{v}_{p}^{n-\frac{1}{2}} \right), \\
\mathbf{B}^{n} &= \frac{1}{2} \left( \mathbf{B}^{n+\frac{1}{2}} + \mathbf{B}^{n-\frac{1}{2}} \right),
\end{align*}
\]  

(5.24)  
(5.25)

where \( \mathbf{B}^{n+\frac{1}{2}}(\mathbf{r}_{p}) = \mathbf{B}^{n+\frac{1}{2}}(\mathbf{r}_{p}^{n}) \) is assumed because \( \mathbf{B} \) is also a function of spatial coordinates. The interpolation scheme of (5.24) and (5.25) is exact when the linear basis functions are used for \( \mathbf{v}_{p}^{n} \) and \( \mathbf{B}^{n} \). Therefore, the basis functions for \( \mathbf{v}_{p}^{n} \) are determined to be the same as the piecewise linear functions used for \( b_{i}(t) \) in (5.17).

The choice of the basis functions for \( \mathbf{v}_{p}^{n} \) are two-fold. First of all, \( \mathbf{v}_{p}^{n} \) and \( \mathbf{B}^{n} \) coincide in the same discrete times. Secondly, as the time derivative of \( \mathbf{v}_{p}^{n} \) and \( \mathbf{B}^{n} \) should be evaluated (see (5.4) and (5.21)), the lowest possible order of the basis functions needs to be the first order. Consequently, \( \mathbf{v}_{p}(t) \) can be expanded as

\[
\mathbf{v}_{p}(t) = \sum_{n} \mathbf{v}_{p}^{n+\frac{1}{2}} \Lambda^{n+\frac{1}{2}}(t).
\]  

(5.26)

Hence, (5.23) is modified to

\[
\begin{align*}
\mathbf{v}_{p}^{n+\frac{1}{2}} - \mathbf{v}_{p}^{n-\frac{1}{2}} &= \frac{q\Delta t}{m} \left[ \mathbf{E}^{n} + \frac{1}{4} \mathbf{v}_{p}^{n+\frac{1}{2}} \times \left( \mathbf{B}^{n+\frac{1}{2}} + \mathbf{B}^{n-\frac{1}{2}} \right) \right. \\
&\quad \left. + \frac{1}{4} \mathbf{v}_{p}^{n-\frac{1}{2}} \times \left( \mathbf{B}^{n+\frac{1}{2}} + \mathbf{B}^{n-\frac{1}{2}} \right) \right].
\end{align*}
\]  

(5.27)
It should be noted that (5.27) is implicit because the cross product is involved. The matrix representation of (5.27) is

\[
\begin{bmatrix}
1 & -\frac{q\Delta t}{2m} B^n_z & \frac{q\Delta t}{2m} B^n_y \\
\frac{q\Delta t}{2m} B^n_z & 1 & -\frac{q\Delta t}{2m} B^n_x \\
-\frac{q\Delta t}{2m} B^n_y & \frac{q\Delta t}{2m} B^n_x & 1
\end{bmatrix}
\begin{bmatrix}
v^n_{x + \frac{1}{2}} \\
v^n_{y + \frac{1}{2}} \\
v^n_{z + \frac{1}{2}}
\end{bmatrix}
= \begin{bmatrix}
v^n_{x - \frac{1}{2}} + \frac{q\Delta t}{m} \left\{E^n_x + \frac{1}{2} \left(v^n_x - \frac{1}{2} B^n_x - v^n_y - \frac{1}{2} B^n_y\right)\right\} \\
v^n_{y - \frac{1}{2}} + \frac{q\Delta t}{m} \left\{E^n_y + \frac{1}{2} \left(v^n_y - \frac{1}{2} B^n_y - v^n_z - \frac{1}{2} B^n_z\right)\right\} \\
v^n_{z - \frac{1}{2}} + \frac{q\Delta t}{m} \left\{E^n_z + \frac{1}{2} \left(v^n_z - \frac{1}{2} B^n_z - v^n_x - \frac{1}{2} B^n_x\right)\right\}
\end{bmatrix}
\]

(5.28)

where

\[
B^n_s = \frac{1}{2} \left(B^n_{s + \frac{1}{2}} + B^n_{s - \frac{1}{2}}\right), \quad s = x, y, \text{ or } z.
\]

The right hand side of (5.28) is rearranged as

\[
\begin{bmatrix}
1 & \frac{q\Delta t}{2m} B^n_z & -\frac{q\Delta t}{2m} B^n_y \\
-\frac{q\Delta t}{2m} B^n_z & 1 & \frac{q\Delta t}{2m} B^n_x \\
\frac{q\Delta t}{2m} B^n_y & -\frac{q\Delta t}{2m} B^n_x & 1
\end{bmatrix}
\begin{bmatrix}
v^n_{x - \frac{1}{2}} \\
v^n_{y - \frac{1}{2}} \\
v^n_{z - \frac{1}{2}}
\end{bmatrix}
+ \frac{q\Delta t}{m}
\begin{bmatrix}
E^n_x \\
E^n_y \\
E^n_z
\end{bmatrix}
\]

(5.30)

Therefore, (5.28) can be succinctly expressed as

\[
v^{n + \frac{1}{2}} = N^{-1} \cdot N^T \cdot v^{n - \frac{1}{2}} + \frac{q\Delta t}{m} N^{-1} \cdot E^n,
\]

(5.31)

where

\[
N = \begin{bmatrix}
1 & -\frac{q\Delta t}{2m} B^n_z & \frac{q\Delta t}{2m} B^n_y \\
\frac{q\Delta t}{2m} B^n_z & 1 & -\frac{q\Delta t}{2m} B^n_x \\
-\frac{q\Delta t}{2m} B^n_y & \frac{q\Delta t}{2m} B^n_x & 1
\end{bmatrix}
\]

(5.32)

Note that N is unitless.
As particles move, grid information of the next positions of particles should be properly updated as well. The grid information not only means the spatial coordinates, but also includes triangle numbers for the new positions and the edge numbers crossed by particles, which is required for the scatter step discussed in Section 5.1.4. To do so, an intelligent mesh for tracking particles needs to be constructed and efficient particle locators can save a large amount of computing resources for unstructured grids because using traditional way of iterative search or lookup tables is too time-consuming. The intelligent mesh contains all connectivity information among grid elements a priori. It is optimal because it starts to search for nearby triangles or edges with which particles are likely to associate next time steps.

**Relativistic Case**

In the relativistic regime [181, 182], the particle update is expressed as

\[
\begin{align*}
\frac{dr_p}{dt} &= v_p, \\
\frac{d\gamma_p v_p}{dt} &= \frac{q}{m_0} (E + v_p \times B),
\end{align*}
\]

(5.33)

(5.34)

where \(m_0\) represents rest mass of the particle and \(\gamma_p\) is the relativistic factor of the \(p\)-th particle defined as

\[
\gamma_p = \frac{1}{\sqrt{1 + (|v_p|/c)^2}},
\]

(5.35)

where \(c\) is the speed of light. To avoid the time derivative in the left hand side of (5.34), the relation \(u_p = \gamma v_p\) is employed, and (5.33) and (5.34) are rewritten as

\[
\begin{align*}
\frac{dr_p}{dt} &= \gamma_p u_p, \\
\frac{du_p}{dt} &= \frac{q}{m_0} \left( E + \frac{u_p}{\gamma_p} \times B \right).
\end{align*}
\]

(5.36)

(5.37)
Similar to the non-relativistic case, (5.36) and (5.37) are discretized in time as

\[
\frac{\mathbf{r}_{p}^{n+1} - \mathbf{r}_{p}^{n}}{\Delta t} = \frac{\mathbf{u}_{p}^{n+\frac{1}{2}}}{\gamma_{p}^{n+\frac{1}{2}}},
\]

(5.38)

\[
\frac{\mathbf{u}_{p}^{n+\frac{1}{2}} - \mathbf{u}_{p}^{n-\frac{1}{2}}}{\Delta t} = \frac{q}{m_{0}} \left( \mathbf{E}^{n} + \frac{\mathbf{u}_{p}^{n}}{\gamma_{p}^{n}} \times \mathbf{B}^{n} \right).
\]

(5.39)

To update \( \mathbf{u}_{p}^{n+\frac{1}{2}} \) in (5.39), the Boris algorithm [183], where electric and magnetic effects are updated separately, is adopted for high efficiency. The particle update can be done as follows [112, 184].

\[
\mathbf{u}_{p}^{-} = \mathbf{u}_{p}^{n-\frac{1}{2}} + \frac{q\Delta t}{2m_{0}} \mathbf{E}^{n},
\]

(5.40)

\[
\mathbf{u}_{p}^{'} = \mathbf{u}_{p}^{-} + \mathbf{u}_{p}^{-} \times \mathbf{t}_{p}^{n},
\]

(5.41)

\[
\mathbf{u}_{p}^{+} = \mathbf{u}_{p}^{-} + \mathbf{u}_{p}^{'},
\]

(5.42)

\[
\mathbf{u}_{p}^{n+\frac{1}{2}} = \mathbf{u}_{p}^{+} + \frac{q\Delta t}{2m_{0}} \mathbf{E}^{n},
\]

(5.43)

where

\[
\mathbf{t}_{p}^{n} = \mathbf{\hat{B}} \tan \left( \frac{q\Delta tB^{n}}{2\gamma_{p}m_{0}} \right),
\]

(5.44)

\[
\mathbf{s}_{p}^{n} = \frac{2\mathbf{t}_{p}^{n}}{1 + |\mathbf{t}_{p}^{n}|^{2}}.
\]

(5.45)

Note that \( \mathbf{u}_{p}^{-}, \mathbf{u}_{p}^{'}, \) and \( \mathbf{u}_{p}^{+} \) are all intermediate vectors. Also, \( \mathbf{\hat{B}} \) and \( B^{n} \) in (5.44) are a unit vector and the magnitude of \( \mathbf{B}^{n} \) in (5.25), respectively. It can be easily shown that the electric and magnetic effects can be separated. When (5.40) and (5.43) are plugged into (5.39), we have

\[
\frac{\mathbf{u}_{p}^{+} - \mathbf{u}_{p}^{-}}{\Delta t} = \frac{q}{2m_{0}} \left( \frac{\mathbf{u}_{p}^{+} + \mathbf{u}_{p}^{-}}{\gamma_{p}^{n}} \times \mathbf{B}^{n} \right),
\]

(5.46)
where the effect of $E^n$ is completely encoded in $u_p^-$ and $u_p^+$. Also, the determination of the relativistic factor at integer times can benefit from the relation

$$
\gamma^n_p = \sqrt{1 + \left(\frac{|u_p^-|}{c}\right)^2} = \sqrt{1 + \left(\frac{|u_p^+|}{c}\right)^2}.
$$

(5.47)

5.1.4 Scatter

In this step, charge and current density are assigned to grid elements from updated particle attributes from the previous step. Whitney basis functions are employed for exact charge conservation because of their consistent relations among various degrees. It is shown below that the proper use of Whitney basis functions not only leads to the satisfaction of the discrete continuity equation, but also provides the geometrical interpretation of charge conservation. It should be stressed that numerical quadratures for time integration are not needed during this step compared to other charge-conserving schemes [153, 154, 155, 156].

First of all, let us discuss the assignment of charge density. The amount of charge assigned to each node (vertex) from the $p$-th particle with total amount of charge $Q$ is determined using Whitney 0-forms as

$$
q_i = Q W^0_i(r_p) = Q \lambda_i(r_p),
$$

(5.48)

where the subscript $i$ is the vertex number, and $W^0_i$ is the Whitney 0-form of vertex $i$, which is simply barycentric coordinate of $r_p$ with respect to vertex $i$, i.e., $W^0_i(r_p) = \lambda_i(r_p)$. For more detailed exposition, refer to Appendix E. By taking summation to (5.48) over all possible $i$’s, we obtain

$$
\sum_i q_i = \sum_i Q \lambda_i(r_p) = Q \sum_i \lambda_i(r_p) = Q,
$$

(5.49)
so that the total charge is preserved. In (5.49), the partition of unity for barycentric coordinates is used, i.e., \( \sum_i \lambda_i(r_p) = 1 \). The geometrical illustration of (5.48) is depicted in Figure 5.3a where vertices and edges are symboled by \( \nu \) and \( e \), respectively. Due to the localization property of Whitney functions, the vertices to which charge is assigned are the ones of the triangle where the particle resides.

Secondly, let us take a look at the assignment of current density. The current flowing along each edge is determined using Whitney 1-forms and the average thereof during \( \Delta t \). The geometrical illustration of the current assignment is shown in Figure 5.3b. It is assumed that the particle moves from \( r_{p,s} \) to \( r_{p,f} \) along a straight path \( L \) during \( \Delta t \). For instance, the current along \( e_1 \) (edge 1) is

\[
i_1 = \frac{Q}{\Delta t} \int_{r_{p,s}}^{r_{p,f}} \mathbf{W}_1^1(r_p) \cdot d\mathbf{L} = \frac{Q}{\Delta t} \left( \lambda^s_1 \lambda^f_2 - \lambda^f_1 \lambda^s_2 \right),
\]

where \( \lambda^s_i \) and \( \lambda^f_i \) mean \( \lambda_i(r_{p,s}) \) and \( \lambda_i(r_{p,f}) \), respectively. In (5.50), the line integral is involved to take the sense of averaging and, very importantly, a key characteristic is that the closed-form for the integral is available. For further discussions about the line integral, refer to Appendix E.3. The currents along other edges could be constructed in a similar way by using appropriate Whitney 1-forms.

During the scatter step, it is possible for the particle to travel more than a single triangle and to cross some edges during \( \Delta t \). Figure 5.4 illustrates one example of this case. The path during \( \Delta t \) is \( L = L_1 + L_2 \) and currents on five edges are updated. Therefore, the crossings should be tracked in the particle update step for proper current assignment. The scatter step above is applied to each line segment.
Figure 5.3: Scatter step: (a) Nodal charge assignment from a charged particle placed at $r_p$ with local numbering of vertices and edges and (b) Current assignment due to charge movement from $r_{p,s}$ to $r_{p,f}$ during $\Delta t$ with default directions of currents.

Figure 5.4: Current assignment during $\Delta t$ when a particle travels two triangles along $L = L_1 + L_2$ by crossing one edge.

### 5.1.5 Charge Conservation

For the verification of charge conservation, we need to take a look at whether the continuity equation is satisfied. The semi-discrete continuity equation is written as
\[ \frac{d}{dt} \mathbf{q} + \mathbf{\tilde{S}} \cdot \mathbf{i} = 0, \]  

(5.51)

where \( \mathbf{\tilde{S}} \) is another incidence matrix representing the discrete divergence operator in the dual grid [175, 177, 179, 185]. It should be pointed out that, similarly to \( \mathbf{C} \), the elements of \( \mathbf{\tilde{S}} \) are -1, 0, or 1. Also, in (5.51), \( \mathbf{q} \) is the array of degrees of freedom representing the amount of charges at all vertices and is defined as

\[ \mathbf{q} = [q_1(t), q_2(t), \cdots, q_{N_v}(t)]^T, \]  

(5.52)

where \( N_v \) is the total number of vertices in the grid. Applying the leap-frog time update to (5.51) gives

\[ \frac{\mathbf{q}^{n+1} - \mathbf{q}^n}{\Delta t} + \mathbf{\tilde{S}} \cdot \mathbf{i}^{n+\frac{1}{2}} = 0. \]  

(5.53)

The first term in (5.53), the charge variation in time, at \( v_1 \) (vertex 1) is expressed as

\[ \frac{q_1^{n+1} - q_1^n}{\Delta t} = \frac{Q \lambda_1^f}{\Delta t} - \frac{Q \lambda_1^s}{\Delta t} = \frac{Q}{\Delta t} (\lambda_1^f - \lambda_1^s). \]  

(5.54)

The second term in (5.53), the net current flowing, out of \( v_1 \) (vertex 1) is expressed as

\[ (\mathbf{\tilde{S}} \mathbf{i}^{n+\frac{1}{2}})_1 = i_1 + i_2 \]

\[ = \frac{Q}{\Delta t} \left[ \int_{r_{p,s}} W_1^1(r_p) \cdot dL + \int_{r_{p,s}} W_2^1(r_p) \cdot dL \right] \]

\[ = \frac{Q}{\Delta t} \left[ \left( \lambda_1^f \lambda_2^f - \lambda_2^s \lambda_2^s \right) + \left( \lambda_1^s \lambda_3^f - \lambda_1^f \lambda_3^s \right) \right] = \frac{Q}{\Delta t} \left[ \lambda_1^s - \lambda_1^f \right], \]  

(5.55)

where the the partition of unity \( \lambda_1 + \lambda_2 + \lambda_3 = 1 \) has been used, and the Whitney edge basis functions are constructed based on an ascending order fashion (instead of
a cyclic order) detailed in Appendix E.2. As the sum of (5.54) and (5.55) is equal to zero, charge conservation is verified analytically.

Charge conservation can be also verified from the geometrical representation of Whitney 0-forms and 1-forms. For detailed discussions, refer to Appendix E.1. The geometrical meaning of (5.54) is illustrated in Figure 5.5a. Therefore,

\[
\frac{Q}{\Delta t} (\lambda_1^f - \lambda_1^s) = \frac{Q}{\Delta t} \frac{A_{q1,n+1} - A_{q1,n}}{A},
\]

(5.56)

where \(A_{q1,n1}\) and \(A_{q1,n+1}\) are the triangle areas depicted in Figure 5.5a, and \(A\) is the whole area of the grid triangle defined by \(\nu_1, \nu_2\) and \(\nu_3\). On the other hand, the geometrical meaning of (5.55) is illustrated in Figure 5.5b. The net current flowing out of \(\nu_1\) (vertex 1) is \(i_1\) plus \(i_2\), which is expressed as

\[
i_1 + i_2 = \frac{Q}{\Delta t} \left[ - \int_{r_{p,s}}^{r_{p,f}} W_1^1(r_p) \cdot dL - \int_{r_{p,s}}^{r_{p,f}} W_2^1(r_p) \cdot dL \right] = -\frac{Q}{\Delta t} \left[ \frac{A_{i1}}{A} + \frac{A_{i2}}{A} \right],
\]

(5.57)

where \(A_{i1}\) and \(A_{i2}\) are the triangle areas depicted in Figure 5.5b, and the minus sign is due to the opposite default directions of the path \(L\) and associated Whitney 1-forms. It is obvious that

\[
A_{q1,n+1} - A_{q1,n} = A_{i1} + A_{i2}.
\]

(5.58)

Hence, the relation (5.53) holds true again from this geometrical interpretation.

5.1.6 Gauss’ Law Preservation

It can be easily demonstrated that Gauss’ law is also satisfied if proper initial conditions are employed. To do so, multiply the both sides of (5.12) by the discrete
Figure 5.5: Geometric representation of charge-conservation identity: (a) Variation of Whitney 0-forms coefficients (barycentric coordinates) associated with $\nu_1$ (vertex 1) during a time interval $\Delta t$, where the solid red region indicates the starting time instant and the striped red region (which includes the solid red region) indicates the finishing time instant. (b) Areas associated with the magnitude of induced currents (as computed by Whitney 1-forms) on adjacent edges $e_1$ (edge 1) and $e_2$ (edge 2) during $\Delta t$. It is clear that $A_{q1,n+1} - A_{q1,n} = A_{i1} + A_{i2}$.

The divergence matrix $\tilde{S}$.

$$\tilde{S} \cdot [\star_\epsilon] \cdot \left( \frac{e^{n+1} - e^n}{\Delta t} \right) = \tilde{S} \cdot C^T \cdot [\star_{\mu-1}] \cdot b^{n+\frac{1}{2}} - \tilde{S} \cdot i^{n+\frac{1}{2}}.$$  \hspace{1cm} (5.59)

The first term of the right-hand side of (5.59) is equal to zero because of the exact sequence property in the dual grid, i.e., $\tilde{S} \cdot C^T = 0$, a discrete version of $\nabla \cdot \nabla \times = 0$ [175, 177, 186, 187, 188, 189, 190]. Therefore, (5.59) is rewritten as using (5.53)

$$\tilde{S} \cdot [\star_\epsilon] \cdot \left( \frac{e^{n+1} - e^n}{\Delta t} \right) = q^{n+1} - q^n.$$ \hspace{1cm} (5.60)

In fact, (5.60) is the discrete version of

$$\frac{\partial}{\partial t} \nabla \cdot D = \frac{\partial}{\partial t} \rho.$$ \hspace{1cm} (5.61)

Therefore, Gauss' law, $\nabla \cdot D = \rho$, is preserved for all discrete times once the proper initial condition $\tilde{S} \cdot [\star_\epsilon] \cdot e^0 = q^0$ is given.
For completeness, Gauss’ law for magnetism is also taken into account. Similarly, multiplying the both sides of (5.11) by another discrete divergence matrix $S$ results in

$$S \cdot \left( \frac{b_{n+\frac{1}{2}} - b_{n-\frac{1}{2}}}{\Delta t} \right) = -S \cdot C \cdot e^n = 0. \quad (5.62)$$

Note that the divergence operator, $\nabla \cdot$, can be discretized to two different matrices ($S$ and $\tilde{S}$) depending on which degrees of freedom it is associated with. While $S$ represents the divergence operator in the primal grid, $\tilde{S}$ represents the divergence operator in the dual grid (See [175, 177]). In (5.62), the exact sequence property in the primal grid, i.e., $S \cdot C = 0$, makes the right hand side of (5.62) vanish. It can be easily seen that (5.62) is the discrete version of

$$\frac{\partial}{\partial t} \nabla \cdot B = 0. \quad (5.63)$$

Therefore, Gauss’ law for magnetism, $\nabla \cdot B = 0$, is also preserved for all discrete times once the proper initial condition $S \cdot b^0 = 0$ is given.

### 5.1.7 Time-Update Sequence

The time update sequence of the present particle-in-cell algorithm is summarized. The appropriate initial conditions for $E^0$, $B^{-\frac{1}{2}}$, $v_p^{-\frac{1}{2}}$, and $r_p^0$, are first pre-determined. During each cycle, $b^{n+\frac{1}{2}}$ is first updated. Then, the field interpolation, $E^n$ and $B^{n+\frac{1}{2}}$, at particle positions is carried out. Using the interpolated field values, the attributes of all particles, $v_p^{n+\frac{1}{2}}$ and $r_p^{n+1}$, are updated. Next, currents $i^{n+\frac{1}{2}}$ are assigned (scattered) to all grid edges. Finally, the update of $e^{n+1}$ is carried out. Note that $v_p$ and $r_p$ are $3 \times 1$ column vectors. The sequence is briefly illustrated in Figure 5.6 and summarized below.
1) B update : $b^{n+\frac{1}{2}} = b^{n-\frac{1}{2}} - \Delta t C \cdot e^n$

2) E gather : $E^n = \sum_{i=1}^{N_e} e_i^n W_1^v (r_p^n)$

3) B gather : $B^{n+\frac{1}{2}} = \sum_{i=1}^{N_f} b_i^{n+\frac{1}{2}} W_2^b (r_p^n)$

4) Particle acceleration : $v_p^{n+\frac{1}{2}} = N^{-1} \cdot N^T \cdot v_p^{n-\frac{1}{2}} + \frac{q \Delta t}{m} N^{-1} \cdot E^n$

5) Particle push : $r_p^{n+1} = r_p^n + \Delta t v_p^{n+\frac{1}{2}}$

6) I scatter : $i_i^{n+\frac{1}{2}} = \frac{Q}{\Delta t} \int_{r_p,s}^r W_1^i (r_p) \cdot dL$

7) E update : $[\kappa] \cdot e^{n+1} = [\kappa] \cdot e^n + \Delta t \left( C^T \cdot [\kappa_{\mu-1}] \cdot b^{n+\frac{1}{2}} - i^{n+\frac{1}{2}} \right)$
5.2 Validation Examples

Several plasma-related simulations are provided in this section for the verification of the present algorithm. In Section 5.2.1, conventional cyclotron motions are simulated in order to examine whether charge conservation is achieved. In Section 5.2.2, several plasma ball simulations are provided. Maxwellian distribution of negatively charged particles with preset thermal velocities is used for initial conditions of the particles. In Section 5.2.3, the particle-in-cell simulations for two-dimensional diodes are provided. In Section 5.2.4, for the relativistic operation, synchrocyclotron motions are provided.

5.2.1 Cyclotron

As the first example, typical cyclotron motions are considered. The principle of the cyclotron is that the $z$-directed magnetic force makes particles lying on the $xy$-plane move along circular trajectories. In this example, the (static) magnetic flux density is determined to be $B_z = 2.275 \times 10^{-3} \text{ Wb/m}^2$ for the gyroradius of 0.25 m using $B_z = (mv)/(rq)$ with $m = 9.1 \times 10^{-31} \text{ kg}$, $v = 10^8 \text{ m/s}$, and $q = -1.6 \times 10^{-19} \text{ C}$.

In Figure 5.7, the movement of a negatively charged particle is illustrated at different time instances. The negatively charge particle represents an electron here. As clarified in Figure 5.7d, the circular motion of 0.25 m radius is observed as expected. The time step is chosen to be $\Delta t = 0.1 \text{ ns}$, which is smaller than the Courant limit $\Delta t_c = 0.14887 \text{ ns}$ for stable field computations. Note that the Courant limit depends on the size of mesh elements and stiffness matrix as discussed in Appendix F. The chosen $\Delta t$ is valid because it is smaller than $\Delta l/|v_p| \approx 0.1/10^8 = 10^{-9} \text{ s}$, where $\Delta l$ is
Figure 5.7: Movement of a negatively charged particle in the uniform static magnetic field at different time instants ($\Delta t = 0.1 \text{ ns}$): (a) $t = 0$, (b) $t = 50\Delta t$, (c) $t = 100\Delta t$, and (d) $t = 200\Delta t$.

The typical length of edges in the grid. For simple simulation settings, it is assumed that net charge density and electric fields are initially set to be zero by assuming a pair of oppositely charged particles at the same locations. Also, the positively charged particle is enforced to be stationary all the time because of its relatively larger mass than the negatively charged particle. Therefore, the positively charged particle is not depicted in Figure 5.7. In Figure 5.8a, the amount of charge at nearby three vertices from the negatively charged particle in the grid is illustrated. As explained in Section 5.1.4, the total amount of charges from the three vertices is constantly equal to the
Figure 5.8: Charge and energy conservation: (a) Distributed amounts of charge to local vertices and their sum at all time instances and (b) Absolute value of the particle velocity at all time instances.

Table 5.1: Charge and energy conservation at larger time instances.

| n  | $q_1$       | $q_2$       | $q_3$       | $Q$        | $|v_p|$ |
|----|-------------|-------------|-------------|------------|--------|
| $10^1$ | $-6.410 \times 10^{-20}$ | $-9.245 \times 10^{-20}$ | $-3.441 \times 10^{-21}$ | $-1.600 \times 10^{-19}$ | $9.999 \times 10^7$ |
| $10^2$ | $-7.635 \times 10^{-20}$ | $-7.154 \times 10^{-20}$ | $-1.210 \times 10^{-20}$ | $-1.600 \times 10^{-19}$ | $9.999 \times 10^7$ |
| $10^3$ | $-6.187 \times 10^{-20}$ | $-7.721 \times 10^{-20}$ | $-2.091 \times 10^{-20}$ | $-1.600 \times 10^{-19}$ | $9.999 \times 10^7$ |
| $10^4$ | $-5.772 \times 10^{-21}$ | $-1.472 \times 10^{-19}$ | $-7.025 \times 10^{-21}$ | $-1.600 \times 10^{-19}$ | $9.999 \times 10^7$ |
| $10^5$ | $-5.766 \times 10^{-20}$ | $-2.809 \times 10^{-20}$ | $-7.423 \times 10^{-20}$ | $-1.600 \times 10^{-19}$ | $1.000 \times 10^8$ |
| $10^6$ | $-1.480 \times 10^{-20}$ | $-1.365 \times 10^{-21}$ | $-1.438 \times 10^{-19}$ | $-1.600 \times 10^{-19}$ | $1.000 \times 10^8$ |

particle charge. Figure 5.8b shows the absolute value of the particle velocity. As the energy of the particle is mostly determined by its kinetic energy, it can be said that energy conservation is also preserved. More results up to $n = 10^6$ time instances are tabulated in Table 5.1 for further verification of the present algorithm.
As the second example, all settings are the same but there are three negatively charged particles each of which represents an electron. Figure 5.9 shows their movement at different time instances. The stationary particles with positive charge initially coincide with the negatively charged particles and are not depicted in Figure 5.9. Again, these particles show the expected circular trajectories. It should be noted that the external magnetic influence is much greater than interactions among the particles. As shown in Figure 5.9a, three vertices are designated for the verification of Gauss’ law, which is discussed in Section 5.1.6. The two terms of the discrete Gauss’ law, \( \bar{S} \cdot [\star \epsilon] \cdot e^n \) and \( q^n \), as shown in (5.60) are computed in double-precision arithmetics and shown in Table 5.2 up to \( 10^6 \) time instances. The difference (residue) between the two terms is also shown in the last column of Table 5.2. There is an excellent agreement, which verifies Gauss’ law all the time. Note that Gauss’ law for magnetism \( \bar{S} \cdot b^n = 0 \) is automatically preserved in these two examples because the magnetic flux density only has the \( z \)-component, which is invariant with respect to \( z \).

### 5.2.2 Plasma Ball

As the next simulations, a blowing-up plasma ball composed of two species, electrons and positive ions, is considered. All the electrons and positive ions are initially placed at the same locations, so that net charge density and electric fields are zero initially. The grid with 2539 edges shown in Figure 5.10 is dense near the center, where the initial plasma ball is placed, and coarse elsewhere. The initial conditions of a total of 4000 electrons are as follows. They are uniformly distributed inside the red circle shown, so particle density is \( n_e = 4 \times 10^3/(0.05^2\pi) = 5.0930 \times 10^5 \text{ m}^{-2} \).
The Debye length for the electrons is computed from

\[
\lambda_D = \sqrt{\frac{\varepsilon_0 kT}{(n_e)^{3/2} q^2}},
\]

which gives \( \lambda_D = 0.1974 \) m. The velocities of the electrons obey the Maxwellian distribution with a thermal velocity \( |v_{th}| = 10^{-3}c \) m/s, where \( c \) is the speed of light in the non-relativistic regime. The velocities of the positive ions are initially zero due to larger mass. In Figure 5.11, the distribution of 4000 electrons is shown at different time instances. It is clearly observed that the plasma ball is expanded as time progresses.
The discrete Gauss’ law is also examined for the verification of charge conservation. Three arbitrarily chosen vertices are illustrated in Figure 5.10. Table 5.3 shows the results and there is a good agreement between the two terms at larger time steps.

To examine energy conservation, the energy balanced equation is taken into account.

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{E} + \frac{1}{2} \mathbf{B} \cdot \mu^{-1} \mathbf{B} \right) + \mathbf{E} \cdot \mathbf{J} = 0,
\]

Table 5.2: Verification of the discrete Gauss’ law at different time instances and at (global) vertices.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>$n$</th>
<th>$\mathbf{S} \cdot \left[ \mathbf{c}_v \right] \cdot \mathbf{e}^n$</th>
<th>$q^n$</th>
<th>$\mathbf{S} \cdot \left[ \mathbf{c}_v \right] \cdot \mathbf{e}^n - q^n$</th>
</tr>
</thead>
<tbody>
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<td>$10^1$</td>
<td>$-6.206610 \times 10^{-36}$</td>
<td>0</td>
<td>$-6.206610 \times 10^{-36}$</td>
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<tr>
<td></td>
<td>$10^2$</td>
<td>$-3.655787 \times 10^{-34}$</td>
<td>0</td>
<td>$-3.655787 \times 10^{-34}$</td>
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<tr>
<td></td>
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<td>$-3.996030 \times 10^{-20}$</td>
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<tr>
<td></td>
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<td>$-3.581126 \times 10^{-20}$</td>
<td>$-3.581126 \times 10^{-20}$</td>
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<td>0</td>
<td>$1.442950 \times 10^{-31}$</td>
</tr>
<tr>
<td></td>
<td>$10^6$</td>
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<td>0</td>
<td>$-2.830713 \times 10^{-30}$</td>
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<td>$-5.373841 \times 10^{-21}$</td>
<td>$7.859451 \times 10^{-30}$</td>
</tr>
</tbody>
</table>
which is discretized as

\[
\frac{d}{dt} \left( \frac{1}{2} e^T \cdot \mathbf{\epsilon} \cdot e + \frac{1}{2} b^T \cdot \mathbf{\mu} \cdot b \right) + e^T \cdot \mathbf{i} = 0. \tag{5.66}
\]

(5.66) can be concisely expressed as

\[
\frac{d}{dt} (W_e + W_m) + P_s = 0, \tag{5.67}
\]

where \( W_e \) and \( W_m \) represent the electric and magnetic energy density, and \( P_s \) represents the electric power from \( \mathbf{J} \) of moving charged particles. Using the leap-frog time update, we obtain

\[
\Delta W_e^{n+\frac{1}{2}} + \Delta W_m^{n+\frac{1}{2}} = -P_s^{n+\frac{1}{2}} \Delta t, \tag{5.68}
\]

where half-integer times are considered to coincide with \( i \). Figure 5.12 shows the comparison of the both sides in (5.68) for all time steps. Therefore, energy conservation is verified as there is an excellent agreement.
Figure 5.11: Distribution of $4 \times 10^3$ electrons with the initial Maxwellian distribution, and zero initial fields, at different time instances ($\Delta t = 0.01$ ns): (a) $t = 10^4 \Delta t$, (b) $t = 2 \times 10^4 \Delta t$, (c) $t = 4 \times 10^4 \Delta t$, and (d) $t = 6 \times 10^4 \Delta t$.

5.2.3 Two-Dimensional Diode

The study of the space-charge-limited (SCL) current through diodes has been of great interest in the application of semiconductors [191, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201]. Typical diodes obey the classical Child-Langmuir law but empirical modifications should be made when many variations to diodes such as the shapes of electrodes, non-zero initial electron velocities, and relativistic electron energies are applied. Therefore, there has been a strong need for simulating the effect of these variations on the SCL current using particle-in-cell algorithms. In this section,
Table 5.3: Verification of the discrete Gauss’ law at different time instances and at three arbitrary (global) vertices.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>$n$</th>
<th>$\mathbf{S} \cdot \left[ \epsilon \right] \cdot \mathbf{e}_n$</th>
<th>$q^n$</th>
<th>$\mathbf{S} \cdot \left[ \epsilon \right] \cdot \mathbf{e}_n - q^n$</th>
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<td>4.216619 $\times 10^{-20}$</td>
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<tr>
<td></td>
<td>$10^3$</td>
<td>5.693133 $\times 10^{-19}$</td>
<td>5.692525 $\times 10^{-19}$</td>
<td>6.087630 $\times 10^{-23}$</td>
</tr>
<tr>
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<td>$6 \times 10^4$</td>
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<td>-1.478877 $\times 10^{-18}$</td>
<td>3.304698 $\times 10^{-22}$</td>
</tr>
</tbody>
</table>

Particle-in-cell simulations for two-dimensional diodes with two different unstructured grids are provided.

The first unstructured grid illustrated in Figure 5.13 is 1 m × 1 m in size and consists of 431 nodes, 1226 edges, and 796 triangles. The left wall represents a cathode whereas the right wall represents an anode. It is assumed that potentials at the cathode and anode are 0 V and 1 V, respectively, in order to push electrons from the left to the right.
Figure 5.12: Numerical verification of energy conservation by comparing the both sides of (5.68) at all time steps.

Figures 5.14a – Figure 5.14c show the particle distribution with the exact charge-conserving scheme at three selected time instances ($t = 80000\Delta t$, $200000\Delta t$, and $400000\Delta t$) with $\Delta t = 4 \times 10^{-11}$ s, which is chosen to be smaller than the Courant limit $\Delta t_c = 4.9091 \times 10^{-11}$ s for stable computation of electromagnetic fields. Each particle represents a single electron with mass $m = 9.1 \times 10^{-31}$ kg and charge $q = -1.6 \times 10^{-19}$ C. Initial velocity of each particle upon emission from the cathode is set to be zero. The total number of particles is around 8000 when the first particle reaches the right wall and leaves out the domain. It should be pointed out that the particle density is smaller near the anode because (i) each particle is accelerated and (ii) current density should be constant over the domain due to charge conservation ($\mathbf{J} = \rho \mathbf{v} =$constant).

Figures 5.14d – Figure 5.14f show the particle distribution with the non-charge-conserving scheme introduced in [156]. It is more obvious in Figure 5.14f that the particle beam becomes narrower near the anode, which is spurious behavior.
Figure 5.13: The first unstructured grid for particle-in-cell simulations for two-dimensional diodes.

Figure 5.15 depicts the self-consistent electric field for both the charge-conserving and non-charge-conserving schemes at two different time instances. Electric field values in each plot are normalized by their respective maximum values for better visualization. It can be observed that electric field from the non-charge-conserving scheme is not aligned near the center of the domain as opposed to the charge-conserving scheme.

Figure 5.16 shows the comparison of the particle distributions with doubled charge density from the cathode at the same time instances. Therefore, the total number of particles is around 16000. Figures 5.16a, 5.16b, and 5.16c are associated with the charge-conserving scheme whereas Figures 5.16d, 5.16e, and 5.16f are associated with the non-charge-conserving scheme. The abnormal alignment of particles are clearly observed in Figure 5.16f due to the violation of charge conservation. Figure 5.17
Figure 5.14: Distribution of a total of 8000 electrons injected from the cathode (left wall) and absorbed to the anode (right wall) on the first unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at three different time instances ($\Delta t = 0.04$ ns): (a) $t = 8000\Delta t$, (b) $t = 200000\Delta t$, (c) $t = 400000\Delta t$, (d) $t = 80000\Delta t$, (e) $t = 200000\Delta t$, and (f) $t = 400000\Delta t$.

shows the self-consistent electric field for this case. Again, the distribution of electric field is not aligned near the center for the non-charge-conserving scheme.
Figure 5.15: Distribution of the self-consistent electric field from a total of 8000 electrons on the first unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at two different time instances ($\Delta t = 0.04 \text{ ns}$): (a) $t = 80000\Delta t$, (b) $t = 400000\Delta t$, (c) $t = 80000\Delta t$, and (d) $t = 400000\Delta t$. 
Figure 5.16: Distribution of a total of 16000 electrons injected from the cathode (left wall) and absorbed to the anode (right wall) on the first unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at three different time instances ($\Delta t = 0.04$ ns): (a) $t = 80000\Delta t$, (b) $t = 200000\Delta t$, (c) $t = 400000\Delta t$, (d) $t = 80000\Delta t$, (e) $t = 200000\Delta t$, and (f) $t = 400000\Delta t$. 

16000 particles, conserving, 80K
16000 particles, conserving, 200K
16000 particles, conserving, 400K
16000 particles, non-conserving, 80K
16000 particles, non-conserving, 200K
16000 particles, non-conserving, 400K
Figure 5.17: Distribution of the self-consistent electric field from a total of 16000 electrons on the first unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at two different time instances ($\Delta t = 0.04$ ns): (a) $t = 80000 \Delta t$, (b) $t = 400000 \Delta t$, (c) $t = 8000 \Delta t$, and (d) $t = 400000 \Delta t$. 

16000 particles, non-conserving, 80K

16000 particles, non-conserving, 400K

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The second unstructured grid is illustrated in Figure 5.18, which is originated to [156]. The grid consists of 411 nodes, 1167 edges, and 757 triangles, and its overall size is again 1 m × 1 m. Again, the left wall represents a cathode. However, there are now two disks with a radius of 0.2 m at the two right corners to represent an anode. The potential difference the cathode and anode is again assumed to be 1 V. Figure 5.19a depicts the potential distribution from this particular configuration and the resultant electric field is depicted in Figure 5.19b.

Figures 5.20a – Figure 5.20c show the particle distribution with the charge-conserving scheme at three selected time instances \((t = 75000\Delta t, 200000\Delta t, \text{and } 400000\Delta t)\) with \(\Delta t = 4 \times 10^{-11} \text{ s}\). The Courant limit for this grid is slightly different; \(\Delta t_c = 4.9098 \times 10^{-11} \text{ s}\). All other simulation settings are the same. Figures 5.20d – Figure 5.20f show the particle distribution with the non-charge-conserving
scheme. It is noted that the non-charge-conserving scheme shows the narrower particle beam. From the comparison of the self-consistent electric fields shown in Figure 5.21, abnormal electric fields are observed near the center of the domain for the non-charge-conserving scheme.

The particle distributions with doubled charge density (a total of 16000 electrons) are illustrated in Figure 5.22 and the self-consistent electric fields are depicted in Figure 5.23. Similar interpretation can be made when charge density is doubled.
Figure 5.20: Distribution of a total of 8000 electrons injected from the cathode (left wall) and absorbed to the anode (right wall) on the second unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at three different time instances ($\Delta t = 0.04$ ns): (a) $t = 75000 \Delta t$, (b) $t = 200000 \Delta t$, (c) $t = 400000 \Delta t$, (d) $t = 75000 \Delta t$, (e) $t = 200000 \Delta t$, and (f) $t = 400000 \Delta t$. 

8000 particles, conserving, 75K
8000 particles, conserving, 200K
8000 particles, conserving, 400K
8000 particles, non-conserving, 75K
8000 particles, non-conserving, 200K
8000 particles, non-conserving, 400K
Figure 5.21: Distribution of the self-consistent electric field from a total of 8000 electrons on the second unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at two different time instances ($\Delta t = 0.04$ ns): (a) $t = 75000\Delta t$, (b) $t = 400000\Delta t$, (c) $t = 75000\Delta t$, and (d) $t = 400000\Delta t$. 

8000 particles, non-conserving, 75K

8000 particles, non-conserving, 400K
Figure 5.22: Distribution of a total of 16000 electrons injected from the cathode (left wall) and absorbed to the anode (right wall) on the second unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at three different time instances ($\Delta t = 0.04$ ns): (a) $t = 75000\Delta t$, (b) $t = 200000\Delta t$, (c) $t = 400000\Delta t$, (d) $t = 75000\Delta t$, (e) $t = 200000\Delta t$, and (f) $t = 400000\Delta t$. 

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Figure 5.23: Distribution of the self-consistent electric field from a total of 16000 electrons on the second unstructured grid with the charge-conserving scheme (top row) and non-charge-conserving scheme (bottom row) at two different time instances ($\Delta t = 0.04$ ns): (a) $t = 75000\Delta t$, (b) $t = 400000\Delta t$, (c) $t = 75000\Delta t$, and (d) $t = 400000\Delta t$. 

16000 particles, non-conserving, 75K

16000 particles, non-conserving, 400K
5.2.4 Synchrocyclotron

Particle-in-cell simulations for a cyclotron motion in the relativistic regime are presented, which can be regarded as an extension of the results in Section 5.2.1. To validate the relativistic particle-in-cell formulations, the simulations of electron accelerators are considered. As electrons usually have velocities near the speed of light in this case, progressively more energy needs to be delivered to the electrons to accelerate due to its relativistic mass $\gamma m_0$. This reduced acceleration results in lower orbital (cyclotron) frequency. Therefore, the driving RF electric field should have variable frequencies matching to the relativistic cyclotron frequency. Figure 5.24 illustrates the selected unstructured grid for the particle-in-cell simulations. In the red region, centripetal force from external magnets is present for the circular motion of electrons. On the other hand, directional force from the RF electric field is present in the blue region to accelerate electrons.
Figure 5.25: Electron trajectories: (a) Non-relativistic, (b) Relativistic without synchronization, and (c) Relativistic with synchronization.

Figure 5.25a shows the cyclotron motion in the non-relativistic regime, where relativistic factor is one. An electron is injected at $(x, y) = (0.52, 0.5)$ with the initial velocity of $|v_0| = v_0 = 1 \times 10^7$ m/s. The static magnetic force is determined to be $B_z = (m_0v_0)/(rq) = 2.84281 \times 10^{-3}$ Wb/m² for initial orbital radius $r$ of 0.02 m, and
the RF electric force is set to be $E_x = 2 \times 10^5$ V/m. The vertical blue strip in the middle of Figure 5.25a represents the region of acceleration by the RF electric force. The thickness of the strip is 0.02 m. As can be seen, the spacing of two adjacent orbits becomes smaller due to increasing velocities.

Figure 5.25b shows the trajectory of the electron with the same initial conditions using the relativistic particle-in-cell formulations discussed in Section 5.1.3. In this case, the frequency of the RF electric force is set to be constant, which results in unsynchronized phase between particle velocity and electric force, and mixed trajectory.

In Figure 5.25c, the frequency of the RF electric force is matched to the orbital frequency of the electron, so it is called synchrocyclotron frequency, which is expressed as $f = (qB)/(2\pi\gamma m_0)$. Therefore, in-phase acceleration is maintained all the time.
and desired circular trajectory is observed with higher velocities. Note that the total distance over which an electron moves in this case is shorter than that for the non-relativistic case because of relatively smaller velocity due to relativistic mass.

Figure 5.26 shows the comparison of magnitudes of electron velocities of the three cases. It is clearly observed that deceleration occurs around 4000 time steps for the second case. Also, the third case shows slightly smaller magnitudes than the first one due to the relativistic mass. Figure 5.27a shows the comparison of orbital frequencies of the electron. As can be seen, the frequency for the relativistic operation decreases as time progresses. In Figure 5.27b, the corresponding relativistic factor is shown.

Tables 5.4, 5.5, and 5.6 show the verification of Gauss’ law and, therefore, charge conservation of the present algorithm. The amount of charge at different time steps and at some vertices is recorded. As the last columns in these tables show, the residues between two terms for the discrete version of Gauss’ law are smaller than the assigned charges, which verifies charge conservation.
Table 5.4: Verification of Gauss’ law at different time instances for the non-relativistic case shown in Figure 5.25a.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Vertex Number</th>
<th>$q^n$</th>
<th>$\tilde{S} \cdot [s] \cdot e^n - q^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>90</td>
<td>$-5.520012 \times 10^{-20}$</td>
<td>$-2.731377 \times 10^{-24}$</td>
</tr>
<tr>
<td>2000</td>
<td>54</td>
<td>$-1.231631 \times 10^{-19}$</td>
<td>$-7.403985 \times 10^{-24}$</td>
</tr>
<tr>
<td>3000</td>
<td>60</td>
<td>$-7.457305 \times 10^{-20}$</td>
<td>$7.298314 \times 10^{-25}$</td>
</tr>
<tr>
<td>4000</td>
<td>137</td>
<td>$-4.121612 \times 10^{-21}$</td>
<td>$-1.447210 \times 10^{-23}$</td>
</tr>
<tr>
<td>5000</td>
<td>309</td>
<td>$-1.208727 \times 10^{-19}$</td>
<td>$7.736219 \times 10^{-24}$</td>
</tr>
</tbody>
</table>

Table 5.5: Verification of Gauss’ law at different time instances for the relativistic case without synchronization shown in Figure 5.25b.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Vertex Number</th>
<th>$q^n$</th>
<th>$\tilde{S} \cdot [s] \cdot e^n - q^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>236</td>
<td>$-6.683617 \times 10^{-20}$</td>
<td>$2.183467 \times 10^{-24}$</td>
</tr>
<tr>
<td>2000</td>
<td>246</td>
<td>$-6.295288 \times 10^{-21}$</td>
<td>$-2.311833 \times 10^{-24}$</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$-1.632915 \times 10^{-24}$</td>
</tr>
<tr>
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<td>$-3.045262 \times 10^{-21}$</td>
<td>$-1.066490 \times 10^{-23}$</td>
</tr>
</tbody>
</table>

Table 5.6: Verification of Gauss’ law at different time instances for the relativistic case with synchronization shown in Figure 5.25c.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Vertex Number</th>
<th>$q^n$</th>
<th>$\tilde{S} \cdot [s] \cdot e^n - q^n$</th>
</tr>
</thead>
<tbody>
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<tr>
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</tr>
<tr>
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</tr>
<tr>
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<td>$-9.923952 \times 10^{-22}$</td>
<td>$-5.328920 \times 10^{-24}$</td>
</tr>
<tr>
<td>5000</td>
<td>290</td>
<td>$-6.641072 \times 10^{-20}$</td>
<td>$-4.877675 \times 10^{-24}$</td>
</tr>
</tbody>
</table>

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Chapter 6

Conclusions

In this dissertation, several numerical algorithms are proposed for computational efficiency. The first algorithm is developed to stabilize the numerical computation of electromagnetic fields from arbitrarily situated Hertzian dipoles in cylindrically-stratified anisotropic media. This problem is related to geophysical and petrophysical exploration where electrical conductivity is primarily measured to characterize surrounding earth formations. The challenge for this problem is the unbound characteristics of standard cylindrical functions comprising solution space under finite arithmetics. As a solution, so-called range-conditioned cylindrical functions are devised in order to tackle the poor scaling of the original cylindrical functions. A key property of the new set of functions is the judicious modification of the existing analytical formulations so as to cancel out the exponential factors of the cylindrical functions wherever possible in any combination of physical parameters such as operating frequencies, constitutive parameters, the number of layers, layer thicknesses, and arbitrary anisotropy ratios. The unconditionally stable computation is therefore achieved. Furthermore, two robust integration paths are proposed to complement to each other to get rid of the possibility of divergent or slow-convergent numerical integration. Depending on the relative positions of a transmitter and receiver, the
adequate integration path is determined. Several practical scenarios are taken into account for the verification of the new algorithm. Through comparison to the Finite Element Method (FEM) results, the new algorithm proves to be robust, stable, and fast.

Computation of the electric potential is also desired when electrode logging is a tool for measuring conductivity (or resistivity) of surrounding earth formations. The accuracy of electrode logging becomes higher when water-based mud rather than oil-based mud is used. Analytical formulations need to first be established because point electrodes rather than coil antennas are used as measuring devices. In contrast to the problem with electromagnetic fields, modified cylindrical functions emerge for valid solutions from several physical constraints regarding the electric potential. Also, a slightly different numerical integration scheme is adopted because the involved Sommerfeld integral is now semi-infinite and singularities are placed near the origin. The use of the weighted-averages method gives fast convergence with acceptable accuracy. Several cases of practical relevance are provided for results validation.

When the transport of space charge is present in plasma media, the conventional approach leaning on Maxwell’s equations to describe electromagnetic phenomena often fails. The reason for the failure can be explained by the limitation of macroscopic viewpoints of the Maxwell system. To revise the numerical model, the interaction between electromagnetic fields along with various densities and the kinetic description of charged particles should be considered. In that sense, particle-in-cell algorithms are very successful to simulate various plasma phenomena. However, there exists long-standing challenge for this type of model, which is the easy violation of charge and energy conservation, especially on unstructured grids. The charge-conserving
formulation motivated by the continuity equation is the best solution to solve the challenge. The proposed algorithm in this dissertation relied on the use of specialized interpolants, Whitney forms, to achieve exact charge conservation. From the algorithm, geometrical interpretation of charge conservation can be obtained in addition to analytical mathematical formulations. A salient feature of the algorithm is that it does not require numerical quadratures for time update of charge and current densities on the unstructured grids. Also, relativistic effect is also incorporated into the algorithm because most relevant applications deal with particles with relativistic energy. Several examples using the algorithm are provided, which verifies the adequacy of the algorithm.
Appendix A

Multiplicative Factors for Modified Integrands

In this Appendix, the multiplicative factors for the modified integrand for computation of electromagnetic fields are provided. In the second to fifth columns of Tables A.1, A.2, A.3, and A.4, S, M, and L represent small, moderate and large, respectively.

For Case 1 and Case 2, \( G_j \) is expressed as

\[
G_j = \frac{1}{n!} \left( \frac{k_{j\rho}}{2} \right)^n. \tag{A.1}
\]

For Case 3 and Case 4, \( G_i \) and \( G_j \) are expressed as

\[
G_i = \frac{1}{n!} \left( \frac{k_{i\rho}}{2} \right)^n, \tag{A.2}
\]

\[
G_j = \frac{1}{n!} \left( \frac{k_{j\rho}}{2} \right)^n. \tag{A.3}
\]

For the definitions of \( P_{ij} \), refer to (2.37a) – (2.37b).
Table A.1: Multiplicative factors in the integrand for all scenarios of Case 1.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$k_{j\rho}a_{j-1}$</th>
<th>$k_{j\rho}'$</th>
<th>$k_{j\rho}$</th>
<th>$k_{j\rho}a_j$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sce. 1</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>$(\frac{\rho}{\rho})^n$</td>
<td>$(\frac{\rho\rho'}{\rho})^n$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{a_{j-1}})^{2n}$</td>
</tr>
<tr>
<td>Sce. 2</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>M</td>
<td>$(\frac{\rho}{\rho})^n$</td>
<td>$G_j P_j^{-1} (\frac{\rho\rho'}{\rho})^n$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{\rho})^{2n}$</td>
</tr>
<tr>
<td>Sce. 3</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>L</td>
<td>$(\frac{\rho}{\rho})^n$</td>
<td>$G_j^2 (\frac{\rho\rho'}{\rho})^n e^{-2k''_{j\rho}a_j}$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{\rho})^{2n}$</td>
</tr>
<tr>
<td>Sce. 4</td>
<td>S</td>
<td>S</td>
<td>M</td>
<td>M</td>
<td>$G_j P_{j3}^{-1} (\frac{\rho}{\rho})^n$</td>
<td>$G_j P_{j3} P_{j4}^{-2} (\frac{\rho\rho'}{\rho})^n$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{\rho})^{2n}$</td>
</tr>
<tr>
<td>Sce. 5</td>
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<td>S</td>
<td>M</td>
<td>L</td>
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<td>$G_j P_{j3} (\frac{\rho}{\rho})^n e^{-2k''_{j\rho}a_j}$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{\rho})^{2n}$</td>
</tr>
<tr>
<td>Sce. 6</td>
<td>S</td>
<td>S</td>
<td>L</td>
<td>L</td>
<td>$G_j (\frac{\rho}{\rho})^n e^{-k''_{j\rho}}$</td>
<td>$G_j (\frac{\rho}{\rho})^n e^{-k''_{j\rho}(2a_j-\rho)}$</td>
<td>1</td>
<td>$(\frac{a_{j-1}}{\rho})^{2n}$</td>
</tr>
<tr>
<td>Sce. 7</td>
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<td>M</td>
<td>M</td>
<td>$P_j P_{j3}^{-1}$</td>
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</tr>
<tr>
<td>Sce. 8</td>
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<td>M</td>
<td>M</td>
<td>L</td>
<td>$P_j P_{j3}^{-1}$</td>
<td>$P_j P_{j3} e^{-2k''_{j\rho}a_j}$</td>
<td>1</td>
<td>$(G_j P_{j2}^{-1})^{2} a_{j-1}^{2n}$</td>
</tr>
<tr>
<td>Sce. 9</td>
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<td>M</td>
<td>L</td>
<td>L</td>
<td>$P_j e^{-k''_{j\rho}(2a_j-\rho)}$</td>
<td>$P_j e^{-k''_{j\rho}(2a_j-\rho-\rho')}$</td>
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<td>$(G_j P_{j2}^{-1})^{2} a_{j-1}^{2n}$</td>
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<td>Sce. 10</td>
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<td>L</td>
<td>L</td>
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<td>$e^{-k''_{j\rho}(2a_j-\rho-\rho')}$</td>
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<td>$(P_j P_{j2}^{-1})^{2}$</td>
</tr>
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<td>M</td>
<td>L</td>
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<td>$P_j P_{j3} e^{-2k''_{j\rho}a_j}$</td>
<td>1</td>
<td>$(P_j P_{j2}^{-1})^{2}$</td>
</tr>
<tr>
<td>Sce. 13</td>
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<td>L</td>
<td>L</td>
<td>$P_j e^{-k''_{j\rho}(2a_j-\rho)}$</td>
<td>$P_j e^{-k''_{j\rho}(2a_j-\rho-\rho')}$</td>
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<td>L</td>
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<td>$e^{-2k''<em>{j\rho}(\rho-\rho')} a</em>{j-1}^{-2n}$</td>
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</table>
Table A.2: Multiplicative factors in the integrand for all scenarios of Case 2.

<table>
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<tr>
<th>Scenarios</th>
<th>$k_{j\rho}a_{j-1}$</th>
<th>$k_{j\rho}\rho'$</th>
<th>$k_{j\rho}a_j$</th>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
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<td>S</td>
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<td>$(\frac{\rho}{\rho'})^n$</td>
<td>$(\frac{a_{j-1}^2}{\rho\rho'})^n$</td>
<td>1</td>
</tr>
<tr>
<td>Sce. 2</td>
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<td>S</td>
<td>S</td>
<td>M</td>
<td>$(\frac{\rho}{\rho'})^n$</td>
<td>$(\frac{a_{j-1}^2}{\rho\rho'})^n$</td>
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</tr>
<tr>
<td>Sce. 3</td>
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<td>S</td>
<td>S</td>
<td>L</td>
<td>$(\frac{\rho}{\rho'})^n$</td>
<td>$(\frac{a_{j-1}^2}{\rho\rho'})^n$</td>
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<td>Sce. 4</td>
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<td>S</td>
<td>M</td>
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</tr>
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<td>$G_j P_{j3}^{-1} (\frac{a_{j-1}}{\rho})^n$</td>
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<td>L</td>
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<td>$G_j (\frac{a_{j-1}^2}{\rho})^n e^{-k_{j\rho}\rho'}$</td>
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<td>Sce. 7</td>
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<tr>
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<td>M</td>
<td>M</td>
<td>L</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
<td>$P_{j2} P_{j3}^{-1} (\frac{a_{j-2}}{a_j})^n e^{-k_{j\rho}\rho'}$</td>
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<td>L</td>
<td>L</td>
<td>$e^{-k_{j\rho}(\rho'-\rho)}$</td>
<td>$G_j 2^{a_{j-1}/n} e^{-k_{j\rho}(\rho'+\rho')}$</td>
<td>1</td>
</tr>
<tr>
<td>Sce. 10</td>
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<td>L</td>
<td>L</td>
<td>$e^{-k_{j\rho}(\rho'-\rho)}$</td>
<td>$G_j 2^{a_{j-1}/n} e^{-k_{j\rho}(\rho'+\rho')}$</td>
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<td>M</td>
<td>M</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
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<tr>
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<td>$P_{j2} P_{j3}^{-1}$</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
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</tr>
<tr>
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<td>M</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>$P_{j2} e^{-k_{j\rho}\rho'}$</td>
<td>$P_{j1} P_{j2}^{-1} e^{-k_{j\rho}\rho'}$</td>
<td>1</td>
</tr>
<tr>
<td>Sce. 14</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$e^{-k_{j\rho}(\rho'-\rho)}$</td>
<td>$P_{j2} e^{-k_{j\rho}(\rho'+\rho')}$</td>
<td>1</td>
</tr>
<tr>
<td>Sce. 15</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$e^{-k_{j\rho}(\rho'-\rho)}$</td>
<td>$e^{-k_{j\rho}(\rho'+\rho' - 2a_{j-1})}$</td>
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</tbody>
</table>
Table A.3: Multiplicative factors in the integrand for all scenarios of Case 3.

<table>
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<tr>
<th>Scenario</th>
<th>$k_{i\rho}a_{i-1}$</th>
<th>$k_{i\rho}$</th>
<th>$k_{i\rho}a_{i}$</th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sce. 1</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>$(\frac{a_{i-1}}{\rho} )^n$</td>
<td>$(\frac{\rho a_{i-1}}{a_{i-1}^2} )^n$</td>
<td>$(\frac{\rho^2}{a_{i-1}^2} )^n$</td>
<td>$(\frac{\rho^2}{a_{i-1}^2} )^n$</td>
</tr>
<tr>
<td>Sce. 2</td>
<td>S</td>
<td>S</td>
<td>M</td>
<td>$(\frac{a_{i-1}}{\rho} )^n$</td>
<td>$(G_i P_{i3}^{-1})^2 (\rho a_{i-1})^n$</td>
<td>$(G_i P_{i3}^{-1})^2 (\rho a_{i-1})^n$</td>
<td>$(G_i P_{i3}^{-1})^2 (\rho a_{i-1})^n$</td>
</tr>
<tr>
<td>Sce. 3</td>
<td>S</td>
<td>S</td>
<td>L</td>
<td>$(\frac{a_{i-1}}{\rho} )^n$</td>
<td>$G_i^2 (\rho a_{i-1})^n e^{-2k''_{i\rho} a_i}$</td>
<td>$G_i^2 (\rho a_{i-1})^n e^{-2k''_{i\rho} a_i}$</td>
<td>$G_i^2 (\rho a_{i-1})^n e^{-2k''_{i\rho} a_i}$</td>
</tr>
<tr>
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<td>S</td>
<td>M</td>
<td>M</td>
<td>$G_i P_{i2}^{-1} a_{i-1}^n$</td>
<td>$G_i P_{i2} P_{i3}^{-2} a_{i-1}^n$</td>
<td>$G_i P_{i2} P_{i3}^{-2} a_{i-1}^n$</td>
<td>$G_i P_{i2} P_{i3}^{-2} a_{i-1}^n$</td>
</tr>
<tr>
<td>Sce. 5</td>
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<td>M</td>
<td>L</td>
<td>$G_i P_{i2}^{-1} a_{i-1}^n$</td>
<td>$G_i P_{i2} a_{i-1} e^{-2k''_{i\rho} a_i}$</td>
<td>$G_i P_{i2} a_{i-1} e^{-2k''_{i\rho} a_i}$</td>
<td>$G_i P_{i2} a_{i-1} e^{-2k''_{i\rho} a_i}$</td>
</tr>
<tr>
<td>Sce. 6</td>
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<td>L</td>
<td>L</td>
<td>$G_i a_{i-1}^n e^{-k''_{i\rho} a_i}$</td>
<td>$G_i a_{i-1}^n e^{-k''_{i\rho} a_i}$</td>
<td>$G_i a_{i-1}^n e^{-k''_{i\rho} a_i}$</td>
<td>$G_i a_{i-1}^n e^{-k''_{i\rho} a_i}$</td>
</tr>
<tr>
<td>Sce. 7</td>
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<td>M</td>
<td>M</td>
<td>$P_{i1} P_{i2}^{-1}$</td>
<td>$P_{i1} P_{i2} P_{i3}^{-2}$</td>
<td>$P_{i1} P_{i2} P_{i3}^{-2}$</td>
<td>$P_{i1} P_{i2} P_{i3}^{-2}$</td>
</tr>
<tr>
<td>Sce. 8</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>$P_{i1} P_{i2}^{-1}$</td>
<td>$P_{i1} P_{i2} e^{-2k''_{i\rho} a_i}$</td>
<td>$P_{i1} P_{i2} e^{-2k''_{i\rho} a_i}$</td>
<td>$P_{i1} P_{i2} e^{-2k''_{i\rho} a_i}$</td>
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<tr>
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<td>L</td>
<td>L</td>
<td>$P_{i1} e^{-k''_{i\rho} a_i}$</td>
<td>$P_{i1} e^{-k''_{i\rho} a_i}$</td>
<td>$P_{i1} e^{-k''_{i\rho} a_i}$</td>
<td>$P_{i1} e^{-k''_{i\rho} a_i}$</td>
</tr>
<tr>
<td>Sce. 10</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''<em>{i\rho} (\rho-a</em>{i-1})}$</td>
<td>$e^{-k''<em>{i\rho} (\rho-a</em>{i-1}-\rho)}$</td>
<td>$e^{-k''<em>{i\rho} (\rho-a</em>{i-1}-\rho)}$</td>
<td>$e^{-k''<em>{i\rho} (\rho-a</em>{i-1}-\rho)}$</td>
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</table>

<table>
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<tr>
<th>Scenario</th>
<th>$k_{j\rho}a_{j-1}$</th>
<th>$k_{j\rho}$</th>
<th>$k_{j\rho}a_{j}$</th>
<th>$C_3$</th>
<th>$C_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sce. 1</td>
<td>S</td>
<td>S</td>
<td>S</td>
<td>$(\frac{\rho}{a_j})^n$</td>
<td>$(\frac{\rho^2}{a_j^2})^n$</td>
</tr>
<tr>
<td>Sce. 2</td>
<td>S</td>
<td>S</td>
<td>M</td>
<td>$G_j P_{j3}^{-1} (\rho')^n$</td>
<td>$(G_j P_{j3}^{-1} (\rho')^n$</td>
</tr>
<tr>
<td>Sce. 3</td>
<td>S</td>
<td>S</td>
<td>L</td>
<td>$G_j (\rho')^n e^{-k''_{j\rho} a_j}$</td>
<td>$G_j (\rho')^n e^{-k''_{j\rho} a_j}$</td>
</tr>
<tr>
<td>Sce. 4</td>
<td>S</td>
<td>M</td>
<td>M</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
<td>$G_j^2 P_{j2}^{-1} P_{j3}^{-1} a_{j-1}^{2n}$</td>
</tr>
<tr>
<td>Sce. 5</td>
<td>S</td>
<td>M</td>
<td>L</td>
<td>$P_{j2} e^{-k''_{j\rho} a_j}$</td>
<td>$G_j^2 P_{j2}^{-1} a_{j-1}^{2n} e^{-k''_{j\rho} a_j}$</td>
</tr>
<tr>
<td>Sce. 6</td>
<td>S</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{j\rho} (a_j-\rho')}$</td>
<td>$G_j^2 a_{j-1}^{2n} e^{-k''_{j\rho} (a_j+\rho')}$</td>
</tr>
<tr>
<td>Sce. 7</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>$P_{j2} P_{j3}^{-1}$</td>
<td>$P_{j1} P_{j2}^{-1} P_{j3}^{-1}$</td>
</tr>
<tr>
<td>Sce. 8</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>$P_{j2} e^{-k''_{j\rho} a_j}$</td>
<td>$P_{j1} P_{j2}^{-1} e^{-k''_{j\rho} a_j}$</td>
</tr>
<tr>
<td>Sce. 9</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{j\rho} (a_j-\rho')}$</td>
<td>$P_{j1} e^{-k''_{j\rho} (a_j+\rho')}$</td>
</tr>
<tr>
<td>Sce. 10</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{j\rho} (a_j-\rho')}$</td>
<td>$e^{-k''<em>{j\rho} (\rho'+a_j-2a</em>{j-1})}$</td>
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</tbody>
</table>
Table A.4: Multiplicative factors in the integrand for all scenarios of Case 4.

<table>
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<tr>
<th>Sce.</th>
<th>$k_{ip}a_{i-1}$</th>
<th>$k_{ip}\rho$</th>
<th>$k_{ip}a_i$</th>
<th>$D_1$</th>
<th>$D_2$</th>
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<tr>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>2</td>
<td>S</td>
<td>S</td>
<td>M</td>
<td>$G_iP_{i3}^{-1}\rho^n$</td>
<td>$G_iP_{i3}^{-1}\left(\frac{a_i}{\rho}\right)^n$</td>
</tr>
<tr>
<td>3</td>
<td>S</td>
<td>S</td>
<td>L</td>
<td>$G_i\rho^n e^{-k''_{ip}a_i}$</td>
<td>$G_i\left(\frac{a_i^2}{\rho}\right)^n e^{-k''_{ip}a_i}$</td>
</tr>
<tr>
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<td>M</td>
<td>M</td>
<td>$P_{i2}P_{i3}^{-1}$</td>
<td>$G_i^2P_{i2}^{-1}P_{i3}^{-1}a_{i-1}^{2n}$</td>
</tr>
<tr>
<td>5</td>
<td>S</td>
<td>M</td>
<td>L</td>
<td>$P_{i2}\rho e^{-k''_{ip}a_i}$</td>
<td>$G_i^2P_{i2}^{-1}a_{i-1}^{2n} e^{-k''_{ip}a_i}$</td>
</tr>
<tr>
<td>6</td>
<td>S</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{ip}(a_i-\rho)}$</td>
<td>$G_i^2a_{i-1}^{2n} e^{-k''_{ip}(a_i+\rho)}$</td>
</tr>
<tr>
<td>7</td>
<td>M</td>
<td>M</td>
<td>M</td>
<td>$P_{i2}P_{i3}^{-1}$</td>
<td>$P_{i1}^2P_{i2}^{-1}P_{i3}^{-1}$</td>
</tr>
<tr>
<td>8</td>
<td>M</td>
<td>M</td>
<td>L</td>
<td>$P_{i2}\rho e^{-k''_{ip}a_i}$</td>
<td>$P_{i1}^2P_{i2}^{-1} e^{-k''_{ip}a_i}$</td>
</tr>
<tr>
<td>9</td>
<td>M</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{ip}(a_i-\rho)}$</td>
<td>$P_{i1}^2e^{-k''_{ip}(a_i+\rho)}$</td>
</tr>
<tr>
<td>10</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>$e^{-k''_{ip}(a_i-\rho)}$</td>
<td>$e^{-k''<em>{ip}(\rho+a_i-2a</em>{i-1})}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sce.</th>
<th>$k_{jp}a_{j-1}$</th>
<th>$k_{jp}\rho$</th>
<th>$k_{jp}a_j$</th>
<th>$D_3$</th>
<th>$D_4$</th>
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</tr>
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<td>S</td>
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<td>M</td>
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<tr>
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<td>S</td>
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<td>L</td>
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<tr>
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<td>S</td>
<td>M</td>
<td>M</td>
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<tr>
<td>5</td>
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<td>M</td>
<td>L</td>
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<tr>
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<td>S</td>
<td>L</td>
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<tr>
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<td>L</td>
<td>L</td>
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</tbody>
</table>

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Appendix B

Definition of Variables for Azimuth Series Folding

In this Appendix, the definitions of some auxiliary variables for the ease of azimuth series folding are provided. It should be noted that $\ddot{Y}$ is a $2 \times 1$ column vector while $\ddot{X}$ and $\ddot{Z}$ are $2 \times 2$ matrices. Also, for $n = 0$, $\ddot{Y}_{z,0} = \ddot{Y}_{\rho,0} = \ddot{Y}_{\phi,0} = 0$ because $\ddot{D}_{j2}'$ is zero. The numbered-subscripts of $\ddot{X}$ and $\ddot{X}$ in the followings represent a specific element of the matrices; i.e., $12$ indicates the element in the first row and second column. This rule also applies to $\ddot{Y}$, $\ddot{Z}$, and $\ddot{Z}$.

For the $z$-components,

$\ddot{X}_{z,n,11} = 2 \cos (n(\phi - \phi')) \ddot{X}_{z,n,11}$, \hspace{1cm} (B.1a)

$\ddot{X}_{z,n,12} = 2i \sin (n(\phi - \phi')) \ddot{X}_{z,n,12}$, \hspace{1cm} (B.1b)

$\ddot{X}_{z,n,21} = 2i \sin (n(\phi - \phi')) \ddot{X}_{z,n,21}$, \hspace{1cm} (B.1c)

$\ddot{X}_{z,n,22} = 2 \cos (n(\phi - \phi')) \ddot{X}_{z,n,22}$, \hspace{1cm} (B.1d)

$\ddot{Y}_{z,n,1} = 2i \sin (n(\phi - \phi')) \ddot{Y}_{z,n,11} \ddot{D}_{j2,1}' + 2 \cos (n(\phi - \phi')) \ddot{Y}_{z,n,12} \ddot{D}_{j2,2}'$, \hspace{1cm} (B.1e)

$\ddot{Y}_{z,n,2} = 2 \cos (n(\phi - \phi')) \ddot{Y}_{z,n,21} \ddot{D}_{j2,1}' + 2i \sin (n(\phi - \phi')) \ddot{Y}_{z,n,22} \ddot{D}_{j2,2}'$, \hspace{1cm} (B.1f)

$\ddot{Z}_{z,n,11} = 2 \cos (n(\phi - \phi')) \ddot{Z}_{z,n,11}$, \hspace{1cm} (B.1g)
\[ \ddot{Z}_{z,n,12} = 2i \sin (n(\phi - \phi')) \, \ddot{Z}_{z,n,12}, \]  
(B.1h)

\[ \ddot{Z}_{z,n,21} = 2i \sin (n(\phi - \phi')) \, \ddot{Z}_{z,n,21}, \]  
(B.1i)

\[ \ddot{Z}_{z,n,22} = 2 \cos (n(\phi - \phi')) \, \ddot{Z}_{z,n,22}. \]  
(B.1j)

For the \( \rho \)-components,

\[ \dddot{X}_{\rho,n,11} = 2 \cos (n(\phi - \phi')) \, \dddot{X}_{\rho,n,11}, \]  
(B.2a)

\[ \dddot{X}_{\rho,n,12} = 2i \sin (n(\phi - \phi')) \, \dddot{X}_{\rho,n,12}, \]  
(B.2b)

\[ \dddot{X}_{\rho,n,21} = 2i \sin (n(\phi - \phi')) \, \dddot{X}_{\rho,n,21}, \]  
(B.2c)

\[ \dddot{X}_{\rho,n,22} = 2 \cos (n(\phi - \phi')) \, \dddot{X}_{\rho,n,22}, \]  
(B.2d)

\[ \dddot{Y}_{\rho,n,1} = 2i \sin (n(\phi - \phi')) \, \dddot{Y}_{\rho,n,1} + 2 \cos (n(\phi - \phi')) \, \dddot{Y}_{\rho,n,1} \]  
(B.2e)

\[ \dddot{Y}_{\rho,n,2} = 2 \cos (n(\phi - \phi')) \, \dddot{Y}_{\rho,n,2} + 2i \sin (n(\phi - \phi')) \, \dddot{Y}_{\rho,n,2}, \]  
(B.2f)

\[ \dddot{Z}_{\rho,n,11} = 2 \cos (n(\phi - \phi')) \, \dddot{Z}_{\rho,n,11}, \]  
(B.2g)

\[ \dddot{Z}_{\rho,n,12} = 2i \sin (n(\phi - \phi')) \, \dddot{Z}_{\rho,n,12}, \]  
(B.2h)

\[ \dddot{Z}_{\rho,n,21} = 2i \sin (n(\phi - \phi')) \, \dddot{Z}_{\rho,n,21}, \]  
(B.2i)

\[ \dddot{Z}_{\rho,n,22} = 2 \cos (n(\phi - \phi')) \, \dddot{Z}_{\rho,n,22}. \]  
(B.2j)

For the \( \phi \)-components,

\[ \dddot{X}_{\phi,n,11} = 2i \sin (n(\phi - \phi')) \, \dddot{X}_{\phi,n,11}, \]  
(B.3a)

\[ \dddot{X}_{\phi,n,12} = 2 \cos (n(\phi - \phi')) \, \dddot{X}_{\phi,n,12}, \]  
(B.3b)

\[ \dddot{X}_{\phi,n,21} = 2 \cos (n(\phi - \phi')) \, \dddot{X}_{\phi,n,21}, \]  
(B.3c)
\[ \ddot{X}_{\phi,n,22} = 2i \sin(n(\phi - \phi')) \dot{X}_{\phi,n,22}, \quad \text{(B.3d)} \]

\[ \ddot{Y}_{\phi,n,1} = 2 \cos(n(\phi - \phi')) \dot{Y}_{\phi,n,11} \dot{D}_{j2,1}' + 2i \sin(n(\phi - \phi')) \dot{Y}_{\phi,n,12} \dot{D}_{j2,2}', \quad \text{(B.3e)} \]

\[ \ddot{Y}_{\phi,n,2} = 2i \sin(n(\phi - \phi')) \dot{Y}_{\phi,n,21} \dot{D}_{j2,1}' + 2 \cos(n(\phi - \phi')) \dot{Y}_{\phi,n,22} \dot{D}_{j2,2}', \quad \text{(B.3f)} \]

\[ \ddot{Z}_{\phi,n,11} = 2i \sin(n(\phi - \phi')) \dot{Z}_{\phi,n,11}, \quad \text{(B.3g)} \]

\[ \ddot{Z}_{\phi,n,12} = 2 \cos(n(\phi - \phi')) \dot{Z}_{\phi,n,12}, \quad \text{(B.3h)} \]

\[ \ddot{Z}_{\phi,n,21} = 2 \cos(n(\phi - \phi')) \dot{Z}_{\phi,n,21}, \quad \text{(B.3i)} \]

\[ \ddot{Z}_{\phi,n,22} = 2i \sin(n(\phi - \phi')) \dot{Z}_{\phi,n,22}. \quad \text{(B.3j)} \]
Appendix C

Electromagnetic Fields in Homogeneous and Isotropic Media

In this Appendix, the analytical expressions of electromagnetic fields due to arbitrarily oriented Hertzian electric dipoles in a homogeneous and isotropic medium are expressed in cylindrical coordinates. These expressions are used to compensate for the subtraction of direct terms in otherwise non-convergent numerical integrals (see Section 2.4.4).

Even though the fields are expressed in cylindrical coordinates, as it will be clear below it is more convenient to first obtain the fields in Cartesian coordinates and then do coordinate transformations. Equations (C.1a) and (C.1b) show the relevant transformations. The superscript \( o \) is used to denote the direct fields.

\[
\begin{bmatrix}
E^o_\rho \\
E^o_\phi \\
E^o_z
\end{bmatrix} =
\begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
E^o_x \\
E^o_y \\
E^o_z
\end{bmatrix}, \quad (C.1a)
\]

\[
\begin{bmatrix}
\alpha_{x'} \\
\alpha_{y'} \\
\alpha_{z'}
\end{bmatrix} =
\begin{bmatrix}
\cos \phi' & -\sin \phi' & 0 \\
\sin \phi' & \cos \phi' & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_{x'} \\
\alpha_{y'} \\
\alpha_{z'}
\end{bmatrix}. \quad (C.1b)
\]

Using the scalar Green's function, the Cartesian field components are

\[
E^o_x = \frac{iIl}{\omega \epsilon} \left[ \hat{x} \cdot \hat{\alpha}' k^2 + \frac{\partial}{\partial x'} \nabla' \cdot \hat{\alpha}' \right] \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad (C.2a)
\]
\[ E_y = \frac{iIl}{\omega \epsilon} \left[ \hat{\gamma} \cdot \hat{\omega} k^2 + \frac{\partial}{\partial y'} \nabla' \cdot \hat{\omega}' \right] \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \text{(C.2b)} \]
\[ E_z = \frac{iIl}{\omega \epsilon} \left[ \hat{\gamma} \cdot \hat{\omega} k^2 + \frac{\partial}{\partial z'} \nabla' \cdot \hat{\omega}' \right] \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \text{(C.2c)} \]
\[ H_x = -\hat{x} \cdot \nabla' \times \hat{\omega}' Il e^{ik|\mathbf{r}-\mathbf{r}'|} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \text{(C.3a)} \]
\[ H_y = -\hat{y} \cdot \nabla' \times \hat{\omega}' Il e^{ik|\mathbf{r}-\mathbf{r}'|} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \quad \text{(C.3b)} \]
\[ H_z = -\hat{z} \cdot \nabla' \times \hat{\omega}' Il e^{ik|\mathbf{r}-\mathbf{r}'|} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \quad \text{(C.3c)} \]

In (C.2a) – (C.2c), common factors for electric field components can be identified as
\[ \left[ \hat{s} \cdot \hat{s}' k^2 + \frac{\partial}{\partial s'} \nabla' \cdot \hat{s}' \right] = k^2 + \frac{\partial^2}{\partial s'^2}, \quad \text{(C.4a)} \]
\[ \left[ \hat{s} \cdot \hat{w}' k^2 + \frac{\partial}{\partial s'} \nabla' \cdot \hat{w}' \right] = \frac{\partial^2}{\partial s' \partial w'}. \quad \text{(C.4b)} \]

where
\[ s = x, y, z \quad \text{and} \quad w = x, y, z. \]

Similarly, common factors for magnetic field components in (C.3a) – (C.3c) are identified. All possible cases are listed below.
\[ -\hat{x} \cdot \nabla' \times \hat{x}' = -\hat{y} \cdot \nabla' \times \hat{y}' = -\hat{z} \cdot \nabla' \times \hat{z}' = 0, \quad \text{(C.5a)} \]
\[ -\hat{x} \cdot \nabla' \times \hat{y}' = \frac{\partial}{\partial z'}, \quad \text{(C.5b)} \]
\[ -\hat{x} \cdot \nabla' \times \hat{z}' = -\frac{\partial}{\partial y'}, \quad \text{(C.5c)} \]
\[ -\hat{y} \cdot \nabla' \times \hat{x}' = -\frac{\partial}{\partial z'}, \quad \text{(C.5d)} \]
\[ -\hat{y} \cdot \nabla' \times \hat{z}' = \frac{\partial}{\partial x'}, \quad \text{(C.5e)} \]
\[ -\hat{z} \cdot \nabla' \times \hat{x}' = \frac{\partial}{\partial y'}, \quad \text{(C.5f)} \]
\[-\hat{z} \cdot \nabla' \times \hat{y}' = -\frac{\partial}{\partial x'}. \tag{C.5g}\]

All partial derivatives can be calculated using the chain rule; i.e.,
\[
\frac{\partial}{\partial s'} = \frac{\partial r}{\partial s'} \frac{\partial}{\partial r}, \tag{C.6}\]
where
\[
r = |\mathbf{r} - \mathbf{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}. \tag{C.7}\]

The necessary partial derivatives are listed below for completeness:
\[
\begin{align*}
\frac{\partial r}{\partial s'} &= s' - s, \tag{C.8a} \\
\frac{\partial^2 r}{\partial s'^2} &= \frac{\partial}{\partial s'} \left( s' - s \right) = \frac{r - (s' - s)s' - s}{r^2} = \frac{r^2 - (s' - s)^2}{r^3}, \tag{C.8b} \\
\frac{\partial^2 r}{\partial w' \partial s'} &= \frac{\partial}{\partial s'} \left( - (s' - s)(w' - w) \right) = \frac{- (s' - s)(w' - w)}{r^3}, \tag{C.8c} \\
\frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) &= \frac{ikre^{ikr} - e^{ikr}}{r^2} = \left( \frac{ikr - 1}{r} \right) \frac{e^{ikr}}{r}, \tag{C.8d} \\
\frac{\partial^2}{\partial r^2} \left( \frac{e^{ikr}}{r} \right) &= \frac{\partial}{\partial r} \left[ \left( \frac{ikr - 1}{r} \right) \frac{e^{ikr}}{r} \right] = \frac{ikr - (ikr - 1)}{r^2} \frac{e^{ikr}}{r} + \left( \frac{ikr - 1}{r} \right)^2 \frac{e^{ikr}}{r} \\
&= \left( \frac{- (kr)^2 - 2ikr + 2}{r^2} \right) \frac{e^{ikr}}{r}. \tag{C.8e}
\end{align*}
\]

Using on (C.8a) – (C.8e), the first and second partial derivatives can be calculated.

The first partial derivative of interest is
\[
\frac{\partial}{\partial s'} \left( \frac{e^{ikr}}{r} \right) = \frac{\partial r}{\partial s'} \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) = \frac{s' - s}{r} \left( \frac{e^{ikr}}{r} \right) = \frac{e^{ikr}}{r} \left[ \frac{ik}{r - \frac{1}{r^2}} \right] (s' - s). \tag{C.9}\]
The second partial derivatives of interest are

\[
\frac{\partial^2}{\partial s'^2} \left( \frac{e^{ikr}}{r} \right) = \frac{\partial}{\partial s'} \left[ \frac{\partial r}{\partial s'} \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) \right] \\
= \frac{\partial^2 r}{\partial s'^2} \cdot \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) + \frac{\partial r}{\partial s'} \frac{\partial}{\partial r} \frac{\partial}{\partial s'} \left( \frac{e^{ikr}}{r} \right) \\
= \frac{\partial^2 r}{\partial s'^2} \cdot \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) + \frac{\partial r}{\partial s'} \frac{\partial}{\partial r} \frac{\partial^2}{\partial r^2} \left( \frac{e^{ikr}}{r} \right) \\
= r^2 - (s' - s)^2 \left( \frac{i kr - 1}{r} \right) \frac{e^{ikr}}{r} + \left( \frac{s' - s}{r} \right)^2 \left( \frac{-(kr)^2 - 2ikr + 2}{r^2} \right) \frac{e^{ikr}}{r} \\
= e^{ikr} \left[ \frac{ikr^3 - r^2}{r^4} + \left( \frac{-ikr + 1 - (kr)^2 - 2ikr + 2}{r^4} \right) (s' - s)^2 \right] \\
= e^{ikr} \left[ \frac{i k^2 - 3i k r^3 + 3}{r^4} (s' - s) (w' - w) \right], \quad (C.10)
\]

\[
\frac{\partial^2}{\partial w' \partial s'} \left( \frac{e^{ikr}}{r} \right) = \frac{\partial}{\partial w'} \left[ \frac{\partial r}{\partial s'} \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) \right] \\
= \frac{\partial^2 r}{\partial w' \partial s'} \cdot \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) + \frac{\partial r}{\partial w'} \frac{\partial}{\partial r} \frac{\partial}{\partial s'} \left( \frac{e^{ikr}}{r} \right) \\
= \frac{\partial^2 r}{\partial w' \partial s'} \cdot \frac{\partial}{\partial r} \left( \frac{e^{ikr}}{r} \right) + \frac{\partial r}{\partial w'} \frac{\partial}{\partial r} \frac{\partial^2}{\partial r^2} \left( \frac{e^{ikr}}{r} \right) \\
= -\frac{(s' - s)(w' - w)}{r^3} \left( \frac{ikr - 1}{r} \right) \frac{e^{ikr}}{r} + \frac{s' - s w' - w}{r} \left( \frac{-(kr)^2 - 2ikr + 2}{r^2} \right) \frac{e^{ikr}}{r} \\
= e^{ikr} \left[ \frac{-ikr + 1 - (kr)^2 - 2ikr + 2}{r^4} \right] (s' - s)(w' - w) \\
= e^{ikr} \left[ -\frac{k^2}{r^2} - \frac{3ikr}{r^3} + 3 \right] (s' - s)(w' - w). \quad (C.11)
\]
Using (C.9), (C.10), and (C.11), the direct fields can be expressed in Cartesian coordinates as

\[
\begin{bmatrix}
E_x^o \\
E_y^o \\
E_z^o
\end{bmatrix} = \frac{iIl}{\omega \epsilon} e^{ikr} \frac{1}{4\pi r} \begin{bmatrix}
k^2 + A + BX^2 & BXY & BXZ \\
BXY & k^2 + A + BY^2 & BYZ \\
BXZ & BYZ & k^2 + A + BZ^2
\end{bmatrix} \begin{bmatrix}
\alpha_x' \\
\alpha_y' \\
\alpha_z'
\end{bmatrix}, \quad (C.12a)
\]

\[
\begin{bmatrix}
H_x^o \\
H_y^o \\
H_z^o
\end{bmatrix} = \frac{Ile}{\omega} ikr \frac{1}{4\pi r} \begin{bmatrix}
0 & AZ & -AY \\
-AZ & 0 & AX \\
AY & -AX & 0
\end{bmatrix} \begin{bmatrix}
\alpha_x' \\
\alpha_y' \\
\alpha_z'
\end{bmatrix}. \quad (C.12b)
\]

After performing the coordinate transformations (C.1a) and (C.1b), the analytical expressions of the direct fields in cylindrical coordinates are given by

\[
\begin{bmatrix}
E_\rho^o \\
E_\phi^o \\
E_z^o
\end{bmatrix} = \frac{iIl}{\omega \epsilon} e^{ikr} \frac{1}{4\pi r} T_1 \cdot M_e \cdot T_2 \cdot \begin{bmatrix}
\alpha_\rho' \\
\alpha_\phi' \\
\alpha_z'
\end{bmatrix}, \quad (C.13a)
\]

\[
\begin{bmatrix}
H_\rho^o \\
H_\phi^o \\
H_z^o
\end{bmatrix} = \frac{Ile}{\omega} e^{ikr} \frac{1}{4\pi r} T_1 \cdot M_m \cdot T_2 \cdot \begin{bmatrix}
\alpha_\rho' \\
\alpha_\phi' \\
\alpha_z'
\end{bmatrix}, \quad (C.13b)
\]

where

\[
T_1 = \begin{bmatrix}
cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad (C.14a)
\]

\[
T_2 = \begin{bmatrix}
cos \phi' & -\sin \phi' & 0 \\
\sin \phi' & \cos \phi' & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad (C.14b)
\]

\[
M_e = \begin{bmatrix}
k^2 + A + BX^2 & BXY & BXZ \\
BXY & k^2 + A + BY^2 & BYZ \\
BXZ & BYZ & k^2 + A + BZ^2
\end{bmatrix}, \quad (C.14c)
\]

\[
M_m = \begin{bmatrix}
0 & AZ & -AY \\
-AZ & 0 & AX \\
AY & -AX & 0
\end{bmatrix} \quad (C.14d)
\]
\[ A = \frac{ik}{r} - \frac{1}{r^2}, \quad (C.14e) \]
\[ B = \frac{-k^2}{r^2} - \frac{3ik}{r^3} + \frac{3}{r^4}, \quad (C.14f) \]
\[ X = x' - x, \quad (C.14g) \]
\[ Y = y' - y, \quad (C.14h) \]
\[ Z = z' - z. \quad (C.14i) \]
Appendix D

Electromagnetic Fields in Homogeneous and Doubly-Uniaxial Media

In this appendix, the analytical expressions used to obtain for electromagnetic fields in homogeneous and doubly-uniaxial media are presented. In such media, Maxwell’s equations with \( e^{-i\omega t} \) time-dependence are written as

\[
\nabla \times \mathbf{E}(\mathbf{r}) = i \omega \mu \cdot \mathbf{H}(\mathbf{r}), \quad (D.1)
\]

\[
\nabla \times \mathbf{H}(\mathbf{r}) = -i \omega \varepsilon \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}). \quad (D.2)
\]

Permittivity values are complex-valued so as to include conductivities. For simplicity, it is assumed that the anisotropy ratio for the permeability tensor is the same as that of the complex permittivity tensor, i.e., \( \kappa_\varepsilon = \kappa_\mu \). To simplify the derivation, we adopt the coordinate stretching techniques [202, 203, 204]. To begin with, let us consider modified Maxwell’s curl equations with stretched coordinates, i.e.,

\[
\tilde{\nabla} \times \tilde{\mathbf{E}}(\tilde{\mathbf{r}}) = i \omega \tilde{\mu} \cdot \tilde{\mathbf{H}}(\tilde{\mathbf{r}}), \quad (D.3)
\]

\[
\tilde{\nabla} \times \tilde{\mathbf{H}}(\tilde{\mathbf{r}}) = -i \omega \tilde{\varepsilon} \cdot \tilde{\mathbf{E}}(\tilde{\mathbf{r}}) + \tilde{\mathbf{J}}(\tilde{\mathbf{r}}), \quad (D.4)
\]

with the modified nabla operator \( \tilde{\nabla} \) defined as

\[
\tilde{\nabla} = \hat{x} \frac{\partial}{\partial \tilde{x}} + \hat{y} \frac{\partial}{\partial \tilde{y}} + \hat{z} \frac{\partial}{\partial \tilde{z}}, \quad (D.5)
\]
where $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are stretched coordinates defined such that

$$u \rightarrow \tilde{u} = \int_0^u s_u(u')du', \quad (D.6)$$

where $s_u$ is the corresponding complex stretching variable, and $u$ indicates $x$, $y$, or $z$. In the above, the fields and sources are non-Maxwellian but $\tilde{\mu}$ and $\tilde{\epsilon}$ are scalars, so the medium is isotropic. Using the technique in [203], (D.3) and (D.4) are rewritten as

$$\nabla \times \left( \bar{S}^{-1} \cdot \tilde{E}(\tilde{r}) \right) = i \omega \tilde{\mu} \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{H}(\tilde{r}), \quad (D.7)$$

$$\nabla \times \left( \bar{S}^{-1} \cdot \tilde{H}(\tilde{r}) \right) = -i \omega \tilde{\epsilon} \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{E}(\tilde{r}) + \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{J}(\tilde{r}), \quad (D.8)$$

where the dyadic $\bar{S}$ is defined as

$$\bar{S} = \hat{x} \hat{x} \left( \frac{1}{s_x} \right) + \hat{y} \hat{y} \left( \frac{1}{s_y} \right) + \hat{z} \hat{z} \left( \frac{1}{s_z} \right). \quad (D.9)$$

Using the relations between the stretched fields to unstretched (Maxwellian) fields,

$$\tilde{E}(\tilde{r}) = \bar{S}^{-1} \cdot \tilde{E}(\tilde{r}), \quad (D.10a)$$

$$\tilde{H}(\tilde{r}) = \bar{S}^{-1} \cdot \tilde{H}(\tilde{r}), \quad (D.10b)$$

$$\tilde{J}(\tilde{r}) = \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{J}(\tilde{r}), \quad (D.10c)$$

we can rearrange (D.7) and (D.8) as

$$\nabla \times \tilde{E}(\tilde{r}) = i \omega \tilde{\mu} \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{H}(\tilde{r}), \quad (D.11)$$

$$\nabla \times \tilde{H}(\tilde{r}) = -i \omega \tilde{\epsilon} \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{E}(\tilde{r}) + \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \tilde{J}(\tilde{r}). \quad (D.12)$$

These two resulting curl equations can be associated with an effective anisotropic medium such that

$$\nabla \times \tilde{E}(\tilde{r}) = i \omega \tilde{\mu} \cdot H(\tilde{r}), \quad (D.13)$$
\[ \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega\mathbf{E}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \quad (D.14) \]

which recover the form of the original curl equations (D.1) and (D.2). Therefore, electromagnetic fields in a homogeneous and uniaxial media with \( \mathbf{E}(\mathbf{r}) \) and \( \mathbf{H}(\mathbf{r}) \) can be easily obtained from \( \tilde{\mathbf{E}}(\tilde{\mathbf{r}}) \) and \( \tilde{\mathbf{H}}(\tilde{\mathbf{r}}) \), which are solutions in isotropic media with coordinate-stretching, by the transformations expressed in (D.10a), (D.10b), and (D.10c). In order to determine the form of the stretching variables relevant to our problem, let us examine the effective anisotropic medium obtained above. The constitutive tensors have the form

\[ \bar{\mu} = \left[ \tilde{\mu} \left( \det S \right)^{-1} S \cdot S \right] = \tilde{\mu} \bar{\Lambda}, \quad (D.15) \]
\[ \bar{\epsilon} = \left[ \tilde{\epsilon} \left( \det S \right)^{-1} S \cdot S \right] = \tilde{\epsilon} \bar{\Lambda}, \quad (D.16) \]

where

\[ \bar{\Lambda} = s_x s_y s_z \begin{bmatrix} s_x^{-2} & 0 & 0 \\ 0 & s_y^{-2} & 0 \\ 0 & 0 & s_z^{-2} \end{bmatrix} = \begin{bmatrix} s_y s_z & 0 & 0 \\ 0 & s_y s_z & 0 \\ 0 & 0 & s_x s_y s_z \end{bmatrix}. \quad (D.17) \]

Using two conditions on the stretching variables for uniaxial anisotropy, and using the wavenumber expression for the modified Maxwell’s equations \( \bar{k} = \omega \sqrt{\bar{\mu} \bar{\epsilon}} \), we can set \( s_x = s_y = 1 \), and \( s_z = \kappa \). Consequently, we obtain \( \bar{\mu} = \frac{\mu h}{\kappa} \) and \( \bar{\epsilon} = \frac{\epsilon h}{\kappa} \). Next, let us consider the source transformation (D.10c). If the source is a point Hertzian electric dipole like \( \mathbf{J}(\mathbf{r}) = Il\hat{e} \delta(\mathbf{r} - \mathbf{r}') \), the coordinate stretching should be carefully treated due to the presence of Dirac delta function. The stretched current density is expressed as \( \tilde{\mathbf{J}}(\tilde{\mathbf{r}}) = Il\hat{\tilde{e}} \delta(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}') \). From the Dirac delta function properties,

\[ \delta(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}') = \frac{1}{s_x s_y s_z} \delta(\mathbf{r} - \mathbf{r}'), \quad (D.18) \]
and from (D.10c),
\[ \hat{\alpha}' \delta(r - r') = \left( \det \bar{S} \right)^{-1} \bar{S} \cdot \hat{\tilde{\alpha}}' \delta(\bar{r} - \bar{r}') = \begin{bmatrix} s_x^{-1} & 0 & 0 \\ 0 & s_y^{-1} & 0 \\ 0 & 0 & s_z^{-1} \end{bmatrix} \cdot \hat{\tilde{\alpha}}' \delta(r - r'). \] (D.19)

Since \( s_x = s_y = 1 \) and \( s_z = \kappa \), we have the source transformation \( \hat{\tilde{\alpha}}' = S^{-1} \cdot \hat{\alpha}' \), and the Cartesian field components due to the Hertzian electric dipole source in homogeneous and isotropic media can be written as

\[
\begin{bmatrix}
\tilde{E}_x \\
\tilde{E}_y \\
\tilde{E}_z
\end{bmatrix} = \frac{iI}{\omega} \frac{e^{i\bar{k}\bar{r}}}{4\pi \bar{r}} \mathbf{M}_e \cdot \begin{bmatrix}
\tilde{\alpha}_x' \\
\tilde{\alpha}_y' \\
\tilde{\alpha}_z'
\end{bmatrix},
\] (D.20a)

\[
\begin{bmatrix}
\tilde{H}_x \\
\tilde{H}_y \\
\tilde{H}_z
\end{bmatrix} = \frac{I}{\omega} \frac{e^{i\bar{k}\bar{r}}}{4\pi \bar{r}} \mathbf{M}_m \cdot \begin{bmatrix}
\tilde{\alpha}_x' \\
\tilde{\alpha}_y' \\
\tilde{\alpha}_z'
\end{bmatrix},
\] (D.20b)

where

\[
\mathbf{M}_e = \begin{bmatrix}
\bar{k}^2 + A + BX^2 & BXY & BXZ \\
BXY & \bar{k}^2 + A + BY^2 & BYZ \\
BXZ & BYZ & \bar{k}^2 + A + BZ^2
\end{bmatrix},
\] (D.21a)

\[
\mathbf{M}_m = \begin{bmatrix}
0 & AZ & -AY \\
-AZ & 0 & AX \\
AY & -AX & 0
\end{bmatrix},
\] (D.21b)

\[
A = i\bar{k}/\bar{r} - 1/\bar{r}^2,
\] (D.21c)

\[
B = -\bar{k}^2/\bar{r}^2 - 3i\bar{k}/\bar{r}^3 + 3/\bar{r}^4,
\] (D.21d)

\[
X = s_x(x' - x) = x' - x,
\] (D.21e)

\[
Y = s_y(y' - y) = y' - y,
\] (D.21f)

\[
Z = s_z(z' - z) = \kappa(z' - z),
\] (D.21g)

\[
\bar{k} = \omega\sqrt{\mu_0\varepsilon_0}/\kappa,
\] (D.21h)
\[
\bar{r} = \left[ (x' - x)^2 + (y' - y)^2 + \kappa^2(z' - z)^2 \right]^{1/2}.
\]  \hspace{1cm} \text{(D.21i)}

Applying field transformations, (D.10a) and (D.10b), and source transformation \( \hat{\alpha}' = \overline{S}^{-1} \cdot \hat{\alpha}' \), we obtain

\[
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix} = \frac{iiL}{\omega \epsilon} \frac{e^{ik\bar{r}}}{4\pi \bar{r}} \overline{S}^{-1} \cdot \overline{M_e} \cdot \overline{S}^{-1} \cdot \begin{bmatrix}
\alpha_{x'} \\
\alpha_{y'} \\
\alpha_{z'}
\end{bmatrix},
\]  \hspace{1cm} \text{(D.22a)}

\[
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix} = \frac{il}{4\pi \bar{r}} \frac{e^{ik\bar{r}}}{4\pi \bar{r}} \overline{S}^{-1} \cdot \overline{M_m} \cdot \overline{S}^{-1} \cdot \begin{bmatrix}
\alpha_{x'} \\
\alpha_{y'} \\
\alpha_{z'}
\end{bmatrix}.
\]  \hspace{1cm} \text{(D.22b)}

Finally, applying the coordinate transformations from Cartesian to cylindrical coordinates, we obtain

\[
\begin{bmatrix}
E_\rho \\
E_\phi \\
E_z
\end{bmatrix} = \frac{iiL}{\omega \epsilon} \frac{e^{ik\bar{r}}}{4\pi \bar{r}} \overline{T}_1 \cdot \overline{S}^{-1} \cdot \overline{M_e} \cdot \overline{S}^{-1} \cdot \overline{T}_2 \cdot \begin{bmatrix}
\alpha_{\rho'} \\
\alpha_{\phi'} \\
\alpha_{z'}
\end{bmatrix},
\]  \hspace{1cm} \text{(D.23a)}

\[
\begin{bmatrix}
H_\rho \\
H_\phi \\
H_z
\end{bmatrix} = \frac{il}{4\pi \bar{r}} \frac{e^{ik\bar{r}}}{4\pi \bar{r}} \overline{T}_1 \cdot \overline{S}^{-1} \cdot \overline{M_m} \cdot \overline{S}^{-1} \cdot \overline{T}_2 \cdot \begin{bmatrix}
\alpha_{\rho'} \\
\alpha_{\phi'} \\
\alpha_{z'}
\end{bmatrix},
\]  \hspace{1cm} \text{(D.23b)}

where

\[
\overline{T}_1 = \begin{bmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix},
\]  \hspace{1cm} \text{(D.24a)}

\[
\overline{T}_2 = \begin{bmatrix}
\cos \phi' & -\sin \phi' & 0 \\
\sin \phi' & \cos \phi' & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]  \hspace{1cm} \text{(D.24b)}
Appendix E

Whitney Forms

E.1 Basic Properties

In this Appendix, the explicit expressions of Whitney forms [205] in 3-D are briefly discussed. In the past, Whitney forms prove to be appropriate for the basis functions of finite elements because they feature some desired properties for representing electromagnetic fields and sources [206, 207, 208]. Although Whitney forms can be expressed compactly and elegantly through the concept of exterior calculus and the language of differential forms [161, 173, 175, 176, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 219], their vector notations are adopted here. For 3-D, there are four Whitney $p$-forms according to their degree $p$.

A Whitney 0-form is a continuous scalar function and its expression is

$$W^0_i(r) = \lambda_i(r),$$

where the subscript $i$ is the index of a vertex, the superscript 0 represents “0-form,” $\lambda_i$ is the barycentric coordinate of a point $r$ associated with vertex $i$ [220]. The illustration of the geometrical construction for barycentric coordinates is provided in Figure E.1. For a 1-D simplex, which is an edge, the barycentric coordinates
associated to the vertices $\nu_1$ and $\nu_2$ of a point $r$ in the simplex are equal to ratios $\lambda_1 = L_2/(L_1 + L_2)$ and $\lambda_2 = L_1/(L_1 + L_2)$, respectively, with $L_1$ and $L_2$ as indicated in Fig. E.1a. For a 2-D simplex, which is a triangle, the barycentric coordinates associated to the vertices $\nu_1$, $\nu_2$ and $\nu_3$ of a point $r$ in the simplex are equal to ratios $\lambda_1 = A_1/A$, $\lambda_2 = A_2/A$, and $\lambda_3 = A_3/A$, respectively, with the areas $A_1$, $A_2$, and $A_3$ as indicated in Figure E.1b, and $A = A_1 + A_2 + A_3$. For a 3-D simplex, which is a tetrahedron, the barycentric coordinates can be now defined as volume ratios. It is obvious that $0 \leq \lambda_i \leq 1$ for all $i$ and that the sum of the barycentric coordinates of any given point $r$ associated to the neighbor vertices equals to one, which is called the partition of unity. Hereinafter, the dependence on $r$ is dropped for notational simplicity, i.e., $\lambda_i(r) = \lambda_i$. For more details about barycentric coordinates, refer to [220].

Figure E.1: Geometric illustration for Whitney 0-forms (equivalently, barycentric coordinates) of a point $r$ in simplices of various degrees: (a) 1-D simplex and (b) 2-D simplex.
Figure E.2: Geometric illustration of the weight assigned to Whitney 1-forms representing a segment $L$ in a 2-D simplex: (a) In red color is the area $A_{e_1}$ associated with the Whitney 1-form on $e_1$ (edge 1) that represents $L$. The associated weight is given by $A_{e_1}/A$, where $A$ is the total area of the triangle composed of $v_1$, $v_2$, and $v_3$. A similar construction can be made for the other two edges $e_2$ and $e_3$. (b) Area represented by a sum of small triangles.

The vector (function) proxy of a Whitney 1-form associated with an arbitrary edge $ij$ consisting of vertices $i$ and $j$ is expressed as [175]

$$W_{ij}^1(r) = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i.$$  \hspace{1cm} (E.2)

Note that the vertex-based indexing is adopted in this Appendix for all other elements for the purpose of clarification. The geometrical illustration of Whitney 1-forms is provided in Figure E.2. As an example, the area $A_{e_1}$ as shown in Figure E.2a is associated with $e_1$ and expressed as

$$A_{e_1} = A \left[ \lambda_1^s \lambda_2^f - \lambda_2^s \lambda_1^f \right],$$  \hspace{1cm} (E.3)
where $\lambda_i^s$ and $\lambda_i^f$ are shorthands of $\lambda_i(r_s)$ and $\lambda_i(r_f)$, respectively. As Figure E.2b shows, $A_{e1}$ is the same as the sum of all small triangles such that

$$A_{e1} = A \sum_n \left[ \lambda_1^n \lambda_2^{n+1} - \lambda_2^n \lambda_1^{n+1} \right]$$

$$= A \sum_n \left[ \lambda_1^n (\lambda_2^n + \Delta \lambda_2^n) - \lambda_2^n (\lambda_1^n + \Delta \lambda_1^n) \right]$$

$$= A \sum_n \left[ \lambda_1^n \Delta \lambda_2^n - \lambda_2^n \Delta \lambda_1^n \right]. \quad (E.4)$$

After taking the limit of infinitesimally small triangles and transforming the summation (E.4) to an integral, we obtain

$$A_{e1} = A \int_{r_s}^{r_f} [\lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1] \cdot dL = A \int_{r_s}^{r_f} W_{12}^1(r) \cdot dL. \quad (E.5)$$

Similarly, the other areas associated with $e_2$ and $e_3$ can be interpreted. The last integral in (E.5) can be regarded as the generalization of barycentric coordinates from 0-dimensional objects (points) to 1-dimensional objects (edges). Therefore, Whitney 0-forms can be used to interpolate any scalar values at a point $r$ using a weighted sum of nearby vertices $\nu_1$, $\nu_2$, and $\nu_3$ with respective weights $A_i/A$. Similarly, Whitney 1-forms can be used to interpolate any vector functions along an edge $[r_s, r_f]$ using a weighted sum of nearby edges $e_1, e_2$, and $e_3$ now with respective weights of $A_{e1}/A$, $A_{e2}/A$, and $A_{e3}/A$. Note that, these weights are indeed the “contraction” of Whitney forms with the corresponding geometric objects [175]. For a Whitney 0-form, the contraction is the evaluation of $W_i^0$ at the point $r$ like $W_i^0(r)$, whereas the contraction is the evaluation of a line integral for a Whitney 1-form as shown in (E.5). The discussion of the line integral is presented in Appendix E.3. For a more general description of the properties of Whitney forms, see [209].
A Whitney 2-form is a vector function and its vector proxy associated with an arbitrary triangle \( ijk \) consisting of vertices \( i, j, \) and \( k \) is expressed as \( [175] \)

\[
W^2_{ijk}(r) = 2 \left[ \lambda_i \nabla \lambda_j \times \nabla \lambda_k + \lambda_j \nabla \lambda_k \times \nabla \lambda_i + \lambda_k \nabla \lambda_i \times \nabla \lambda_j \right]. \quad (E.6)
\]

Finally, a Whitney 3-form is piecewise constant scalar function and its vector proxy associated with an arbitrary tetrahedron \( ijk\ell \) in 3-D is written as \( [175] \)

\[
W^3_{ijkl}(r) = 6 \left[ \lambda_i \nabla \lambda_j \cdot (\nabla \lambda_k \times \nabla \lambda\ell) + \lambda_j \nabla \lambda_k \cdot (\nabla \lambda\ell \times \nabla \lambda_i) + \lambda_k \nabla \lambda\ell \cdot (\nabla \lambda_i \times \nabla \lambda_j) \right]. \quad (E.7)
\]

Despite the complicated expression in \( (E.7) \), \( W^3_{ijkl} \) can be succinctly expressed as

\[
W^3_{ijkl}(r) = \begin{cases} 
\frac{1}{V}, & \text{if } r \text{ is in the tetrahedron } ijk\ell, \\
0, & \text{otherwise},
\end{cases} \quad (E.8)
\]

where \( V \) is the volume of the tetrahedron \( ijk\ell \) \([175]\).

One of the properties of Whitney forms is the localization property. In other words, Whitney forms are interpolants in the precise sense that they are equal to one when they are evaluated on their corresponding grid elements (vertices, edges, triangles, and tetrahedra) and to zero on all remaining elements, which can be summarized as

\[
\int_{\sigma^p_i} W^p_j = \delta_{ij} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j
\end{cases}, \quad (E.9)
\]

where \( \sigma^p_i \) is the \( p \)-dimensional simplex, \( W^p_j \) is Whitney \( p \)-forms, and \( \delta_{ij} \) is the Kronecker delta.

Furthermore, Whitney forms inherit the desired type of continuity of what they represent. This conformity property makes Whitney forms useful. Using the vector proxies, \( W^0_i(r) \) is a continuous scalar function representing scalar potentials, \( W^1_\ell(r) \) is a tangentially continuous vector function representing intensity vector fields, \( W^2_{ijk}(r) \)
Figure E.3: Whitney edge basis functions in the cyclic order for two possible vertex (node) numberings.

is a normally continuous vector function representing either flux density vector fields or volumetric current densities, and $W^{3}_{ijkl}(r)$ is a discontinuous scalar field representing volumetric charge densities. For more properties of Whitney forms, readers are referred to [171, 172, 209, 210, 211] and references therein.

### E.2 Notations of Whitney 1-forms

Two notations of Whitney edge basis functions (Whitney 1-forms) are presented in this section. As conventional, the functions can be defined in the sense of a cyclic order. This notation is straightforward and easy to remember. Figure E.3 briefly illustrates the edges with their orientations and the basis functions. All three basis functions are in the same sense of rotation, clockwise or counterclockwise, and are expressed as

\begin{align}
W_1 &= \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1, \\
W_2 &= \lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2, \\
W_3 &= \lambda_3 \nabla \lambda_1 - \lambda_1 \nabla \lambda_3.
\end{align}
Another notation can be based on the sense of an ascending order illustrated in Figure E.4. This notation is convenient to deal with global node and edge numbers. In other words, for any triangle in 2-D unstructured grids, we don’t have to seek local node and edge numbers to define the basis functions. Advantage of the ascending order notation is that the orientation of each basis function only depends on the two nodes of the edge irrespective of the third node. The functions in the ascending order are expressed as

\begin{align*}
W_1 &= \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1, \\
W_2 &= \lambda_1 \nabla \lambda_3 - \lambda_3 \nabla \lambda_1, \\
W_3 &= \lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2.
\end{align*}

(E.11)

### E.3 Line Integral of Whitney 1-forms

In this section, the detailed derivation of the line integral of Whitney 1-forms is provided. Figure E.5 illustrates an arbitrary path \( \mathbf{L} \) from \( \mathbf{x}_s \) to \( \mathbf{x}_f \) in a triangle. The
path can be decomposed into $a$ and $b$. $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are barycentric coordinates associated with $\nu_1$ (vertex 1) and $\nu_2$ (vertex 2). Hereinafter, several shorthand notations are used for the barycentric coordinates: $\lambda_1(x_s) = \lambda_1^s$, $\lambda_1(x_f) = \lambda_1^f$, $\lambda_2(x_s) = \lambda_2^s$, and $\lambda_2(x_f) = \lambda_2^f$. $h_1$ and $h_2$ are the heights of the triangle for the base of $e_3$ and $e_2$, respectively. The edge vectors $e_1$, $e_2$, and $e_3$ are oriented in an ascending fashion. Note that the edge numbers do not coincide with the vertex numbers.

One of the easiest ways to solve this type of line integral,

$$\int_L W_i^1 \cdot dL,$$  \hspace{1cm} (E.12)  

is to use parametric equations such that

$$W_i^1 = W_i^1(t) \quad \text{and} \quad dL = dL(t).$$  \hspace{1cm} (E.13)
$W_1^i(t)$ and $dL(t)$ are simply assumed to be linear functions of $t$ and the range of $t$ is set to be $0 \leq t \leq 1$. As an example, the line integral of the Whitney edge basis function associated with $e_1$ (edge 1), $W_1^1 = \lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1$, is derived below. Vectors $a$ and $b$ can be expressed as

$$a = -\left(\lambda_2^f - \lambda_2^s\right)e_3 = -\Delta_2 e_3, \quad (E.14)$$

$$b = -\left(\lambda_1^f - \lambda_1^s\right)e_2 = -\Delta_1 e_2. \quad (E.15)$$

First, $L$ and $dL$ are parameterized through $t$ such that

$$L = L(t) = (a + b)t = -(\Delta_2 e_3 + \Delta_1 e_2)t, \quad (E.16)$$

$$dL = \frac{dL}{dt} dt = (a + b) dt = -(\Delta_2 e_3 + \Delta_1 e_2) dt. \quad (E.17)$$

Next, the barycentric coordinates and their gradients are parameterized through $t$ such that

$$\lambda_1(t) = \left[\lambda_1^f - \lambda_1^s\right] t + \lambda_1^s = \Delta_1 t + \lambda_1^s, \quad (E.18)$$

$$\lambda_2(t) = \left[\lambda_2^f - \lambda_2^s\right] t + \lambda_2^s = \Delta_2 t + \lambda_2^s. \quad (E.19)$$

The gradients of the barycentric coordinates are constant, so they are not the function of $t$.

$$\nabla \lambda_1 = \frac{1}{e_3 h_1} \hat{z} \times e_3 = \frac{1}{2A} \hat{z} \times e_3, \quad (E.20)$$

$$\nabla \lambda_2 = \frac{1}{e_2 h_2} e_2 \times \hat{z} = \frac{1}{2A} e_2 \times \hat{z}, \quad (E.21)$$

where $A$ is the area of the triangle. Another representation of $A$ is

$$A = \frac{1}{2} \hat{z} \cdot (e_2 \times e_3), \quad (E.22)$$

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which will be used later. Some dot products used for the line integral are summarized below.

\[ \nabla \lambda_1 \cdot e_2 = \frac{1}{2A} (\hat{z} \times e_3) \cdot e_2 = \frac{1}{2A} \hat{z} \cdot (e_3 \times e_2) = -1, \]  
(\text{E.23a})

\[ \nabla \lambda_1 \cdot e_3 = 0, \]  
(\text{E.23b})

\[ \nabla \lambda_2 \cdot e_2 = 0, \]  
(\text{E.23c})

\[ \nabla \lambda_2 \cdot e_3 = \frac{1}{2A} (e_2 \times \hat{z}) \cdot e_3 = \frac{1}{2A} \hat{z} \cdot (e_3 \times e_2) = -1. \]  
(\text{E.23d})

Therefore, the line integral is modified to

\[
\int_L W_1 \cdot dL = \int_L (\lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1) \cdot dL \\
= \int_0^1 [(\Delta_1 t + \lambda_1^s) \nabla \lambda_2 - (\Delta_2 t + \lambda_2^s) \nabla \lambda_1] \cdot (-\Delta_2 e_3 - \Delta_1 e_2) \, dt \\
= -\Delta_2 (\nabla \lambda_2 \cdot e_3) \int_0^1 (\Delta_1 t + \lambda_1^s) \, dt + \Delta_1 (\nabla \lambda_1 \cdot e_2) \int_0^1 (\Delta_2 t + \lambda_2^s) \, dt \\
= \Delta_2 \left[ \frac{\Delta_1}{2} + \lambda_1^s \right] - \Delta_1 \left[ \frac{\Delta_2}{2} + \lambda_2^s \right] \\
= \Delta_2 \lambda_1^s - \Delta_1 \lambda_2^s \\
= (\lambda_2^f - \lambda_2^s) \lambda_1^s - (\lambda_1^f - \lambda_1^s) \lambda_2^s \\
= \lambda_1^s \lambda_2^f - \lambda_1^f \lambda_2^s. \]  
(\text{E.24})

Similarly, the other two line integrals can be obtained.

\[
\int_L W_2 \cdot dL = \lambda_1^s \lambda_3^f - \lambda_1^f \lambda_3^s, \]  
(\text{E.25})

\[
\int_L W_3 \cdot dL = \lambda_2^s \lambda_3^f - \lambda_2^f \lambda_3^s. \]  
(\text{E.26})
Appendix F

Trade-offs for Unconditional Stability in Finite-Element Time-Domain

In this Appendix, the basic trade-offs in unconditionally-stable time updates for the Finite-Element Time-Domain (FETD) method are discussed. Similar discussion can be found in [221]. This discussion is particularly focused on highly refined meshes because unconditionally stable schemes are conventionally considered to surpass conditionally stable schemes.

To begin with, let us consider the vector wave equation

\[ \nabla \times \mu^{-1} \nabla \times \mathbf{E} + \varepsilon \frac{\partial}{\partial t} \mathbf{E} = 0, \]

(F.1)

where \( \mathbf{E} \) is assumed to be expanded using Whitney edge basis functions such that [171]

\[ \mathbf{E} = \sum_{i=1}^{N_e} e_i \mathbf{W}^1_i, \]

(F.2)

where \( N_e \) is the total number of edges in the unstructured grid. With the aid of Galerkin’s testing, we obtain

\[ \mathbf{M} \frac{\partial}{\partial t} \mathbf{e} + \mathbf{S} \mathbf{e} = 0, \]

(F.3)

where the array of degrees of freedom \( \mathbf{e} \) is defined as

\[ \mathbf{e} = [e_1, e_2, \cdots, e_{N_e}]^T, \]

(F.4)
the mass $M$ and stiffness $S$ matrix elements are defined as [161]

\[
M_{ij} = \int_\Omega \epsilon W_i^1 \cdot W_j^1 \, dV, \quad (F.5)
\]

\[
S_{ij} = \int_\Omega \frac{1}{\mu} (\nabla \times W_i^1) \cdot (\nabla \times W_j^1) \, dV, \quad (F.6)
\]

and $\Omega$ represents the integration volume. By backward-differencing, (F.3) is approximated to

\[
(M + \Delta t^2 S) e^{n+1} = 2Me^n - Me^{n-1}, \quad (F.7)
\]

which is an unconditionally stable scheme [161]. As an alternative, by central-differencing, (F.3) is approximated to

\[
Me^{n+1} = (2M - \Delta t^2 S) e^n - Me^{n-1}, \quad (F.8)
\]

which is now a conditionally stable scheme [161] with the Courant-Friedrichs-Lewy (CFL) limit

\[
\Delta t \leq \Delta t_c = \frac{2}{\sqrt{\max[\zeta_X]}}, \quad (F.9)
\]

where $\zeta_X$ is the set of eigenvalues of $X = M^{-1}S$ [157, 222]. Furthermore, the Newmark scheme [166, 208] can be used for unconditionally stable schemes under the constraint of $\theta \geq 1/4$ as shown below.

\[
(M + \theta \Delta t^2 S) e^{n+1} = [2M - (1 - 2\theta)\Delta t^2 S] e^n - (M + \theta \Delta t^2 S)e^{n-1}, \quad (F.10)
\]

For other conditionally stable schemes, the mixed E-B FETD can be used, which is built from two coupled Maxwell’s curl equations, can be used [159, 161, 223, 224, 225]

\[
\frac{\partial}{\partial t} B = -\nabla \times E, \quad (F.11)
\]

\[
\frac{\partial}{\partial t} eE = \nabla \times \mu^{-1} B. \quad (F.12)
\]
The discrete representations of (F.11) and (F.12) are expressed as

\[ b^n = b^{n-1} - \Delta t \mathbf{C} e^{n-\frac{1}{2}}, \]  
(F.13)

\[ [\star_x] e^{n+\frac{1}{2}} = [\star_x] e^{n-\frac{1}{2}} + \Delta t \mathbf{C}^T [\star_{\mu^{-1}}] b^n, \]  
(F.14)

where the array of degrees of freedom \( b = [b_1, \cdots, b_{N_f}]^T \) represents magnetic flux density, which is expanded using Whitney face basis functions such that

\[ \mathbf{B} = \sum_{i=1}^{N_f} b_i \mathbf{W}_i^2, \]  
(F.15)

where \( N_f \) is the total number of faces in the grid. Note that the two conditionally stable schemes (F.8) and (F.13)–(F.14) are equivalent because \( \mathbf{S} = \mathbf{C}^T [\star_{\mu^{-1}}] \mathbf{C} \) with the same CFL limit [161].

Although the two types of the FETD schemes are similar that a linear solve is necessary at each time step, they differ in terms of the associated linear systems. The linear systems for the conditionally stable schemes are characterized by \( \mathbf{M}(= [\star_x]) \), whereas the linear systems for the unconditionally stable schemes are characterized by \( \mathbf{A} = \mathbf{M} + \theta \Delta t^2 \mathbf{S} \) with either \( \theta = 1 \) or \( \theta \geq 1/4 \), respectively. It should be stressed that both \( \mathbf{M} \) and \( \mathbf{A} \) are symmetric-sparse matrices but they possess distinct properties. \( \mathbf{M} \) encodes the local metric of nearby elements in the grid, so its inverse \( \mathbf{M}^{-1} \) can be well approximated by a sparse matrix because the corresponding inverse operator only affects in a short range in the continuum limit [161]. On the other hand, \( \mathbf{A}^{-1} \) inherently possesses the long-range interactions, i.e., Green’s functions, because \( \mathbf{A} \) represents the discrete version of the operator \( \nabla \times \mu^{-1} \nabla \times \) and affected by \( \mathbf{S} \). Therefore, \( \mathbf{A}^{-1} \) cannot be discretized by a sparse matrix anymore.

To visualize the difference, the sparsity patterns of \( \mathbf{M}^{-1} \) and \( \mathbf{A}^{-1} \) are illustrated in Figure F.1 with a small mesh with 1,226 DoFs. For proper visualization, any
Figure F.1: Sparsity patterns of (a) $M^{-1}$ and (b) $A^{-1}$. Here, it is assumed $A = M + (10\Delta t_c)^2S$ and $r=0.001$.

normalized elements by their respective maximum values are set to be zero when they are smaller than $r=0.001$ threshold.

Another difference between the two linear systems is the behavior of their condition numbers $\kappa$. It is well known that the cost of a linear solve is heavily affected by condition numbers $\kappa$ of the associated system matrix. Figure F.2 shows $\kappa(M)$ and $\kappa(S_{\text{rel}})$ for various refinement levels. For each refinement level, the lowest possible condition numbers that are usually obtained from uniformly triangulated grids are compared. The regularized stiffness matrix $S_{\text{rel}}$ is obtained by excluding zero eigenvalues, which are responsible for the static fields, $E = -\nabla \phi$. As can be seen in Figure F.2, it is obvious that $\kappa(M)$ is almost constant irrespective of mesh refinement levels but $\kappa(S_{\text{rel}})$ increases without bound.

In Figure F.3, various $\kappa(A)$ values for different mesh refinement levels and CFL numbers (CFLN), $c_f = \Delta t/\Delta t_c$, are shown. $c_f$ rather than the mesh refinement
level plays an important role in $\kappa(A)$. The increasing rate of $\kappa(A)$ for $c_f > 1$ is proportional to $c_f^2$.

This interpretation can be also provided from the analysis in the Fourier domain, where (F.3) can be transformed into

$$ Se(\omega) = \omega^2 Me(\omega). $$

Note that (F.16) is a generalized eigenvalue problem. It should be pointed out that the eigenvalues $\omega^2 = \omega_r^2$, $r = 1, \cdots, N_e$, of $S$ relative to $M$ in (F.16) are related to resonant frequencies of the cavity, which is the unstructured grid. Also, the associated eigenvectors $e(\omega_r) = e_r$ to the eigenvalues are the eigenmodes of the cavity. The lowest eigenvalue corresponds to the lowest-order eigenmode, which behaves as $\min_r [\omega_r] \sim v_p/D$, where $D$ is the biggest dimension of the cavity and $v_p$ is the phase velocity of electromagnetic waves. Moreover, the highest eigenvalue corresponds to the highest-order eigenmode that can be captured by the mesh refinement level, which behaves.
as $\max_r[\omega_r] \sim v_p/d$, where $d$ is the typical edge length in the grid. With refined grids, the ratio $D/d$ increases, which leads to

$$\kappa(M^{-1}S) = \frac{\max_r[\omega_r^2]}{\min_r[\omega_r^2]} \sim \left(\frac{D}{d}\right)^2.$$  \hspace{1cm} (F.17)

Similarly, combining $A = M + \Delta t^2S$ and (F.16) gives

$$Ae_r = \lambda_r Me_r,$$  \hspace{1cm} (F.18)

which is another generalized eigenvalue problem. In (F.18), $\lambda_r = 1 + \omega_r^2\Delta t^2$. Using (F.9) and (F.16), we have

$$\max_r[\omega_r] = \sqrt{\max_r[\zeta_X]} = \frac{2}{\Delta t_c}.$$  \hspace{1cm} (F.19)

Therefore, $\max_r[\lambda_r] = 1 + 4c_f^2$. Furthermore, since $\min_r[\omega_r] = 0$,

$$\kappa(M^{-1}A) = \frac{\max_r[\lambda_r]}{\min_r[\lambda_r]} = 1 + 4c_f^2.$$  \hspace{1cm} (F.20)
Note that the computational cost of the linear solve in (F.7) is influenced by $\kappa(M^{-1}A)$ instead of $\kappa(A)$ because $(M^{-1}A)e^{n+1} = 2e^n - e^{n-1}$ by rearranging (F.7). It can be concluded that both $\kappa(M^{-1}A)$ and $\kappa(A)$ are the functions of $c_f^2$ when $c_f > 1$. Therefore, when the grid is refined, the cost associated with unconditionally stable schemes due to the linear solve increases without bound. This is certainly not observed in conditionally stable schemes.
References


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