Robust Numerical Electromagnetic Eigenfunction Expansion Algorithms

Dissertation

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Abstract

This thesis summarizes developments in rigorous, full-wave, numerical spectral-domain (integral plane wave eigenfunction expansion [PWE]) evaluation algorithms concerning time-harmonic electromagnetic (EM) fields radiated by generally-oriented and positioned sources within planar and tilted-planar layered media exhibiting general anisotropy, thickness, layer number, and loss characteristics. The work is motivated by the need to accurately and rapidly model EM fields radiated by subsurface geophysical exploration sensors probing layered, conductive media, where complex geophysical and man-made processes can lead to micro-laminate and micro-fractured geophysical formations exhibiting, at the lower (sub-2MHz) frequencies typically employed for deep EM wave penetration through conductive geophysical media, bulk-scale anisotropic (i.e., directional) electrical conductivity characteristics. When the planar-layered approximation (layers of piecewise-constant material variation and transversely-infinite spatial extent) is locally, near the sensor region, considered valid, numerical spectral-domain algorithms are suitable due to their strong low-frequency stability characteristic, and ability to numerically predict time-harmonic EM field propagation in media with response characterized by arbitrarily lossy and (diagonalizable) dense, anisotropic tensors. If certain practical limitations are addressed, PWE can robustly model sensors with general position and orientation that probe generally numerous, anisotropic, lossy, and thick layers.
The main thesis contributions, leading to a sensor and geophysical environment-robust numerical modeling algorithm, are as follows: (1) Simple, rapid estimator of the region (within the complex plane) containing poles, branch points, and branch cuts (“critical points”) (Chapter 2), (2) Sensor and material-adaptive azimuthal coordinate rotation, integration contour deformation, integration domain sub-region partition and sub-region-dependent integration order (Chapter 3), (3) Integration partition-extrapolation-based (Chapter 3) and Gauss-Laguerre Quadrature (GLQ)-based (Chapter 4) evaluations of the deformed, semi-infinite-length integration contour “tails”, (4) Robust “in-situ”-based (i.e., at the spectral-domain integrand level) “direct”/homogeneous-medium field contribution subtraction and analytical curbing of the source current spatial spectrum function’s ill behavior (Chapter 5), and (5) Analytical re-casting of the direct-field expressions when the source is embedded within a NBAM, short for non-birefringent anisotropic medium (Chapter 6). The benefits of these contributions are, respectively, (1) Avoiding computationally intensive critical-point location and tracking (computation time savings), (2) Sensor and material-robust curbing of the integrand’s oscillatory and slow decay behavior, as well as preventing undesirable critical-point migration within the complex plane (computation speed, precision, and instability-avoidance benefits), (3) sensor and material-robust reduction (or, for GLQ, elimination) of integral truncation error, (4) robustly stable modeling of scattered fields and/or fields radiated from current sources modeled as spatially distributed (10 to 1000-fold compute-speed acceleration also realized for distributed-source computations), and (5) numerically stable modeling of fields radiated from sources within NBAM layers.
Having addressed these limitations, are PWE algorithms applicable to modeling EM waves in tilted planar-layered geometries too? This question is explored in Chapter 7 using a Transformation Optics-based approach, allowing one to model wave propagation through layered media that (in the sensor’s vicinity) possess tilted planar interfaces. The technique leads to spurious wave scattering however, whose induced computation accuracy degradation requires analysis. Mathematical exhibition, and exhaustive simulation-based study and analysis of the limitations of, this novel tilted-layer modeling formulation is Chapter 7’s main contribution.
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Second, I thank my high school physics teacher Mr. Donald Crowell. His enthusiasm for the fascinating subject of electricity and magnetism led me to pursue undergraduate, and then graduate, education in electromagnetics (EM).

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Finally, copyright acknowledgments: The content in chapters 2, 3, 4, 5, 6, and 7 has largely been copied (with minor modifications) from, respectively, the following references published by me: References [1], [2], [3], [4], [5], and [6]. Chapter one's content is largely taken from [6] too. Copyright of these six cited documents remains with the respective copyright holders.
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K. Sainath and F. L. Teixeira “Spectral-Domain Computation of Fields Radiated by
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K. Sainath and F. L. Teixeira “Interface-Flattening Transform for EM Field Model-
ing in Tilted, Cylindrically Stratified Geophysical Media”, *IEEE Antennas Wireless


**Fields of Study**

Major Field: Electrical and Computer Engineering
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A.1 $\epsilon_{rm}$, $<n_m^2>$, and $L_{em}$ are the average dielectric constant, refractive index variance ($n_m = n_m' + in_m''$), and refractive index correlation length characterizing layer $m$’s dielectric fluctuation statistics ($m = 1, 2, ..., N$). $\sigma_{hm}^2$ and $L_m$ are the height roughness variance and correlation length (resp.) characterizing the statistics of layer $m$’s bottom interface at depth $-d_m$. $r_p$ is the range from antenna $p$ to some arbitrarily chosen SAR image pixel’s reference location $r_0 = (0, 0, 0)$. Note: Our model allows flexible prescription of statistics (Gaussian, exponential, etc.) for each volume and rough surface. Figure/caption copied from [19]. ........................................... 238

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Chapter 1: Introduction

Long-standing and sustained interest has been directed towards the numerical evaluation of electromagnetic (EM) fields produced by sensors embedded in complex, layered-medium environments [20]. In particular, within the context of geophysical exploration (of hydrocarbon reserves, for example), there exists great interest to computationally model the response of sub-2MHz induction tools that can remotely sense the electrical and structural properties of complex geological formations (and consequently, their hydrocarbon productivity) [9, 21]. Indeed, high-fidelity, rapid, and geometry-robust computational forward-modeling aids fundamental understanding of how factors such as the formation’s global inhomogeneity structure, conductive anisotropy in formation bed layers, induction tool geometry, exploration borehole geometry, and drilling fluid type (among other factors) affect the sensor’s responses. This knowledge informs both effective and robust geophysical parameter retrieval algorithms (inverse problem), as well as sound data interpretation techniques [11, 9]. Developing forward-modeling algorithms which not only deliver rapid, accuracy-controllable results, but also simulate the effects of a greater number of dominant, geophysical features without markedly increased computational burden, represents a high priority in subsurface geophysical exploration and motivated our
work in Chapters 2-6. Moreover, the need to efficiently simulate the effects, on subsurface sensor responses, due to a greater range of dominant geophysical features motivated our preliminary explorations of modeling relative tilt between cylindrical layers [22], and subsequently modeling relative tilt between planar layers (c.f. Chapter 7).

In the interest of obtaining a good trade-off between the forward modeler’s solution speed while still satisfactorily modeling the EM behavior of the environment’s dominant geophysical features, a layered-medium approximation of the geophysical formation often proves very useful. Indeed cylindrical layering, planar layering, and a combination of the two (for example, to model the cylindrical exploratory borehole and invasion zone embedded within a stack of planar formation beds) are arguably three of the most widely used layering approximations in subsurface geophysics [13, 8, 9, 23, 7, 11, 24, 25, 14, 26, 27, 21, 28, 29, 30, 31, 32, 33, 34], for both onshore and offshore geophysical exploration modeling [35, 36, 37, 38, 39, 40, 41, 42, 43, 44]. The prevalence of layered-medium approximations stems in large part, at least from a computational modeling standpoint, due to the typical availability of closed-form eigenfunction expansions to compute the EM field [45][Ch. 2-3]. These full-wave techniques are quite attractive since they can robustly deliver rapid solutions with high, user-controlled accuracy under widely varying conditions with respect to anisotropy and loss in the formation’s layers, orientation and position of the electric or (equivalent) magnetic current-based sensors (viz., electric loop antennas), and source frequency. The robustness to physical parameters is highly desirable in geophysics applications since geological structures are known to exhibit a wide range of inhomogeneity profiles with respect to conductivity, anisotropy, and geometrical layering [25, 9, 22].
For example, with respect to formation conductivity properties, diverse geological structures can embody macro-scale conductive anisotropy in the induction frequency regime, such as (possibly deviated) sand-shale micro-laminate deposits, clean-sand micro-laminate deposits, and either natural or drilling-induced fractures. The electrical conduction current transport characteristics of such structures indeed are often mathematically described by a uniaxial or biaxial conductivity tensor exhibiting directional electrical conductivities whose value range can span in excess of four orders of magnitude [26, 9, 13]. Documenting our efforts to address robustly accurate and efficient computational modeling needs in subsurface geophysics, Chapters 2-6 document novel, developed numerical techniques and algorithms to accurately and rapidly compute geophysical exploration sensor responses (both in the marine and subsurface borehole contexts) in a manner robust to the type, orientation, and anisotropy ratio of each (parallel) planar layer’s electrical conductivity properties in addition to its thickness, as well as the number of layers, (non-zero) sensor frequency, transmitter current distribution, and sensor orientation (i.e., even horizontal sensors, whose response computation using ordinary methods would often lead to oscillatory- or monotone-divergent integrals).

When employing planar and cylindrical layer approximations one almost always assumes that the interfaces are parallel, i.e. exhibit common central axes (say, along \( z \)) in the case of cylindrical layers [33, 34], or interfaces that are all parallel to a common plane in the case of planar layers [25, 21]. However, it may be more appropriate in many cases to admit layered media with material property variation along the
direction(s) conventionally presumed homogeneous. For example, in cylindrically-layered medium problems involving deviated drilling, gravitational effects may induce a downward diffusion of the drilling fluid that leads to a cylindrical invasion zone angled relative to the cylindrical exploratory borehole [22]. Similarly, formations that locally (i.e., in the proximity of the EM sensor) appear as a “stack” of beds with tilted (sloped) planar interfaces can appear (for example) due to temporal discontinuities in the formation’s geological record. These temporal discontinuities in turn can manifest as commensurately abrupt spatial discontinuities, known as unconformities (especially, angular unconformities) [46, 47, 48]. Indeed, the effects of unconformities and other complex formation properties (such as fractures) have garnered increasing attention over the past ten years [49, 50, 51], particularly in light of the relatively recent availability of induction sensor systems offering a rich diversity of simultaneous measurement information with respect to radiation frequency, transmitter and receiver orientation (“directional” diversity), and transmitter/receiver separation [52, 53, 54, 55, 51]. Subsequently turning our attention, then, to tilted planar-layered media, in Chapter 7 we propose a pseudo-analytical method based on EM plane wave eigenfunction expansions that manifest mathematically as two-dimensional (2-D) Fourier integrals. This is in contrast to faster, but more restrictive (with respect to allowed media) 1-D Fourier-Bessel (“Sommerfeld”) and Fourier-Hankel integral transforms that express EM fields in planar-layered media as integral expansions of EM conical wave eigenfunctions [45][Ch. 2]. Our choice rests upon robust error control capabilities and speed performance of the 2-D integral transform with respect to source radiation frequency, source distribution, and
material properties, as documented in Chapters 2-6. The use of eigenfunction expansions for modeling EM behavior of non-parallel layers is enabled here by the use of Transformation Optics (T.O.) techniques \cite{16, 56, 57, 17, 58} to effectively replace the original problem (with tilted interfaces) by an equivalent problem with strictly parallel interfaces, where additional “interface-flattening” layers with anisotropic response are inserted into the geometry to mimic the effect of the original tilted geometry. We remark that the 2-D Fourier integral is capable of modeling propagation and scattering behavior of these interface-flattening media, which possess azimuthal non-symmetric material tensors, while the 1-D integral transforms (restricted to modeling azimuthal-symmetric media) lack this capability.

Throughout this work we adopt the $\exp(-i\omega t)$ convention, as well as assume all EM media are spatially non-dispersive, time-invariant, and are representable by diagonalizable anisotropic $3\times3$ material tensors. Note that diagonalizability of the material tensors, which physically corresponds to a medium having a well-defined response for any direction of applied electric and magnetic field, is required for completeness of the plane wave basis. All naturally-occurring media, as well as the interface-flattening slabs introduced in Chapter 7 (and, more broadly, NBAM media in general \cite[c.f. Ch. 6]{6}), are characterized by diagonalizable material tensors. Important Note: The conventions, abbreviations, and notation within each of the following chapters are self-contained.
Chapter 2: Robust Computation of Dipole Electromagnetic Fields in Arbitrarily Anisotropic, Planar-Stratified Environments

2.1 Introduction

2.1.1 Chapter Summary and Contributions

We develop a general-purpose formulation, based on two-dimensional spectral integrals, for computing electromagnetic fields produced by arbitrarily-oriented dipoles in planar-stratified environments, where each layer may exhibit arbitrary and independent anisotropy in both the (complex) permittivity and permeability. Among the salient features of our formulation are (i) computation of eigenmodes (characteristic plane waves) supported in arbitrarily anisotropic media in a numerically robust fashion, (ii) implementation of an $hp$-adaptive refinement for the numerical integration to evaluate the radiation and weakly-evanescent spectra contributions, and (iii) development of an adaptive extension of an integral convergence acceleration technique to compute the strongly-evanescent spectrum contribution. While other semi-analytic techniques exist to solve this problem, none have full applicability to media exhibiting arbitrary double anisotropies in each layer, where one must account for the whole range of possible phenomena such as mode coupling at interfaces and non-reciprocal
mode propagation. Brute-force numerical methods can tackle this problem but only at a much higher computational cost. The present formulation provides an efficient and robust technique for field computation in arbitrary planar-stratified environments. We demonstrate the formulation for a number of problems related to geophysical exploration.\footnote{NOTE: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.}

### 2.1.2 Background

The study of electromagnetic fields produced by dipole sources in planar-stratified environments with anisotropic layers is pertinent to many applications such as geophysical prospection \cite{8, 9, 7, 11, 24, 14, 59}, microwave remote sensing \cite{60}, ground-penetrating radar \cite{61, 62}, optical field focusing \cite{63}, antenna design \cite{64, 65}, microwave circuits \cite{66}, and plasma physics \cite{67}. For this problem class, one can exploit the planar symmetry and employ pseudo-analytical approaches based upon embedding spectral Green’s Function kernels within Fourier-type integrals to compute the space-domain fields \cite{20, 68, 69}. A crucial aspect then becomes how to efficiently compute such integrals \cite{70, 71, 12, 72, 73}. Based on the specific characteristics of the planar-stratified environment(s) considered, efficient, case-specific methods arise. For example, when one assumes isotropic layers so that no TE\textsubscript{z}/TM\textsubscript{z} mode-coupling occurs at the planar interfaces, the original vector problem can be reduced to a set of scalar problems whose mixed domain Green’s functions (i.e. those functions having \((k_x, k_y, z)\) dependence) are either the primary kernels in integral representations of the Green’s dyads (e.g. “transmission-line”-type Green’s functions \cite{70, 68, 69}) or the
field components themselves (e.g. free-space Green’s function [45][Ch. 2]). Alternatively, when each layer exhibits azimuthal symmetry in its material properties, one can transform two-dimensional, infinite-range Fourier integrals into one-dimensional, semi-infinite range Sommerfeld integrals [71, 68, 12, 69, 20, 72]. For layers with arbitrary anisotropy, however, neither of the above simplifications apply, and a more general formulation is required.

Irrespective of the integral representation used, the following challenges exist concerning their numerical evaluation [70] [45][Ch. 2]: (1) The presence of branch-points/branch-cuts associated with semi-infinite and infinite-thickness layers, (2) the presence of poles associated with slab- and interface-guided modes, and (3) an oscillatory integrand that demands adequate sampling and whose exponential decay rate reduces with decreasing source-observer depth separation [12]. Among the approaches to address these issues one can cite (1) direct numerical evaluation, possibly combined with integral acceleration techniques [74, 70, 71, 12, 75, 72], (2) asymptotic approximation of the space-domain field [45][Ch. 2], and (3) approximation of the mixed-domain integrand via a sum of analytically invertible “images” [70, 73, 76]. While image-approximation and asymptotic methods exhibit faster solution time, they are fundamentally approximate methods that either (resp.) (1) require user intervention in performing a-priori “fine-tuning”, have medium-dependent applicability, and lack tight error-control [70, 72], or (2) have a limited range of applicability in terms of admitted medium classes and source/observer locations [45][Ch. 2].

Since our focus is on the general applicability and robustness of the algorithm (and not on the optimality for a specific class of layer arrangements, medium parameters,
and source-observer geometries), we adopt a direct numerical integration methodology based on 2-D, infinite-range Fourier-type integrals. Some key ingredients of the present formulation are:

1. A numerically-balanced recasting of the state matrix [45][Ch. 2] to enable the accurate computation of the eigenmodes supported in media exhibiting arbitrary anisotropy (e.g. isotropic, uniaxial, biaxial, gyrotropic).

2. Closed-form eigenmode formulations for isotropic and reciprocal, electrically uniaxial media that significantly reduce eigenmode solution time (versus the state matrix method), obviate numerical overflow, and yield higher-precision results versus prior (canonical) formulations in [77][Ch. 7] [45][Ch. 2].

3. A numerically stable method to decompose degenerate modes produced by sources in isotropic layers.

4. A multi-level, error-controlled, adaptive $hp$ refinement procedure to evaluate the radiation/weakly evanescent spectral field contributions, employing nested Kronrod-Gauss quadrature rules to reduce computation time.

5. Adaptive extension of the original Method of Weighted Averages (MWA) [71, 75] and its application to accelerating the numerical evaluation of infinite-range, 2-D Fourier-type integrals concerning environments containing media with arbitrary anisotropy and loss.

Section 2.2 overviews the formulation. Section 2.3 contains an analytical derivation of the mixed-domain, vector-valued integrand$^2$ $W_L(k_x, k_y; z)$ of the 2-D Fourier integral.

$^2$Vector, matrix, and tensor quantities have boldface script. Furthermore, field quantities with $(k_x, k_y, z)$ dependence are denoted mixed-domain quantities.
Section 2.4 exhibits an efficient numerical algorithm to compute the (inner) $k_x$ integral in Eq. (2.2.13) (note that this discussion applies, in dual fashion, to the $k_y$ integral).

### 2.2 Formulation Overview

![Diagram](image)

Figure 2.1: Layer $M$ contains the source point $r' = (x', y', z')$ and layer $L$ contains the observation point $r = (x, y, z)$. The dipole source $\mathcal{L}$ can be either electric or magnetic.

Our problem concerns computing the electromagnetic field at $r = (x, y, z)^3$ produced by an elementary/Herztian dipole source which radiates at frequency $\omega$ within a planar-stratified, anisotropic environment at location $r' = (x', y', z')$. We assume $N$ layers stratified along the $z$ axis as depicted in Figure 2.1, each with (complex-valued)

$^3$Note: $z$ can refer to the observation depth or the coordinate, depending on context.
3 × 3 material tensors\(^4\) \(\bar{\epsilon}_c\) and \(\bar{\mu}_c\) exhibiting independent and arbitrary anisotropy\(^5\), that is\(^6\)

\[
\bar{\epsilon}_c = \epsilon_0 \bar{\epsilon}_r = \epsilon_0 \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix}, \quad \bar{\mu}_c = \mu_0 \bar{\mu}_r = \mu_0 \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix}
\] (2.2.1)

being simultaneous full, complex-valued tensors that can be different for each layer.

With this in mind, Maxwell’s equations in a homogeneous region with impressed electric and (equivalent) magnetic current densities\(^7\) \(\mathcal{J}\) and \(\mathcal{M}\) (resp.), as well as impressed volumetric electric and (equivalent) magnetic charge densities \(\rho_v\) and \(\rho_m\) (resp.), write as\(^8\)

\[
\nabla \times \mathbf{E} = i\omega \bar{\mu}_c \cdot \mathbf{H} - \mathcal{M}
\] (2.2.2)
\[
\nabla \times \mathbf{H} = \mathcal{J} - i\omega \bar{\epsilon}_c \cdot \mathbf{E}
\] (2.2.3)
\[
\nabla \cdot (\bar{\epsilon}_c \cdot \mathbf{E}) = \rho_v
\] (2.2.4)
\[
\nabla \cdot (\bar{\mu}_c \cdot \mathbf{H}) = \rho_m
\] (2.2.5)

After multiplying Eq. (2.2.2) by \(\nabla \times \bar{\mu}_c^{-1}\cdot\mathbf{E}\) and using Eq. (2.2.3), one has\([45][Ch. 1]\):

\[
[\nabla \times (\bar{\mu}_c^{-1} \cdot \nabla \times) - \omega^2 \bar{\epsilon}_c] \mathbf{E} = i\omega \mathcal{J} - \nabla \times \bar{\mu}_c^{-1} \cdot \mathcal{M}
\] (2.2.6)

Alternatively, defining the tensor-valued vector wave operator as

\[
\bar{\mathbf{A}} = \nabla \times \bar{\mu}_r^{-1} \cdot \nabla \times -k_0^2 \bar{\epsilon}_r.
\] (2.2.7)

\(^4\)Matrix and tensor quantities are denoted by an over-bar.

\(^5\)We assume the material tensors to be diagonalizable, as this facilitates using plane wave fields as a basis to synthesize the field solution. Since all naturally occurring media possess diagonalizable material tensors, this constraint is not a practical concern and thus warrants no further discussion.

\(^6\)\(c_0\) (m/s) is the speed of light in free space, \(\mu_0\) (H/m) is the free space magnetic permeability, and \(\epsilon_0 = \frac{1}{\mu_0 c_0^2}\) (F/m) is the free space electric permittivity.

\(^7\)Field quantities exhibiting purely spatial dependence have calligraphic script and are denoted spatial quantities.

\(^8\)\(i\) is the unit-magnitude imaginary number, \(\omega = 2\pi f\) (rad/sec) is the angular frequency at which the source radiates, and the time convention \(\exp(-i\omega t)\) is assumed and suppressed.
one can re-express Eq. (2.2.6) as

\[
\vec{\mathcal{A}} \cdot \mathcal{E} = i k_o \eta_o \mathcal{J} - \nabla \times \vec{\mu}_r^{-1} \cdot \mathcal{M}
\]  

(2.2.8)

where \( k_o = \omega \sqrt{\varepsilon_o \mu_o} \) (m\(^{-1}\)) and \( \eta_o = \sqrt{\mu_o / \varepsilon_o} \) (Ω) are the wave number and wave impedance of free space (resp.). Now, define a three-dimensional Fourier Transform (FT) pair as\(^9\):

\[
\tilde{\mathcal{E}}(k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{E}(r) e^{-ik \cdot r} \, dx \, dy \, dz \quad (2.2.9)
\]

\[
\mathcal{E}(r) = \left( \frac{1}{2\pi} \right)^3 \int_{-\infty}^{+\infty} \tilde{\mathcal{E}}(k) e^{ik \cdot r} \, dk_x \, dk_y \, dk_z \quad (2.2.10)
\]

with \( r = (x, y, z) \) and \( k = (k_x, k_y, k_z) \), and similarly for all other field and source quantities. Now, assuming an electric or magnetic dipole source (resp.), one has\(^10\)

\[
\mathcal{J} = \hat{a} J_o \delta(r - r') \quad \text{or} \quad \mathcal{M} = \hat{a} M_o \delta(r - r')
\]

in the space domain and \( \tilde{\mathcal{J}} = \hat{a} J_o \) or \( \tilde{\mathcal{M}} = \hat{a} M_o \) in the Fourier domain. To determine the spectral-domain fields, we first write the inverse of \( \tilde{\mathcal{A}} \) as \( \text{inv}(\tilde{\mathcal{A}}) = \text{adj}(\tilde{\mathcal{A}}) / \text{det}(\tilde{\mathcal{A}}) \), where \( \text{adj}(\tilde{\mathcal{A}}) \) is the adjugate matrix (not the conjugate-transpose matrix) [78]. The determinant \( \text{det}(\tilde{\mathcal{A}}) = g_o(k_z - \tilde{k}_{1z})(k_z - \tilde{k}_{2z})(k_z - \tilde{k}_{3z})(k_z - \tilde{k}_{4z}) \), where \( g_o = \varepsilon_{zz} k_o^2 (\tau_{xy} \tau_{yx} - \tau_{xx} \tau_{yy}) \), is a fourth-order polynomial in \( k_z \)\(^11\). Next, define the spectral Green’s dyad operators \( \tilde{\mathcal{G}}_{ee}(k; r') = e^{-ik' \cdot r'} \text{inv} \left( \tilde{\mathcal{A}} \right) \) and \( \tilde{\mathcal{G}}_{em}(k; r') = e^{-ik' \cdot r'} \text{inv} \left( \tilde{\mathcal{A}} \right) \cdot \nabla \times \) that (resp.) map electric and magnetic sources to the spectral electric field as follows: \( \tilde{E}(k) = i k_o \eta_o \tilde{\mathcal{G}}_{ee} \cdot \tilde{J} \) and \( \tilde{E}(k) = -\tilde{\mathcal{G}}_{em} \cdot \vec{\mu}_r^{-1} \cdot \mathcal{M} \).

In a homogeneous medium, the integral along \( k_z \) in Eq. (2.2.10) can be performed analytically using the Residue Theorem. The vector-valued residues are the four

\(^9\)Field quantities (besides \( k \)) exhibiting purely spectral dependence have an over-tilde and are denoted spectral quantities. Furthermore, modal (non-modal) spectral field quantities appear in lower (upper) case.

\(^10\)\( \delta(r - r') = \delta(x - x') \delta(y - y') \delta(z - z') \) is the three-dimensional Dirac delta function.

\(^11\)\( \vec{\tau}_r = \vec{\mu}_r^{-1} \)
supported eigenmode electric fields having propagation constants corresponding to the four roots of \( \det(\tilde{\mathbf{A}}) \), in terms of which we have the following generic expression for the space-domain (direct) electric field \( \mathbf{E}_d(r) \)\(^{12} \):

\[
\mathbf{E}_d(r) = \frac{i}{(2\pi)^2} \int \int \left\{ u(z - z') \sum_{n=1}^{2} \tilde{a}_n \tilde{e}_n e^{i\tilde{k}_{nz}(z-z')} + u(z' - z) \sum_{n=3}^{4} \tilde{a}_n \tilde{e}_n e^{i\tilde{k}_{nz}(z-z')} \right\} \times e^{ik_x(x-x') + ik_y(y-y')} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi}
\]

where the \( \{\tilde{e}_n(k_x, k_y)\} \) are eigenmode electric field vectors and the \( \{\tilde{a}_n(k_x, k_y)\} \) are (source dependent) modal amplitudes associated with the four eigenvalues (i.e. poles of \( \text{inv}(\tilde{A}) \)) \( \{\tilde{k}_{nz}\} \). In the multi-layer case, with \( r' \) in layer \( M \) and \( r \) in layer \( L \), a scattered field contribution \( \mathbf{E}_L^s(r) \) is added to \( \mathbf{E}_d(r) \) so that the total electric field in layer \( L \) writes as

\[
\mathbf{E}_L(r) = \delta_{LM} \mathbf{E}_d(r) + \mathbf{E}_L^s(r),
\]

where

\[
\mathbf{E}_L^s(r) = \frac{i}{(2\pi)^2} \int \int \left\{ (1 - \delta_{LN}) \sum_{n=1}^{2} \tilde{a}_{L,n} \tilde{e}_{L,n} e^{i\tilde{k}_{L,nz}z} + (1 - \delta_{L1}) \sum_{n=3}^{4} \tilde{a}_{L,n} \tilde{e}_{L,n} e^{i\tilde{k}_{L,nz}z} \right\} \times e^{ik_x(x-x') + ik_y(y-y')} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi}
\]

an additional subscript is introduced to denote the layer number (e.g. \( L \) in this case), \( \delta_{pq} \) denotes the Kronecker delta, and the \( \{\tilde{a}_{L,n}(k_x, k_y)\} \) represent the (source-dependent) scattered-field modal amplitudes. The four modal terms inside both the direct and scattered field integrals above can be classified into two upward and two downward propagation modes, distinguished according to the signs of \( \{\text{Im}(\tilde{k}_{L,nz})\} \)\(^{13} \).

To expedite propagating the source fields to \( r \), which requires enforcing continuity of the tangential EM field components throughout the environment, instead of

\(^{12}\) \( u(\cdot) \) represents the Heaviside unit step function.

\(^{13}\) The eigenvalues \( \{\tilde{k}_{L,1z}, \tilde{k}_{L,2z}, \tilde{k}_{L,3z}, \tilde{k}_{L,4z}\} \) correspond to the propagation constants of the (resp.) Type I up-going, Type II up-going, Type I down-going, and Type II down-going plane wave modes of layer \( L \), and so on for the other \( N - 1 \) layers [45][Ch. 2].
working with Eq. (2.2.12) directly it is more convenient to work with a $4 \times 1$ vector composed of the four tangential EM field components (see [45][Ch. 2]): $\mathbf{V} = [\mathcal{E}_x \mathcal{E}_y \mathcal{H}_x \mathcal{H}_y]$. The two longitudinal field components can be subsequently obtained from the transverse components [45][Ch. 2]. An equation analogous to Eq. (2.2.12) thus arises, with $\mathcal{E}_L$ replaced by $\mathbf{V}_L$, which writes as

$$
\mathbf{V}_L(\mathbf{r}) = \frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \mathbf{W}_L(k_x, k_y; z) e^{ik_x(x-x')+ik_y(y-y')} \, dk_x \, dk_y \quad (2.2.13)
$$

### 2.3 Integrand Manipulations

For some $(k_x, k_y)$ that defines the transverse phase variation $\exp[ik_x(x-x') + ik_y(y-y')]$ common to all the plane wave modes within the environment, one desires the total modal contribution $\mathbf{W}_L(k_x, k_y; z) \exp[ik_x(x-x') + ik_y(y-y')]$ at $\mathbf{r}$. Assuming this transverse phase variation $\exp[ik_x(x-x') + ik_y(y-y')]$, Maxwell’s equations for a homogeneous medium can be manipulated [45][Ch. 2] to yield the state matrix shown in Eq. (2.3.2). After substituting in a given layer’s constitutive properties, its solution yields the four eigenmodes supported in that layer along with the corresponding modal (axial) propagation constants; this process, repeated for all $N$ layers, is the starting point of procuring $\mathbf{W}_L(k_x, k_y; z)^{14}$. Subsequently, knowledge of the transverse modal fields in each layer combined with enforcement of tangential field continuity across layer interfaces allows one to propagate the radiated fields to $\mathbf{r}$ in layer $L$. Note that given the transverse EM fields of the $n$th mode, the complete six-component, $z$-independent modal field vector $\{\tilde{e}_n, \tilde{h}_n\}$ is completely determined [45][Ch. 2].

---

14 The form of Eq. (2.3.2) differs slightly from formula (2.10.10) in [45][Ch. 2]. The $-i$ factor on both sides of Eq. (2.3.2), which is embedded into $\tilde{\mathbf{H}}$ on the left side and explicitly shown on the right side, facilitates an eigenvalue/eigenvector problem in which the propagation constants $\{k_{m,nz}\}$ are the sought-after values rather than the $\{ik_{m,nz}\}$ values procured in [45][Ch. 2].
2.3.1 Modal Eigenvectors and Eigenvalues

The characteristic plane wave modes for an arbitrarily anisotropic layer $m$ are summarily described by the four eigenvalues ($\tilde{k}_{m,1z}$, $\tilde{k}_{m,2z}$, $\tilde{k}_{m,3z}$, $\tilde{k}_{m,4z}$) and the four corresponding $4 \times 1$ eigenvectors $[\tilde{s}_{m,1} \quad \tilde{s}_{m,2} \quad \tilde{s}_{m,3} \quad \tilde{s}_{m,4}]$ of the $4 \times 4$ state matrix $\tilde{H} = \tilde{H}(k_x, k_y)$. Defining the $n$th eigenvector as

$$\tilde{s}_{m,n} = \tilde{s}_{m,n}(k_x, k_y) = \begin{bmatrix} \tilde{e}_{m,nx} \\ \tilde{e}_{m,ny} \\ \tilde{h}_{m,nx} \\ \tilde{h}_{m,ny} \end{bmatrix}$$

and noting that the corresponding $n$th characteristic solution $v_{m,n}$ to

$$\tilde{H} \cdot v_{m,n} = -i \frac{\partial}{\partial z} v_{m,n}$$

has the form $v_{m,n} = \tilde{s}_{m,n} e^{ik_{m,nz}(z - z^*)}$, one can show that the eigenvector/eigenvalue problem $\tilde{H} \cdot \tilde{s}_{m,n} = \tilde{k}_{m,nz} \tilde{s}_{m,n}$ results.

To facilitate accurate and rapid numerical eigenmode computation, the following relations comprise analytical changes made to the canonical eigenmode formulations for isotropic media [45][Ch. 2], reciprocal, electrically uniaxial media [77][Ch. 7], and generally anisotropic media (i.e. via the state matrix $\tilde{H}$) [45][Ch. 2]:

$$k_x \rightarrow k_o(k_x/k_o), \quad k_y \rightarrow k_o(k_y/k_o) = k_o k_{yr}, \quad \omega \mu_o \rightarrow k_o \eta_o, \quad \text{and} \quad \omega \epsilon_o \rightarrow k_o/\eta_o$$

(2.3.3)

Accurate computation of the eigenvectors and eigenvalues is of paramount importance to achieving high-precision results. This is because, as will be seen throughout this section, every mixed-domain field quantity is dependent upon the eigenvectors and/or eigenvalues.
2.3.2 Intrinsic Reflection and Transmission Matrices

We next calculate the $2 \times 2$ intrinsic reflection and transmission matrices\textsuperscript{15}. If down-going incident fields in layer $m$ are phase-referenced\textsuperscript{16} to $z = z_m$, then $\tilde{R}_{m,m+1}$ and $\tilde{T}_{m,m+1}$ are easily procured \cite{45}[Ch. 2]; similar holds for $\tilde{R}_{m+1,m}$ and $\tilde{T}_{m+1,m}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{The incident modes ($i,I$ and $i,II$ subscripts), Type I/II reflected modes due to the incident Type I ($s,I,I$ and $s,II,II$ subscripts) and Type II modes ($s,II,I$ and $s,II,II$ subscripts), and Type I/II transmitted modes due to the incident Type I ($t,I,I$ and $t,I,II$ subscripts) and Type II modes ($t,II,I$ and $t,II,II$ subscripts) are shown.}
\end{figure}

2.3.3 Generalized Reflection/Three-Layer Transmission Matrices

With the intrinsic reflection/transmission matrices now available, we derive the generalized reflection matrices (GRM) and three-layer transmission matrices (3TM). The 3TM yields the total down (up) going fields in the slab layer of the canonical three-layer medium problem for incident downward (upward) fields, while the GRM yields the reflected fields in the top (bottom) layer (see Figure 2.3).

\textsuperscript{15}“Intrinsic” refers to reflection/transmission matrix quantities associated with only two media present (see Figure 2.2).

\textsuperscript{16}Fields “phase-referenced” to $z^*$ possess a $\exp(i\tilde{k}_z(z - z^*))$ $z$-dependence.
The GRM assuming down-going incident fields can be determined by looking down into the three bottom-most layers of an $N$ layer medium (resp. labeled as 1’ (top), 2’ (middle), and 3’ (bottom) in Figure 2.3) and assuming that the scattered fields in region 2’ and down-going incident fields in region 1’ are phase-referenced to $z_{2'}$ and $z_{1'}$ (resp.). Following [45][Ch. 2], one imposes two “constraint conditions” that result in two matrix-valued equations\(^{17}\)

\[
\tilde{\Lambda}^-_{2'}(z_{1'} - z_{2'}) \cdot \tilde{a}^-_{2'} = \tilde{T}_{1'2'} \cdot \tilde{a}^-_{1'} + \tilde{R}_{2'1'} \cdot \tilde{\Lambda}^+_{2'}(z_{1'} - z_{2'}) \cdot \tilde{R}_{2'3'} \cdot \tilde{a}^-_{2'} \tag{2.3.4}
\]

\[
\tilde{R}_{1'2'} \cdot \tilde{a}^-_{1'} = \tilde{R}_{1'2'} \cdot \tilde{a}^-_{1'} + \tilde{T}_{2'1'} \cdot \tilde{\Lambda}^+_{2'}(z_{1'} - z_{2'}) \cdot \tilde{R}_{2'3'} \cdot \tilde{a}^-_{2'} \tag{2.3.5}
\]

By rearranging Eqs. (2.3.4)-(2.3.5), one has\(^{18}\)

\[
\tilde{M} = [\tilde{I}_2 - \tilde{\Lambda}^-_{2'}(z_{2'} - z_{1'}) \cdot \tilde{R}_{2'1'} \cdot \tilde{\Lambda}^+_{2'}(z_{1'} - z_{2'}) \cdot \tilde{R}_{2'3'}] \tag{2.3.6}
\]

and the 3TM

\[
\tilde{T}_{1',2'} = \tilde{M}^{-1} \cdot \tilde{\Lambda}^-_{2'}(z_{2'} - z_{1'}) \cdot \tilde{T}_{1'2'} \tag{2.3.7}
\]

with which one has

\[
\tilde{a}^-_{2'} = \tilde{T}_{1',2'} \cdot \tilde{a}^-_{1'} \tag{2.3.8}
\]

Substituting the right hand side of Eq. (2.3.8) for $\tilde{a}^-_{2'}$ in Eq. (2.3.5), one obtains the GRM

\[
\tilde{R}_{1'2'} = \tilde{R}_{1'2'} + \tilde{T}_{2'1'} \cdot \tilde{\Lambda}^+_{2'}(z_{1'} - z_{2'}) \cdot \tilde{R}_{2'3'} \cdot \tilde{T}_{1',2'} \tag{2.3.9}
\]

This procedure can be repeated for layers $N-3$, $N-2$, and $N-1$ by labeling them as layers 1’, 2’, and 3’ (resp.) and replacing $\tilde{R}_{2'3'}$ in Eq. (2.3.9) with $\tilde{R}_{2'3'}$ [45][Ch. 2].

\(^{17}\)The up-going mode eigenvalues are block-represented as the $2 \times 2$ diagonal matrix $\tilde{\Lambda}^+_{m}(z) = \exp(\text{diag} [i\tilde{k}_{m,1}z, i\tilde{k}_{m,2}z])$, while the down-going mode eigenvalues are block-represented as the $2 \times 2$ diagonal matrix $\tilde{\Lambda}^-_{m}(z) = \exp(\text{diag} [i\tilde{k}_{m,3}z, i\tilde{k}_{m,4}z])$.

\(^{18}\)The $n \times n$ identity matrix is denoted $\tilde{I}_n$. 
The process is recursively performed up to the top three layers. A similar procedure can be used to find the GRM and 3TM looking up into each interface, whose expressions are found by using Eq. (2.3.8) and Eq. (2.3.9), labeling the bottom, middle, and top layers as 1′, 2′, and 3′ (resp.), and making the following two variable interchanges in the modified GRM/3TM relations:

\[ \tilde{\Lambda}^+_2(z_{1'} - z_{2'}) \leftrightarrow \tilde{\Lambda}^+_2(z_{2'} - z_{1'}) \] (2.3.10)

\[ \tilde{a}^+_m \leftrightarrow \tilde{a}^-_{m'}(m = 1, 2, 3) \] (2.3.11)

While the procedure above is analytically exact, to avoid the risk of numerical overflow one should shift the reference depth of the slab’s transmitted fields to the observation point depth \( z \) when the slab contains \( r \). This avoids propagating downward the up-going modes (or vice versa) at the final stage of assembling the total mixed-domain field \( W_L(k_x, k_y; z) \). Otherwise, exponentially increasing propagators would be present, which may cause numerical overflow. To find the numerically stable 3TM and GRM expressions, we perform similar manipulations as before to obtain:

\[ \tilde{M} = [\tilde{I}_2 - \tilde{\Lambda}^-_2(z - z_{1'}) \cdot \tilde{R}_{21'} \cdot \tilde{\Lambda}^+_2(z_{1'} - z_{2'}) \cdot \tilde{R}_{23'} \cdot \tilde{\Lambda}^-_2(z_{2'} - z)] \] (2.3.12)

\[ \tilde{a}^-_{2'} = \tilde{M}^{-1} \cdot \tilde{\Lambda}^-_2(z - z_{1'}) \cdot \tilde{T}_{12'} \cdot \tilde{a}^-_{1'} = \tilde{T}_{12'} \cdot \tilde{a}^-_{1'}, \] (2.3.13)

\[ \tilde{R}_{12'} = \tilde{R}_{12'} + \tilde{T}_{21'} \cdot \tilde{\Lambda}^+_2(z_{1'} - z_{2'}) \cdot \tilde{R}_{23'} \cdot \tilde{\Lambda}^-_2(z_{2'} - z) \cdot \tilde{T}_{12'} \] (2.3.14)
2.3.4 Direct Field Modal Amplitudes

We next procure the direct field modal amplitudes. For simplicity, the layer-number notation is omitted in this sub-section with the understanding that all field quantities are associated with layer $M$.

If the eigenvalues are unique, we first obtain $\tilde{H}$ from $\tilde{E}$ to form the four-component vector $\tilde{V} = \begin{bmatrix} \tilde{E}_x & \tilde{E}_y & \tilde{H}_x & \tilde{H}_y \end{bmatrix}$. With this, we perform the analytic $k_z$ integration of $\tilde{V} e^{i k_z r}$ to obtain

$$ e^{i k_z (x-x') + i k_y (y-y')} 2\pi i \sum_{l=l_1}^{l_2} \left[ \left( k_z - \tilde{k}_{l_2} \right) \tilde{V} e^{i k_z (z'-z')} \right] \bigg|_{k_z=\tilde{k}_{l_2}} \tag{2.3.15} $$

Equivalently, by setting $\tilde{V}' = \tilde{V} e^{i k_z (z^*-z')}$, one obtains

$$ e^{i k_z (x-x') + i k_y (y-y')} 2\pi i \sum_{l=l_1}^{l_2} \left[ \left( k_z - \tilde{k}_{l_2} \right) \tilde{V}' e^{i k_z (z-z')} \right] \bigg|_{k_z=\tilde{k}_{l_2}} \tag{2.3.16} $$

where the sum runs over the two up-going modes (denoted by the substitutions $(l_1, l_2) \rightarrow (1, 2)$ and $z^* \rightarrow z^*_{M-1}$) or two down-going modes (denoted by the substitutions $(l_1, l_2) \rightarrow (3, 4)$ and $z^* \rightarrow z^*_M$, $z^*_M = \delta_{1M} z' + (1 - \delta_{1M}) z_{M-1}$, and
\[ z^*_M = \delta_{NM} z' + (1 - \delta_{NM}) z_M. \] Note that Eq. (2.3.15) was redefined as Eq. (2.3.16) to facilitate subsequently calculating reflected and transmitted fields.

Next, defining for up-going mode \( l (l = 1, 2) \) the tangential fields, obtained after \( k_z \) integration followed by suppression of the propagators, as

\[
\hat{u}_l^* = \hat{u}_l^*(k_x, k_y) = \left[ \left( k_z - \tilde{k}_lz \right) \tilde{V} \right]_{k_z = \tilde{k}_lz} \quad (2.3.17)
\]

\[
\hat{u}_l = \hat{u}_l(k_x, k_y) = \left[ \left( k_z - \tilde{k}_lz \right) \tilde{V}' \right]_{k_z = \tilde{k}_lz} \quad (2.3.18)
\]

one can define the amplitudes \( \tilde{a}_{l,D}^* \) and \( \tilde{a}_{l,D} \) (the \( D \) subscript stands for “direct”), corresponding to this mode, which satisfy

\[
\hat{u}_l^* = \tilde{a}_{l,D}^* \tilde{\hat{s}}_l \quad (2.3.19)
\]

\[
\hat{u}_l = \tilde{a}_{l,D} \tilde{\hat{s}}_l \quad (2.3.20)
\]

If the eigenvalues are degenerate (i.e. when layer \( M \) is isotropic), one instead uses the analytically simplified spectral Green’s Dyads devoid of double-poles [45][Chs. 1,7] when employing Eqs. (2.3.15)-(2.3.20). Since the resulting degenerate field is a linear combination of the \( \text{TE}_z \) and \( \text{TM}_z \) modes, one follows its evaluation with a \( \text{TE}_z/\text{TM}_z \) modal decomposition. One decomposition example is

\[
\begin{bmatrix}
\tilde{e}_{x}^I + \\
\tilde{e}_{y}^I + \\
\tilde{e}_{x}^I + \\
\tilde{e}_{y}^I + \\
\tilde{a}_{x}^I + \\
\tilde{a}_{x}^I + \\
\tilde{a}_{y}^I + \\
\tilde{a}_{y}^I + \\
\end{bmatrix}
\begin{bmatrix}
\tilde{e}_{x}^I + \\
\tilde{e}_{y}^I + \\
\tilde{a}_{x}^I + \\
\tilde{a}_{y}^I + \\
\end{bmatrix} = \begin{bmatrix}
\tilde{e}_{x}^I + \\
\tilde{e}_{y}^I + \\
\tilde{e}_{x}^I + \\
\tilde{e}_{y}^I + \\
\end{bmatrix}
\]

(2.3.21)

where \( \tilde{a}_{D}^I \) and \( \tilde{a}_{D}^{II} \) are the up-going \( \text{TE}_z \) and \( \text{TM}_z \) modal amplitudes (resp.). If using the transverse components leads to an ill-conditioned system, one can use relations in [45][Ch. 2] to find \( \tilde{e}_{x}^I, \tilde{e}_{y}^I \) and then solve Eq. (2.3.21) using \( \tilde{e}_{x} \) and \( \tilde{e}_{y} \) (or \( \tilde{e}_{x} \) and \( \tilde{e}_{y} \)). Note that since Eq. (2.3.21) is a second-rank linear system, its inversion is trivial; therefore, only the system’s conditioning limits the accuracy of the computed amplitudes [79].

\[^{19}\text{Unit-magnitude vectors have an over-hat symbol.}\]
2.3.5 Scattered Mode Calculation and Field Transmission

Now, the total field impinging upon the interfaces \( z = z_{M-1} \) and \( z = z_M \) must be calculated; this is done via exhibiting and solving the *vectorial generalization* of relations in [45][Ch. 2] accounting for arbitrary anisotropy (i.e. including inter-mode coupling at planar interfaces). All field quantities exhibited below through Eq. (2.3.31) are associated with layer \( M \).

Define \( \tilde{a}_{D}^+ = (\tilde{a}_{D}^{I_+}, \tilde{a}_{D}^{II_+}) \), \( \tilde{a}_{S1}^- \), and \( \tilde{a}_{S1}^+ \) as \( 2 \times 1 \) vectors containing (resp.) the amplitudes of the direct up-going, scattered up-going, and scattered down-going modes phase-referenced to \( z = z_{M-1} \). Similarly, define \( \tilde{a}_{D}^- = (\tilde{a}_{D}^{I_-}, \tilde{a}_{D}^{II_-}) \), \( \tilde{a}_{S2}^+ \), and \( \tilde{a}_{S2}^- \) for the same modes but phase-referenced to \( z = z_M \). With this, one defines the following quantities:*20:

\[
\begin{align*}
    f_D^+(k_x, k_y; z) &= \tilde{S}_M^+ \cdot \tilde{A}_M^+(z - z_{M-1}) \cdot \tilde{a}_D^+ \\
    f_D^-(k_x, k_y; z) &= \tilde{S}_M^- \cdot \tilde{A}_M^-(z - z_M) \cdot \tilde{a}_D^- \\
    f_{S1}^+(k_x, k_y; z) &= \tilde{S}_M^+ \cdot \tilde{A}_M^+(z - z_{M-1}) \cdot \tilde{a}_{S1}^+ \\
    f_{S2}^+(k_x, k_y; z) &= \tilde{S}_M^+ \cdot \tilde{A}_M^-(z - z_M) \cdot \tilde{a}_{S2}^+ \\
    f_{S1}^-(k_x, k_y; z) &= \tilde{S}_M^- \cdot \tilde{A}_M^-(z - z_{M-1}) \cdot \tilde{a}_{S1}^- \\
    f_{S2}^-(k_x, k_y; z) &= \tilde{S}_M^- \cdot \tilde{A}_M^-(z - z_M) \cdot \tilde{a}_{S2}^- 
\end{align*}
\]

Subsequently, in layer \( M \) we can represent the tangential fields \( W_M(k_x, k_y; z) \) as

\[
W_M(k_x, k_y; z) = \begin{cases} 
    f_D^+ + f_{S1}^+ + f_{S2}^+, & z > z' \\
    f_D^- + f_{S2}^+, & z < z'
\end{cases}
\]

(2.3.28)

Armed with relations Eqs. (2.3.22)-(2.3.28), one now imposes two “constraint conditions” [45][Ch. 2] that yield the relations (1) \( \tilde{a}_{S1}^- = \tilde{R}_{M,M-1} \cdot (\tilde{a}_D^+ + \tilde{a}_{S1}^+) \) and

*20The up-going mode eigenvectors are block-represented as the \( 4 \times 2 \) matrix \( \tilde{S}_m^+ = [\tilde{s}_m^+, \tilde{s}_m^-] \), while the down-going mode eigenvectors are block-represented as the \( 4 \times 2 \) matrix \( \tilde{S}_m^- = [\tilde{s}_m^-, \tilde{s}_m^+] \).
(2) $\mathbf{a}_{S2}^+ = \mathbf{R}_{M,M+1} \cdot (\mathbf{a}_D^+ + \mathbf{a}_{S1}^+)$

Using these two constraints along with (1) $\mathbf{a}_{S1}^+ = \mathbf{A}_M^+(z_{M-1} - z_M) \cdot \mathbf{a}_{S2}^+$ and (2) $\mathbf{a}_{S2} = \mathbf{A}_M^+(z_M - z_{M-1}) \cdot \mathbf{a}_{S1}^+$, which arise from enforcing continuity of the scattered fields at $z = z'$, upon performing algebraic manipulation one has $\mathbf{a}_{S1}^+$ and $\mathbf{a}_{S2}^+$ as functions of $\mathbf{a}_D^+$ and $\mathbf{a}_D^-$:

$$\tilde{\mathbf{M}}_1 = \mathbf{A}_M^-(z_M - z_{M-1}) \cdot \mathbf{R}_{M,M-1}, \quad \tilde{\mathbf{M}}_2 = \mathbf{A}_M^+(z_{M-1} - z_M) \cdot \mathbf{R}_{M,M+1}$$

$$\mathbf{a}_{S1}^+ = \left[ \mathbf{I}_2 - \tilde{\mathbf{M}}_2 \cdot \tilde{\mathbf{M}}_1 \right]^{-1} \cdot \tilde{\mathbf{M}}_2 \cdot \left[ \mathbf{a}_D^+ + \tilde{\mathbf{M}}_1 \cdot \mathbf{a}_D^+ \right]$$

$$\mathbf{a}_{S2}^- = \left[ \mathbf{I}_2 - \tilde{\mathbf{M}}_1 \cdot \tilde{\mathbf{M}}_2 \right]^{-1} \cdot \tilde{\mathbf{M}}_1 \cdot \left[ \mathbf{a}_D^- + \tilde{\mathbf{M}}_2 \cdot \mathbf{a}_D^- \right]$$

For $L \neq M$, one then uses the sum $\mathbf{a}_D^+ + \mathbf{a}_{S1}^+$ ($\mathbf{a}_D^- + \mathbf{a}_{S2}^-$) and the 3TM matrices to find $\mathbf{a}_L^+(\mathbf{a}_L^-)$ for $L < M$ ($L > M$), which write as (resp.)

$$\tilde{T}_{L+1,L} \cdots \left[ \mathbf{A}_M^+(z_{M-3} - z_{M-2}) \cdot \tilde{T}_{M-1,M-2} \right] \left[ \mathbf{A}_M^+(z_{M-2} - z_M) \cdot \tilde{T}_{M,M-1} \right] \left( \mathbf{a}_D^+ + \mathbf{a}_{S1}^+ \right)$$

$$\tilde{T}_{L-1,L} \cdots \left[ \mathbf{A}_M^-(z_{M+2} - z_{M+1}) \cdot \tilde{T}_{M+1,M+2} \right] \left[ \mathbf{A}_M^-(z_{M+1} - z_M) \cdot \tilde{T}_{M,M+1} \right] \left( \mathbf{a}_D^- + \mathbf{a}_{S2}^- \right)$$

where for some intermediate layer $m \neq L$, $z_m^\text{ref}$ is the user-defined phase-reference depth\textsuperscript{21}. Given $\mathbf{a}_L^+(\mathbf{a}_L^-)$ for $L < M$ ($L > M$), one then finds $\mathbf{a}_L^-(\mathbf{a}_L^+)$ as (resp.)

$$\mathbf{a}_L^- = \mathbf{A}_L^-(z - z_{L-1}) \cdot \mathbf{R}_{L,L-1} \cdot \mathbf{A}_L^+(z_{L-1} - z) \cdot \mathbf{a}_L^+$$

$$\mathbf{a}_L^+ = \mathbf{A}_L^+(z - z_L) \cdot \mathbf{R}_{L+1,L} \cdot \mathbf{A}_L^-(z_{L} - z) \cdot \mathbf{a}_L^-$$

With the above in mind, we have the following expressions for $W_L(k_x,k_y;z)$ when either $L < M$ or $L > M$ (resp.):

$$\left( \mathbf{A}_L^+([z - z_1] \delta_{L1}) \cdot \mathbf{S}_L^+ + (1 - \delta_{L1}) \mathbf{S}_L^- \cdot \mathbf{A}_L^-(z - z_{L-1}) \cdot \mathbf{R}_{L,L-1} \cdot \mathbf{A}_L^+(z_{L-1} - z) \right) \cdot \mathbf{a}_L^+$$

\textsuperscript{21}If layer $L$ corresponds to a slab, we compute the 3TM $\tilde{T}_{L+1,L}$ in (2.3.32) according to the numerically stable 3TM/GRM formulation presented in Section 2.3.3. If instead layer $L$ corresponds to the top layer, $\tilde{T}_{L+1,L}$ reduces to the intrinsic transmission matrix. Similar holds for $\tilde{T}_{L-1,L}$ in (2.3.33).
\[
\left( \tilde{\mathbf{A}}_L^\dagger([z-z_{N-1}]\delta_{LN}) \cdot \tilde{\mathbf{S}}^\dagger_N + (1-\delta_{LN})\tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}^\dagger_L (z-z_L) \cdot \tilde{\mathbf{R}}_{L,L+1} \cdot \tilde{\mathbf{A}}^\dagger_L (z_L-z) \right) \cdot \tilde{\mathbf{a}}_L^- (2.3.37)
\]

If \( L = M \), then for \( N < M < 1 \), one instead obtains \( \tilde{\mathbf{a}}_{S1}^- \) and \( \tilde{\mathbf{a}}_{S2}^+ \) and propagates these to \( z \). Note that this method obviates propagating downward (upward) \( \tilde{\mathbf{a}}_{S1}^- \) (\( \tilde{\mathbf{a}}_{S2}^+ \)), thereby preventing another potential source of numerical overflow. The up-going (down-going) direct fields, as phase-referenced to \( z' \), can be propagated to \( z \) for \( z > z' \) (\( z < z' \)). Now recall Eq. (2.3.19) and define \( \tilde{\mathbf{a}}^\dagger \). Furthermore, \( \tilde{\mathbf{a}}^\dagger \) is propagated to \( z \) when \( z > z' \) (\( z < z' \)).

With this, for \( M = 1 \) (\( M = N \)) we have (resp.) for \( \mathbf{W}_L(k_x,k_y;z) \)

\[
u(z-z')\tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z') \cdot \tilde{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z') \cdot \tilde{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z_{N-1}) \cdot \tilde{\mathbf{a}}^\dagger (2.3.40)
\]

\[
u(z-z')\tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z') \cdot \tilde{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z') \cdot \tilde{\mathbf{a}}^\dagger \cdot \tilde{\mathbf{S}}^\dagger_L \cdot \tilde{\mathbf{A}}_L^\dagger (z-z_{N-1}) \cdot \tilde{\mathbf{a}}^\dagger (2.3.41)
\]

Note that in all expressions obtained throughout this section, no exponentially rising terms are present since down-going (up-going) modes are always propagated downward (upward), leading to a stable numerical implementation.
2.4 Integration Methodology

In the numerical evaluation of Eq. (2.2.12), one repeats the steps in Section 2.3 for every sampled \((k_x, k_y)\) point, approximating Eq. (2.2.12) as the double sum

\[
\mathcal{V}(\mathbf{r}) \simeq \frac{i}{(2\pi)^2} \sum_{p=-P_1}^{P_2} \sum_{q=-Q_1}^{Q_2} W_L(k_{xq}, k_{yp}; z) e^{ik_{xq}(x-x')} e^{ik_{yp}(y-y')} w(k_{xq}) w(k_{yp}) \tag{2.4.1}
\]

In Section 2.4.1, we describe an efficient methodology to compute the contribution from the “pre-extrapolation” region \(-\xi_1 < \text{Re}(k_x) < \xi_1\), (see Figure 2.4). In section 2.4.2, we detail an adaptive implementation of the MWA [66, 75, 71] tailored for this problem to compute the contribution from the “extrapolation” region \(|k_x| > |\xi_1|\) [71].

Figure 2.4: Typical \(k_x\) plane features present when evaluating Eq. (2.2.12). “Radiation BC Map” and “Program BC Map” refer to the branch cuts associated with the radiation/boundedness condition at infinity and the computer program’s square root convention (resp.). The encircled “X” symbols represent the branch points and the red “X” symbols represent slab-/interface-guided mode poles. For \(K\) extrapolation intervals used, the red contour represents the integration path extending to \(k_x = \pm \xi_{K+1}\).
2.4.1 Pre-Extrapolation Region

The presence of critical points (i.e. branch points/cuts and slab/interface mode poles) near the \( \text{Re}(k_x) \) axis in the pre-extrapolation region requires a detoured contour to yield a robust numerical integration [45][Ch. 2]. Furthermore, the oscillatory nature of \( W_L(k_x, k_y; z)\exp[i k_x(x-x') + i k_y(y-y')] \) and the potentially close proximity of critical points to the detoured contour warrants adaptively integrating to ensure accurate results [70, 71] (see Figure 2.4).

First we discuss the integration path’s initial sub-division and parameterization. Similar to [70], we define: a maximum detour height \( d_x \), the two points bounding the detour as \( k_x = \pm P_k \), and the two points within which one adaptively integrates as \( k_x = \pm \xi_1 \). All these points are indicated in Figure 2.4. The detour path can be parameterized, using the real-valued variable \( r \), as \( k_x = r - i \sin(\pi r/P_k) \) and \( dk_x = (\partial k_x/\partial r) \, dr \), where \( \partial k_x/\partial r = 1 - i(\pi/P_k) \cos(\pi r/P_k) \) and \(-\xi_1 \leq r \leq \xi_1 \) [70].

To compute \( \pm P_k \), we adapt the procedure described in [70] to arbitrarily anisotropic media. For a layer \( p (p=1,2,...,N) \), we calculate the three eigenvalues of its relative material tensors \( \bar{\epsilon}_{r,p} \) and \( \bar{\mu}_{r,p} \) (\( \{\epsilon_{pi}\}, \{\mu_{pi}\}, i=1,2,3 \)), find \( \sqrt{\epsilon_{pi}\mu_{pj}} \) for \( i,j=1,2,3 \), and take the \( p \)-th layer “effective” refractive index \( n_p (p = 1, 2, ..., N) \) to be the \( \sqrt{\epsilon_{pi}\mu_{pj}} \) value having the real part with the largest magnitude but with imaginary part below a user-defined threshold \( T \). Subsequently we compute \( n^+ = \max(|\text{Re}\{n_p\}|) \), which yields a “worst-case” scenario for the maximum magnitude of the real part of any poles or branch points near the \( \text{Re}(k_x) \) axis. Finally, we set \( P_k = l_o k_o (n^+ + 1) \), where \( l_o \geq 1 \) is a user-defined pre-extrapolation region magnification constant.
Furthermore, defining $\Delta x = |x - x'|$, $\Delta y = |y - y'|$, and $\Delta z = |z - z'|$, we compute the following integration path parameters [70]:

$$d_x = \begin{cases} \frac{1}{\Delta x}, & \Delta x > 1 \\ 1, & \text{otherwise} \end{cases} \quad (2.4.2)$$

$$\Delta \xi_x = \begin{cases} \frac{\pi}{\Delta x}, & \Delta x > 1 \\ \frac{\pi}{\pi}, & \text{otherwise} \end{cases} \quad (2.4.3)$$

$$\xi_1 = \left( \text{Int} \left( \frac{P_k}{\Delta \xi_x} \right) + 1 \right) \Delta \xi_x \quad (2.4.4)$$

where $\text{Int}(\cdot)$ truncates its argument to an integer number. Next, we splice the regions $(0, P_k)$ and $(-P_k, 0)$ each into $P$ regions. Letting $T_1$ and $T_2$ be two user-defined constants, one has

$$\Delta k = \begin{cases} \frac{\pi}{T_1 \Delta x}, & \Delta x + \Delta z > 0 \\ \frac{\pi}{T_1 \Delta y}, & \text{otherwise} \end{cases} \quad (2.4.5)$$

$$N_{\text{node}} = \text{Int} \left( \frac{P_k}{\Delta k} \right) + 1 \quad (2.4.6)$$

resulting in $P = \text{Int}(1 + N_{\text{node}}/T_2)$. This empirical methodology for parameterizing and splicing the pre-extrapolation region relies upon the conservative assumption of equidistant sampling.

We utilize a nested Patterson-Gauss/Kronrod-Gauss quadrature scheme [70] throughout the pre-extrapolation region. Such schemes sacrifice algebraic degrees of precision, yielding only $3n + 1$ ($3n + 2$) degrees of precision for $n$ odd (even) when adding on $n + 1$ nested quadrature nodes [80, 81], in contrast to $4n + 1$ degrees of precision for a $(2n+1)$-point Gauss quadrature formula [82]. However, considering the extensive calculations involved at each sampled $(k_x, k_y)$ node (see Section 2.3), a Patterson-Gauss scheme significantly reduces the overall computation time [83].

Finally, one folds the integral results from $(0, \xi_1)$ and $(-\xi_1, 0)$ to yield $I_{x0}' = I_{x0}(k_y)$. 

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2.4.2 Extrapolation Region

Subsequently, one must approximate the integral over the path’s tails ($\xi_1, \infty$) and $(-\infty, -\xi_1)$ along the Re($k_x$) axis. For a robust computation, so that both approximation error and convergence rate are good for different geometries and ranges of layer constitutive properties, an integral acceleration/extrapolation technique is required. Here we adopt the MWA [66, 75, 71]$^{22}$ and briefly summarize below the extensions and adaptations made to our problem$^{23}$:

1. Splice the path ($\xi_1, \xi_1 + N\Delta \xi_x$) into $N$ sub-intervals$^{24}$ with bounding breakpoints $\xi_{xn} = \xi_1 + (n - 1)\Delta \xi_x$ [70, 71, 75].

2. Integrate each sub-interval using (for example) a 15- or 20-point Legendre-Gauss quadrature rule [70, 75].

3. Store the these results as $I_{xp}^+(p = 1, 2, ... N)$.

4. Repeat steps 1-3 for the path ($-\xi_1 - N\Delta \xi_x, -\xi_1$) to procure $I_{xp}^-$. 

5. Fold $I_{xp}^+$ and $I_{xp}^-$ together to form $I_{xp}' = I_{xp}^+ + I_{xp}^-$ ($p=1,2,...,N$).

6. Obtain cumulative integrals $I_{xp,c}'$ via update: $I_{xp,c}' = I_{xp,c} + I_{xp(p-1),c}'$ ($p=2,3,...,N$)

   (Note: $I_{x1,c} = I_{x1}'$).

7. Use the $\{I_{xp,c}'\}$ to estimate the non-truncated tail integral $I_{x}'$ as $I_{x}^{''(N)}$.

$^{22}$More specifically, we employ the the “Mosig-Michalski algorithm” variant of MWA [84] (MMA for short).

$^{23}$It is assumed that (1) one has detoured sufficiently far past any branch points/poles near to the Re($k_x$) axis [70, 71, 75] and (2) as $|k_x| \to \infty$, $ik_z(z - z') \to -f(k_x)\Delta z$, where $f(k_z) = f(-k_x)$ and Re($f(k_x)$) $> 0$.

$^{24}$This $N$ is unrelated to the number of layers.
8. Compute the complete $k_x$ integral $I' = I'(k_y) = I'^{(N)} + I'_{x0}$.

The MWA accelerates convergence of integrals like Eq. (2.2.12) via estimating the tail integral’s truncation error followed by combining two or more estimates, exemplified by

$$I'^{(N)} = \sum_{n=1}^{N} w_n I'_{x,n,c}$$

(2.4.7)

to accelerate the truncation error’s decay. First, denote the true truncation error of $I'_{x,n,c}$ as $R_{x,n}$ such that $I'_{x} = I'_{x,n,c} + R_{x,n}$. Then, defining $\gamma_{1,2} = w_2/w_1$ and setting $N = 2$, one can re-write Eq. (2.4.7) as [71, 75]:

$$I'^{(2)} = \frac{w_1 [I'_{x} - R_{x,1}] + w_2 [I'_{x} - R_{x,2}]}{w_1 + w_2} = I'_{x} - R_{x,2} \frac{R_{x,1}}{1 + \gamma_{1,2}}$$

(2.4.8)

Next, setting $\gamma_{1,2} = -R_{x,1}/R_{x,2}$ yields $I'^{(2)} = I'_{x}$ despite using only two finite-length tail integrals. However, in reality one must estimate the $\{R_{x,n}\}$ (thus yielding estimated error ratios $\{-\gamma_{n,n+1}^{est}\}$) via approximation of the truncation error integral’s asymptotic behavior [71]. By folding the asymptotic form of the $k_x$ integral’s tail section one has

$$\int_{\xi_1}^{\infty} k_x^q e^{-f(k_x)\Delta z} e^{ik_x(x-x')} dk_x + \int_{-\infty}^{-\xi_1} k_x^q e^{-f(k_x)\Delta z} e^{ik_x(x-x')} dk_x =$$

$$\int_{\xi_1}^{\infty} 2k_x^q \left\{ \cos k_x(x-x') \right\} e^{-f(k_x)\Delta z} dk_x$$

(2.4.9)

with the sine (cosine) factor for $q$ odd (even). Furthermore, the factor $e^{-f(k_x)\Delta z} k_x^q$ above can be rewritten as $(e^{-f(k_x)\Delta z} k_x^{q+1})/k_x$ to conservatively ensure that in the multi-layer case, one can satisfy the assumption [71] that the integrand has the form

$h(k_x; z, z') = g(k_x; z, z') p(k_x)$, where $p(k_x)$ is an oscillatory function with period $2T = 2\pi/\Delta x$ and (asymptotically) $g(k_x)$ has the form

$$g(k_x; z, z') \sim \frac{e^{-f(k_x)\Delta z}}{k_x^{\alpha}} [C + O (k_x^{-1})] \sim \frac{e^{-f(k_x)\Delta z}}{k_x^{\alpha}} \sum_{l=0}^{\infty} \frac{c_l}{k_x^l}$$

(2.4.10)
Adapted to our problem, the analytic remainder estimate takes the form (for $\Delta x > 0$) \[ R_{\text{rest}}(x_n) = \left( -1 \right)^n e^{-f(k_x) \Delta z} \xi_{n+1}^{q+1}, \] where $R_{xn}$ has the asymptotic form $R_{xn,a} \sim R_{zn}^{\text{est}(1)} \sum_{l=0}^{\infty} \alpha_l z^{-l} \xi_{n+1}$ [71]. Subsequently, assuming that $R_{xn}/R_{x(n+1)}$ has the asymptotic form $R_{xn}^a \sim R_{x(n+1)}^a \left[ 1 + O \left( \xi_{n+1}^{-2} \right) \right]$ one can insert $R_{x1,a}^e$ and $\gamma_{1,2}^{\text{est}(1)} = -R_{x1,a}^{\text{est}(1)}/R_{x2}^{\text{est}(1)}$ (in place of $\gamma_{1,2}$) into Eq. (2.4.8) to obtain [71] 

\[ I_{x}^{t}(2) = I_{x}^{t} + R_{x2} \left[ 1 + O \left( \xi_{2}^{-2} \right) \right] - 1 \]

\[ = I_{x}^{t} + R_{x2} \frac{O \left( \xi_{2}^{-2} \right)}{1 + 1/\gamma_{1,2}^{\text{est}(1)}} = I_{x}^{t} - R_{x1}^{(2)} \]

with remainder $R_{x1}^{(2)} = -R_{x2}O \left( \xi_{2}^{-2} \right) / (1 + 1/\gamma_{1,2}^{\text{est}(1)})$. It is seen that $R_{x1}^{(2)}$ is asymptotically equal to $R_{x2}$ except for being scaled by the factor $\xi_{2}^{-2}$; similarly, its corresponding remainder estimate $R_{x1}^{\text{est}(2)}$ is also scaled by $\xi_{2}^{-2}$ [71, 75]. The above procedure can be applied recursively to estimate $I_{x}^{t}$ using $N$ cumulative integrals [71, 75]. By defining

\[ \gamma_{n,n+1}^{\text{est}(r-1)} = \gamma_{n,n+1}^{\text{est}(1)} \left( \xi_{n+2}/\xi_{n+1} \right)^{2(r-2)} \quad (r = 3, 4, ..., N + 1) \] (2.4.12)

\[ I_{x,n,c}^{t(1)} = I_{x,n,c}^{t} \quad (n = 1, 2, ..., N) \] (2.4.13)

\[ I_{x1}^{t(N)} = I_{x}^{t(N)} \] (2.4.14)

the following expression is obtained in place of Eq. (2.4.11) [71, 75]:

\[ I_{x,n,c}^{t(r)} = \frac{I_{x,n,c}^{t(r-1)} + I_{x,n,c}^{t(r-1)} \gamma_{n,n+1}^{\text{est}(r-1)}}{1 + \gamma_{n,n+1}^{\text{est}(r-1)}}, \quad 2 \leq r \leq N, 1 \leq n \leq N - r + 1 \] (2.4.15)

Note from Eq. (2.4.3) that as $|x - x'|$ increases, $\Delta \xi_x$ is reduced. This is done to keep the interval break-points at the extrema (nulls) of the cosine (sine) function in Eq. (2.4.9) [71], and sample the integrand at an adequate rate. However, simultaneously shrinking the region $(\xi_1, \xi_{N+1})$ may cause an undesirable degradation in accuracy. This can be solved via adaptive integration of the tail integral, using additional extrapolation intervals combined with successively higher-order weighted average schemes until convergence ensues.
For implementing an adaptive version of the MMA, one could in principle utilize an \( N \)-tier recursive function call chain to evaluate Eq. (2.4.15). However, this is not efficient since the number of active, simultaneous calls to the function carrying out extrapolation would peak at \( N(N+1)/2 \). Instead, pre-computing the weights for each desired \( N \)-tier scheme prior to integration such that one can simply compute 

\[
I^t(N) = I^t_{x1} = \sum_{n=1}^{n=N} w_{n,N} I'_{xn,c}
\]

(2.4.16)

where \( w_{n,N} \) is the \( n \)th weight\(^{25} \) (\( n = 1, 2, ..., N \)) of the tier-\( N \) MMA scheme, is preferred. The three advantages of this strategy are that it (1) obviates extensive recursive function calls, (2) eliminates the redundancy of re-computing tier \( N \) weights for each new \( k_y \) node (this is markedly important for 2-D integration), and (3) requires only one weighted average (i.e. Eq. (2.4.16)), thereby drastically reducing the arithmetic operations associated with each of the \( \{I'_{xn,c}\} \) to one multiplication and one final summation versus \( O(2^{N-1}) \) total multiplications and additions required to compute \( I^t(N) \) via the recursive function call chain approach. Assuming \( N_{max} > 1 \) tiers are sought, the pre-computation of the weights proceeds as follows (\( N = 2, 3, ..., N_{max} \)):

1. In computing \( w_{n,N} \) (\( 1 < n \leq N \)), admit \( n \) “intermediate” values \( \{w^{(1)}_{n,N}, ..., w^{(n)}_{n,N}\} \), where \( w^{(1)}_{N,N} = 1 \).

2. Recall Eq. (2.4.15) and set \( r = 2 \). Comparing this with Eq. (2.4.16), we find 

\[
w_{1,2} = 1/(1 + \gamma^{est(1)}_{1,2}) \quad \text{and} \quad w_{2,2} = 1/(1 + 1/\gamma^{est(1)}_{1,2}).
\]

We also set \( w^{(1)}_{2,2} = 1 \) and \( w^{(2)}_{2,2} = w_{2,2} \).

3. Recursively compute the \( \{w_{1,N}\} \) as 

\[
w_{1,m} = \frac{w_{1,m-1}}{1+\gamma^{est(m)}_{1,2}} \quad (m = 3, 4, ..., N_{max}).
\]

\(^{25}\)For a given \( N \), these weights are related to the weights shown in Eq. (2.4.7) via the relation 

\[
w_{n,N} = w_n/\sum_{n=1}^{n=N} w_n, \quad \text{where the} \quad \{w_n\} \quad \text{here tacitly exhibit dependence on} \quad N.\]
4. To compute \( w_{n,N} \) \( (2 \leq n \leq N, \ N > 2) \), first note the \( \{w_{n,N}^{(m)}\} \) initially update as

\[
w_{n,N}^{(m)} = \frac{w_{n,N}^{(m-1)}}{1 + \gamma_{est}(N+m-n-1)} \quad (m = 1, 2, \ldots, n; n \neq N) \tag{2.4.17}
\]

\[
w_{n,N}^{(1)} = 1, w_{n,N}^{(2)} = w_{n,N}^{(3)} = \ldots = w_{n,N}^{(N)} = 0 \quad (n = N) \tag{2.4.18}
\]

5. Update the \( \{w_{n,N}^{(m)}\} \) again as

\[
w_{n,N}^{(m)} = w_{n,N}^{(m)} + \frac{w_{n,N}^{(m-1)}}{1 + \gamma_{est}(N+m-n-1)} \quad (m = 2, 3, \ldots, n) \tag{2.4.19}
\]

set \( w_{n,N} = w_{n,N}^{(n)} \) to obtain the desired weight, and store the intermediate values for recursive re-application of steps 4-5.

To clarify steps 4-5, let us take a simple example and outline the process of obtaining the third cumulative integral’s weights corresponding to the three-tier, four-tier, and five-tier MMA (i.e. \( w_{3,3}, w_{3,4}, \) and \( w_{3,5} \)). Starting with \( N = 3 \) and noting that \( n = N = 3 \), we apply Eq. (2.4.18) to obtain \( w_{3,3}^{(1)} = 1 \) and \( w_{3,3}^{(2)} = w_{3,3}^{(3)} = 0 \). Second, we apply Eq. (2.4.19) to obtain \( w_{3,3}^{(2)} = 0 + w_{3,3}^{(1)}/(1 + 1/\gamma_{est}(1)) \) and use this updated \( w_{3,3}^{(2)} \) value to compute \( w_{3,3} = w_{3,3}^{(3)} = 0 + w_{3,3}^{(2)}/(1 + 1/\gamma_{est}(2)) \), yielding one of our desired weights. Third, we use these three updated intermediate values as the input to another application of step four with \( N = 4 \), using Eq. (2.4.17) to obtain \( w_{3,4}^{(1)} = w_{3,4}^{(1)}/(1 + \gamma_{est}(1)), w_{3,4}^{(2)} = w_{3,4}^{(2)}/(1 + \gamma_{est}(2)), \) and \( w_{3,4}^{(3)} = w_{3,4}^{(3)}/(1 + \gamma_{est}(3)) \). Finally, use Eq. (2.4.19) to obtain \( w_{3,4}^{(2)} = w_{3,4}^{(2)} + w_{3,4}^{(1)}/(1 + 1/\gamma_{est}(3)) \) and \( w_{3,4} = w_{3,4}^{(3)} = w_{3,4}^{(3)} + w_{3,4}^{(2)}/(1 + 1/\gamma_{est}(3)) \), giving the second desired weight.

The above procedure lends two practical improvements to the original MMA by (1) significantly reducing the operation count involving the \( \{I'_{x,m,c}\} \) and (2) devising a numerically stable scheme to efficiently update the weights. After the tail integral has converged, one computes \( I'_x = I'_{x_1}^{(N)} + I'_{x_0} \) to yield the final result.
2.5 Results

We now present a series of numerical results using the formulation presented above for the analysis of (1) well-logging induction (resistivity) tools for geophysical prospection (compared against [8, 9, 7]) and (2) the field pattern generated by electric current sources supported on grounded dielectric substrates (compared against [10]). The layers are numbered starting with the layer at the highest elevation and \( z_B \) contains the interface depth values.

Induction tools are generally composed of a system of transmitter and receiver loop antennas that can be modeled as Hertzian magnetic dipoles. The parameter \( L_m \) denotes the separation between the transmitter and \( m \)-th receiver (if all receivers are co-located, then \( L = L_1 \)).

The environmental parameter of interest is the resistivity of the surrounding Earth media, which can exhibit electrical anisotropy and planar-stratified inhomogeneity. Earth layers exhibiting reciprocal, electrical uniaxial anisotropy possess different resistivities on and transverse to their respective bedding planes, which are equal to \( R_{hn} = 1/\sigma_{hn} \) and \( R_{vn} = 1/\sigma_{vn} \) in layer \( n \) (resp.). Furthermore, each such layer has a bedding plane with arbitrary misalignment w.r.t. to the \( z \) axis, which for layer \( n \) is characterized by a dip angle and a strike angle that are denoted as \( \alpha_n \) and \( \beta_n \) (resp.). \( \alpha \) (\( \beta \)) refers to the tool’s polar (azimuthal) rotation relative to the \( z \) axis; see [9] for the formation dip/strike angle convention, which is the same as the tool dip/strike angle convention.

Note that for homogeneous formations characterized by this type of anisotropy, we use the variable \( \alpha \) to refer to the tool inclination angle relative to the \( z \)-directed optic axis or the tilting of the optic axis relative to the \( z \) axis (with a \( z \)-directed
tool) interchangeably; these definitions are equivalent in homogeneous formations exhibiting isotropy or reciprocal, electrical uniaxial anisotropy [14].

When displacement currents are non-negligible compared to induction currents, the anisotropy ratio of layer \( n \), \( \kappa_n \), is defined as

\[
\kappa_n = \sqrt{\frac{k_0 \epsilon_{h_n,r} + i \eta_0 \sigma_{h_n}}{k_0 \epsilon_{v_n,r} + i \eta_0 \sigma_{v_n}}} \quad (2.5.1)
\]

where \( \epsilon_{h_n,r} \) (\( \epsilon_{v_n,r} \)) is the complex-valued dielectric constant parallel (orthogonal) to the layer’s bedding plane [7]. This reduces to \( \kappa_n = \sqrt{R_{v_n}/R_{h_n}} = \sqrt{\sigma_{h_n}/\sigma_{v_n}} \) [9] when displacement currents are negligible compared to induction currents (i.e. at sufficiently low frequencies). For later reference, we also state the approximate formula predicting the formation resistivity estimated by a standard coaxial induction tool in a homogeneous, uniaxial medium [14]:

\[
R_{ap} = \frac{\kappa R_h}{\sqrt{\sin^2 \alpha + \kappa^2 \cos^2 \alpha}} \quad (2.5.2)
\]

2.5.1 Arrayed Coaxial Sonde

The first logging scenario simulated here is an arrayed, coaxial induction sonde with one transmitter and two receivers immersed in a homogeneous, uniaxial medium with \( z \)-directed optic axis [7]. We vary i) \( \alpha \) and ii) \( \kappa \) (i.e. fix \( R_h, \epsilon_h \) and vary \( R_v, \epsilon_v \)).

To extract effective, homogeneous-medium resistivity information from the observed magnetic field data, we follow the approach explained in [7], which we summarize here. First define the ratio of the two axial-directed magnetic field values, observed at the two receiver loop antennas spaced at distances \( L_1 \) and \( L_2 \) from the transmitter loop antenna (i.e. \( H_{z1} \) and \( H_{z2} \), resp.), as \( g_{12} = H_{z1}/H_{z2} \). Also, for

\[26\]That is, the magnetic field component directed along the sonde axis, normal to the area of the coaxial receiver loop antenna.
some complex-valued phasor quantity $F$, define its phase as $\angle F$ and its magnitude as $|F|$. Phase-apparent resistivity $R_{ap,ph}$ is obtained by first generating a look-up table of $\angle g_{12}$, at a specified transmitter radiation frequency, as a function of conductivity present in a homogeneous, isotropic medium. Subsequently, when the sonde is immersed in a heterogeneous environment that may contain anisotropic media, one compares the actual observed $\angle g_{12}$ to the look-up table and extracts the effective conductivity. This is finally inverted to obtain phase-apparent resistivity. Similar applies for magnitude-apparent resistivity $R_{ap,Amp}$, except now working with $|g_{12}|$ rather than $\angle g_{12}$. NB: The reference 3 curve label in Figures 2.5-2.6, corresponding to reference 3 in the original publication [1] describing this chapter’s presented algorithm, corresponds to reference [7] within this thesis.

We see that throughout Figures 2.5-2.6, agreement is consistently strong. Note that in Figures 2.5a and 2.6a, where $\alpha = 0^\circ$, the sensed resistivity is insensitive to $\kappa$. This is because when $\alpha = 0^\circ$ in a homogeneous, uniaxial medium, the coaxial sonde produces only $H$-mode plane wave spectra with electric field confined to the bedding plane [77][Ch. 7]. Furthermore, since the anisotropy ratio $\kappa$ is swept by keeping $R_h$ and $\epsilon_h$ constant while varying $R_v$ and $\epsilon_v$, it is expected that the received signal is independent of $\kappa$. 

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Figure 2.5: Phase-apparent resistivity log comparison with Figure 2 of [7] (homogeneous medium): \( R_h = 10 \Omega \text{m}, \beta = 0^\circ, f = 2\text{MHz}, L_2 = 31\text{in} \). In Figures 2.5a-2.5f the respective tool dip angles are as follows: 0\(^\circ\), 30\(^\circ\), 45\(^\circ\), 60\(^\circ\), 75\(^\circ\), 90\(^\circ\).
Figure 2.6: Magnitude-apparent resistivity log comparison with Figure 3 of [7] (homogeneous medium): $R_h = 10 \Omega \text{m}$, $\beta = 0^\circ$, $f = 2 \text{MHz}$, $L_1 = 25 \text{in}$, $L_2 = 31 \text{in}$. In Figures 2.6a-2.6f the respective tool dip angles are as follows: $0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, $75^\circ$, $90^\circ$. 
2.5.2 Triaxial Induction Sonde

The next logging scenarios involve a triaxial induction sonde with three mutually orthogonal, co-located transmitters and, spaced apart by a distance $L$, three mutually orthogonal, co-located receivers (see [11] and Fig. 1 of [8]). To invert apparent conductivity from the received magnetic field, formula (18) of [24] is used. NB: The reference 1 curve label in Figures 2.7-2.8, corresponding to reference 1 in the original publication [1] describing this chapter’s presented algorithm, corresponds to reference [8] within this thesis.

Figure 2.7 corresponds to the sonde in a homogeneous, uniaxial medium with varying $\alpha$; agreement is excellent. Figure 2.8 corresponds to a thirteen-layer environment with $\alpha = \beta = 0^\circ$. Note that our depth convention here corresponds to the half-way depth between the transmitters and receivers. Excellent agreement is observed between the results. For the coil separation used, $L=0.4m$, we notice that the coaxial ($\sigma_{a,z'z'}$) and co-planar ($\sigma_{a,x'x'}$) measurements provide marked resolution of even the thinnest bed present (0.2m thick); see the first spike and first valley from the left edge of Figures 2.8a and 2.8b (resp.).
Figure 2.7: Apparent conductivity log comparison with Figure 2 of [8] (homogeneous medium). $\kappa = \sqrt{5}$, $R_h = 1\,\Omega\,m$, $\beta = 0^\circ$, $f = 25\,\text{kHz}$, $L = 1\,\text{m}$. 
Figure 2.8: Apparent conductivity log comparison with Figure 3 of [8]. \( \kappa_n = \sqrt{5} \) and \( \alpha_n = \beta_n = 0^\circ \) in all beds; \( f = 25 \text{kHz} \), \( L = 0.4 \text{m} \), \( \sigma_h = \{1.0, 0.1, 1.0, 0.1, 1.0, 0.1, 1.0, 0.1, 1.0, 0.1, 1.0, 0.1, 1.0\} \text{S/m} \), \( z_B = \{0.0, 0.2, 4.2, 4.7, 8.7, 9.7, 13.7, 15.7, 19.7, 22.7, 26.7, 31.7\} \text{m} \).
2.5.3 Coaxial Sonde and Cross-bedding Anisotropy

The next logging scenario simulated corresponds to a 2MHz coaxial sonde vertically traversing inhomogeneous environments. We compare our results against those presented in [9]. It is important to note that there is an ambiguity in the resistivity inversion method and data post-processing used in [9] and hence only a qualitative comparison is made here. Since the inversion method was not stated explicitly in [9], we tried different inversion methods and found that the method corresponding to magnitude-apparent effective resistivity, specified in [7] and summarized above in section 2.5.1, produced the best-matching results with [9]. Also, [9] does not specify the depth convention in their plots (e.g. the transmitter depth). To render our data symmetric with respect to zero depth ($D = 0\text{ft}$) in this case, we define the depth $D$ as mid-way between the receiver and transmitter. NB: The reference 2 curve label in Figures 2.9-2.14, corresponding to reference 2 in the original publication [1] describing this chapter’s presented algorithm, corresponds to reference [9] within this thesis.

In Figure 2.9, where there is a low resistivity contrast of $R_{h1} = 4R_{h2}$, observe that the agreement between the two data sets is good. Note that for Figures 2.9a-2.9b, the effective resistivity in the top isotropic half-space levels off to that of the half space's actual resistivity, as is expected. Furthermore, note in Figure 2.9a that deep within the bottom uniaxial half-space, the effective resistivity levels off to $R_{h2} \sim 0.5\Omega\text{m}$, which is consistent with Eq. (2.5.2). This is because the transmitter antenna produces a primary (i.e. if $\vec{\sigma} = \vec{0}$) $\hat{\phi}'$-oriented electric field\textsuperscript{27}. Being oriented perpendicular to the uniaxial medium's bedding plane, the loop only produces $H$-mode plane wave spectra [77][Ch. 7] and thus induces azimuthal currents parallel to the bedding plane.

\textsuperscript{27}The prime denotes the tool system [11].
possessing intensity affected solely by $R_{h2}$ and the top formation’s resistivity. On the other hand, when $\alpha_2 = 60^\circ$, the transmitter loop’s primary electric field now induces currents both parallel and perpendicular to the bedding plane. As a result, now the induced current and sensed resistivity $R_{ap,Amp}$ is also affected by $R_{v2} = \kappa_2^2 R_{h2}$, leading to a higher value of $R_{ap,Amp}$ (as qualitatively corroborated by Eq. (2.5.2)).

In Figure 2.10, where there is a high resistivity contrast of $R_{h2} = 12.5R_{h1}$, we notice a greater level of discrepancy. This is particularly so just beneath the interface at $z_B = 0\text{ft}$, where the reflected fields are strongest. In the well-logging community, one refers to the phenomenon where conductive formations adversely reduce the apparent resistivity sensed in their resistive neighbors as the “shoulder bed effect” [85].

In Figure 2.11, we again note a high resistivity contrast of $R_{h1} = 200R_{h2}$. Comments dual to those made on Figure 2.10 apply here regarding (1) the resistivity log’s notable deviation in the top isotropic region from the true resistivity of 100Ω m and (2) the greater disagreement versus [9].

In Figure 2.12, the resistivity contrast is low ($R_{h1} = 4R_{h2}$). Like in Figure 2.9, we note that there is excellent agreement.

Now we comment upon Figures 2.13-2.14. The data from [9] suggest a very strong shoulder bed effect present in the top and bottom isotropic half-spaces when $\alpha_2 = 0^\circ$, leading to notable disagreement for Figures 2.13a and 2.14a. There is also notable discrepancy in modeling the formation interface “horns” and resistivity valleys (see, in particular, the infinite-resistivity spike in Figure 2.13f). However, the data sets in [9] are not free of infinite-resistivity spikes either (see Figures 14 and 23 in [9]), suggesting that the resistivity inversion and data post-processing methods used (and their differences between here and [9]) are causing the observed discrepancies. These
quantitative discrepancies aside, however, we notice excellent *qualitative* agreement in modeling the shoulder bed effect, as well as the interface horns and valleys due to the high-dipping-angle uniaxial bed.

![Magnitude-apparent resistivity log comparison with Figure 6 of [9]: $\kappa_1 = 1, \kappa_2 = \sqrt{20}, R_{h1} = 2\Omega \text{m}, R_{h2} = 0.5\Omega \text{m}, \beta_2 = 0^\circ, f = 2\text{MHz}, L = 40\text{in}, z_B = 0\text{ft. In Figures 2.9a-2.9b the respective dip angles of the bottom formation are } 0^\circ \text{ and } 60^\circ.$]
Figure 2.10: Magnitude-apparent resistivity log comparison with Figure 7 of [9]: $\kappa_1 = 1, \kappa_2 = \sqrt{20}, R_{h1} = 20 \Omega \text{m}, R_{h2} = 25 \Omega \text{m}, \beta_2 = 0^\circ, f = 2 \text{MHz}, L = 40 \text{in}, z_B = 0 \text{ft}$. In Figures 2.10a-2.10b the respective dip angles of the bottom formation are $0^\circ$ and $60^\circ$.

Figure 2.11: Magnitude-apparent resistivity log comparison with Figure 8 of [9]: $\kappa_1 = 1, \kappa_2 = \sqrt{20}, R_{h1} = 100 \Omega \text{m}, R_{h2} = 0.5 \Omega \text{m}, \beta_2 = 0^\circ, f = 2 \text{MHz}, L = 40 \text{in}, z_B = 0 \text{ft}$. In Figures 2.11a-2.11b the respective dip angles of the bottom formation are $0^\circ$ and $60^\circ$. 
Figure 2.12: Magnitude-apparent resistivity log comparison with Figure 9 of [9]: $\kappa_1 = 1, \kappa_2 = \sqrt{20}, R_{h1} = 100\Omega\text{m}, R_{h2} = 25\Omega\text{m}, \beta_2 = 0^\circ, f = 2\text{MHz}, L = 40\text{in}, z_B = 0\text{ft}$. In Figures 2.12a-2.12b the respective dip angles of the bottom formation are $0^\circ$ and $60^\circ$. 
Figure 2.13: Magnitude-apparent resistivity log comparison with Figure 11 of [9]: $\kappa_1 = \kappa_3 = 1, \kappa_2 = 5, R_{h1} = R_{h3} = 40\, \Omega \, m, R_{h2} = 2\, \Omega \, m, \beta_2 = 0^\circ, f = 2\, MHz, L = 40\, in, z_B = \{2.5, -2.5\} \, ft$. In Figures 2.13a-2.13f the respective dip angles of the center formation are as follows: 0°, 45°, 60°, 70°, 80°, 90°.
Figure 2.14: Magnitude-apparent resistivity log comparison with Figure 13 of [9]: \( \kappa_1 = \kappa_3 = 1, \kappa_2 = 5, R_{h1} = R_{h3} = 40 \Omega \text{ m}, R_{h2} = 2 \Omega \text{ m}, \beta_2 = 0^\circ, f = 2 \text{MHz}, L = 40 \text{in}, z_B = \{10, -10\} \text{ ft}. \) In Figures 2.14a-2.14f the respective dip angles of the center formation are as follows: 0°, 45°, 60°, 70°, 80°, 90°.
2.5.4 Dipole Fields Near a PEC-Backed Microwave Substrate

The last validation result concerns a $y$-directed Hertzian electric dipole on top of a dielectric substrate supported by a metallic ground plane [10]). The ground is modeled as a semi-infinite layer with conductivity $\sigma = 10^9$ S/m. We compute the radiated $H_x$, $H_z$, and $E_y$ components. This environment is meant to highlight the algorithm’s ability to simulate magnetic fields produced by an electric, rather than magnetic source and thus (from the duality theorem) its ability to compute magnetic and electric fields from both electric and magnetic sources. By simulating a case with $4\lambda_o \leq |x - x'| \leq 14\lambda_o$ ($\lambda_o = 37.5$m), we also provide here an example of the general-purpose nature of the algorithm in regards to the $r - r'$ geometry. We emphasize that this flexibility is primarily attributed to the adaptive extension of the original MMA as discussed in Section 2.3. NB: The reference 39 curve label in Figure 2.15b, corresponding to reference 39 in the original publication [1] describing this chapter’s presented algorithm, corresponds to reference [10] within this thesis.

Figure 2.15b below shows excellent agreement, in the range $4.25\lambda_o \leq |x - x'| \leq 13.6\lambda_o$, with the available data from [10]. Figures 2.15a and 2.15c shows similar results for the other two components. The oscillatory behavior results from interference effects caused by the ground plane. To facilitate easier comparison with [10] and exhibit the three field magnitude variations on identical scales, all three data sets were scaled such that their maximum magnitudes correspond to the maximum magnitude of $H_z$ in [10].
Figure 2.15: Field component intensities from a $y$-directed Horizontal Electric Dipole (HED), which is radiating at $f = 8$MHz ($\lambda_o = 37.5$m), centered at the origin, and supported on a grounded dielectric substrate $4\lambda_o$ thick with free space above. The substrate’s dielectric constant is $\epsilon_r = 3.3(1 + 0.01i)$, while $y - y' = 0$m and $z - z' = 3$m. Only $|H_z|$ reference data were published in [10].
2.6 Convergence Characteristics

To characterize our numerical formulation’s ability to converge towards the field solution, we present two case studies concerning the $z$-directed magnetic field component $H_z$ produced by a $z$-directed magnetic dipole radiating at $f=2\text{MHz}$ in free space. The first case comprises a benign scenario in which $\mathbf{r} - \mathbf{r}' = (1, 1, 1)\text{m}$, while the second case represents a much more challenging scenario where $\mathbf{r} - \mathbf{r}' = (500, 500, 1)\text{m}$ in which the integrand oscillates on the order of 500 times more rapidly than the first case. For both cases, we choose $x - x' = y - y'$ to ensure the code faces the same convergence challenges when evaluating both the $k_x$ and $k_y$ integrals. Furthermore, we set the pre-extrapolation region magnification factor $l_o$ (see section 2.4.1) equal to ten and artificially set $\xi_1 = 2P_k$ to facilitate characterization of the interval subdivision factor $h$, with which one quantifies the sub-interval lengths after full interval sub-division, as $h\xi_1$.

For each case, we present results related to both the pre-extrapolation and extrapolation domain characteristics. To avoid mixing the numerical formulation’s handling of the pre-extrapolation and extrapolation region sections of the $k_x$ and $k_y$ integration paths, the “pre-extrapolation domain” (termed “Region 1” below) refers to the region $(-\xi_1 < k'_x < \xi_1) \cup (-\xi_1 < k'_y < \xi_1)$. Similarly, the “extrapolation domain” (termed “Region 2” below) refers to the region $(k'_x > \xi_1) \cup (k'_x > -\xi_1) \cup (k'_y > \xi_1) \cup (k'_y < -\xi_1)$. Since one cannot obtain closed-form solutions to the pre-extrapolation and extrapolation domain contributions, reference field values from which one measures accuracy must be chosen; their computation details are provided in Figures 2.16-2.17 below.

For the pre-extrapolation domain study, we exhibit the accuracy obtained versus $(h)$ and the Patterson-Gauss quadrature order $(p)$ used to integrate each sub-interval.
We notice that for both cases, there is the expected increase in accuracy both as one reduces $h$ and increases $p$.

For the extrapolation domain contribution, we make the typical assumption [12, 71, 70] that the integrand is well-behaved in this portion of the spectral domain and thus do not perform interval sub-division. Instead, we set the $k_x$ and $k_y$ plane extrapolation region interval lengths to be half the spectral period of the Fourier kernels $\exp[i k_x (x - x')]$ and $\exp[i k_y (y - y')]$ (resp.), as suggested in [71], and examine the variation of accuracy versus the number of extrapolation region intervals employed ($B$) and the Legendre-Gauss quadrature order used ($LGQ$) to integrate each interval\(^{28}\). For the extrapolation domain field contribution, we notice that as one increases $LGQ$ and $B$ there is the expected decay in error. In particular, for small $B$ ($B \sim 3$ for both cases) we notice that tail integral truncation effects dominate the region two error. On the other hand, after a certain value of $B$ ($B \sim 10$ for case 1 and $B \sim 6$ for case 2), we find that aliasing/sampling effects dominate the error.

Note that for all figures below, errors below -150dB were coerced to equal -150dB. This is because error levels below approximately -150dB do not represent error levels attained due to the convergence characteristic of the formulation itself, but instead represent instances wherein the given and reference answers are equal in all the digits available using finite, double-precision arithmetic.

\(^{28}\) $B$ intervals are used in both the $k'_{x} > 0$ and $k'_{y} < 0$ integration path half-tails; the same applies for the $k_y$ path half-tails.
Figure 2.16: Convergence towards the solution comprising the field contribution from “Region 1”. The reference field values are computed using $p=31$ for both figures, as well as $-\log_2(h)=9$ for Figure 2.16a and $-\log_2(h)=11$ for Figure 2.16b. The reference field values computed for Figures 2.16a and 2.16b use different $h$ because in the latter scenario, $H_z$ converges more slowly and thus necessitates smaller $h$ values in the non-reference field results to show a meaningful decay in error. As a result, one also requires an even smaller $h$ for the reference field result from which the relative error is computed.
Figure 2.17: Convergence towards the solution comprising the field contribution from “Region 2”. The reference field values are computed using LGQ=30 for both figures, as well as $B = 150$ for Figure 2.17a and $B = 1000$ for Figure 2.17b. The reference field values computed for Figures 2.17a and 2.17b use different $B$. This is because in the latter scenario, as can be observed, $H_z$ converges more slowly; indeed, while $H_z$ in case two levels off more rapidly than in case one, it fails to reach accuracy near to machine precision within the same range of $B$ exhibited for both cases. Thus similar reasoning applies as that behind using smaller $h$ for the reference and non-reference field results in Figure 2.16b (versus Figure 2.16a).
2.7 Conclusion

We have presented a general-purpose and efficient pseudo-analytical formulation to compute electromagnetic fields from dipole sources in planar-stratified environments with arbitrary anisotropy, loss, and \( r - r' \) geometries. The formulation is based on embedding spectral Green’s Function kernels within Fourier-type integrals to compute the space-domain fields. Some of the salient features that are combined here to yield a robust algorithm are: (a) judicious selection of a numerically robust integration path, (b) re-casting of critical formulae to facilitate accurate field computations and obviate numerical overflow, (c) adaptive integration along the pre-extrapolation region of the integrals, and (d) adaptive extension of the original MMA, applied to environments containing media with anisotropy and loss, both to accelerate the tail integral’s convergence and to endow error control to its evaluation. The formulation’s accuracy has been validated through four sets of numerical data and its convergence properties characterized.
Chapter 3: Complex-Plane Generalization of Scalar Levin
Transforms: A Robust, Rapidly Convergent Method to
Compute Potentials and Fields in Multi-Layered Media

3.1 Introduction

3.1.1 Chapter Summary and Contributions

We propose the complex-plane generalization of a powerful algebraic sequence
acceleration algorithm, the Method of Weighted Averages (MWA), to guarantee
exponential-cum-algebraic convergence of Fourier and Fourier-Hankel (F-H) integral
transforms. This “complex-plane” MWA, effected via a linear-path detour in the com-
plex plane, results in rapid, absolute convergence of field/potential solutions in multi-
layered environments regardless of the source-observer geometry and anisotropy/loss
of the media present. In this work, we first introduce a new integration path used to
evaluate the field contribution arising from the radiation spectra. Subsequently, we
(1) exhibit the foundational relations behind the complex-plane extension to a gen-
eral Levin-type sequence convergence accelerator, (2) specialize this analysis to one
member of the Levin transform family (the MWA), (3) address and circumvent restric-
tions, arising for two-dimensional integrals associated with wave dynamics problems,
through minimal complex-plane detour restrictions and a novel partition of the integration domain, (4) develop and compare two formulations based on standard/real-axis MWA variants, and (5) present validation results and convergence characteristics for one of these two formulations.\textsuperscript{29}

### 3.2 Definitions and Conventions

We state the following regarding notation used in this chapter:

1. $i$ is the unit-magnitude imaginary number.

2. The time-harmonic field convention used and suppressed throughout this paper is $\exp(-i\omega t)$, where $\omega$ is the angular frequency at which the source distribution radiates.

3. $c$ is the speed of light in free space.

4. $k_o = \omega/c$ is the characteristic wave number of free space.

5. $r = (x, y, z)$ denotes the observer location, while $r_t = (x, y)$ denotes the transverse observer location with magnitude $\rho = \sqrt{x^2 + y^2}$.

6. $r' = (x', y', z')$ denotes the source location, while $r'_t = (x', y')$ denotes the transverse source location with magnitude $\rho' = \sqrt{x''^2 + y''^2}$.

7. $k = (k_x, k_y, k_z)$ denotes the wave vector, while $k_t = (k_x, k_y)$ denotes the transverse wave vector with complex amplitude $k_\rho = \sqrt{k_x^2 + k_y^2}$. It is implicitly understood that one evaluates $\sqrt{k_x^2 + k_y^2}$ such that $H_n^{(1)}(k_\rho|\rho - \rho'|)$ (see definition

\textsuperscript{29}NOTE: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.
10 below) corresponds to an exponentially decaying function versus increasing $|\rho - \rho'|$.

8. The axial wave number component for the $n$th characteristic mode supported in layer $M$, $\tilde{k}_{M,nz}$, is coupled to the transverse wave numbers $k_x$ and $k_y$ via the $n$th mode’s dispersion relation.

9. The up-going or down-going mode’s axial wave number component in some isotropic layer is denoted by $\tilde{k}_z^+$ or $\tilde{k}_z^-$ (resp.) when the particular layer is not critical to understanding the discussion.

10. $H_n^{(1)}(k_\rho |\rho - \rho'|)$ is the order-$n$ Hankel function of the first kind, corresponding to an out-going cylindrical wave.

11. Quantities dependent only on one or more spectral variables $\{k_x, k_y, k_z\}$ are denoted spectral quantities and are distinguished with an over-tilde (e.g. $\tilde{f}(k_x, k_y, k_z)$).

12. Quantities dependent on $k_t$, $z$, and $z'$ are denoted mixed-domain quantities and have no over-symbol.

13. Numbers expressed as $\binom{A}{B}$ correspond to the binomial coefficients.

14. $\epsilon_{m,r}$ and $\mu_{m,r}$ represent the relative electric permittivity (including conductive and polarization losses) and relative magnetic permeability of isotropic layer $m$.

15. Re$[k_x]$ and Im$[k_x]$ are used interchangeably with $k'_x$ and $k''_x$ (resp.) to denote the real and imaginary part of $k_x$ (resp.). Analogous definitions apply for $k_y$ and other complex-valued quantities.
3.2.1 Background

In many application areas concerning time-harmonic electromagnetic (EM) fields, one encounters environments containing media of varying and arbitrary anisotropy\textsuperscript{30} whose inhomogeneity can be approximated as multi-layered in nature. Examples include geophysical prospection [13, 8, 9, 7, 11, 24, 14], plasma physics [67], antenna design [64, 65], optical field control [63], microwave remote sensing [60], ground-penetrating radar [61, 62], and microwave circuits [66], among others. Such applications regularly encounter integrals of the form\textsuperscript{31}

\[ f(\mathbf{r}) \sim \int_{-\infty}^{+\infty} \tilde{f}(k_x, k_y)e^{ik_x(x-x')+ik_y(y-y')+ik_z(z-z')} \, dk_x \, dk_y \]  

and/or

\[ f(\mathbf{r}) \sim \int_{-\infty}^{+\infty} \tilde{f}(k_\rho)H_n^{(1)}(k_\rho|\rho-\rho'|)e^{ik_z(z-z')} \, dk_\rho \]  

which express space-domain field/potential functions as Fourier and Fourier-Hankel (F-H) integral transforms (resp.).

In many practical applications, these integrals must often be rapidly evaluated for a wide range of longitudinal and transverse source-observer separation geometries \( \mathbf{r} - \mathbf{r}' \neq \mathbf{0} \) (e.g. for potential or field profile reconstruction). However, when using standard integration paths that run on/close to the real axis such as (1) the classic Sommerfeld Integration Path (SIP) [45][Ch. 2] and (2) paths detouring around the branch points, branch cuts, and poles followed by real-axis integration [70, 71] (also, c.f. Ch. 2), the convergence rate of these integrals is strongly dependent upon the

\textsuperscript{30}We assume each medium’s anisotropy manifests in diagonalizable constitutive material tensors to ensure completeness of the plane wave basis. Since all naturally-occurring media possess diagonalizable material tensors, in practical applications this assumption is always true.

\textsuperscript{31}The previous sub-section (3.2) summarizes the notation, terminology, and conventions used here.
transverse \((r_t - r'_t)\) and longitudinal \((z - z')\) separations. \(r_t - r'_t\) determines the rapidity of the integrand’s oscillation due to the Fourier and/or Hankel kernels in (3.2.1)-(3.2.2), with rising \(|\rho - \rho'|\) leading to an integrand that traditionally requires increasingly finer sampling to limit spatial aliasing and thus leads to undesirably long computation times. Furthermore, the longitudinal separation \(z - z'\) governs the rate at which the evanescent spectrum’s field contribution decays with increasing transverse wave number magnitudes\(^{32}\), with rising \(|z - z'|\) effecting more rapid decay (and hence faster convergence) [12]. On the other hand, as \(|z - z'| \to 0\) the convergence rate lessens, with the limiting case \(z - z' = 0\) yielding integrals of the form

\[
f(r) \sim \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{f}(k_x, k_y) e^{ik_x(x-x') + ik_y(y-y')} \, dk_x \, dk_y
\]

(3.2.3)

and

\[
f(r) \sim \int_{-\infty}^{+\infty} \tilde{f}(k_\rho) H_n^{(1)}(k_\rho |\rho - \rho'|) \, dk_\rho
\]

(3.2.4)

that lead to divergent results when numerically evaluated, using these standard paths, without convergence acceleration.

See Figure 3.1 for typical application scenarios wherein these standard paths either succeed or fail to deliver accurate field results. Observing Figure 3.1, one immediately realizes that devising an evaluation method for these integrals exhibiting robustness with respect to all ranges of \(r - r' \neq 0\) and medium classes (e.g. isotropic, uniaxial, bi-axial) is highly desirable. This robustness criterion inherently excludes fundamentally approximate methods such as image and asymptotic methods due to their geometry-specific applicability and lack of rigorous error control [86, 70, 73, 76] [45][Ch. 2]. As a result, to reliably ensure accurate field results for arbitrary environmental medium

\(^{32}\)i.e. \(|k_x|\) and \(|k_y|\) for Fourier double-integrals, or \(|k_\rho|\) for F-H integrals.

58
composition/source-observer geometry combinations, we choose a direct numerical integration method.

In this vein, one option involves pairing standard integration methods with (real-axis path based) algebraic convergence acceleration techniques such as the standard MWA which, based on published numerical results, successfully imparts algebraic convergence acceleration even when $|z - z'| = 0$ [12, 71]. However, it is desirable to (1) guarantee absolute, exponential convergence in the classical/Riemann sense for any $r - r' \neq 0$ separation geometry (in contrast to only guaranteeing algebraic convergence in the Abel sense when $|z - z'| = 0$ [12]) and (2) endow error control to the evanescent-zone field contribution associated with the tail integral, whose relative importance (compared to the radiation-zone contribution) to the field solution grows as $|r - r'|$ decreases, to ensure that both the radiation-zone and evanescent-zone contributions are accurately evaluated\(^{33}\). To this end, we propose a novel numerical integration method, representing a complex-plane generalization of a specific member of the “scalar Levin-type sequence transform” (SLST) family [84] (i.e. the MWA), that:

1. bends the “extrapolation region”/tail (c.f. Ch. 2) integration path sections to guarantee absolute, exponential convergence of integrals like (3.2.1)-(3.2.4),

2. imparts added, robust algebraic convergence acceleration to the tail integrals, which compounds with the exponential convergence acceleration to effect absolute, exponential-cum-algebraic convergence, via use of a linear path bend combined with our novel, complex-plane generalization of the MWA [71, 12],

\(^{33}\)One cannot rely upon a-posteriori error checking, as was done in [71, 12], for general environment/source-observer scenarios.
3. adjusts the detour bend angles to account for the presence of branch points, branch cuts, and poles (summarily referred to here as “critical points”), and

4. addresses the added challenges associated with evaluating \textit{two-dimensional} integral transforms arising as solutions to the wave equation in planar-stratified environments lacking azimuthal symmetry.

We note that other path deformation techniques, such as the Steepest Descent Path (SDP) and one comprising the enclosure of the first/fourth quadrants of the $k_p$ plane involving an imaginary-axis integration, have been investigated and used \cite{87, 88, 74} \cite{45}\cite{Ch. 2}. However, we seek a \textit{robust} integration method, valid for all $r - r' \neq 0$ geometries, that obviates having to separately account for discrete poles while possessing applicability to multi-layered environments containing media with arbitrary anisotropy and loss. Thus while our method may result in longer solution times versus above-mentioned methods, it touts general applicability and minimal necessary book-keeping as its defining virtues.

Furthermore, a robust detour path within the pre-extrapolation region (c.f. Ch. 2) maintaining a near-constant separation between the path and critical points near/on the real axis would be preferred over more traditional paths used with the MWA \cite{70} (also, c.f. Ch. 2). To address this, the paper’s second contribution entails a trapezoidal integration path paired with adaptive $hp$ refinement\textsuperscript{34}.

In Section 3.3 we present and discuss our revision to the radiation-zone integration path\textsuperscript{35}. In Sections 3.4 and 3.5 we develop the detoured linear integration path and complex-plane generalization to SLST for efficiently evaluating the tail sections

\textsuperscript{34}The adaptive $hp$ refinement integration methodology is the same as that discussed in Ch. 2.

\textsuperscript{35}We present this secondary contribution first for fluidity in the narrative.
of (3.2.1)-(3.2.4), as well as exhibit and compare two possible candidate formulations to implement the resulting modified-MWA. These developments are formulated in the context of two-dimensional integrals such as (3.2.1) to simultaneously address herein their additional issues versus one-dimensional integrals. However, the formulation applies equally to one-dimensional F-H transforms like (3.2.2) appearing in field/potential computations within cylindrically- and (azimuthal-symmetric) planar-stratified environments and, after converting the Fourier-Bessel (F-B) transform to a F-H transform [33] [45][Chs. 2-3], to F-B transforms as well\(^{36}\). Section 3.6 presents validation results using one of the two new formulations. In Section 3.7 we present a study on the convergence characteristics of our algorithm as concerning the same formulation used to generate the results in Section 3.6. Finally, Section 3.8 contains our concluding remarks.

In the ensuing discussion, we assume appropriate transformations to the material tensors and source vector have already been performed to effect a coordinate rotation such that in the resultant (azimuthal-rotated) coordinate frame, within which all integration is performed, one has \(x - x' = y - y' \geq 0\) \(^{37}\). Discussed in detail at the end of Section 3.4, this is done to guarantee absolute convergence and maximize exponential decay of both the \(k_x\) and \(k_y\) integrals.

\(^{36}\)One can accommodate the logarithmic branch-cut, manifest on the \(-\text{Re}[k_{\rho}]\) axis for F-H transforms [45][Ch. 2], through a slight perturbation of the \(\text{Re}[k_{\rho}] < 0\) half-plane path into the second quadrant.

\(^{37}\)More generally, if \(|x - x'| = |y - y'|\) in the rotated frame the method will work. Of course, rotating such that \(x - x' \leq 0\) forces one to alter the \(k_x\) plane extrapolation region path such that it now incurs into the \(\text{Im}[k_x] < 0\) half-plane (and similarly for \(k_y, y - y'\)).
Figure 3.1: Figure 3.1a depicts a “triaxial” hydrocarbon sensor system [11] of three loop antenna transmitters \( \{ M_T \} \) and three loop antenna receivers \( \{ M_R \} \) traversing a vertical/moderately-inclined logging path bounded by a borehole (dark gold lines). Here, one typically finds \( |z - z'| \) large enough to use standard numerical integration methods, based on real-/near real-axis paths, without convergence acceleration. On the other hand, Figure 3.1b shows the same sensor system traversing a horizontal path while Figure 3.1c exhibits a micro-strip geometry in which the user requests the field distribution at the air-substrate interface. The two latter geometries exhibit \( 0 \leq |z - z'| \ll 1 \) and represent scenarios for which these standard methods typically yield divergent results due to the oscillatory-divergent nature of integrals like (3.2.3)-(3.2.4).
3.3 Pre-Extrapolation Region Path Revision

First we discuss the parameterization and initial sub-division of the $k_x$ plane pre-extrapolation region; discussion of the $k_y$ plane follows identically due to our assuming $x - x' = y - y' \geq 0$. Applying a parameterization similar to that in [70] (also, c.f. Ch. 2), define $\pm P_k$ as the points on the Re[$k_x$] axis within which one detours, $d_x$ as the maximum height of the trapezoid-shaped detour, and $\pm \xi_1$ as the points on the Re[$k_x$] axis within which one adaptively integrates (see Figure 3.2).

To compute $\pm P_k$, first define $n^+$ as the magnitude of the real part of the global “effective” refractive index among all the layers (see Ch. 2 on computing $n^+$). One then computes $P_k$ analogously to [70] and sets $P_k = l_0k_o(n^+ + 1)$, where $l_0 \geq 1$ is a user-defined pre-extrapolation region magnification constant. Next, define $\Delta x = |x - x'| = \Delta y = |y - y'|$, $\Delta z = |z - z'|$, $a'$, and $b'$, where $0 < a' < 1/2$ and $b' > 1$. Now compute the following pre-extrapolation region integration path parameters [70] (also, c.f. Ch. 2):

\[
Q_k = a'k_o(n^+ + 1) \quad (3.3.1)
\]
\[
d_x = \ln(b')/\max(T_0, \Delta x) \quad (3.3.2)
\]
\[
\beta_x = \tan^{-1}\frac{d_x}{Q_k}, 0 < \beta_x < \pi/2 \quad (3.3.3)
\]
\[
\Delta \xi_x = \pi/\max(T_0, \Delta x) \quad (3.3.4)
\]
\[
\xi_1 = (\text{Int}(P_k/\Delta \xi_x) + 1) \Delta \xi_x \quad (3.3.5)
\]

38This detour allows magnification of $P_k$ without compromising the detour height near critical points, which represents one of two primary benefits compared to the half-sine-shaped contour [70] (also, c.f. Ch. 2).

39$T_0 > 0$ limits the extrapolation region sub-interval length $\Delta \xi_x$ when $\Delta x \ll 1$ to ensure the extrapolation intervals (see Sections 3.3-3.4) are adequately sampled, thereby limiting spatial aliasing.
where Int(·) converts its argument to an integer via fractional truncation. Now parameterize the pre-extrapolation region integration path, for Re[k_x] > 0, as

\[
k_x = \begin{cases} 
  r (\cos \beta_x - i \sin \beta_x), & 0 < r < |Q_k + i d_x| \\
  r - i d_x, & Q_k < r < P_k - Q_k \\
  P_k - Q_k - i d_x + r (\cos \beta_x + i \sin \beta_x), & 0 < r < |Q_k + i d_x| 
\end{cases}
\]

(3.3.6)

\[
\frac{\partial k_x}{\partial r} = \begin{cases} 
  \cos \beta_x - i \sin \beta_x, & 0 < r < |Q_k + i d_x| \\
  1, & Q_k < r < P_k - Q_k \\
  \cos \beta_x + i \sin \beta_x, & 0 < r < |Q_k + i d_x| 
\end{cases}
\]

(3.3.7)

for the trapezoidal contour (used to integrate up to k_x = P_k) combined with a real-axis path to integrate within the section P_k ≤ k_x ≤ ξ_1. An analogous parameterization holds for the Re[k_x] < 0 pre-extrapolation region path. Note that ∂k_x/∂r is independent of k_x and thus can be computed prior to integration, unlike other commonly used detours\(^{40}\). This is the trapezoidal path’s second benefit in addition to that mentioned in footnote 38.

Now we splice the regions (0, P_k) and (−P_k, 0) each into P regions, where P is calculated as follows. First define

\[
d' = \text{abs} \left( \frac{\partial e^{ik_x \Delta x}}{\partial r} \bigg|_{r=|Q_k+i d_x|} \right) = \sin(\beta_x) \Delta x e^{d_x \Delta x}
\]

(3.3.8)

as the largest magnitude assumed by ∂e^{ik_x \Delta x}/∂r along the trapezoidal path, c’ as the user-defined maximum allowed magnitude change of e^{ik_x \Delta x} between two sampling points, and T_1 and T_2 as two user-defined parameters. Subsequently, define the

\(^{40}\)such as, e.g., the half-sine-shaped detour [70] (also, c.f. Ch. 2)
quantities

\[ \Delta k_1 = \min \left( \frac{\pi}{(T_{1\max}(\Delta x, \Delta z))}, \frac{c'}{d'} \right) \]  \hspace{1cm} (3.3.9)

\[ \Delta k_2 = \frac{\pi}{(T_{1\max}(\Delta x, \Delta z))} \]  \hspace{1cm} (3.3.10)

\[ N_{\text{node}, 1} = \text{Int} \left( \frac{|Q_k + id_x|}{\Delta k_1} \right) + 1 \]  \hspace{1cm} (3.3.11)

\[ N_{\text{node}, 2} = \text{Int} \left( \frac{(P_k - 2Q_k)}{\Delta k_2} \right) + 1 \]  \hspace{1cm} (3.3.12)

which are used to yield \( P_m = \text{Int}(1 + N_{\text{node}, m}/T_2) \) \((m=1,2)\) with the corresponding final result \( P = 2P_1 + P_2 \). Note that this method of parameterizing the pre-extrapolation region path is empirical in nature and based on the pessimistic assumption of equidistant sampling (c.f. Ch. 2).
Figure 3.2: Figures 3.2a and 3.2b depict the new and old integration $k_x$ plane integration paths used in this chapter and chapter two (resp.). “Radiation BC Map” and “Program BC Map” refer to the branch cuts associated with the radiation/boundedness condition at infinity ($\text{Im}[\tilde{k}_z^2] = 0$, $\text{Re}[\tilde{k}_z^2] > 0$) and the computer program’s square root convention ($\text{Im}[\tilde{k}_z^2] = 0$, $\text{Re}[\tilde{k}_z^2] < 0$) (resp.). The encircled “X” symbols represent branch points and the red “X” symbols represent guided mode poles. For $K$ extrapolation intervals used in the bottom or top method, the red contour represents the integration path connecting the end-points $k_x = (-\xi_1 - K\Delta\xi_x, \xi_1 + K\Delta\xi_x)$ or $k_x = (-\xi_1 - t_o^- K\Delta\xi_x^{-'}, \xi_1 + t_o^+ K\Delta\xi_x^{'+'})$ (resp.); see Sections 3.3-3.4 for definitions of $\Delta\xi_x$, $\Delta\xi_x^{-'}$, $\Delta\xi_x^{'+'}$, $t_o^-$, and $t_o^+$.
3.4 Extrapolation Region Path Revision

The MWA, initially constructed in [66, 75] with further variants developed in [71] and [12], has also demonstrated the ability to accelerate convergence of infinite-range Fourier double-integrals in high-loss, planar-stratified environments containing anisotropic media (c.f. Ch. 2). However, due to the highly oscillatory behavior of the mixed-domain integrand in integrals such as (3.2.1) arising from the Fourier kernels $e^{ik_x\Delta x}$ and $e^{ik_y\Delta y}$ when one has large $\Delta x$ and $\Delta y$ (resp.), the solution times (in our experience) became inordinately long. Therefore, it would be desirable to also deform the $k_x$ and $k_y$ plane “extrapolation” region contours to lend additional exponential decay via these two kernels, thereby dramatically accelerating convergence of the Fourier tail integrals and guaranteeing their absolute convergence even in the “worst-case” scenario $z - z' = 0$. A cursory analysis reveals an apparent severe drawback, however: one can no longer employ the MWA, which was derived assuming a real axis integration path [12, 66, 75, 71]. However, choosing a linear deformed path retains the MWA’s algebraic convergence acceleration, as we show below\textsuperscript{41}. For this analysis, take $k_x$ ($k_y$) as the inner (outer) integration variable.

We first exhibit the foundational relations needed to implement the complex-plane extension to a general SLST followed by exhibiting the specific case arising from modeling the tail integral truncation error using the function family stipulated in the “Mosig-Michalski Algorithm” (MMA) [71, 84]. Subsequently, we naively compute the optimal extrapolation region path detour angles without consideration for

1. the presence of critical points in the $k_x$ and $k_y$ planes and

\textsuperscript{41}The MWA is retained for robustness in field solution acceleration; the mathematical and environmental constraints present typically prevent one from integrating along the ideal Constant Phase Path, as discussed below.
2. **two-dimensional** integrals, associated with wave propagation phenomena, imparting a transitory nature to these critical points in the $k_x$ plane (i.e. their locations now depend on the fixed $k_y$ value for which the $k_x$ integral is evaluated).

To address the first concern, we pessimistically estimate the locations of critical points and reduce the $k_x$ plane departure angle of the deformed paths to ensure these features are not crossed. To address the latter concern, we (1) adjust the departure angles of the $k_y$ plane integration path and (2) partition the $k_x - k_y$ integration domain to ensure that the critical points

1. possess real parts with magnitude decaying as $|k_y|$ increases, leading to a bounded pre-extrapolation region, and

2. do not extend into the second/fourth quadrants, as this would require a) tracking their locations and b) adjusting the $k_x$ integration path, both of which would become functions of $k_y$ and lead to a non-robust integration path.

For simplicity, the analysis developing the complex-plane SLST generalization assumes isotropic planar layers\(^{42}\). Fix $k_y$ at some (generally complex) value $k_{y0}$ and assume $z - z' \ge 0$\(^{43}\); we see then that the inner integral of (3.2.1) writes as

$$
\int_{-\infty}^{\infty} \tilde{f}(k_x, k_{y0}) e^{ik_x \Delta x} e^{ik_{y0} \Delta y} e^{ik_{y0}^+ \Delta z} \, dk_x
$$

(3.4.1)

where $\tilde{k}_y^+$ is the up-going mode propagation constant, which for our time convention has positive imaginary part. Assuming $N$ extrapolation intervals are used [71] (also,\(^{42}\)We justify this assumption based on previous analysis and results discussed in Ch. 2.

\(^{43}\)In Sections 3.6-3.7 we demonstrate that the linear detour assures rapid convergence even in the regime $0 \le |z - z'| \ll 1$.\)
c.f. Ch. 2), the linear path detour used in the \(k_x\) integration path’s extrapolation region is parameterized as

\[
k_x = \begin{cases} 
\xi_1 + r_x (\cos \gamma^+ + i \sin \gamma^+) & , 0 \leq r_x \leq N\Delta_\xi^+, \Re[k_x] > 0 \\
-\xi_1 + r_x (\cos \gamma^- - i \sin \gamma^-) & , -N\Delta_\xi^- \leq r_x \leq 0, \Re[k_x] < 0
\end{cases}
\]

(3.4.2)

\[
\frac{\partial k_x}{\partial r_x} = \begin{cases} 
\cos \gamma^+ + i \sin \gamma^+ & , 0 \leq r_x \leq N\Delta_\xi^+\gamma, \Re[k_x] > 0 \\
\cos \gamma^- - i \sin \gamma^- & , -N\Delta_\xi^- \leq r_x \leq 0, \Re[k_x] < 0
\end{cases}
\]

(3.4.3)

where one defines \(\Delta_\xi^+ = \Delta_\xi/\cos \gamma^+\), \(\Delta_\xi^- = \Delta_\xi/\cos \gamma^-, \{\gamma^+, \gamma^-\} \geq 0\), and \(r_x\) as real-valued. \(\xi_1\) is assumed large enough to ensure that we have sufficiently detoured past any critical points near the real axis [12]. Now recall that plane wave propagation in a homogeneous, unbounded, isotropic medium with wave number \(k\) is governed by the dispersion relation \((\tilde{k}_z^+)^2 = (\tilde{k}_z^-)^2 = (k' + ik'')^2 - k_z^2 - k_{y0}^2\) [45][Ch. 1]. For large \(|k_x|\) this relation becomes\(^\text{44}\)

\[
\tilde{k}_z^+ \rightarrow i \left\{ \begin{align*}
\xi_1 + r_x (\cos \gamma^+ + i \sin \gamma^+) & , \Re[k_x] > 0 \\
\xi_1 - r_x (\cos \gamma^- - i \sin \gamma^-) & , \Re[k_x] < 0
\end{align*} \right.
\]

(3.4.4)

Next, assuming that (asymptotically) \(\tilde{f}(k_x, k_{y0}) \rightarrow k_x^q \sum_{m=0}^\infty \tilde{a}_m(k_{y0})\tilde{\phi}_m(k_x)\) [71, 84], one can substitute this series expression into the extrapolation region section of (3.4.1) to obtain

\[
\int_{\text{ext}} k_x^q \sum_{m=0}^\infty \tilde{a}_m(k_x)\tilde{\phi}_m(k_x)e^{ik_x\Delta_x}e^{ik_{y0}\Delta_y}e^{i\tilde{k}_z^+\Delta_z}dk_x
\]

(3.4.5)

where “ext” denotes the \(k_x\) plane extrapolation region integration path section and the \(\{\tilde{\phi}_m\}\) comprise a family of functions used to asymptotically model \(\tilde{f}(k_x, k_{y0})\) and the truncation error (discussed below) [84]. Setting \(t_o^+ = \cos \gamma^+ + i \sin \gamma^+, t_o^- = \cos \gamma^- - i \sin \gamma^+, l^+ = t_o^+e^{ik_{y0}\Delta_y}e^{\xi_1(i\Delta_x+\Delta_z)}, \text{ and } l^- = t_o^-e^{ik_{y0}\Delta_y}e^{-\xi_1(i\Delta_x+\Delta_z)},\)

(3.4.5)

\(^{44}\)In arriving at (3.4.4), the large-\(|k_x|\) form of the dispersion relation, the proper square root sign is taken to assure exponential decay of the Fourier kernel \(e^{i\tilde{k}_z^+\Delta_z}\) in accordance with the radiation condition.
becomes the union of (3.4.6) and (3.4.7):

\[
I_{\text{ext}}^+ = I_{\text{ext}}^- = I = \sum_{m = 0}^{+\infty} \int_{\xi_1}^{\infty} \tilde{a}_m \tilde{\phi}_m (\xi_1 + r_x t^+_o) (\xi_1 + r_x t^+_o) e^{r_x t^+_o (i\Delta x - \Delta z)} dr_x
\] (3.4.6)

with the respective truncation error integrals of 3.4.6-3.4.7 manifesting as

\[
I_{\text{tr}}^+ = I_{\text{tr}}^- = I - \sum_{m = 0}^{-\infty} \int_{-\infty}^{0} \tilde{a}_m \tilde{\phi}_m (-\xi_1 + r_x t^+_o) (-\xi_1 + r_x t^+_o) e^{r_x t^+_o (i\Delta x + \Delta z)} dr_x
\] (3.4.7)

Performing a change of variables on (3.4.8)-(3.4.9) subsequently yields the following relations:

\[
I_{\text{tr}}^+ = I_{\text{tr}}^- = I = \sum_{m = 0}^{+\infty} \int_{N\Delta \xi^+_x}^{\infty} \tilde{a}_m \tilde{\phi}_m (\xi_1 + r_x t^+_o) (\xi_1 + r_x t^+_o) e^{r_x t^+_o (i\Delta x - \Delta z)} dr_x
\] (3.4.8)

\[
I_{\text{tr}}^- = I_{\text{tr}}^- = I = \sum_{m = 0}^{-\infty} \int_{-\infty}^{-N\Delta \xi^-_x} \tilde{a}_m \tilde{\phi}_m (-\xi_1 + r_x t^-_o) (-\xi_1 + r_x t^-_o) e^{r_x t^-_o (i\Delta x + \Delta z)} dr_x
\] (3.4.9)

Next, one evaluates (3.4.10)-(3.4.11) for \(M + 1\) different values of \(N\) (e.g. \(N = 1, 2, ..., M+1\)), truncates these \(M+1\) relations after the \(m = (M-1)\) error series term (i.e. retain the first \(M\) series terms), defines the \(m\)th truncation error series coefficient pair as \(\tilde{c}_m^+, \tilde{c}_m^-\) (e.g. see (3.4.12)-(3.4.13) below), and solves the corresponding \((M + 1)\)-order system to estimate \(I_{\text{ext}}^+\) and \(I_{\text{ext}}^-\). This procedure represents the complex-plane SLST generalization, applicable to the sequence of \(M + 1\) successive “cumulative tail integral” estimates (c.f. Ch. 2), to accelerate evaluation of \(I_{\text{ext}}^+\) and \(I_{\text{ext}}^-\).
Let us now examine the specific case of modeling \( \tilde{f}(k_x, k_{y0}) \) using the family of \((N,m)\)-parameterized functions \( \tilde{\phi}_m = \tilde{\phi}_m^+(N, s) = (\xi_1 + (s + N\Delta\xi_x^t)\tau_o^+)^{-m} \) for \( \text{Re}[k_x] > 0 \) and \( \tilde{\phi}_m = \tilde{\phi}_m^-(N, s) = (-\xi_1 - (s + N\Delta\xi_x^t)\tau_o^-)^{-m} \) for \( \text{Re}[k_x] < 0 \). Performing Maclaurin expansions of the \( \{\tilde{\phi}_m^+\} \) and \( \{\tilde{\phi}_m^-\} \), retaining only their respective zeroth-order expansion terms, setting \( \omega_+^N = \exp(-N\Delta\xi_x^t \cos \gamma^+ \Delta x + \sin \gamma^+ \Delta z) \) and \( \omega^-_N = \exp(-N\Delta\xi_x^- \cos \gamma^- \Delta x + \sin \gamma^- \Delta z) \), and defining \( \omega_+^N \) and \( \omega^-_N \) as (resp.)

\[
\omega_+^N \tilde{\phi}_m^+(N, 0), \tilde{c}_m^+ \sim t^+ \int_0^\infty \tilde{a}_m e^{st^+ (i\Delta x - \Delta z)} e^{iN\Delta\xi_x^t (\cos \gamma^+ \Delta x - \sin \gamma^+ \Delta z)} ds \quad (3.4.12)
\]

\[
\omega^-_N \tilde{\phi}_m^-(N, 0), \tilde{c}_m^- \sim t^- \int_0^\infty \tilde{a}_m e^{-st^- (i\Delta x + \Delta z)} e^{-iN\Delta\xi_x^- (\cos \gamma^- \Delta x - \sin \gamma^- \Delta z)} ds \quad (3.4.13)
\]
yields a pair of expressions capturing the dominant behavior of the truncation error \( I_{tr}^+ \cup I_{tr}^- \):

\[
I_{tr}^+ \sim \omega_+^N \sum_{m=0}^\infty \tilde{c}_m^+ \tilde{\phi}_m^+(N, 0) \quad (3.4.14)
\]

\[
I_{tr}^- \sim \omega_-^N \sum_{m=0}^\infty \tilde{c}_m^- \tilde{\phi}_m^-(N, 0) \quad (3.4.15)
\]

which comprises the complex-path generalization, as concerning infinite-range Fourier integrals, to the error expression developed in Section 2 of [71]. The corresponding truncation error expressions associated with F-H transforms like (3.2.2) follow in analogous fashion.

With the foundational expressions available, we now seek to maximize exponential convergence acceleration of (3.4.5) through a suitable choice of the detour departure angles \( \gamma^+ \) and \( \gamma^- \). Differentiating the exponent expressions \((\cos \gamma^+ \Delta x + \sin \gamma^+ \Delta z)\) and \((\cos \gamma^- \Delta x + \sin \gamma^- \Delta z)\) with respect to \( \gamma^+ \) and \( \gamma^- \) (resp.) and setting the resulting expressions equal to zero leads us to initially (naively) choose \( \gamma^+ = \gamma^- = \)}
\[\tan^{-1}\left(\frac{\Delta x}{\Delta z}\right)\], which (asymptotically) corresponds to the path of most rapid exponential decay\(^{45}\) or (equivalently) the Constant Phase Path (CPP). This detour angle choice can be likened to a compromise between the so-called “z-transmission representation” and “radial transmission representation” \(^{46}\) of the space-domain field, which were discussed therein in the context of F-H and F-B transforms. Identical expressions hold for the \(k_y\) plane detour departure angles \(\alpha^+\) and \(\alpha^-\). Next we consider the detour constraints imposed by these two phenomena.

To this end, first define the branch point as \(k_y^2 = k^2 - k^2_{y_0}\) and temporarily assume that the \(k_y\) path was chosen so that \(k_x\) plane critical points neither manifest in the second/fourth quadrants nor migrate towards \(\text{Re}[k_x] = \pm\infty\) with increasing \(|k_y|\). Recalling the effective refractive indices \(\{\sqrt{\epsilon_{pm} \mu_{pn}}\}\) \((m, n = 1, 2, 3)\) for layer \(p\)\(^{47}\) and how we subsequently computed \(\xi_1\) in Ch. 2, set \(\gamma_{m,n,p}\) equal to either (1) the angle between \(\xi_1\) and the \(p\)th layer’s \((m, n)\)th “effective wave number”\(^{48}\) \(k_{m,n,p} = k_o \sqrt{\epsilon_{pm} \mu_{pn}}\), if \(\text{Re}[k_{m,n,p}] \geq \xi_1\), or (2) \(\pi/2\) if \(0 \leq \text{Re}[k_{m,n,p}] < \xi_1\). Then, \(\gamma^+\) is updated as \(\gamma^+ = \min(\gamma^+, \{\gamma_{m,n,p}\})\). No critical points are located in the second/fourth quadrants by assuming (for simplicity) the absence of “double-negative”/meta-material and active/gain media. Therefore, we do not have to constrain \(\gamma^-\). However, these calculations can be readily adjusted to appropriately constrain both \(\gamma^+\) and \(\gamma^-\) if such media are present so that our assuming their absence represents a trivial constraint in our methodology.

\(^{45}\)For F-H integrals, use the asymptotic form of the Hankel function and replace \(\Delta x\) with \(|\rho - \rho'|\) when computing \(\gamma^+ = \gamma^-\).

\(^{46}\)For reasons discussed in [89], this path appears similar to, but is not always necessarily, the SDP.

\(^{47}\)These are used to compute, but are not the same as, the global effective refractive index \(n^+\) mentioned above (c.f. Ch. 2).

\(^{48}\)After coercing the wave number’s real part to be positive, if need be.
Now we justify the assumptions above about the $k_y$ path, and constrain it to avoid the two issues stated earlier regarding two-dimensional integral transforms arising as the solution to wave-dynamics problems in planar-stratified environments lacking azimuthal symmetry. To this end, for some arbitrary $k_y$ value along the $k_y$ plane integration path first expand the branch point $k_a^2$ as

$$k_a^2 = \left[ (k'^2 - k''^2) - (k_y'^2 - k_y''^2) \right] + 2i \left[ k'k'' - k'_yk'' \right]$$

(3.4.16)

and recall that the radiation branch cut is jointly defined by the conditions $\text{Im}[\tilde{k}_z^2] = 0$ and $\text{Re}[\tilde{k}_z^2] > 0$ [45][Ch. 2]. To ensure that critical points in the first (third) quadrant of the $k_x$ plane do not migrate towards $\text{Re}[k_x] = +\infty$ ($\text{Re}[k_x] = -\infty$) for large $|r_y|^50$, one must ensure that asymptotically $\text{Re}[k_a^2] \to -\infty$ as $|r_y| \to \infty$. Observing the real part of (3.4.16), we see that one must constrain $\alpha^-$ and $\alpha^+$ to the interval $0 \leq \{\alpha^-, \alpha^+\} \leq \pi/4$. Furthermore, to prevent critical points from migrating into the second/fourth $k_x$ plane quadrants, we require that $\text{Im}[k_a^2] \geq 0$ as $|r_y| \to \infty$. Observing the imaginary part of (3.4.16) and noting in the region $\{k'_y > 0 \cup k''_y > 0\}$ that $k'_yk''_y > 0$, a cursory analysis suggests that one cannot safely choose a non-zero value of $\alpha^+$ without risking this migration, which would force one to dynamically re-define the $k_x$ integral’s pre-extrapolation region path, now a function of $k_y$, to ensure that one (1) encloses all the quadrant one critical points that migrated into quadrant four while (2) avoiding the encirclement of quadrant three critical points that migrated into quadrant two.

49One can verify that the $k_x$ plane discrete poles will exhibit similar behavior as the branch points [45][Ch. 2]. Therefore, our analysis based on examining the branch point’s behavior sheds analogous insight into the behavior of the poles.

50$r_y$ is the $k_y$ plane dual of $r_x$, exhibited earlier.
As a result, it appears that one must set the additional, more restrictive constraint
\( \alpha^+ = \min(0, \pi/4) = 0 \), which in theory may lead to an outer integral exhibiting
monotonic-divergent behavior when \( 0 \leq |z - z'| \ll 1 \) [12]. However, this limitation
can be overcome via clever partition of the two-dimensional integration domain; see
Figure 3.3, which summarizes the proposed partition. Integrating first over Regions
I, IIa, and IIb in Figure 3.3 followed by integrating in Region III, which encompasses
the intersection of the \( k_x \) and \( k_y \) plane extrapolation regions, renders the Region III
integration’s result *immune* to the migration of critical points into the second/fourth
quadrants. This is because one had already stipulated a domain partitioning and
completed integration over Regions I, IIa, and IIb.

We conclude that so long as one conforms to the restrictions \( 0 \leq \{\alpha^-, \alpha^+\} \leq \pi/4 \)
and \( \gamma^+ = \min(\tan^{-1}(\Delta x/\Delta z), \{\gamma_{m,n,p}\}) \), one can detour in all four spectral “quadrants” \( \{k_x' > 0 \cup k_y' > 0\}, \{k_x' < 0 \cup k_y' > 0\}, \{k_x' > 0 \cup k_y' < 0\}, \text{ and } \{k_x' < 0 \cup k_y' < 0\} \) (see Figure 3.3) through
which the stipulated integration path proceeds\(^{51}\). Indeed, our proposed partition of
the \( k_x - k_y \) integration domain ensures that for any \( \mathbf{r} - \mathbf{r}' \neq \mathbf{0} \) geometry, the double-
integral (3.2.1) exhibits *absolute-convergent* behavior in the classical/Riemann sense.

This discussion also brings to light the benefit of our starting assumption in this
analysis, made at the end of Section 3.1, that \( |x - x'| = |y - y'| \geq 0 \): a compromise
is reached that ensures exponential-cum-algebraic convergence of both the \( k_x \) and
\( k_y \) integrals throughout the integration domain. As an alternative we could have,
for example, performed an azimuthal rotation such that \( |x - x'| = 0 \) and \( |y - y'| = \)

\(^{51}\)Due to our assuming \( x - x' = y - y' \geq 0 \), it is implicitly understood that in all four “quadrants”
\( \text{Im}[k_x] \geq 0 \) and \( \text{Im}[k_y] \geq 0 \) (excepting the minor pre-extrapolation region detour made into quadrant
four).
$|\rho - \rho'|$ to maximize convergence acceleration of the $k_y$ integral. However, when
$0 \leq |z - z'| \ll 1$, the $k_x$ integral may exhibit *monotonic-divergent* behavior. In contrast
to oscillatory-divergent behavior [12], the MWA variants (including the generalized
version developed herein) cannot curb monotonic-divergent behavior due to the lack
of oscillations that must be present for the MWA to “average out” the oscillatory-
divergent sequence of cumulative tail integral estimates to obtain a final, convergent
result.

Note that for integration in Regions IIa, IIb, and III in Figure 3.3, one performs a
separate integration and extrapolation of the individual $k_x$ and/or $k_y$ half-tail integral
sections. This is in contrast to the method developed in Ch. 2 wherein we folded
the half-tail integrals in the $\pm \text{Re}[k_x]$ half-planes to yield a cosine or sine oscillatory
kernel, based on assuming spectral symmetry in the environment’s plane wave reflec-
tion/transmission properties, prior to performing tail integral extrapolation along the
positive $\text{Re}[k_x]$ axis (and similarly for the $k_y$ plane). Our present method, in bending
both half-tail $k_x$ paths into the upper-half $k_x$ plane, forbids such folding due to the
now-absent lack of reflection symmetry (about the $\text{Im}[k_x]$ axis) with respect to the
two halves of the extrapolation region path. The resulting penalty paid in using the
complex-plane MWA manifests in having to use twice the number of weight sets ver-
sus when one can perform half-tail integral folding followed by cumulative tail integral
sequence extrapolation, leading to increased memory requirement and computation
time in regards to procuring the MWA weight sets. As a practical consideration, then,
we wish to reduce the number of extrapolation weight sets that must be evaluated$^{52}$.

$^{52}$Nominally, there are twelve weight sets one must pre-compute and store to implement the
complex-plane MWA: three field components, each with differing combinations of $(k_x, k_y)$ monomial
power dependencies, multiplied by up to four distinct extrapolation region detour angles $\alpha^+, \alpha^-, \gamma^+, \gamma^-$. 

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To this end, we take two steps to halve this number to the six weight sets originally required when performing folding followed by extrapolation. First, we set $\alpha^+ = \gamma^+$ and $\alpha^- = \gamma^-$. Second, we make the approximation (for each field component) that the asymptotic monomial power dependence on both $|k_x|$ and $|k_y|$ [45][Chs. 1,7] equals the average monomial power dependencies on $k_x$ and $k_y$. For example, if we determine the integrand for one field component has asymptotic monomial dependencies of $O(k_x^{q_1})$ and $O(k_y^{q_2})$, then we take (as both our $k_x$ and $k_y$ monomial dependence factors) $q_o = \text{Nint}((q_1 + q_2)/2)$, where $\text{Nint}(\cdot)$ converts its argument to an integer via rounding. Furthermore, to ensure stability of the accelerator weight expressions and minimize aliasing effects due to inordinately long extrapolation region intervals, we neglect the integrand oscillation due to $\Delta z$ in the exponential kernels of (3.4.12)-(3.4.13). This allows one to update the truncation error estimates $\{\omega^+_N, \omega^-_N\}$ as $\omega^+_N = (-1)^N \omega^+_N$ when $\Delta x > T_0$ [71].

Beyond the concern of weight computation stability, we also ignore the phase variation associated with $\Delta z$ due to $\Delta z$, in general, being ill-defined. Indeed, in (1) an anisotropic homogeneous environment or (2) a stratified environment containing isotropic and/or anisotropic media, several phenomena typically obfuscate a univocal, clear definition for the effective longitudinal distance traversed by the characteristic plane wave fields when traveling from $r'$ in layer $M$ to $r$ in layer $L$. These are (1) multi-bounce within slab layers, (2) the layer and (for anisotropic media) mode dependence of the longitudinal propagation constants, (3) interface reflections in layer $L$ causing both up-going and down-going modal fields (four total modes in general) to contribute to the observed field at $r$, and (4) inter-mode coupling at the interfaces. In

\[53\] Stability and aliasing considerations also motivated our choice of the break-point spacings $\Delta \xi_x$, $\Delta \xi^+_x$, and $\Delta \xi^-_x$.
fact these four considerations, along with the inherently asymptotic nature of the CPP parameterization and the constraints associated with critical points/two-dimensional integrals addressed above, lead one in practice to not integrate exactly along the CPP. As a result, one typically finds the integrands of extrapolation-region integrals still exhibiting undesirable residual oscillation due to the complex exponential factors. While, for $\Delta x \geq 0$, one still always has a non-zero detour angle for both the $k_x$ and $k_y$ extrapolation region paths, these practical considerations are what demand the inclusion of an algebraic convergence accelerator like the MWA that exactly acts upon the very types of oscillatory integrals that will typically result. Therefore, while one does not typically realize the ideal situation of maximized exponential convergence acceleration (the strongest acceleration theoretically available here outside of the SDP), we largely mitigate this pitfall with the robust algebraic acceleration afforded by the MWA, which is agnostic to the environment/source-observer scenario (so long as $\Delta x > 0$). Indeed, for an order-$N$ MWA method used (see below) one realizes a reduction in truncation error between $O(k_x^{-N})$ and $O(k_x^{-2N})$ [71].

54Recall from Ch. 2 that the pre-extrapolation region serves to detour around those critical points within a certain distance from the real axis. Therefore, the presence of critical points cannot force $\gamma^+$ and $\gamma^-$ to equal zero exactly.
3.5 **Revised Accelerator Weight Computation**

The MWA, both in its form as the MMA [71, 84] and its more recent variant the new/“revisited” MWA [12], each offer different, desirable attributes. The latter version offers a straightforward methodology to unambiguously define arbitrary-order
accelerator weight sets and recursively compute higher-order weight sets upon demand. While we showed (c.f. Ch. 2) for the MMA how one can reduce the FLOP\textsuperscript{55} count involving the cumulative integrals themselves, the weight computations (1) depended on whether the series of successive extrapolation region sub-interval integrals exhibited oscillating or monotone behavior \cite{71}, and (2) the FLOP count to compute the weight sets rapidly grows for successive weight sets, placing a practical limit on obtainable accuracy in the weights (and thus the estimated tail integral) due to roundoff error accumulation in the computed weights. On the other hand, the computation of the new MWA weight sets is (1) a numerically unstable process rapidly leading to numerical overflow (when using the form exhibited in \cite{12}) and (2) directly linked to procuring the estimated tail integral \cite{12}, which is the solution to a highly ill-conditioned linear system (shown in Section 3.5.2), which previously led us to use the MMA in Ch. 2. Nevertheless, both flavors of MWA offer useful mathematical developments for the weights that are couched in the framework of SLST, using a family of functions in a series representation to model the spectral portion of the mixed-domain Green’s Function\textsuperscript{56} and resulting tail integral truncation error. For the MMA, the proposed series \cite{71}

\[ \tilde{f}(k_x, k_{y0}) \sim k_x^{q} \sum_{m=0}^{\infty} \tilde{a}_m(k_{y0}) \frac{k_x^m}{k_{y}^m} \quad (3.5.1) \]

is intuitive in its form, and we confirm below the validity (and in fact \textit{optimality}) of using this approximating series by a straightforward mathematical analysis\textsuperscript{57}. It will

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\textsuperscript{56}i.e. the fundamental spectral kernel embedded in the integrands of (3.2.1)-(3.2.4).

\textsuperscript{57}When assuming the spectral portion of the integrands in (3.2.1)-(3.2.4) asymptotically behave as monomial powers of $k_x$ and $k_y$, the new MWA’s error-modeling functions reduce to this function family too \cite{12}.
be shown that this optimality arises due to the error modeling series (see (3.4.14)-(3.4.15) above and [71]) being entirely consistent with the closed-form expression of the truncation error both in the absence and (via linear superposition) presence of stratified inhomogeneity. The practical consequence of this function family’s modeling optimality manifests in minimizing the number of cumulative tail integral estimates required to accurately estimate $I_{\text{ext}}^+$ and $I_{\text{ext}}^-$, as demonstrated in Section 3.7.

In summary, we seek a revised, complex-plane MWA that combines the best aspects of both the MMA [71] and more recent MWA variant [12], in tandem with incorporating the added exponential convergence acceleration afforded by bending the extrapolation region integration path, to effect robust and powerful field solution convergence acceleration. To this end, in this section we (1) analyze and justify using (3.5.1) as the approximating series for $\tilde{f}(k_x, k_{y0})$ and (2) exhibit and compare two proposed formulations for implementing the complex-plane generalization of the MWA, using the new “remainder estimates” $\{\omega_N^+, \omega_N^-\}$ [71] and the asymptotic series expansion (3.5.1).

### 3.5.1 The Optimal Error-Modeling Function Family

Herein we examine the inner spectral ($k_x$) integral for some fixed $k_{y0}$ in the region $\text{Re}[k_x] > 0$. Furthermore, assume $\tilde{f}(k_x, k_{y0})$ has an asymptotic $k_x$ monomial dependence of $k_x^q$ [12, 71]. One then has the asymptotic truncation error

$$I_{\text{tr}}^+ = \omega_N^+ I_{\text{tr}}^{+'} = \omega_N^+ \int_0^\infty \left[ \xi_1 + (s + N\Delta \xi_x^+)^t_o^+ \right] e^{s t_o^+ (i \Delta x - \Delta z)} ds$$

which just equals (3.4.10) with the asymptotic series expansion for $\tilde{f}(k_x, k_{y0})$ replaced by the dominant series term $k_x^q$. Next note that $I_{\text{tr}}^{+'}$ has a closed-form, convergent solution for $\text{Re}[t_o^+ (i \Delta x - \Delta z)] < 0$; setting $\tilde{a} = \tilde{a}(N) = \phi_0^+ (N, 0) = \tilde{\xi}_1 + t_o^+ N \Delta \xi_x^+$ and
\[ \chi = -t_o^+(i\Delta x - \Delta z), \] one obtains

\[ I^+_\text{tr} = \omega^+_N I^+_\text{tr} = \omega^+_N \begin{cases} \frac{1}{\chi} & , q = 0 \\ \frac{\bar{a}(t_o^+ + \bar{a}\chi)}{\chi^2} & , q = 1 \\ \frac{2t_o^2 + 2\bar{a}t_o + \bar{a}^2\chi^2}{\chi^3} & , q = 2 \end{cases} \tag{3.5.3} \]

and so on for other values of \( q \). Examining the asymptotic limit for these three illustrative cases, we find:

\[ \lim_{|\tilde{a}| \to \infty} I^\prime_{\text{tr}} = \begin{cases} \frac{1}{\chi} & , q = 0 \\ \frac{\bar{a}}{\chi} & , q = 1 \\ \frac{\bar{a}^2}{\chi} & , q = 2 \\ \frac{\bar{a}^q}{\chi} & , q \in \mathcal{N}^+ \end{cases} \tag{3.5.4} \]

where \( \mathcal{N}^+ \) represents the set of positive natural numbers. Similarly, one expects that the reflected/transmitted field terms will also have an asymptotic monomial dependence [12]. For example, consider a two-layer, planar-stratified environment containing isotropic media. The TE\(_z\)/TM\(_z\) reflection and transmission coefficients for a plane wave, incident from half-space number one upon half-space number two, write as [45][Ch. 2]

\[ R_{\text{TM}} = \frac{\epsilon_{2,r}\tilde{k}_{1,z}^+ - \epsilon_{1,r}\tilde{k}_{2,z}^+}{\epsilon_{2,r}\tilde{k}_{1,z}^+ + \epsilon_{1,r}\tilde{k}_{2,z}^+}, \quad R_{\text{TE}} = \frac{\mu_{2,r}\tilde{k}_{1,z}^+ - \mu_{1,r}\tilde{k}_{2,z}^+}{\mu_{2,r}\tilde{k}_{1,z}^+ + \mu_{1,r}\tilde{k}_{2,z}^+}, \quad T_{\text{TM}} = 1 + R_{\text{TM}}, \quad T_{\text{TE}} = 1 + R_{\text{TE}} \tag{3.5.5} \]

where

\[ (\tilde{k}_{m,z}^+) = (\tilde{k}_{m,z}^-)^2 = k_o^2\epsilon_{m,r}\mu_{m,r} - k_o^2 \quad (m = 1, 2), \quad \lim_{|k_o| \to \infty} \tilde{k}_{z}^+ = ik_o \tag{3.5.6} \]

Indeed, we see that for large \( |k_o| \) the reflection/transmission coefficients have a monomial power dependence \( O(k_o^2) \).

Pulling out the dominant monomial term \( \tilde{a}^q \) in (3.5.3), setting the \( N \)th cumulative remainder estimate as \( \omega^{+}_{N} = \omega^{+}_{N}\tilde{a}^q \), and invoking superposition (see footnote 58), one

\[ ^{58}\text{For the multi-layered scenario, use (3.4.10) and invoke superposition of the closed-form results for different monomial powers} \ q - m \ (m = 0, 1, 2, \ldots). \]
now has $\omega_N^{+} I_N^{+} = \omega_N^{+} \sum_{m=0}^{q} \tilde{b}_m^+/\tilde{a}^m$, which recovers the dominant contribution to the complex-path extension (3.4.14) of the error expression derived (using (3.5.1)) in [71]. The same procedure shown above, using instead $t_o^-, \tilde{a} = \tilde{\phi}_o^-(N, 0) = -\xi_1 - t_o^- N \Delta \xi_x^-$, and $\chi = t_o^- (i \Delta x + \Delta z)$, can be repeated for the Re$[k_x] < 0$ tail integral to obtain a dual set of expressions that recover (3.4.15). Based on this analysis, when proposing two revised MWA methods we will use the $(N,m)$-parameterized function families $\{\tilde{\phi}_m(N) = \tilde{\phi}_m^+(N, 0)\}$ and $\{\tilde{\phi}_m(N) = \tilde{\phi}_m^-(N, 0)\}$ to model the tail integral truncation error.

### 3.5.2 Two Proposed Formulations

For the first formulation we take inspiration from [12]. To this end, for the Re$[k_x] > 0$ tail integral first define $I_N^{+}$ and $I_{N+1}^{+}$ as two input cumulative tail integral estimates and the under-determined linear system, with respect to which the non-truncated tail integral $I_{\text{ext}}^{+}$ is defined, as [12, 84]

$$I_{\text{ext}}^{+} = I_N^{+} + \omega_N^{+} \sum_{m=0}^{\infty} \frac{\tilde{b}_m^+}{\tilde{a}(N)^m}$$

(3.5.7)

$$I_{\text{ext}}^{+} = I_{N+1}^{+} + \omega_{N+1}^{+} \sum_{m=0}^{\infty} \frac{\tilde{b}_m^+}{\tilde{a}(N+1)^m}$$

(3.5.8)

whose equations are subsequently truncated after the $m = 0$ term [12]. This truncation yields a second-order linear system solved for an improved estimate $I_N^{+(2)}$ of $I_{\text{ext}}^{+}$ that is free of the $\tilde{a}^0$ term in its truncation error series [12, 71, 84]:

$$\eta_N^{+(1)} = -\frac{\omega_N^{+}}{\omega_{N+1}^{+}}$$

(3.5.9)

$$I_N^{+(2)} = \frac{I_N^{+} + \eta_N^{-1} I_{N+1}^{+}}{1 + \eta_N^{(1)}} = I_{\text{ext}}^{+} + \omega_N^{+} \sum_{m=1}^{\infty} \frac{\tilde{b}_m^+ [\tilde{a}(N+1)^{-m} - \tilde{a}(N)^{-m}]}{1 + \eta_N^{(1)}}$$

(3.5.10)

The coefficients $\{\tilde{b}_m^+\}$, however, are computed exactly; contrast this to the $\{\tilde{c}_m^+\}$ of (3.4.14).
Similarly, using $M+1$ ($M = 1, 2, ...$) cumulative tail integral estimates \( \{I_1^{+, \prime}, I_2^{+, \prime}, ..., I_{M+1}^{+, \prime}\} \) to eliminate the first $M$ terms of $\sum_{m=0}^{\infty} \tilde{b}_m^+ / \bar{a}^m$, one has for the $P$th truncated linear equation ($P = 1, 2, ..., M + 1$) \( I_1^{+(M+1)} = I_P^{+, \prime} + \omega_P^{+, \prime} \sum_{m=0}^{M-1} \tilde{b}_m^+ / \bar{a}(P)^m \). Subsequently, one procures the weights via solving the associated order-($M+1$) linear system for the best $I_{\text{ext}}^+$ estimate (i.e. $I_N^{+(M+1)}$), whose solution implicitly contains the expressions for the weights [12]. However, obtaining all desired weight tier sets by directly solving the associated linear systems (1) is very costly and (2) possibly exacerbates weight accuracy degradation due to the poor conditioning of these systems (see below). Instead, one can obtain closed-form solutions to the weight sets using the methodology outlined in [12] as adapted to our choice of (1) error-modeling functions \( \{\tilde{a}(N)^{-m}\} \) and (2) truncation error estimates \( \{\omega_N^{+, \prime}\} \). In [12] it was assumed that $\tilde{f}(k_x, k_y)$ asymptotically exhibited a monomial power dependence of the form $C k_x^q$ ($C$ being some constant), with the obvious consequence that $\partial^n \tilde{f}(k_x, k_y) / \partial k_x^n$ corresponds to a new function asymptotically behaving as $\sim k_x^{q-n}$ [80]. Rearranging the order-($M+1$) linear system thus yields a similar (but not yet identical) system to equation (22) in [12]:

\[
\begin{bmatrix}
-1/\omega_1^{+, \prime} & 1 & \tilde{a}(1)^{-1} & \cdots & \tilde{a}(1)^{-(M-1)} \\
-1/\omega_2^{+, \prime} & 1 & \tilde{a}(2)^{-1} & \cdots & \tilde{a}(2)^{-(M-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1/\omega_{M+1}^{+, \prime} & 1 & \tilde{a}(M + 1)^{-1} & \cdots & \tilde{a}(M + 1)^{-(M-1)} \\
\end{bmatrix}
\begin{bmatrix}
I_1^{+(M+1)} \\
\tilde{b}_0^+ \\
\tilde{b}_1^+ \\
\vdots \\
\tilde{b}_{M-1}^+ \\
\end{bmatrix}
= 
- \begin{bmatrix}
I_1^{+, \prime}/\omega_1^{+, \prime} \\
I_2^{+, \prime}/\omega_2^{+, \prime} \\
I_3^{+, \prime}/\omega_3^{+, \prime} \\
\vdots \\
I_{M+1}^{+, \prime}/\omega_{M+1}^{+, \prime} \\
\end{bmatrix}
\]

Noting that the weight sets in [12] were computed for arbitrary monomial power dependence $k_x^q$, one can cross-multiply the $\tilde{a}^q$ factors in the $\{\omega_p^{+, \prime}\}$ across the respective rows of (3.5.11) to obtain an analogous system, where now the $\{\tilde{a}(p)^{(q-n)}\}$ factors in

\[\text{Examining } [45][\text{Chs. 2,7}] \text{ confirms the asymptotic monomial power dependence of } \tilde{f}(k_x, k_y).\]
the modified form of (3.5.11) represent (up to a constant) successive \( s \) derivatives of \((\xi_1 + t_0^+ s)^q\) evaluated at \( s = 0 \). Having now matched our linear system to [12], the \( n \)th weight \((n = 1, 2, ..., M + 1)\) for the tier-(\( M + 1 \)) complex-plane generalization of the new MWA writes as [12]

\[
w_n^{(M+1)} = (-1)^{n+1} \binom{M}{n-1} \tilde{a}(n)^{M-1-q}/\omega_n^+ \tag{3.5.12}
\]

with the expression for our best tail integral estimate given as

\[
I_1^{(M+1)} = \frac{\sum_{n=1}^{M+1} w_n^{(M+1)} I_n^{+}}{\sum_{n=1}^{M} w_n^{(M+1)}} \tag{3.5.13}
\]

The expressions for the \( \{w_n^{(M+1)}\} \) corresponding to Re\([k_x]\) < 0 tail integral follows analogously. Furthermore, one expects that with a different choice of \( \{\tilde{\phi}_m(N, s)\} \), this derivation can be repeated to develop complex-plane extensions to other SLST algorithmic members.

From an analytic standpoint, the derivation of the weights for this formulation is complete. However, despite the analytic form of the new MWA weights shown in (3.5.12) and [12], in a finite-precision, numerical implementation this casting leads to arithmetic overflow. This drawback, along with the numerically unstable means to recursively update the weights to procure higher-order weight sets, can be easily remedied as follows:

1. Starting at some tier-\( N \) weight set (e.g. set \( N = M + 1 \)), multiply all the weights by \( \omega_N^+ / \tilde{a}(N)^{N-2-q} \). This ensures that the weights remain bounded for all \( n \) and \( N \).

2. To subsequently obtain a tier-(\( N + 1 \)) weight set from the tier-\( N \) set:

\(^{61}\)One does not have the \( q \)-dependent constants in columns three to \( M + 1 \) of the matrix in (3.5.11). However, one can include these constants and only affect the unneeded coefficients \( \{\tilde{b}_1^+, ..., \tilde{b}_{M-1}^+\} \).
(a) Set $w_{N+1}^{(N+1)} = (-1)^{N+2}/N$.

(b) For the remaining $N$ weights, set $w_n^{(N+1)} = d_{n,N}w_n^{(N)}$

where

$$d_{n,N} = e^{-\Delta t^+ (\Delta x \sin \gamma + \Delta z \cos \gamma)} \frac{a(N)}{N - n + 1} \left( \frac{a(N)}{a(N+1)} \right)^{N-2-q} \frac{a(n)}{a(N+1)} \quad (3.5.14)$$

The second proposed formulation employs the MMA [71], as extended in Ch. 2 to facilitate adaptive tail integral evaluation, in conjunction with our complex extension to the truncation error estimates $\{\omega^+, \omega^-\}$. The formulae to compute arbitrary-order weight sets is given in [71], while the method to recursively find higher-order weight sets is exhibited in Ch. 2. Therefore, the reader is referred to these two references for the elementary details.

Between these two formulations, we opt to implement and show validation results for the second formulation based on the MMA. This is because of the first formulation’s poor suitability for an adaptive tail integral evaluation scheme, which in turn is due to increasingly higher-order weight sets being the solutions to increasingly ill-conditioned linear systems. Even though we now have available the analytically recast, numerically stable, closed-form expressions for the first formulation’s weights and their update scheme (which obviates any potential algorithmic instability exacerbating computed weight errors), the relative accuracy of the computed weights is still fundamentally capped by the linear system’s conditioning\textsuperscript{62}. To illustrate the

\textsuperscript{62}Indeed, as is well-known, for a condition number $CN$ one can expect to lose $O(\log_{10}(CN))$ digits of precision in computing the weights [79].
ill-conditioning of the weight computation, we show in Figure 3.4 below, for four different $r - r'$ geometries, the two-norm condition number $CN$ of (3.5.11) as a function of its rank $M' = M + 1$:

1. $(x - x', z - z') = (1, 0)m$
2. $(x - x', z - z') = (1000, 0)m$
3. $(x - x', z - z') = (1, 10)m$
4. $(x - x', z - z') = (1000, 10)m$

To confirm that the system matrix ill-conditioning is not due to the complex-plane generalization of the new MWA, in Figure 3.4a we show the two-norm condition number for $\gamma^+ = \pi/4$ while in Figure 3.4b we show, for the same four $r - r'$ geometries, the conditioning for $\gamma^+ = 0$ (i.e. as if we performed the standard, real-axis MWA from [12]).

$^{63}$We set $T_0 = 0.25m$, $q=0$, and $\xi_1 = \Delta \xi_x = \cos \gamma^+ \Delta \xi^+_x = \pi/\Delta x$ for all cases in both figures.
Figure 3.4: Base-10 logarithm of the two-norm system matrix condition number used to compute the new MWA weights, as specified in (3.5.12) and [12], for Figures 3.4a and 3.4b (resp.). The vertical axis displays the number of digits of precision lost in the weights, when numerically computing them, due to the conditioning of (3.5.11). The solid horizontal curve corresponds to $\log_{10}(CN) = 16$; weights arising as solutions to a rank-$M'$ linear system with condition number greater than this are expected to be just numerical “noise” when computed using IEEE double-precision arithmetic.

One readily observes from Figure 3.4 that accurate weight computation is unrealistic as $M'$ increases; in fact, the situation is downright prohibitive for an adaptive MWA implementation (e.g., see Ch. 2). Even for the best-conditioned geometry (i.e. $(x - x', z - z') = (1, 0)\text{m}$), one cannot realistically expect even a single digit of precision in the weights for $M'$ equalling or exceeding approximately seven and ten in Figures 3.4a and 3.4b (resp.), as can be seen from the intersection of the corresponding curves in Figures 3.4a-3.4b with the solid horizontal curve corresponding to $\log_{10}(CN) = 16$. As a result, we choose the second proposed MWA formulation, based on the MMA [71], for computing validation results in Section 3.6. Based on our previous work (c.f. Ch. 2) using the standard, real-axis MMA for environments
containing high loss and conductively-uniaxial layers, one can expect its success in again producing high-precision results. Indeed, the validation results in Section 3.6 speak to this effect\textsuperscript{64}.

### 3.6 Results and Discussion

In this section we exhibit validation results in scenarios involving the modeling of induction sondes for geophysical prospection of hydrocarbons (i.e. induction well logging \[11\]). In Ch. 2, we demonstrated numerous simulated resistivity logs pertaining to environments containing a combination of isotropic and reciprocal, electrically uniaxial media \[77\][Ch. 7] as probed by longitudinally-oriented induction sondes. For those case studies, the adaptive, real-axis MMA was successfully incorporated into our algorithm to yield high-precision results exhibiting excellent agreement with data from previous literature \[9, 8, 7\].

Herein, we exhibit a case study involving a near-horizontal tool orientation where the tool axis dip angle $\alpha = 89^\circ$, tool axis strike angle $\beta = 0^\circ$ \textsuperscript{65}, and source-observer separation $L_{\text{tool}} = |r - r'| = 40'' = 1.016\text{m}$, corresponding to a source-observer depth separation $|z - z'| = L_{\text{tool}} \cos \alpha \sim 17.7\text{mm}$. Consequently, this study serves to validate the efficacy of our new algorithm and its ability to impart absolute, exponential-cum-algebraic convergence on Fourier double-integrals like (3.2.1) even for the traditionally prohibitive regime $|z - z'| \ll 1$. Furthermore, to exemplify the general-purpose nature of our new algorithm in regards to the media present, we generate synthetic resistivity

\textsuperscript{64}Due to the intricate, recursively-related nature of the MMA weight set tiers, a straightforward definition and analysis of the conditioning of the problem related to procuring these weights proves elusive and therefore is not pursued here. Rather, its use herein is strictly based on, and justified by, its \textit{empirically}-demonstrated efficacy in \[71\] and Ch. 2

\textsuperscript{65}The tool thus rotates and is confined within the $xz$ plane \[14\].
logs for a two-layer, planar-stratified environment containing reciprocal, electrically biaxial media\textsuperscript{66}. In this scenario, wherein all four characteristic plane wave modes in the anisotropic layer containing $\mathbf{r}$ can (in general) contribute to the observed field, the definition of an exact “$\Delta z$” and thus CPP is ill-defined (see Section 3.4). Therefore, this set of results also justifies our retaining the MWA’s robust environment/source-observer geometry convergence acceleration characteristic, yielding an overall robust and rapid electromagnetic field solution method.\textsuperscript{67} NB: The reference 1 curve label in Figures 3.5-3.8, corresponding to reference 1 in the original publication \cite{2} describing this chapter’s presented algorithm, corresponds to reference \cite{13} within this thesis.

Note that save for Figure 3.8f, there exists strong agreement across the full logging path in each plot. Even for Figure 3.8f, with some discrepancy in the upper half-space $D > 0$, overall there is strong qualitative agreement and (in the bottom half-space) quantitative agreement too\textsuperscript{68}.

\textsuperscript{66}For Figures 3.5-3.8, the frequency of operation is $f=2$MHz, the interface is located at $z_B = 0$m, and the resistivity tensor for layer $n$ (with respect to the standard Earth system) is described in the figure headings by the diagonal matrix $R_n = [R_{x'x'}, R_{y'y'}, R_{z'z'}]=\text{diag}[R_{x'x'}, R_{y'y'}, R_{z'z'}]$.

\textsuperscript{67}The results of the reference publication’s presented algorithm (hatched red curves) and this chapter’s presented algorithm (solid blue curves), presented in Figures 3.5-3.8 of this chapter, are repeated, in the solid red curves (reference publication’s results) and dotted-green curves (this chapter’s presented algorithm) of Figures 4.3-4.6 within the next chapter. This repetition is done to compare the numerical precision of results generated using the algorithms presented in this chapter and the next chapter.

\textsuperscript{68}For the figures shown below, the material formation parameter labeling is reversed versus the labeling in the reference paper such that the material scenario $\{R_1 = [100, 200, 500]\Omega m, R_2 = [1, 2, 5]\Omega m\}$ corresponds to $\{R_1 = [100, 50, 500]\Omega m, R_2 = [1, 0.5, 5]\Omega m\}$ in \cite{13} and vice versa. Due to the strong agreement between the simulation data produced here and by the reference source after this labeling reversal, we suspect this apparent data discrepancy is attributed to a simple typographical error on the part of the authors of \cite{13}.
Figure 3.5: Comparison of simulated magnetic field $H_{x'x'}$ with Figure 4 of [13].
Figure 3.6: Comparison of simulated magnetic field $H_{x'z'}$ with Figure 4 of [13].
Figure 3.7: Comparison of simulated magnetic field $H_{x'}\times$ with Figure 4 of [13].
Figure 3.8: Comparison of simulated magnetic field $H_{z'z'}$ with Figure 4 of [13].
3.7 Convergence Characteristics

To characterize our numerical formulation’s ability to converge towards the field solution, we present two case studies concerning the \( z \)-directed electric field component \( E_z \) produced by a \( z \)-directed electric dipole radiating at \( f=2\text{MHz} \) in free space. The first case comprises a benign scenario in which \( x - x' = y - y' = z - z' = 1\text{m} \), while the second case represents a very challenging scenario wherein \( x - x' = 500\text{m} \) and \( y - y' = z - z' = 0\text{m} \). The latter scenario’s prohibitive challenges, when using a standard numerical integration method, are that the integrand \(^{69} \) oscillates on the order of \( 500/\sqrt{2} \) times more rapidly than the integrand in case one and \(^{69} \) exhibits absolutely no exponential decay due to the annihilation of the \( \exp(ik_x'|z - z'|) \)-type factors. If we were to use a traditional numerical integration methodology, we emphasize that one would obtain a divergent result.

For each case, we present results related to the Region III field contribution (see Figure 3.3). Since one cannot obtain a closed-form solution to this field contribution, reference field values from which one measures relative accuracy must be appropriately chosen; their computation details are provided in Figure 3.9 below. As in Ch. 2 and \([12, 71, 70]\), we assume the integrand is well-behaved in Region III and thus do not perform adaptive interval sub-division. Instead, we set the \( k_x \) and \( k_y \) plane extrapolation region interval lengths as per Section 3.4 and examine the accuracy versus \(^{70} \) the number of extrapolation region intervals employed \( (B) \) and \(^{70} \) the Legendre-Gauss quadrature order used \( (LGQ) \) to integrate each interval.

\(^{69} \)Recall the azimuthal coordinate rotation performed such that in the rotated basis \( x - x' = y - y' \geq 0 \).

\(^{70}B \) intervals are used in both the \( k_z^' > 0 \) and \( k_z^' < 0 \) integration path half-tails; the same applies for the \( k_y \) path half-tails.

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Our comments on the relative importance of aliasing and truncation error are analogous to our comments in Ch. 2: Up to approximately $B = 3$ the truncation error dominates the total relative error, while using more than approximately $B = 6$ or 7 intervals effects no noticeable decrease in the error for a fixed $LGQ$. Beyond this point aliasing error dominates the total relative error, which is evidenced by the error decreasing versus increasing $LGQ$ but remaining flat versus increasing $B$. However, we notice the following two remarkable characteristics about the algorithm’s convergence for case two:

1. The $LGQ = 30$ curve reaches within 25dB of case one’s $LGQ = 30$ curve despite representing a scenario wherein the field solution would ordinarily have *diverged* using standard numerical integration techniques.

2. Despite this case representing a far more prohibitive scenario (if traditionally evaluated) versus case two presented in Ch. 2, wherein $r - r' = (500, 500, 1)m$, at $B = 10$ the $LGQ = 30$ curve here levels off at an error approximately 23dB lower than its case two counterpart in Ch. 2.

Note that (akin to, and for the same reasons stated in, Ch. 2) relative errors below -150dB were coerced to -150dB.

Since the pre-extrapolation region formulation in this paper is not radically different from that in Ch. 2, we expect similar convergence characteristics when using the trapezoidal detour (versus those presented in Ch. 2) and thus omit the Region I convergence study for brevity. Furthermore, the Region IIa/IIb convergence studies are omitted as well since the field convergence results would be affected by the algorithm’s handling of both the pre-extrapolation and extrapolation region sections of
the \( k_x \) and \( k_y \) plane integration paths. Equivalently, presenting information on the Region I and Region III field convergence characteristics sheds insight into the Region Ia/Iib convergence characteristics. This is because if the respective algorithms handling the Region I and Region III integrations robustly yield accurate, rapidly convergent results, one can expect similar behavior for the Region Ia/Iib results.

![Figure 3.9](image.png)

Figure 3.9: Convergence towards the solution comprising the \( E_z \) contribution from Region III. The reference field values are computed using \( LGQ=30 \) and \( B = 500 \) for both figures.

### 3.8 Conclusion

In this work, we have presented a novel integration scheme composed of (1) a complex-plane, adaptive/error-controlling extension to the standard real-axis MMA in conjunction with (2) a more robust pre-extrapolation region integration path to effect fast, absolute, and \textit{exponential-cum-algebraic} convergence of Fourier- and F-H-type integral transforms such as (3.2.1)-(3.2.4). Due to combining the detour with the
MMA and its robust algebraic convergence acceleration characteristic, this is indeed the case irrespective of the source-observer geometry and loss/anisotropy characteristics of the stratified media present. Furthermore, this is accomplished without the added complication of having to separately account for slab/interface mode contributions whose poles may be crossed when otherwise deforming to more well-known, rapidly-convergent paths such as the SDP [74] [45][Ch. 2], resulting in a numerically robust and easily-implemented integration methodology.

The algorithm’s ability to accurately simulate the observed fields for classically “worst-case” scenarios $0 \leq |z - z'| \ll 1$, and that too in complex, planar-stratified environments containing biaxial-conductive media, has been verified through numerous validation checks against [13]. Finally, the algorithm’s convergence characteristics in the strongly-evanescent spectral zone have been explored, analyzed, and shown to be superior compared to an older methodology exhibited in Ch. 2 that was based on an adaptive extension to the real-axis MMA.

We conclude that the present algorithm’s robustness with respect to source-observer geometries and medium types present, as well as its straight-forward nature and ease of implementation, makes it very useful for the analysis of electromagnetic wave propagation and scattering in multi-layered environments containing media of arbitrary anisotropy and loss.
Chapter 4: Tensor Greens Function Evaluation in Arbitrarily Anisotropic, Layered Media Using Complex-Plane Gauss-Laguerre Quadrature

4.1 Introduction

4.1.1 Chapter Summary and Contributions

We discuss the application of Complex-Plane Gauss-Laguerre Quadrature (CGLQ) to efficiently evaluate two-dimensional Fourier integrals arising as the solution to electromagnetic fields radiated by elementary dipole antennas embedded within planar-layered media with arbitrary material parameters. More specifically, we apply CGLQ to the long-standing problem of rapidly and efficiently evaluating the semi-infinite length “tails” of the Fourier integral path while simultaneously and robustly guaranteeing absolute, exponential convergence of the field solution despite diversity in the doubly anisotropic layer parameters, source type (i.e., electric or equivalent magnetic dipole), source orientation, observed field type (magnetic or electric), (non-zero) frequency, and (non-zero) source-observer separation geometry. The proposed algorithm exhibits robustness despite unique challenges arising for the fast evaluation of such two-dimensional integrals. Herein, we develop the mathematical treatment to rigorously evaluate the tail integrals using CGLQ and discuss and address the specific
issues posed to the CGLQ method when anisotropic, layered media are present. To empirically demonstrate the CGLQ algorithm’s computational efficiency, versatility, and accuracy, we perform a convergence analysis along with two case studies related to modeling of electromagnetic resistivity tools employed in geophysical prospection of layered, anisotropic Earth media and validating the ability of isoimpedance substrates to enhance the radiation performance of planar antennas placed in close proximity to metallic ground planes.71

4.1.2 Background

A long-standing need exists to efficiently, robustly, and accurately solve time-harmonic electromagnetic (EM) radiation and scattering problems in layered media [20]. Applications regularly encountering problem scenarios approximated by planar-layered, anisotropic media include hydrocarbon well/logging using radar and induction instruments [13, 8, 9, 23, 7, 11, 24, 14, 90, 29, 26, 27], analysis and design of both microwave circuits and antennas [66, 64, 65], plasma physics [67], atmospheric studies [60], ground penetrating radar (GPR) [61, 62], and optical field manipulation [63]. Illustrations of application areas requiring algorithms with such features can be found in Figure 4.1 below and Figure 3.1 of Ch. 3. Computational cost is a critical aspect in many cases, such as when attempting to solve inverse EM problems via “direct search” based techniques, due to the need for solving the forward EM problem many times to effect successful extraction of the desired environmental parameters [91]. To address the simultaneous needs to solve layered-media problems both rigorously and efficiently, pseudo-analytic approaches, which consist of posing

71NOTES: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.
the EM field solution as an inverse Fourier-type integral that synthesizes the required
Green’s Tensor components as a spectral superposition of modal fields (e.g., character-
istic plane waves), often represent the preferred numerical solution method [45, 77].
Such integrals typically assume the form, for some tensor Green’s function compo-
nent \( \Psi(\mathbf{r}) \), as either a two-dimensional Fourier integral (4.1.1) or one-dimensional
Fourier-Hankel integral (4.1.2)\textsuperscript{72}:

\[
\Psi_1(\mathbf{r}) \sim \int_{C_1} \tilde{\Psi}_1(k_x, k_y) e^{ik_z(x-x'+i k_y(y-y')+i k_z(z-z')} dk_x dk_y
\]

\[
\Psi_2(\mathbf{r}) \sim \int_{C_2} \tilde{\Psi}_2(k_\rho) H_n^{(1)}(k_\rho |\rho - \rho'|) e^{ik_z(z-z')} dk_\rho
\]

where \( H_n^{(1)}(k_\rho |\rho - \rho'|) \) is the \( n \)th order Hankel function of the first kind, representing
an outgoing cylindrical wave\textsuperscript{73}, \( \tilde{\Psi} \) is the spectral domain analog of the space domain
function \( \Psi \), \( \mathbf{k} = (k_x, k_y, \bar{k}(k_x, k_y)) \) is the wave vector, \( k_\rho = \sqrt{k_x^2 + k_y^2}, \rho = \sqrt{x^2 + y^2}, \)
\( \mathbf{r} = (x, y, z) \) is the field observation point, and \( \mathbf{r}' = (x', y', z') \) is the source location.
Note that the first integral (4.1.1), but not the second one (4.1.2), is suitable for
evaluating fields in either isotropic or arbitrarily anisotropic\textsuperscript{74} planar-layered media
and thus represents the class of integrals we examine further\textsuperscript{75}.

Despite the rigor of the solution method provided by such integrals, in prac-
tice when subject to direct numerical evaluation using a path on or near the real

\textsuperscript{72}The formulation presented herein is readily applicable to Sommerfeld integrals (i.e., Fourier-
Bessel transforms [20]) and also to fields evaluated in cylindrically-layered media employing similar
integral representations [45][Chs. 2,4].

\textsuperscript{73}The exp(–i\omega t) time harmonic convention is assumed and suppressed throughout.

\textsuperscript{74}We assume, to ensure the completeness of the plane wave basis, that the material tensors are
diagonalizable. However, this constraint is not limiting in practical problems since all natural media
are characterized by diagonalizable permittivity and permeability tensors.

\textsuperscript{75}The methodology developed below can be applied to integrals of the form (4.1.2), and by exten-
sion to Sommerfeld integrals via an appropriate transformation [45][Ch. 2], by setting the optimal
path detour angle as \( \gamma = \tan^{-1}(|\rho - \rho'|/|z - z'|) \) in the case of planar-layered media (see Section
4.2.2). A similar formula for \( \gamma \) applies for cylindrically-layered media.
axis, the integrand may exhibit highly oscillatory or weakly convergent behavior for various types of source-observer separation geometries \( \mathbf{r} - \mathbf{r}' \) of interest (i.e., \( |\rho - \rho'| = \sqrt{(x-x')^2 + (y-y')^2} \gg 1 \) or \( 0 \leq |z - z'| \ll 1 \) resp.). To address these challenges, various approaches have been developed over the years. On one side are techniques aimed at circumventing the need to perform direct numerical integration altogether. Prominent among this class of methods are closed-form asymptotic solutions [45][Ch. 2][77] and image methods [70, 73, 76]. Asymptotic techniques typically feature geometry-specific applicability and accuracy depending on an asymptotic value of one or more parameters (e.g., frequency, observation distance, etc.) [45][Ch. 2]. On the other hand, image methods are known to typically lack robust error-control mechanisms [86] in addition to also exhibiting geometry-specific applicability [70]. In contrast to these techniques, a different strategy consists in attempting the (efficient) direct numerical integration. Among these techniques are the so-called “weighted average”-type techniques [71, 70, 12, 75] (also, c.f. Ch. 2), which belong to the broader family of scalar Levin sequence transforms [84]. These methods treat the full, non-truncated Sommerfeld, Fourier-Hankel and Fourier tail integrals as a sum of integrals, each of whose paths span a finite section of the tail, and devise a “weighted average” formulation that, in effect, aptly guesses, compensates for, and thereby reduces the truncation error associated with evaluating only a finite section of the integration path tail. A recent extension to this method discussed in Ch. 3, and denoted as the “Complex-Plane Method of Weighted Averages” (CPMWA), consists of deforming the Fourier integral tail path into a linear path impinging into the upper-half of the complex plane, partitioning the deformed path into finite-length intervals,
and adaptively taking weighted averages of a successively greater number of estimations of the non-truncated tail integral. Through validation and convergence studies, it was demonstrated in Ch. 3 that this strategy could rigorously guarantee absolute, exponential-cum-algebraic convergence for a wide range of planar-layered problems, a significant improvement over the (real-axis) extrapolation methods employed in the past [71, 12] (also, c.f. Ch. 2).

We should also note the possibility of numerically evaluating the integral along the Steepest Descent Path (SDP). However, the possibility of intersecting and (or) deforming past critical points on the complex-plane\textsuperscript{76}, along with the requirement to identify and integrate through the saddle point, the book-keeping necessary to track all critical-point-crossing occurrences, and having to analytically account for these problem-dependent critical-point-crossings at the post-integration stage makes such a method less desirable. Due to similar book-keeping needs and problem-dependent characteristics, we also avoid use of the integration path suggested in [87].

Despite its robustness, the CPMWA (discussed in Ch. 3) still presents some drawbacks associated with the large number of integrand evaluations necessary to evaluate the full integral tail, the need to pre-compute the set of weights required for an adaptive implementation, as well as residual aliasing and numerical stability considerations. In particular, the need to mitigate aliasing caused by unduly long extrapolation region intervals can force the choice of suboptimal path deformation detour angles (see Ch. 3 for details). Additionally, the efficiency and numerical stability of all the extrapolation methods discussed above implicitly relies upon the

\textsuperscript{76}That is, branch points, branch cuts, or poles.
oscillatory behavior of the integrand [70, 12] (also, c.f. Chs. 2-3). Indeed, this oscillatory characteristic of the integrand was assumed in Ch. 3 due to the (practical) inability to (in general) construct a rigorous, mode-independent Constant-Phase Path (CPP) as a result of the presence of different locations for the critical points according to the individual anisotropic layer parameters, layer thicknesses, and so on. However, when the numerical integration does occur along or very near to the asymptotic CPP, computation of the $k_x$ integral weights becomes a numerically unstable procedure for $0 \leq |x - x'| \ll 1$ [92], which necessitates an ad-hoc adjustment to the MWA-type weight computation methodology\(^{77}\). Therefore, a new method eliminating the (1) excessive integrand evaluations and pre-computation of multiple weight sets, (2) potential numerical instability and subsequent need for ad-hoc adjustment of the weight computation method, and (3) artificial (i.e., algorithm-dependent) added constraints placed upon the path deformation detour angles to mitigate aliasing and the number of pre-computed weight sets, and instead offering a direct integration procedure with minimal integrand evaluations, no required pre-computation and use of weight sets potentially resulting from an ad-hoc computation scheme, and minimal constraints imposed upon the departure angles\(^{78}\), while simultaneously guaranteeing absolute, exponential convergence for all ranges of anisotropic, planar-layered problems is highly desirable.

The solution method introduced here to effect these changes is the complex-plane extension of Gauss-Laguerre Quadrature (CGLQ) [89, 93], which in its traditional form (i.e., integration along the real axis) approximates semi-infinite range integrals.

\(^{77}\)An analogous statement holds for the weights used to compute the $k_y$ tail integrals.

\(^{78}\)The term “minimal constraints” refers to those constraints imposed by the fundamental behavior of the wave dynamics solution as manifest in a Fourier, Fourier-Hankel, or Sommerfeld integral representation.

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of the form
\[ \int_0^\infty e^{-x} f(x) dx \] (4.1.3)
via an order-\( P \) numerical quadrature formula \( \sum_{m=1}^{P'} f(x_m) w_m \), where both the nodes \( \{x_m\} \) and weights \( \{w_m\} \) are real valued. On the other hand for a general path deformation into the complex plane, parameterized in terms of spanning the semi-infinite range of a real-valued variable, the nodes and weights can both be complex-valued.

The deformed path we decide to use is identical in shape to that shown in Figure 3.2 of Ch. 3, and (ideally) spans (asymptotically) the CPP\(^{79} \) along which the exponential phase factors exhibit no oscillation while simultaneously imparting maximum exponential decay to the integrand \[89\]. As a result, the exponential decay combined with minimized integrand oscillation makes this integral type an ideal candidate for accurate \textit{and} efficient numerical evaluation by CGLQ.

We note, however, that in the present CGLQ method one removes non-adaptive integration path sub-division, which was used in prior MWA variants \[70, 71, 12\] (also, c.f. Chs. 2-3) to increase tail integral accuracy via limiting integrand oscillation. Instead, one now relies solely upon the sufficiently well-behaved nature of the integrand \( f(x) \) along the deformed path to facilitate its interpolation via Laguerre polynomials, along with adaptively refining the solution using successively higher-order CGLQ quadrature rules (i.e., \( p \) refinement). To minimize any fast integrand variations and thereby facilitate successfully modeling \( f(x) \) via these Laguerre polynomials, undesirable integrand oscillations that (dominantly) arise from the exponential

\(^{79}\)As pointed out in \[89\], the CPP is not necessarily equivalent to the SDP \[45\][Ch. 2]. In particular, we note that the presence of a saddle point, through which the SDP would proceed, is neither stipulated nor solved for here, and no asymptotic dependence in regards to the observation point is assumed or implied in our present formulation.
complex-phase factors of the form $\exp(ik_x \Delta x + ik_y \Delta y + ik_z \Delta z)$ are suppressed here. Note that since we initially perform adaptive $hp$ integration refinement within and sufficiently past the neighborhood of any critical points near the real axis, we assume the critical points themselves do not cause appreciably abrupt variations of $f(x)$ along the tail integral path [71, 12]. The validation results and convergence study presented here indicate that major gains in both computational efficiency and accuracy robustness, with respect to diverse problem parameters, are realized with only a marginal penalty in accuracy compared to CPMWA.

Before proceeding, we remark that (akin to our remark in Ch. 3) it is assumed that one has already performed an azimuthal basis rotation such that in the rotated basis $x - x' = \Delta x = y - y' = \Delta y \geq 0$ while $-\infty < (z - z' = \Delta z) < \infty$, where $r = (x, y, z)$ is the observation point and $r' = (x', y', z')$ is the dipole source location in the rotated basis. This rotation is performed to ensure absolute, exponential convergence of both the outer \textit{and} inner integral regardless of the transverse source-observer separation geometry $\rho - \rho'$ while streamlining the formulation dictating the shape of the integration path. Knowledge of all required vector and tensor field transformations done as part of the azimuthal basis rotation is implicitly assumed and not discussed further herein.
Figure 4.1: Schematic illustration of two application areas frequently encountering environments well-approximated and modeled as planar-layered media containing one or more anisotropic layers. Figure 4.1a illustrates usage of ground-penetrating radar (GPR) in subsurface material profile retrieval (i.e., an example of solving the inverse EM problem), while Figure 4.1b illustrates radio-wave propagation through and distortion by an inhomogeneous, dispersive atmosphere potentially containing one or more anisotropic layers. Note: Contrary to what Figure 4.1 suggests, our algorithm also admits arbitrarily anisotropic material parameters in layer one.
4.2 Formulation

4.2.1 Propagation Spectra Contribution

Henceforth we discuss the two-dimensional Fourier integral, rather than the one-dimensional Fourier-Hankel integral, to raise and address specific concerns regarding the former. To this end, let \( k_x \) and \( k_y \) be the inner and outer integration variables (resp.). Any discussion pertaining to the inner integral, which we assume is being evaluated for a fixed \( k_y \) value \( k_{y0} \), applies analogously to the outer integral (and vice-versa) unless explicitly stated otherwise. First we briefly summarize treatment of the propagation spectra contributions to the observed field, which mirrors that in Ch. 3 (see Ch. 2 for details on evaluating the integrand), before proceeding to the primary content of this article.

To robustly estimate and avoid the region wherein critical points may lie near the real axis, we employ a conservative estimate for the multilayered environment’s “effective refractive index” \( n^+ \) (c.f., Ch. 2) and a trapezoidal detour, terminating at \( k_x = \pm P_k \) on the real axis, that is parameterized identically to its counterpart in Ch. 3 except for setting \( P_k = k_0 (n^+ + 2) \), where \( k_0 = \omega/c \) is the free space wave number, \( \omega = 2\pi f \) is the angular frequency of radiation, and \( c \) is the speed of light in free space\(^80\). Adjoined to this trapezoidal path are real-axis segments spanning the interval \((-\xi_1 \leq \text{Re}[k_x] \leq -P_k) \cup (P_k \leq \text{Re}[k_x] \leq \xi_1) \) \(^81\). Within the region \((-\xi_1 \leq \text{Re}[k_x] \leq \xi_1)\), labeled herein as ”Propagation Region,” an error-controllable

\(^80\)The presence of the constant \( T_0 \) in the CGLQ method, as seen upon examining the similar propagation spectrum evaluation methodology in Ch. 3, is simply to bound the detour height. This stands in contrast to its dual usage in Ch. 3 to bound both the detour height and the length of the extrapolation intervals. Extrapolation intervals are, of course, absent in the present CGLQ formulation.

\(^81\)See Ch. 3 for calculation of \( \xi_1 \).
numerical integration is done through a multi-level \emph{hp} adaptive refinement. On the other hand, the region containing the integral tails is denoted as “Evanescent Region”\superscript{82}.

### 4.2.2 Evanescent Spectra Contribution

The geometry of the path and spectral domain partition used here follows Ch. 3; see Figure 3.2a and 3.3 therein for illustrations. From Ch. 3, it was determined that the optimal departure angle $\gamma$ to asymptotically maximize decay of the complex exponential $\exp(ik_x \Delta x + ik_z^+ \Delta z)$ is given by $\gamma = \tan^{-1}(\Delta x / \Delta z)\superscript{83}$; similarly, choosing $\alpha = \tan^{-1}(\Delta y / \Delta z)$ asymptotically maximizes decay of the complex exponential $\exp(ik_y \Delta y + ik_z^+ \Delta z)$.

Upon setting $t^+_o = \cos \gamma^+ + i \sin \gamma^+$, $t^-_o = \cos \gamma^- - i \sin \gamma^-$, and parameterizing the tail integral path as

$$
k_z = \begin{cases} 
\xi_1 + t^+_o r_x, & r_x > 0 \\
-\xi_1 + t^-_o r_x, & r_x < 0 
\end{cases} \quad (4.2.1)
$$

the half-tail integrals $\Psi^\pm_e$ in the $\pm \text{Re}[k_x]$ half-planes, corresponding physically to evanescent spectra contributions to the observed field, asymptotically behave as

\superscript{82}Note that these two labels are loosely employed; indeed, there exists (in general) no sharp boundary in wave number space delineating propagating modes from evanescent modes in the presence of planar inhomogeneity and lossy layers.

\superscript{83}As in Ch. 3, we estimate the departure angle assuming the environment were homogeneous and isotropic. Under this approximation one has $\hat{k}_z^\pm = \pm \sqrt{k^2 - k_x^2 - k_y^2}$, where $k$ is the characteristic wave number of the medium, while for large $|k_x|$ one has the asymptotically-valid relations, i.e., for $|k_x| \gg 0$, $\hat{k}_z^\pm \rightarrow \pm ik_x$ ($\text{Re}[k_x] \geq 0$) and $\hat{k}_z^\pm \rightarrow \mp ik_x$ ($\text{Re}[k_x] \leq 0$).
\[ \Psi^+_e = t^+_o e^{i k y_0 y + \xi_1 (i \Delta x - \Delta z)} \int_0^{\infty} \tilde{g}(\xi_1 + r_x t^+_o, k_y) e^{r_x t^+_o (i \Delta x - \Delta z)} dr_x \]  
\[ (4.2.2) \]

\[ \Psi^-_e = t^-_o e^{i k y_0 y - \xi_1 (i \Delta x + \Delta z)} \int_{-\infty}^{0} \tilde{g}(-\xi_1 + r_x t^-_o, k_y) e^{r_x t^-_o (i \Delta x + \Delta z)} dr_x \]  
\[ (4.2.3) \]

where without loss of generality it is assumed that \( \Delta z = z - z' \geq 0 \). Recalling the definitions of \( t^\pm_0 \), setting \( \tau^\pm = \Delta x \sin \gamma^\pm + \Delta z \cos \gamma^\pm \), and defining \( \beta^\pm = \Delta x \cos \gamma^\pm - \Delta z \sin \gamma^\pm \), we find that (4.2.2)-(4.2.3) asymptotically become

\[ \Psi^+_e = t^+_o e^{i k y_0 y + \xi_1 (i \Delta x - \Delta z)} \int_0^{\infty} \tilde{g}(\xi_1 + r_x t^+_o, k_y) e^{-r_x (\tau^+ - i \beta^+)} dr_x \]  
\[ (4.2.4) \]

\[ \Psi^-_e = t^-_o e^{i k y_0 y - \xi_1 (i \Delta x + \Delta z)} \int_{-\infty}^{0} \tilde{g}(-\xi_1 + r_x t^-_o, k_y) e^{-r_x (\tau^- + i \beta^-)} dr_x \]  
\[ (4.2.5) \]

By making the change of variable \( r_x = -r_x' \) in (4.2.5) and subsequently dropping the prime, one has

\[ \Psi^+_e = t^+_o e^{i k y_0 y + \xi_1 (i \Delta x - \Delta z)} \int_0^{\infty} \tilde{g}(\xi_1 + r_x t^+_o, k_y) e^{-r_x (\tau^+ - i \beta^+)} dr_x \]  
\[ (4.2.6) \]

\[ \Psi^-_e = t^-_o e^{i k y_0 y - \xi_1 (i \Delta x + \Delta z)} \int_{0}^{\infty} \tilde{g}(-\xi_1 + r_x t^-_o, k_y) e^{-r_x (\tau^- + i \beta^-)} dr_x \]  
\[ (4.2.7) \]

Next, by making the substitution \( r_x^\pm = r_x \tau^\pm \), subsequently dropping the “\( \pm \)” superscripts in \( r_x^\pm \), and defining \( t^\pm = t^\pm_o / \tau^\pm \), one obtains the following pair of integrals

84 We use the convention \( \tau^\pm = \Delta x \sin \gamma^\pm + \Delta z \cos \gamma^\pm \) to compactly denote the relations \( \tau^+ = \Delta x \sin \gamma^+ + \Delta z \cos \gamma^+ \) and \( \tau^- = \Delta x \sin \gamma^- + \Delta z \cos \gamma^- \) simultaneously. An analogous comment applies for other expressions bearing this plus-minus superscript type of convention.
suitable for evaluation by complex-plane Gauss-Laguerre quadrature:

\[
\Psi^+_e = l^+ e^{ik_0 \Delta y + \xi_1 (i \Delta x - \Delta z)} \int_0^\infty e^{-r_x} \tilde{g}(\xi_1 + l^+ r_x, k_{y0}) e^{ir_x \beta^+ / \tau^+} dr_x
\] (4.2.8)

\[
\Psi^-_e = l^- e^{ik_0 \Delta y - \xi_1 (i \Delta x + \Delta z)} \int_0^\infty e^{-r_x} \tilde{g}(-\xi_1 - l^- r_x, k_{y0}) e^{-ir_x \beta^- / \tau^-} dr_x
\] (4.2.9)

where the \( k_x \) plane nodes and weights (\( \{k_{xp}\} \) and \( \{w_{xp}\} \)) are related to the real-valued \( r_x \) plane nodes and weights (\( \{r_{xp}\} \) and \( \{w_{rp}\} \)), used to evaluate (4.2.8) and (4.2.9), as (resp.)

\[
k_{xp} = \xi_1 + l^+ r_{xp}, \quad w_{xp} = w_{rp}
\] (4.2.10)

\[
k_{xp} = -\xi_1 - l^- r_{xp}, \quad w_{xp} = w_{rp}
\] (4.2.11)

such that one can now efficiently compute (4.2.8)-(4.2.9), with zero tail integral truncation error, as

\[
\Psi^+_e \sim l^+ e^{ik_0 \Delta y + \xi_1 (i \Delta x - \Delta z)} \sum_{p=1}^P e^{ir_x \beta^+ / \tau^+} \tilde{g}(\xi_1 + l^+ r_{xp}, k_{y0}) w_{rp}
\] (4.2.12)

\[
\Psi^-_e \sim l^- e^{ik_0 \Delta y - \xi_1 (i \Delta x + \Delta z)} \sum_{p=1}^P e^{-ir_x \beta^- / \tau^-} \tilde{g}(-\xi_1 - l^- r_{xp}, k_{y0}) w_{rp}
\] (4.2.13)

using a \( P \)-point Gauss-Laguerre numerical quadrature formula.

However, since (4.2.8)-(4.2.9) is only asymptotically true, there will be a residual error associated with approximating \( ik_z^+ \) as \( -\cos \gamma^z r_x - \xi_1 \mp i \sin \gamma^z r_x \), where the top and bottom signs of this expression’s “±” and “∓” symbols hold for \( r_x > 0 \) and \( r_x < 0 \) (resp.). Therefore, in having extracted the term \( -r_x \Delta z \cos \gamma^z \) in (4.2.6)-(4.2.7) to create the exponential Laguerre polynomial weight factor \( \exp(-r_x \tau^\pm) \), to ensure analytical exactness in the formulation one must account for this extraction via “adding back in” the term \( +r_x \Delta z \cos \gamma^z \) that is expected to (asymptotically) cancel
with \( \text{Re}[i\tilde{k}_z^\pm \Delta z] \) up to the factor \((-\xi_1 \mp i \sin \gamma \pm r_x) \Delta z \) \footnote{When planar inhomogeneity or anisotropy is involved, naturally the extent of asymptotic cancelation that occurs in reality can exhibit great variation with respect to mode type and the media involved. To account for such uncertainty, one can robustly mitigate exponentially rising terms via placing the natural logarithm of the CGLQ numerical quadrature weights in the argument of the exponential, seen in the integrand of (4.2.14), prior to evaluating the exponent.}. Recalling the final variable transform made in deriving (4.2.8)-(4.2.9) from (4.2.6)-(4.2.7), one finally arrives at the exact expressions

\[
\Psi_e^\pm = l^\pm e^{iky_0 \Delta y \pm i \xi_1 \Delta x} \int_0^\infty e^{-r_x \tilde{g}(\pm \xi_1 \pm l^\pm r_x, k_y)} \times
eq e^{\Delta z(i\tilde{k}_z^\pm + (r_x/\tau^\pm) \cos \gamma \pm i(r_x/\tau^\pm) \Delta x \cos \gamma \pm dr_x} \quad (4.2.14)
\]

### 4.2.3 Comments on the Constant Phase Path

The above analysis shows that the (ideal) detour angle maximizing the integrand’s exponential decay is given by \( \gamma = \tan^{-1}(\Delta x/\Delta z) \), with the associated function providing the decay asymptotically expressed as \( \exp(-r_x \sqrt{(\Delta x)^2 + (\Delta z)^2}) \) in the event of the actual and ideal detour angles coinciding. Furthermore, one can easily show that along this path the phase associated with the complex exponential is (asymptotically) non-varying with respect to \( r_x \) \footnote{When planar inhomogeneity or anisotropy is involved, naturally the extent of asymptotic cancelation that occurs in reality can exhibit great variation with respect to mode type and the media involved. To account for such uncertainty, one can robustly mitigate exponentially rising terms via placing the natural logarithm of the CGLQ numerical quadrature weights in the argument of the exponential, seen in the integrand of (4.2.14), prior to evaluating the exponent.}, hence the name “Constant-Phase Path”. However, in practice one may not actually be able to deform (asymptotically) onto the exact CPP due to the presence of critical points, as well as the necessity to preclude their migration into the second and fourth quadrants of the \( k_x \) plane and (or) towards \( \text{Re}[k_x] = \pm \infty \). Although the impact of these requirements on the detour angles can be mitigated, in both the CPMWA and present CGLQ algorithms, through a suitable partitioning of the integration domain (see Figure 3.3 in Ch. 3), these requirements still can prevent deforming onto the optimal path that asymptotically maximizes numerical accuracy and convergence speed (compared to other tail path deformation
angles). Furthermore, in general two main factors prevent one from even defining a unique, common longitudinal propagation distance traversed by the modal fields (and hence a unique, common CPP associated with the four modal contributions to the observed field [c.f. Chs. 2-3]) [45][Ch. 2]:

1. Planar stratification leads to the presence of characteristic modes exhibiting layer-dependent longitudinal propagation constants. Furthermore, due to reflections at layer interfaces, both up-going and down-going modes are typically present, which in general travel different effective longitudinal distances before reaching the observation point \( r \). The planar stratification also produces, in general, multi-bounce of fields within each slab layer.

2. Anisotropy leads to mode-dependent longitudinal propagation constants along with cross-coupling of characteristic modes at the planar interfaces.

The inability, in practice, to robustly define a unique CPP departure angle and to deform onto the estimated (asymptotically) optimal path leads to unwanted, residual integrand oscillation along the actual integration path. In principle, these residual oscillations are expected to restrict the practical range of applicability within which accurate results can be delivered. Indeed, due to evaluating the semi-infinite tail integrals without path pre-partitioning, the CGLQ algorithm’s only error control mechanism for evaluating the evanescent spectra contributions consists of adaptive \( p \) refinement. However, the empirical results presented in the following section show, nevertheless, that the CGLQ method delivers excellent accuracy in strong accord with the data both from our previously developed CPMWA algorithm (c.f. Ch. 3) and the results published in [13] (see Figures 4.3-4.6). We emphasize that this excellent
agreement manifests despite the presence of multiple anisotropic layers (both uniaxial and biaxial) in these examples, which in Figures 4.3-4.6 also exhibit significant conductive loss. The algorithm’s accuracy is also distinctly manifest in its ability to confirm standard, expected results from the employment of Transformation Optics media (see Figures 4.7-4.9 below).

4.3 Validation Results

To verify the accuracy and efficiency of the proposed CGLQ algorithm, and to numerically assess the impact of the issues considered above, we now exhibit results concerning computation of fields radiated by elementary dipole sources (tensor Greens’ function components) embedded in planar-layered, anisotropic media. The results illustrate the algorithm’s performance in a wide range of environments with respect to layer material parameters, source-observer geometrical configurations, and a wide range of frequencies spanning 1kHz to 13.56MHz (i.e., five decades of frequency).

4.3.1 Resistivity Well-Logging: Induction Sondes’ Response

First, we show a data set related to the use of induction-regime electromagnetic sondes for resistivity well-logging (hydrocarbon prospection) in layered geologic formations exhibiting uniaxial or biaxial resistivity in their effective [9, 23, 26] resistivity tensors. A schematic illustration of the geophysical sonde is depicted in Figure 4.2 below. To facilitate comparison of accuracy of the computed field solution between the CGLQ and CPMWA methods, we choose the same set of results used
for the CPMWA algorithm in Ch. 3. Details of the simulation problem parameters can be found in [13] and Ch. 3, which are summarized here as follows: the induction tool axis dip and strike angles are $\alpha = 89^\circ$ and $\beta = 0^\circ$ (resp.), the tool’s frequency of electromagnetic emissions is $f = 2$ MHz, the distance $L$ between the (co-located) loop antenna receivers and (co-located) loop antenna transmitters is equal to 1.016 m (i.e., as measured along the sonde axis; see Figure 4.2), the interface partitioning the two-layer formation is located at $D = 0$ m, and the diagonal matrices $\mathbf{R}_1 = \text{diag}[100, R_{y'y',1}, 500] \Omega \text{m}$ and $\mathbf{R}_2 = \text{diag}[1, R_{y'y',2}, 5] \Omega \text{m}$ describe the resistivity tensors for layer one (top layer, $D > 0$ m) and layer two (bottom layer, $D < 0$ m), respectively. For the top, middle, and bottom rows (resp.) of each plot set concerning Figures 4.3-4.6, $\{R_{y'y',1}, R_{y'y',2}\} = \{200, 2\} \Omega \text{m}$, $\{100, 1\} \Omega \text{m}$, and $\{50, 0.5\} \Omega \text{m}$ (resp.). Furthermore, note that $H_{p'q'} (p, q = x, y, z)$ denotes the magnetic field observed by a $q'$-directed receiver due to a $p'$-directed transmitter (prime denotes coordinates with respect to the tool axis; see Figure 4.2), while $\text{Re}(H_{p'q'})$ and $\text{Im}(H_{p'q'})$ refer to the real and imaginary parts of the time-harmonic magnetic field (resp.). Finally, we alert the reader to the reversal (versus that in [13]) in the resistivity tensor labels assigned, on each of the four pages containing the induction logging plots, between the two plots on the first row and two plots on the third row. NB: The reference 23 and reference 2 curve labels (resp.) within Figures 4.3-4.6, corresponding to reference 23 and reference 2 in the original publication [3] (resp.) describing this chapter’s presented algorithm, corresponds to reference [2] (or, c.f. Ch. 3) and reference [13] (resp.) within this thesis.

86 We scale all results from our algorithm by the constant multiplicative factor $-i\omega\mu_0$ before plotting them against reference data in [13]. This is done to compensate for the $-i\omega\mu_0$ scaling of the magnetic current performed in [13]; indeed, compare equation (2.2.2) in Ch. 2 and equation (1) of [13].
Besides Figure 4.6f (where there is still acceptable accord for the intended application), we observe excellent agreement between the CGLQ algorithm (blue hatched line curves) and the results in [13] (solid red line curves). Moreover, across all the plots in Figures 4.3-4.6 there is a strong accord between the CGLQ and CPMWA (dotted green line) algorithms, suggesting that observed discrepancies versus the data in [13] can perhaps trace down to the inaccuracies of the algorithm utilized in generating the initially published reference data [13].

Figure 4.2: Schematic description of a standard triaxial electromagnetic sonde, which consists of a system of electrically small loop antennas that are modeled as Hertzian dipoles supporting an equivalent magnetic current (i.e., three orthogonally-oriented, co-located transmitters $M^T_{x'}$, $M^T_{y'}$, and $M^T_{z'}$ spaced a distance of $L=1.016\text{m}$ from three orthogonally-oriented, co-located receivers $M^R_{x'}$, $M^R_{y'}$, and $M^R_{z'}$) [13]. The “tool coordinate” $x'y'z'$ system, rotated by an angle $\alpha$ with respect to the standard $xyz$ coordinate system, is such that the $z'$ axis is parallel to the “tool axis” [13, 14].
Figure 4.3: Comparison of computed magnetic field $H_{x'x'}$ against results from Figure 4 of [13]. The top, middle, and bottom rows of plots concern material geometries of $\{R_{y'y'}, 1\} = \{200, 2\} \Omega m$, $\{100, 1\} \Omega m$, and $\{50, 0.5\} \Omega m$, respectively.
Figure 4.4: Comparison of computed magnetic field $H_{x'z'}$ against results from Figure 4 of [13]. The top, middle, and bottom rows of plots concern material geometries of $\{R_{y'y'}, 1\} = \{200, 2\} \Omega$m, $\{100, 1\} \Omega$m, and $\{50, 0.5\} \Omega$m, respectively.
Figure 4.5: Comparison of computed magnetic field $H_{z',x'}$ against results from Figure 4 of [13]. The top, middle, and bottom rows of plots concern material geometries of $\{R_y', R_y', 1\} = \{200, 2\} \Omega m$, $\{100, 1\} \Omega m$, and $\{50, 0.5\} \Omega m$, respectively.
Figure 4.6: Comparison of computed magnetic field $H_{zz'}$ against results from Figure 4 of [13]. The top, middle, and bottom rows of plots concern material geometries of $\{R_{y'y}', R_{y'y''}\} = \{200, 2\} \Omega m$, $\{100, 1\} \Omega m$, and $\{50, 0.5\} \Omega m$, respectively.
4.3.2 Planar Antenna Above Doubly-Anisotropic Isoimpedance Substrates

Next, we illustrate the application of the proposed algorithm to the modeling of planar radiators on top of isoimpedance anisotropic substrates backed by metallic ground planes. Isoimpedance substrates are substrates utilized to minimize the antenna profile by reducing substrate thickness. Conventionally, to reduce the strong field cancelation effect caused by the presence of a ground plane, substrates made of (for example) dielectric or ferrite material are used [94, 15]. Such conventional substrates typically exhibit various disadvantages such as high ohmic loss, unwanted surface waves (and hence reduced radiation efficiency and realized antenna gain), reduced bandwidth [15, 94], and the need for large thickness to yield a useful radiation resistance. On the other hand, isoimpedance substrates can facilitate a miniaturized longitudinal profile by mimicking the effect of a thicker substrate [94, 16]. Furthermore, since isoimpedance media are inherently impedance-matched to free space for all wave incidence angles [17], they do not support surface waves [94]. The problem under consideration is illustrated in Figures 4.7a, 4.7b, and 4.7c, which show a lateral view of the geometry.

The field distributions are presented in Figures 4.8-4.9. The scenario corresponding to Figures 4.8a-4.8b and 4.9a-4.9b is depicted in Figure 4.7a; similarly, the scenario corresponding to Figures 4.8c-4.8d and 4.9c-4.9d is depicted in Figure 4.7b while the scenario corresponding to Figures 4.8e-4.8f and 4.9e-4.9f is depicted in Figure 4.7c. See the captions below Figures 4.7-4.9 for the problem scenario descriptions.
First, by comparing the second row to the first row of plots in Figures 4.8-4.9, a significant weakening of the electric and magnetic field distributions can be observed by the overall darker tone, manifest in regions away from the source, in each plot. This is caused by the metallic ground’s field cancellation effect [15]. The third row of plots in each figure set corresponds to placing the dipole on top of a \( d = 5\text{mm} \) thick isoimpedance substrate, with properly chosen material tensors \( \bar{\epsilon}_r = \bar{\mu}_r = \text{Diag}[5, 5, 1/5] \), that mimics the case of a thicker, 25mm free-space buffer separating the source and ground. This leads to a field distribution, for a fixed source-observer depth separation \( z - z' > 0 \), that is exactly identical to that obtained if the source resided in free space 25mm above ground. This result can be also established analytically [94, 16], and is confirmed numerically upon observing the full agreement between the following pairs of figures: 4.8a versus 4.8e, 4.8b versus 4.8f, 4.9a versus 4.9e, and 4.9b versus 4.9f.

We also make a minor remark concerning the mottled blue annular “ring”, visible around the central region of intense electric field near the source in Figures 4.9a, 4.9c, and 4.9e. Since the \( xy \) plane field distribution cuts are in fact taken one meter above the plane on which the source resides, the \( xy \) observation planes intersect the intense main beam of the dipole as well as the deep nulls in the radiation pattern surrounding the main beam. Indeed, observing Figures 4.8a, 4.8c, and 4.8e at the elevation \( z - z' \sim 1\text{m} \), one observes that deep nulls in the dipole’s electric field distribution occur at approximately \( |x - x'| \sim 2\text{m} \), which corresponds to the annular region \( |ho - \rho'| \sim 2\text{m} \) in the \( xy \) plane electric field plots.

\(^{87}\)Please consult the color version of this article online for optimal interpretation of these graphical results.
Figure 4.7: Schematic illustration of the three scenarios simulated. The Hertzian electric dipole is always oriented in the +x direction, radiates at f = 13.56MHz, and is located a distance d above the ground plane, which has a conductivity $\sigma = 10^9$ S/m. The light brown region indicates the region of observation in free space for the exhibited $xz$ plane ($y - y' = 0$) electric field distribution plots in Figures 4.8a, 4.8c, and 4.8e, while the region of observation for the magnetic field distribution plots in Figures 4.8b, 4.8d, and 4.8f is obtained upon rotating this light brown-colored plane by ninety degrees about the z axis, yielding the $yz$ plane ($x - x' = 0$). Finally, the constant-z plane indicated by the green line in each sub-figure of Figure 4.7 indicates the location of the $xy$ plane cut on which $|E_z|$ and $|H_z|$ are plotted in Figure 4.9. Note that, contrary to the situation suggested in Figure 4.7, the ground plane is assumed infinite in its lateral extent while the observation plane is laterally bounded.
Figure 4.8: Electric field $|E_z|$ distribution (first column) and magnetic field $|H_z|$ distribution (right column) due to a Hertzian electric dipole located at $(0, 0, d)$ m.
Figure 4.9: Each row of plots corresponds to the same respective environment scenarios as Figure 4.8, except both $|E_z|$ and $|H_z|$ are plotted on an $xy$ plane cut (see Figure 4.7).
4.3.3 Convergence and Accuracy Comparison: CPMWA and CGLQ

Finally, we perform a study comparing the ability of the CPMWA and CGLQ algorithms to converge to the field contribution due to the evanescent spectra. Since the treatment of the propagation spectra is virtually identical to that in CPMWA, convergence results for the CGLQ algorithm in this latter region are omitted. By demonstrating the CGLQ algorithm’s ability to converge to the field contribution from evanescent spectra, we also demonstrate, by extension, the algorithm’s ability to converge to the field contribution arising from hybrid spectra. By “hybrid spectra” we refer to those characteristic plane wave modes that exhibit propagation behavior along the $x$ direction but evanescent behavior along the $y$ direction (or vice-versa); for reference, these two hybrid spectrum regions were denoted Regions IIa and IIb in Figure 3.3 of Ch. 3 and consist of the regions $(|\text{Re}[k_x]| < \xi_1) \cup (|\text{Re}[k_y]| > \xi_1)$ and $(|\text{Re}[k_y]| < \xi_1) \cup (|\text{Re}[k_x]| > \xi_1)$.

In Figures 4.10a-4.10b below, we plot the residual error in the evanescent spectrum field contribution from using the CGLQ and CPMWA algorithms for two representative scenarios. Figure 4.10a represents a relatively benign scenario, with small transverse source-observer separation $|x - x'| = |y - y'| = 1$ m and moderate depth separation $z - z' = 1$ m. In this case, even if $k_x$ and $k_y$ were real-valued along their respective integration paths, the integrand would exhibit low oscillation and fast exponential decay with respect to increasing $|k_x|$ and $|k_y|$. This scenario computes $H_z$ due to a Hertzian (equivalent) magnetic dipole source oriented parallel to the optical axis of a uniaxial medium characterized by the conductivity tensor.
\( \bar{\sigma} = \text{diag}[\sigma_{xx}, \sigma_{yy}, \sigma_{zz}] \text{S/m} = \text{diag}[1, 1, 1/10] \text{S/m} \). On the other hand, Figure 4.10b represents a more challenging scenario if evaluated by standard real-axis integration due to the large \( |x - x'| = |\rho - \rho'| = 500 \text{m} \) source-observer transverse separation (i.e., \( \sim 16.7 \) free-space wavelengths) and vanishing \( |z - z'| = 0 \text{m} \) source-observer depth separation.

In this case, we compute \( H_y \) radiated by a Hertzian vertical electric dipole in vacuum. Both results are compared against available analytical solutions\(^88\). To illustrate the applicability of the CGLQ algorithm over a wide frequency range, the source radiates at \( f = 1 \text{kHz} \) in the scenario of Figure 4.10a and at \( f = 10 \text{MHz} \) in the scenario of Figure 4.10b.

For the CPMWA, we vary the Gauss-Legendre quadrature order \( P \) used to integrate each of the \( B \) extrapolation intervals on a given Fourier integral half-tail, whose successive “cumulative” integration results\(^89\) were employed as the input into the CPMWA weighted average computation detailed in Ch. 3. For the CGLQ, we only vary the Gauss-Laguerre quadrature order (also denoted \( P \) in the Figures) used to evaluate each Fourier integral half-tail. To facilitate plotting the results, we keep the accuracy of the CGLQ results constant versus increasing \( B \) (obviously, there is no integration path splicing in CGLQ).

We observe that in both Figures 4.10a and 4.10b the CGLQ algorithm successfully converges to the correct evanescent field spectrum contribution. Not surprisingly, based on results in Ch. 3, the CPMWA also exhibits good convergence characteristics for both scenarios. In Figure 4.10a, we observe that while CPMWA has a slight better accuracy than CGLQ, the difference is very small. In exchange for this small difference

\(^{88}\) Note that the first scenario admits, as its closed-form solution, the equivalent magnetic dipole fields in an isotropic medium with effective conductivity \( \sigma = \sigma_{xx} = \sigma_{yy} = 1 \text{S/m} \) [77][Ch. 7].

\(^{89}\) By “cumulative” integrals we mean the unprocessed estimates of the non-truncated tail integral obtained by simply integrating over an increasingly longer path (c.f. Ch. 2).
in accuracy, a significant reduction in computational cost is obtained. Observing that the CPMWA method delivers a result with maximum accuracy (relative to the range of $B$ tested and shown in Figure 4.10) within approximately $B=6$ intervals used for each half-tail path, one realizes that $6 \times 30 = 180$ integrand evaluations are necessary when using CPMWA; on the other hand, compare this to thirty integrand evaluations using CGLQ. Similarly, a savings factor of about four in computational cost results from comparing CGLQ against CPMWA with $B = 6, P = 20$ (with a 1-2dB better accuracy exhibited by CPMWA).

In Figure 4.10b we again notice that both the CPMWA and CGLQ algorithms converge well to the true evanescent field contribution solution, tailing off with a residual error of about -95dB (or approximately nine to ten digits of accuracy) for $B \geq 6$ using either the 20-point or 30-point CGLQ variant and either the 20-point or the 30-point CPMWA variant. Thus, comments concerning computational efficiency gains in this scenario parallel those from the more benign case, with one realizing a factor of four to six in computational cost savings. We also make a side remark, for Figure 4.10b, that the two CPMWA curves do in fact run very close to each other, and likewise concerning the CGLQ curves for $P=20$ and 30; combined with the plot line thickness, the visual distinction for these two plot line pairs may prove difficult.\footnote{To aid in interpreting Figures 4.10a-4.10b, the reader is encouraged to obtain the online version of this article, which is displayed in full color.}

$90$
Figure 4.10: Convergence rate and accuracy characteristics for the CGLQ and CPMWA algorithms. To compute the reference evanescent spectrum field contribution values against which the algorithm’s results were measured for accuracy, the propagation and hybrid spectrum field contributions were computed with an adaptive integration error tolerance of 1.2d-15 (i.e., precision goal of approximately fifteen digits), summed together, and subtracted from the closed-form, space domain Hertzian dipole field solution available from [15].

4.4 Conclusion

In this work, we have detailed the mathematical formulation behind a novel application of complex-plane Gauss-Laguerre quadrature (CGLQ) to the evaluation of spectral integrals arising in the computation of the tensor Green’s function components for planar-stratified media containing layers of arbitrary anisotropy and loss. The proposed CGLQ algorithm touts the ability to robustly guarantee absolute, exponential convergence for the tail integrals for a wide range of frequencies, layer medium properties, source and field type, source orientation, and $\mathbf{r} - \mathbf{r}' \neq \mathbf{0}$ separation geometry. Compared to prior leading algorithms used for this type of problems, the computational burden in computing the hybrid and evanescent spectra has been
significantly reduced, computer storage requirements for the numerical quadrature algorithm have also been reduced, algorithm-dependent constraints on the path deformation detour angles have been eliminated, and the numerical instability of weight computations (along with the resultant need for ad-hoc adjustment of the weighted average-type extrapolation schemes) has been eliminated. Furthermore, by replacing the prior CPMWA algorithm’s cumbersome interval partition-cum-extrapolation methodology with a highly streamlined process involving one simple Gauss-Laguerre numerical quadrature, the present CGLQ method proves far easier to implement.

To validate the new algorithm’s accuracy and convergence properties, two case studies of practical interest involving layered anisotropic media, as well as a convergence study, were performed. The CGLQ algorithm was shown to effect a fast and robust computation of spectral integrals needed for the evaluation of Green’s Tensor components in layered anisotropic media. Based on the results shown, we can state that CGLQ stands, at the very least, as a viable competitor to extrapolation-based methods previously touted as the most robust means by which one can robustly compute Fourier-type integrals susceptible to rapid oscillation and small decay rate [71, 12]. The present contribution has significantly mitigated, in one stroke, both the convergence and computational efficiency bottlenecks associated with the evaluation of the evanescent spectra field contributions that have plagued the direct numerical evaluation of such layered-media Green’s Tensor integrals in the past.
Chapter 5: Spectral-Domain-Based Scattering Analysis of Fields Radiated by Distributed Sources in Planar-Stratified Environments with Arbitrarily Anisotropic Layers

5.1 Introduction

5.1.1 Chapter Summary and Contributions

We discuss the numerically stable, spectral-domain computation and extraction of the scattered electromagnetic field excited by distributed sources embedded in planar-layered environments, where each layer may exhibit arbitrary and independent electrical and magnetic anisotropic response and loss profiles. This stands in contrast to many standard spectral-domain algorithms that are restricted to computing the fields radiated by Hertzian dipole sources in planar-layered environments where the media possess azimuthal-symmetric material tensors (i.e., isotropic, and certain classes of uniaxial, media). Although computing the scattered field, particularly when due to distributed sources, appears (from the analytical perspective, at least) relatively straightforward, different procedures within the computation chain, if not treated carefully, are inherently susceptible to numerical instabilities and (or) accuracy limitations due to the potential manifestation of numerically overflown and (or) numerically unbalanced terms entering the chain. Therefore, primary emphasis
herein is given to effecting these tasks in a numerically stable and robust manner for all ranges of physical parameters. After discussing the causes behind, and means to mitigate, these sources of numerical instability, we validate the algorithm’s performance against closed-form solutions. Finally, we validate and illustrate the applicability of the proposed algorithm in case studies concerning active remote sensing of marine hydrocarbon reserves embedded deep within lossy, planar-layered media.\textsuperscript{91}

5.1.2 Background

Spectral-domain based computation and analysis of electromagnetic (EM) fields radiated by current distributions, embedded within planar-stratified environments with generally anisotropic media characterized by arbitrary (diagonalizable\textsuperscript{92}) $3 \times 3$ relative permeability and permittivity tensors $\bar{\mu}_r$ and $\bar{\epsilon}_r$ (resp.), finds application in myriad areas. Some examples are geophysical prospection in subterranean [9, 85, 90, 29] and sub-oceanic [35, 37, 41, 42, 44] environments, microstrip antennas [94, 64, 65, 95], planar waveguides [67], transionospheric EM propagation studies [60], ground penetrating radar [91], and so on. To facilitate field computation in such problems, which can possess domains exhibiting length scales on the order of hundreds or even thousands of wavelengths, spectral/Fourier-domain based EM field calculation methods exhibit both robustness and speed as defining virtues, making them oftentimes indispensable [70, 12, 71] (also, c.f. Chs. 2-4). For example, as demonstrated in Ch. 4, through use of Complex-Plane Gauss-Laguerre Quadrature (CGLQ) and adaptive $hp$ refinement one can rapidly and accurately evaluate, without

\textsuperscript{91}NOTE: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.

\textsuperscript{92}The diagonalizability constraint ensures completeness of the plane wave basis; naturally-occurring media are always characterizable by diagonalizable material tensors, however.
analytical-stage\textsuperscript{93} approximations, the radiated EM field via direct numerical integration without major concern about slow integrand decay (and hence slow convergence) or rapid integrand oscillation (necessitating fine sampling and high computational cost). It is upon these and other previous works (c.f. Chs. 2-3) that we build to create a robust, error-controllable, and rapid direct integration algorithm directed specifically at achieving two objectives concerning EM radiation and scattering in planar-layered, generally anisotropic media, which comprise the main contributions of this paper: (1) An “in-situ” scattered EM field extraction method, applicable to both point-like/Hertzian and distributed radiators (e.g., wire and aperture antennas), and (2) Direct Fourier-domain evaluation of the radiation integral used to compute fields radiated by distributed sources. Elaborated upon in detail below, these contributions add to the extensive body of work concerning spectral-domain based calculation of EM fields in planar-layered media which dominantly focus on radiation of Hertzian dipole sources in planar-layered media where the layers possess azimuthal-symmetric material tensors [70, 61, 70, 12, 71]. In contrast, in the spirit of previous work (c.f. Chs. 2-4), our proposed scattered-field and distributed radiator computation algorithms are applicable to planar-layered media where the layers can possess arbitrary (diagonalizable) material tensors.

It is often the case that the time-harmonic scattered field \( \mathbf{E}_s(\mathbf{r}) = \mathbf{E}(\mathbf{r}) - \delta_{L,M} \mathbf{E}_d(\mathbf{r}) \)\textsuperscript{94} (or, via Fourier synthesis, the time domain scattered field) constitutes the signal of

\textsuperscript{93}As opposed to when using, for example, discrete image methods [73] which can involve approximating the spectral integrand as a sum of analytically invertible “images” possessing closed-form integral solutions.

\textsuperscript{94}Rather than the time-harmonic homogeneous medium/“direct” field \( \mathbf{E}_d(\mathbf{r}) \) or time-harmonic total field \( \mathbf{E}(\mathbf{r}) \). The Kronecker delta \( \delta_{L,M} \) equals either one or zero when the source and observation layers \( (M \text{ and } L, \text{ resp.}) \) either coincide or differ, respectively.
interest as it carries information about the inhomogeneity of the medium under interrogation. For example, in geophysical borehole prospection it is well known that planar inhomogeneity can contribute to erroneous extraction of the resistivity tensor of the local earth formation in which the sonde is presently embedded [85]. Therefore, being able to extract and analyze only the scattered field contribution may facilitate mitigating formation inhomogeneity effects in induction sonde measurements. Similarly concerning radars, one is usually only interested in the scattered field as it carries information about the surrounding environment’s parameter(s) of interest [96, 97].

Two straightforward ways to effect scattered field extraction are (1) a-posteriori subtraction of $\mathbf{E}_d(\mathbf{r})$ (computed in closed form) from $\mathbf{E}(\mathbf{r})$ (numerically evaluated with spectral methods) and (2) temporal discrimination between the time domain (TD) direct and scattered field signals.

There are important drawbacks with each of these two methods, however. The subtraction method suffers from lack of general applicability when the source is embedded in generally anisotropic media wherein the time-harmonic space domain tensor Green’s functions may not be available in closed form. Furthermore, even when $\mathbf{E}_d(\mathbf{r})$ is available in closed form, a posteriori direct field subtraction lacks robustness in the numerical evaluation of $\mathbf{E}(\mathbf{r})$ since $\{|\mathbf{E}(\mathbf{r})|, |\mathbf{E}_d(\mathbf{r})|\} \to \infty$ as the observation point $\mathbf{r} = (x, y, z)$ approaches a source point $\mathbf{r}' = (x', y', z')$ (e.g., time-harmonic scattered field received at a mono-static radar), leading to the subtraction of two numerically overflown results. Time-gating, on the other hand, is feasible subject to temporal resolvability between the direct and scattered fields; this is fundamentally absent in time-harmonic fields, however, which are oftentimes the quantities of interest. Furthermore, the time-gating method also suffers from the same numerical instability.
issue when simulating “mono-static”-like scenarios. Indeed when the TD signal, in
such eigenfunction expansion techniques, is synthesized through a superposition of
frequency domain signals, obviously one requires here too a numerically stable and
robust scheme to compute the total (frequency domain) field $\mathbf{E}(\mathbf{r})$ at each desired fre-
quency to facilitate TD windowing of the synthesized TD signal. Therefore, both the
subtraction and TD windowing techniques return us, in general, back to the question
of how to compute the frequency-domain field across a wide range of source/observer
distribution and position scenarios.

In contrast to the above two techniques, the proposed scattered field extraction
approach proposed relies upon “in-situ” subtraction of the direct field during the
modal field synthesis (i.e., spectral integration) process itself. This “in-situ” subtrac-
tion approach, constituting the first of our two proposed contributions, sports the
following advantages:

1. Applicability to time-harmonic fields and, through Fourier temporal harmonic
   synthesis, TD fields.

2. Does not require the space-domain tensor Green’s functions (either in the fre-
   quency or time domain) in closed form.

3. Robustness and numerical stability even as $|\mathbf{r} - \mathbf{r}'| \to 0$, rendering it applicable
even to “mono-static”-like radiation and reception scenarios.

4. Imposes no additional computational burden versus computing $\mathbf{E}(\mathbf{r})$ (c.f. Chs.
   2-4).

5. Imparts added exponential decay to the spectral integrand that further accel-
erates convergence of the field solution.
6. Automatically and rigorously effects the time-windowing function ordinarily performed after synthesis of the TD total field signal, removing any need for additional processing to discriminate between the TD direct and scattered field signals.

7. Applicability to general source geometries possessing a closed form Fourier domain representation.

Beyond extracting the scattered field, we propose a rapid, robust algorithm to compute the spectral domain integral representation of the field produced by distributed sources embedded in planar-layered, generally anisotropic media. This strategy is based on the spectral representation of compactly-supported, otherwise arbitrary distributed sources in terms of spatial (sinusoidal) current harmonics and finds applicability where realistic modeling of current sources (whether they be physical antennas or equivalent current distributions) is otherwise prohibitive due to the computationally expensive task of either repeatedly computing the (space-domain) tensor Green’s function and/or having to perform spatial discretization of the source distribution. One such example includes computing the received scattered field at a spaceborne radar platform in such a way that captures the effects of an inhomogeneous atmosphere and (or) subsurface environment. This approach may also prove desirable in aperture field synthesis, where it can separately compute the field pattern of (orthogonal) Fourier current modes radiating in a given inhomogeneous, anisotropic environment, and thus constitute an efficient forward engine for aperture synthesis optimization algorithms seeking to solve the inverse problem of procuring an aperture current distribution leading to a desired, pre-defined field pattern.
The relative efficiency of the spectral domain method, concerning distributed sources, arises primarily from two factors. The first factor is the lower sampling requirement needed in the spectral domain as compared to the spatial domain (e.g., spatial sampling using a Hertzian dipole/“pulse” basis) to represent a given harmonic current distribution and its radiated field. That is to say, for each harmonic current (requiring, self-evidently, only one spectral domain sample) and its radiated field that is simulated, one must use (based on our numerical experiments) approximately ten Hertzian dipole samplings per half-cycle variation of current amplitude. This sampling efficiency in turn amounts to approximately one order of magnitude solution speed acceleration for one-dimensional, wire-like source distributions and approximately two orders of magnitude solution speed acceleration for two-dimensional, aperture-like source distributions. The second contributing factor towards efficiency is the sparse representation of many commonly encountered current distributions in terms of mutually orthogonal spatial harmonic current distributions.\footnote{As mathematically demonstrated in Section 5.4 a secondary phenomenon, relating to the “tapering” of the distributed source’s field spectrum (as compared to the spectrum of the fields from a Hertzian dipole), also imparts added efficiency in evaluating the fields in the spectral domain.} Although in the spectral domain the distributed characteristic of the source enters into the field spectrum as a (deceptively simple) multiplicative factor augmenting the Hertzian dipole field spectrum, serious numerical instability issues can arise in a practical implementation due to the manifestation of exponentially rising field terms. This issue must be addressed to realize the computational efficiency benefits of the spectral domain evaluation of distributed source fields.

We discuss the above-mentioned stability and robustness issues, along with the proposed solutions to them, in Sections 5.3 and 5.4. First for convenience, we briefly
summarize some fundamentals behind the underlying formulation; notation information and details of the underlying formulation can be found in Chs. 2-4.

5.2 Formulation Fundamentals: Overview

Initially assume a homogeneous medium possessing material tensors\(^{96}\) \(\bar{\varepsilon}_c = \varepsilon_0 \bar{\varepsilon}_r\) (permittivity, including losses) and \(\bar{\mu}_c = \mu_0 \bar{\mu}_r\) (permeability, including losses) exhibiting arbitrary and independent anisotropy and loss, in which there are impressed (i.e., causative) electric and (equivalent) magnetic current densities \(\mathbf{J}(\mathbf{r})\) and \(\mathbf{M}(\mathbf{r})\) (resp.), as well as impressed volumetric electric and (equivalent) magnetic charge densities \(\rho_v\) and \(\rho_m\) (resp.). From Maxwell’s equations, one obtains [45] (also, c.f. Ch. 2):

\[
\bar{\mathbf{A}} = \nabla \times \bar{\mu}_r^{-1} \cdot \nabla \times -k_0^2 \bar{\varepsilon}_r. \tag{5.2.1}
\]

\[
\bar{\mathbf{A}} \cdot \mathbf{\varepsilon} = i k_0 \eta_0 \mathbf{J} - \nabla \times \bar{\mu}_r^{-1} \cdot \mathbf{M}. \tag{5.2.2}
\]

where the \(\exp(-i\omega t)\) convention is assumed and suppressed. Subsequently defining the three-dimensional Fourier Transform (FT) pair, for some generic vector field \(\mathbf{L}\), as (c.f. Ch. 2)

\[
\tilde{\mathbf{L}}(\mathbf{k}) = \iiint_{-\infty}^{+\infty} \mathbf{L}(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} \, d\mathbf{x} \, d\mathbf{y} \, d\mathbf{z}. \tag{5.2.3}
\]

\[
\mathbf{L}(\mathbf{r}) = \left(\frac{1}{2\pi}\right)^3 \iiint_{-\infty}^{+\infty} \tilde{\mathbf{L}}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} \, dk_x \, dk_y \, dk_z. \tag{5.2.4}
\]

with \(\mathbf{r} = (x, y, z)\) and \(\mathbf{k} = (k_x, k_y, k_z)\), one can take the FT of (5.2.2) to yield its Fourier domain version followed by multiplying \(\bar{\mathbf{A}}^{-1}\) on both sides of the resultant

\(^{96}\)\(\varepsilon_0\), \(c\), and \(\mu_0 = 1/(\varepsilon_0 c^2)\) are the vacuum permittivity, speed of light, and permeability, respectively. Furthermore, \(\omega = 2\pi f\), \(k_0 = \omega/c\), and \(\eta_0 = \sqrt{\mu_0 / \varepsilon_0}\) are the angular radiation frequency, vacuum wavenumber, and intrinsic impedance [15, 45], respectively, while \(i\) denotes the unit imaginary number.
Fourier-domain expression. Further manipulations, upon assuming a single Hertzian dipole source at \( r' \) and denoting the observation point as \( r \), leads to the expression for the time-harmonic direct electric field \( E_d(r) \) radiated by said distribution in this layer possessing material properties of (what is, in the multi-layered medium scenario, defined as) layer \( M \) (c.f. Ch. 2):

\[
\left. \frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ u(z-z') \sum_{n=1}^{2} \tilde{a}_{M,n} \tilde{e}_{M,n} e^{ik_{M,n}z'} + u(z'-z) \sum_{n=3}^{4} \tilde{a}_{M,n} \tilde{e}_{M,n} e^{ik_{M,n}z'} \right] \times e^{ik_x \Delta x + ik_y \Delta y} \, dk_x \, dk_y \right) (5.2.5)
\]

where \( \Delta z = z - z' \), \( \Delta x = x - x' = \Delta y = y - y' \geq 0 \), \( \{ \tilde{e}_{P,n}, \tilde{k}_{P,nz}, \tilde{a}_{P,n} \} \) stand for the modal electric field vector, longitudinal propagation constant, and (source dependent) direct field amplitude of the \( P \)th layer’s \( n \)th mode (\( 1 \leq P \leq N \)) (resp.), and \( u(\cdot) \) denotes the Heaviside step function. Similarly, the time-harmonic scattered electric field \( E_s(r) \) writes as (c.f. Ch. 2):

\[
\left. \frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ (1 - \delta_{L,N}) \sum_{n=1}^{2} \tilde{a}_{L,n} \tilde{e}_{L,n} e^{ik_{L,nz}} + (1 - \delta_{L,1}) \sum_{n=3}^{4} \tilde{a}_{L,n} \tilde{e}_{L,n} e^{ik_{L,nz}} \right] \times e^{ik_x \Delta x + ik_y \Delta y} \, dk_x \, dk_y \right) (5.2.6)
\]

where \( \tilde{a}_{P,n}^s \) is the scattered field amplitude in layer \( P \).

Before proceeding, we note that when referring to the \( n \)th modal field in layer \( P \) being “phase-referenced” to a particular \( z = z_o \) plane, this means that its longitudinal propagator has been cast in the form \( e^{ik_{P,nz}(z-z_o)} \).

\(^{97}\)An azimuthal coordinate rotation is assumed to have been performed such that \( \Delta x = \Delta y \geq 0 \) (c.f. Chs. 3-4).
5.3 Direct Field Subtraction

5.3.1 Modal Field Representation Modifications

We now exhibit the formulation to extract the scattered electric field observed at \( r \) in layer \( L \) due to a source at \( r' \) in layer \( M = L \) for \( 1 \leq (M = L) \leq N \). We use here the same notation and nomenclature as in Ch. 2 concerning scattered fields, whose expressions we briefly review next. First define \( \tilde{a}^+_D \) (\( \tilde{a}^-_D \)) as the direct field \( 2 \times 1 \) modal amplitude vector associated with up-going (down-going) characteristic modes in layer \( M \) phase-referenced to the top (bottom) bounding interface at depth \( z = z_{M-1} \) (\( z = z_M \)) (c.f. Ch. 2). Second, define the \( 2 \times 1 \) vectors \( \tilde{a}^+_S \) and \( \tilde{a}^-_S \) as the up-going and down-going scattered field modal amplitudes (resp.) in layer \( M \) whose respective modal fields are phase-referenced to the interface at \( z = z_{M-1} \); likewise, \( \{ \tilde{a}^+_S, \tilde{a}^-_S \} \) are the amplitudes for scattered modal fields that are phase-referenced to the interface \( z = z_M \) (c.f. Ch. 2). Third, denote the \( 2 \times 2 \) generalized reflection matrix from layer \( P \) to adjacent layer \( P' \) as \( \tilde{\bar{R}}_{P,P'} \). One then obtains the standard formulae below for the scattered field amplitudes as a function of the direct field amplitudes (c.f. Ch. 2):\(^{99}\)

\[
\tilde{\Lambda}^+_M(z_o) = \text{Diag} \begin{bmatrix} e^{i\tilde{k}_M,1z_o} e^{i\tilde{k}_M,2z_o} \end{bmatrix}, \quad \tilde{\Lambda}^-_M(z_o) = \text{Diag} \begin{bmatrix} e^{i\tilde{k}_M,3z_o} e^{i\tilde{k}_M,4z_o} \end{bmatrix} \tag{5.3.1}
\]

\[
\tilde{M}_1 = \tilde{\Lambda}^+_M(z_{M-1} - z_M) \cdot \tilde{R}_{M,M+1}, \quad \tilde{M}_2 = \tilde{\Lambda}^-_M(z_M - z_{M-1}) \cdot \tilde{R}_{M,M-1} \tag{5.3.2}
\]

\[
\tilde{a}^-_{S1} = \left[ \mathbb{I}_2 - \tilde{R}_{M,M-1} \cdot \tilde{M}_1 \cdot \tilde{\Lambda}^-_M(z_M - z_{M-1}) \right]^{-1} \cdot \tilde{R}_{M,M-1} \cdot \left[ \tilde{a}^+_D + \tilde{M}_1 \cdot \tilde{a}^-_D \right] \tag{5.3.3}
\]

\[
\tilde{a}^+_{S2} = \left[ \mathbb{I}_2 - \tilde{R}_{M,M+1} \cdot \tilde{M}_2 \cdot \tilde{\Lambda}^+_M(z_{M-1} - z_M) \right]^{-1} \cdot \tilde{R}_{M,M+1} \cdot \left[ \tilde{a}^-_D + \tilde{M}_2 \cdot \tilde{a}^+_D \right] \tag{5.3.4}
\]

\(^{98}\)That is, \( P' \) equals either \( P + 1 \) or \( P - 1 \) when \( P' \) corresponds to the layer below or above layer \( P \), respectively, where layer \( P \) is the layer containing the incident modal fields.

\(^{99}\)\( \mathbb{I}_\nu \) is the \( \nu \times \nu \) identity matrix.
which are required when \( M = L \) (i.e., when the observation and source layers coincide). Then the observed scattered field amplitudes, when \( L = M \), write as

\[
\tilde{a}_L^+ = \tilde{A}_L^+(z - z_L) \cdot \tilde{s}_2, \quad \tilde{a}_L^- = \tilde{A}_L^-(z - z_{L-1}) \cdot \tilde{s}_1
\]  

(5.3.5)

with the subsequent scattered amplitude-weighted superposition of the observed scattered modal fields following the prescription described in Ch. 2. Note that if \( M = L = 1 \) or \( M = L = N \), (5.3.3) or (5.3.4) (resp.) reduce to \( 0 \) and derive from the fact that no down-going or up-going reflected fields are present in layer \( L \) (resp.) [45][Ch. 2].

Now that the direct fields have served their purpose of exciting the scattered fields, their subtraction from the total field solution enters via coercion of the Kronecker delta \( \delta_{L,M} \) in the expression \( \mathcal{E}(r) = \delta_{L,M} \mathcal{E}_d(r) + \mathcal{E}_s(r) \) to zero. Indeed, one uses the direct fields to excite the scattered fields, but does not include the direct field contributions themselves when assembling the total observed modal field for some \((k_x, k_y)\) doublet, as evidenced by the expressions for \( \tilde{a}_L^+ \) and \( \tilde{a}_L^- \) in (5.3.5) being devoid of explicit dependence on \( \tilde{a}_D^+ \) and \( \tilde{a}_D^- \).\(^{100}\) Such a scattered-field extraction procedure is independent of the type of excitation involved; that is to say, this procedure sports applicability to electric and (equivalent) magnetic currents of arbitrary polarization, (bounded) amplitude profile, and (compact) spatial support region subject to possessing a valid Fourier (wave-number) domain representation. Finally, we remark that due to the concept of a “scattered” field becoming more ambiguous when \( M \neq L \), the observed modal field amplitudes are computed identically to the procedure used in Ch. 2 to compute the total field \( \mathcal{E}(r) \).

\(^{100}\)Implicitly, of course, \( \tilde{a}_L^+ \) and \( \tilde{a}_L^- \) do depend on the direct field excitation.
5.3.2 Additional Remarks

The spectral integral in (5.2.6) is evaluated along properly chosen integration contours in the $k_x$ and $k_y$ complex planes. For details, the reader is referred to Chs. 3-4; at present it suffices to recall from Chs. 3-4 that for the semi-infinite $k_x$ and $k_y$ “tail” integrals one detours into the upper-half plane, parameterized by the detour angle $\gamma = \tan^{-1}(|\Delta x/\Delta z|)$ (c.f. Ch. 3), where one now replaces $\Delta z = z - z'$ with $\Delta z_{\text{eff}}$:

$$
\Delta z_{\text{eff}} = \begin{cases} 
|z - z'| & , M \neq L \\
(z - z_1) + (z' - z_1) & , M = L = 1 \\
(z_{N-1} - z) + (z_{N-1} - z') & , M = L = N \\
\min[(z_{M-1} - z) + (z_{M-1} - z'), (z - z_M) + (z' - z_M)] & , 1 < (M = L) < N 
\end{cases}
$$

(5.3.6)

In the first and fourth cases above, for which $M \neq L$ and $1 < (M = L) < N$ (resp.), one typically encounters (excepting when $M \neq L$, with $L = 1$ or $N$) both up-going and down-going scattered fields. Therefore, we are obliged to make conservative (small) assumptions for $\Delta z_{\text{eff}}$ to minimize the residual\textsuperscript{141} $\exp(r_x \gamma_x \Delta z_{\text{eff}})$ that we add back into the integrand (and similarly for the $k_y$ integration). Indeed, this is especially important due to the asymptotic Constant Phase Path (CPP), in general, not being well defined due to anisotropy and planar inhomogeneity (c.f. Chs. 3-4). Therefore, conservatively assigning $\Delta z_{\text{eff}}$ avoids situations where (for example) $1 < (M = L) < N$, the source and observation points are both very close to the interface at $z = z_{M-1}$, and one uses an alternative effective propagation distance such as $\Delta z_{\text{eff}} = [(2z_{M-1} - z - z') + (z + z' - 2z_M)]/2$ that may over-estimate the effective longitudinal propagation distance of (the dominant contribution to) the down-going scattered

\textsuperscript{141}$r_x$ is the real-valued variable in Ch. 4 in terms of which the $k_x$ plane integration contour path “tail” is parameterized. The residual factor $\exp(r_x \gamma_x \Delta z_{\text{eff}})$ arises from using Complex-Plane Gauss-Laguerre Quadrature (CGLQ).
fields ("single-bounce reflection term"). This may lead to exponential kernels of the form \(10^2 \exp(\tilde{k}_z \Delta z' + \Delta z_{\text{eff}} r_x \cos \gamma_x)\), corresponding to the down-going scattered fields whose actual effective longitudinal propagation distance \(\Delta z'\) has been overestimated as \(\Delta z_{\text{eff}}\). Such exponential residuals may lead to unbounded solutions for increasing \(|r_x|\), rather than asymptotically tending to a constant magnitude and contributing towards a numerically stable computation process (c.f. Ch. 4).

5.3.3 Validation Results: Scattered Field Extraction

To validate the algorithm’s ability to accurately extract the scattered field, we use it to verify the following well-known results concerning the effect of placing Hertzian dipole radiators infinitesimally close to a perfectly conducting ground plane of infinite lateral extent [15]:

1. The direct EM field of a vertical electric dipole (VED) will be reinforced by the field scattered off the ground. That is, the scattered and direct fields should be equal.

2. The direct EM field of a vertical magnetic dipole (VMD) will be canceled by the ground-scattered field. That is, the scattered and direct contributions to any given field component should be equal in magnitude and opposite in sign.

To avoid (1) numerical instability due to entering an infinite conductivity for the ground plane and (2) inaccuracy stemming from a ground plane with finite conductivity, the presence of a perfectly reflecting ground plane is equivalently effected via manually coercing, within the code, the ground plane’s intrinsic reflection coefficients

\(^{102}\Delta z'\) loosely denotes the correct mode-dependent effective longitudinal distance. Of course, in reality \(\Delta z'\) is elusive to accurately quantify due to anisotropy and/or, when finite-thickness slabs are present, internal “multi-bounce” effects.
for the incident TE\(_z\) and TM\(_z\) modes (c.f. Ch. 2). We emphasize that this coercion is done only to facilitate the present image theory study and does not fundamentally alter any of the other computations.

Prior to discussing results, we note the following conventions used for all numerical results discussed in the paper: (1) All errors are displayed as field component-wise relative error 10log\(_{10}\)\(|(L_{num} - L_{exact})/L_{exact}|\) (dB units); (2) all computations are performed in double precision; (3) any relative errors below -150dB are coerced to -150dB; (4) An adaptive integration tolerance of 1.2 \(\times\) 10\(^{-t}\) denotes a precision goal of approximately \(t\) digits \([98]\); and (5) the error is coerced to -150dB whenever the computed and reference solution magnitudes are (within machine precision equal to) zero.

Figure 5.1a shows the error in computing the reflected electric field \(E_z\) due to a VED, while Figure 5.1b shows the error in computing the reflected magnetic field \(H_z\) due to a VMD, where both sources are radiating at \(f=2\)MHz. The observation point is kept at a fixed radial distance \(|r - r'|=10\)m from the source, the observation angle in azimuth is set to \(\phi = 0^\circ\), and the polar angle \(\theta\) is swept from \(-89^\circ \leq \theta \leq 89^\circ\).

To test the scattered-field extraction for all possible scenarios concerning the source and observation points being in the same layer,\(^\text{103}\) we perform the scattered-field extraction in the following four cases referred to in the legends of Figures 5.1a-5.1b.

Case 1: Vacuum half-space above perfect electric conductor (PEC) half-space, with the source placed infinitesimally above the PEC ground. Case 2: Vacuum half-space below PEC half-space, with the source placed infinitesimally below the PEC ground. Case 3: Vacuum half-space, fictitiously partitioned into two layers such

\(^\text{103}\)Recall that if \(L \neq M\), the scattered-field extraction algorithm reduces to computing the total field.
that both source and observer reside in a “slab” of vacuum, above PEC half-space (source placed infinitesimally above the PEC ground). Case 4: Vacuum half-space, fictitiously partitioned into two layers such that both source and observer reside in a “slab” of vacuum, below PEC half-space (source placed infinitesimally below the PEC ground). By being placed “infinitesimally” above or below the ground plane, we mean to say that the longitudinal distance between the source and ground plane is set to $1.0 \times 10^{-15}$m. Needless to say, this distance (nuclear scale) is many orders of magnitude below the length scales of this problem; rather, it is simply used as a numerical means to test the proposed scattered-field extraction algorithm’s accuracy.

We observe that approximately between the angles $-60^\circ \leq \theta \leq 60^\circ$ the algorithm delivers at least eleven digits of accuracy, which is consistent with the adaptive integration tolerance set as $1.2 \times 10^{-12}$. However, accuracy declines to approximately four digits as the observation point tends toward the surface of the ground plane. The cause behind this degradation of accuracy as the polar observation angle tends towards horizon, which is also evident in the wire and aperture antenna studies in Sections 5.4.5 and 5.4.6 below, is a topic of ongoing investigation.
5.4 Distributed-Source Field Computation

5.4.1 Introduction

The process of evaluating fields from distributed sources traditionally involves discretization of the space-domain radiation integrals concerning the (for example) electric field produced by either an electric or (equivalent) magnetic current source distribution (resp.) [45, 77]:

$$\mathbf{E}(\mathbf{r}) = ik_0\eta_0 \iint_{V'} \mathbf{G}_{ee}(\mathbf{r};\mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')dV' \quad (5.4.1)$$

$$\mathbf{E}(\mathbf{r}) = - \iint_{V'} \mathbf{G}_{em}(\mathbf{r};\mathbf{r}') \cdot \mathbf{\mu}_r^{-1}(\mathbf{r}') \cdot \mathbf{M}(\mathbf{r}')dV' \quad (5.4.2)$$

where $\mathbf{G}_{ee}(\mathbf{r};\mathbf{r}')$ and $\mathbf{G}_{em}(\mathbf{r};\mathbf{r}')$ are the space domain tensor Green’s functions describing electric fields radiated by Hertzian electric and magnetic dipole sources (resp.), and $dV' = d\mathbf{z}'d\mathbf{y}'d\mathbf{z}'$ is the differential volume element on the emitter antenna manifold occupying volume $V'$ in layer $M$. Now admit either an electric or magnetic
source distribution and assume it is contained in one layer for simplicity. Furthermore, let $N_{\text{avg}}$ denote the average number of points on $V'$ whose equivalent Hertzian dipole contributions, to the electric field at observation point $r$, need to be sampled to re-construct the observed field with some pre-prescribed accuracy level. In this case, one must (in general) evaluate a total of $N_{\text{avg}}$ two-dimensional Fourier integrals due to the space domain tensor Green's functions being translation-\emph{variant} along the longitudinal direction. In addition, one must then evaluate the space domain radiation integrals (5.4.1)-(5.4.2) themselves, which for an electrically large radiator with rapid variation in the current amplitude and/or polarization profile may itself also be a non-trivial task.

It turns out that for simple antenna geometries whose space-domain Fourier transforms are readily available in closed form, one can feasibly eliminate the intermediate step of evaluating the space-domain radiation integrals in (5.4.1)-(5.4.2) by directly computing the radiation integrals in the spectral domain itself. Indeed, recall that for a homogeneous medium (5.4.1)-(5.4.2) reduce to three-dimensional convolution integrals which can equivalently be computed in the Fourier domain [45][Ch. 7]:

$$\mathbf{E} (r) = i k_0 \eta_0 \left( \frac{1}{2 \pi} \right)^3 \int \int \int_{-\infty}^{\infty} \tilde{\mathbf{G}}_{ee} (k; r') \cdot \tilde{\mathbf{J}} (k) e^{i k_0 x_0 + i k_y y_0 + i k_z z_0} \, dk_x \, dk_y \, dk_z$$

(5.4.3)

$$\mathbf{E} (r) = - \left( \frac{1}{2 \pi} \right)^3 \int \int \int_{-\infty}^{\infty} \tilde{\mathbf{G}}_{em} (k; r') \cdot \tilde{\mathbf{M}} (k) e^{i k_0 x_0 + i k_y y_0 + i k_z z_0} \, dk_x \, dk_y \, dk_z$$

(5.4.4)

whose more generalized manifestation, in the case of planar-layered media, writes as shown in (5.2.5)-(5.2.6). Of course, in the case of homogeneous isotropic media, the spectral-domain implementation may not be advantageous since the space-domain tensor Green’s functions are available in closed form [45][Chs. 1,7]. However, in either homogeneous media exhibiting arbitrary anisotropy and/or planar-stratified
media as considered here, wherein space-domain tensor Green’s functions are typically unavailable in closed form, the spectral domain evaluation of the radiation integrals can offer a significant advantage in terms of solution speed and computational resource demand.

5.4.2 Generalized Source Distribution: Formulation and Analyticity Considerations

First we start with the vector wave equation (5.2.2) under the assumption of a homogeneous medium, as usual. Assuming the source distribution to have a valid FT, i.e., $\tilde{J}(k)$ and $\tilde{M}(k)$ are well-defined spectral quantities, one can exhibit the spectral-domain version of (5.2.2) as

$$\tilde{A}(k) \cdot \tilde{E}(k) = ik_0\eta_0 \tilde{J}(k) - \nabla \times \tilde{\mu}^{-1} \cdot \tilde{M}(k)$$  \hspace{1cm} (5.4.5)

Inverting $\tilde{A}(k)$ and taking the three-dimensional inverse Fourier integral on both sides of (5.4.5) yields the space domain electric field $E(r)$. For a homogeneous medium, we note that (5.2.5) (with $\mathcal{L}$ and $\tilde{\mathcal{L}}$ replaced by $\mathcal{E}$ and $\tilde{\mathcal{E}}$ in (5.2.4), resp.) is just the spectral-domain implementation of convolution for a general source distribution (5.4.3)-(5.4.4). Despite the conceptually straightforward task of computing the radiation integral in the spectral domain, the presence of a distributed source presents a practical challenge to the numerically robust and stable evaluation of the EM field.

This is because the analyticity properties of the spectral integrand in regards to the $k_x$, $k_y$, and $k_z$ spectral variables in (5.2.4) are now obfuscated. That is, one often encounters the following scenario with distributed radiators: Regardless of whether
one deforms the integration path into the upper- or lower-half of the \( k_x \) or \( k_y \) complex plane,\(^{104}\) one may encounter a numerically overflown result.

To illustrate this, consider the modified FT pair below for the electric source current distribution \( \mathcal{J}(r) \):

\[
\tilde{\mathcal{J}}(k) = \mathcal{F}\left\{ \mathcal{J}(r) \right\} = \iiint_{-\infty}^{\infty} \mathcal{J}(r') e^{-ik\cdot r'} dx'dy'dz' 
\]

\[(5.4.6)\]

\[
\mathcal{J}(r) = \mathcal{F}^{-1}\left\{ \tilde{\mathcal{J}}(k) \right\} = \left( \frac{1}{2\pi} \right)^3 \iiint_{-\infty}^{\infty} \iiint_{-\infty}^{\infty} \mathcal{J}(r') e^{ik(r-r')} dk_x dk_y dk_z dx' dy' dz' 
\]

\[(5.4.7)\]

where one can draw a physical association between \( r \) and the observation point, as well as between \( r' \) and an equivalent Hertzian dipole source belonging to \( \mathcal{J}(r) \).

Now, assume fixed \( x \) and \( x' \) values for which one evaluates the inner (i.e., along \( k_x \)) spectral integral. One promptly realizes that depending on whether \((x - x') > 0\) or \((x - x') < 0\) the spectral-domain integrand’s region of analyticity, and hence the region in which one can apply Jordan’s lemma in the \( k_x \) plane \([45]\)[Ch. 2], depends on the sign of \( x - x' \). Physically, this corresponds to a situation of the observer witnessing incoming radiation from sources placed on either side of the observer along \( x \). An analogous observation can be made with respect to more general volumetric sources distributed along \( x, y, \) and \( z \). To better understand the analyticity issue, consider the example of the current distribution \( \mathcal{J}(r) = [\delta(x - L/2) + \delta(x + L/2)] \delta(x)\delta(y)\hat{z} \),\(^{105}\) possessing FT \( \tilde{\mathcal{J}}(k) = [e^{-ik_zL/2} + e^{ik_zL/2}] \hat{z} = 2\hat{z} \cos (k_z L/2) \), which radiates in free space. Evaluating (analytically) the \( k_z \) integral of the spectral field and detouring in

\(^{104}\)This deformation is performed mainly to minimize integrand oscillation and accelerate integrand decay along the deformed Fourier tail integral paths, thereby resulting in rapid convergence of the evanescent spectrum field contribution (c.f. Chs. 3-4).

\(^{105}\)\( \delta(x) \) is the Dirac delta function.
the \( k_x \)-plane integration path’s “tail” section (c.f. Chs. 3-4) results in \(|\tilde{J}(k)| \to \infty \) as \(|\text{Im}(k_x)| \to \infty \) whenever \(|x| < L/2 \); see Figure 5.2.

![Figure 5.2](image)

Figure 5.2: Source-location-dependent region of analyticity of the spectral EM field in the \( k_x \) plane regarding the discussed example of two dipole sources. When \(|x| < L/2 \), the real-axis path is equivalent to enclosing either the upper-half or lower-half \( \text{Im}[k_x] \) plane for the source located at \( x' = -L/2 \) or \( x' = +L/2 \) (resp.).

To mitigate the risk of numerical overflow for arbitrary \( r \neq r' \), we make the conservative judgment to only allow observation of \( \mathbf{E}_s(r) \).\(^{106}\) To further suppress any exponentially rising terms, we purposefully incorporate the real-valued, numerical

\(^{106}\)Restriction to calculating only the scattered field is done to lend exponential damping to the spectral integrand, helping to offset exponentially rising terms due to distributed sources.
Laguerre-Gauss quadrature weights\textsuperscript{107} directly into the power of the complex exponentials prior to evaluating the exponentials themselves. It warrants pointing out that the importance of incorporating these quadrature weights directly into the exponentials should not be underestimated in comparison to the importance of restricting calculation to only the scattered field when it comes to distributed sources. Indeed, the weights themselves rapidly decay with respect to the real-valued variable $r_x$ used to parameterize the deformed path along which we evaluate the Fourier “tail” integrals in the $k_x$ plane (c.f. Ch. 4). As a result, they serve to mitigate $r_x$-dependent exponential increase due to the presence of a distributed source and its $\exp(ik_x L/2)$-like terms. This is illustrated below in Figure 5.3, where we plot $\ln(w_x)$ versus the “normalized” quadrature node number $N'' = n'/N'$ for various Laguerre-Gauss quadrature rules ($n'$ is the actual quadrature node number and $N'$ is the quadrature rule order); we indeed observe a rapid decrease in the weights as $n'$, and hence the $k_x$ plane integration path parameterizing variable $r_x$, increases. An analogous discussion likewise holds for integration within the $k_y$ plane and the corresponding Laguerre-Gauss quadrature weights $w_y$.

\textsuperscript{107}The constant, complex valued factors $l^+$ and $l^-$ manifest in the expressions $k_x = l^\pm r_x \pm \xi_1$ used to parameterize the linear path deformation, appearing in Eqs. (4.2.10)-(4.2.11) of Ch. 4, are placed outside the double Fourier integral and thus allow the weights to be real-valued. See Ch. 4 for details.
Referring again to the notation and terminology used in Ch. 4, letting \( w_x \) be the Laguerre-Gauss quadrature weight multiplying into the evaluation of the integrand for some \((k_x, k_y)\) doublet \((k_{x0} = \xi_1 + l^+ r_{x0}, k_{y0})\), and (for the sake of illustration) assuming one is presently integrating in the intersection of the a) evanescent spectrum zone of the \(k_x\) plane (let \( \text{Re}[k_x] > 0 \)) and b) propagation spectrum zone of the \(k_y\) plane, then upon defining (let \( \tilde{k}_x^+ = \tilde{k}_{M,1z} = \tilde{k}_{M,2z} \))

\[
\tau = i\tilde{k}_x^+ \Delta z' + r_{x0} \cos(\gamma_x)(i\Delta x + \Delta_{\text{eff}}) + ik_{y0} \Delta y
\]  

(5.4.8)

for our twin vertical electric dipole radiation example, one can simply set

\[
\left( e^{ik_{x0}L/2} + e^{-ik_{x0}L/2} \right) e^\tau w_x w_y \rightarrow \left( e^{ik_{x0}L/2 + \tau + \ln(w_x)} + e^{-ik_{x0}L/2 + \tau + \ln(w_x)} \right) w_y
\]

(5.4.9)
Similarly, if one detours in the upper-half $k_y$ plane to evaluate the plane wave spectra evanescent with respect to $k_y$, then place the weights \{\ln(w_y)\} into the exponentials. Detouring into both the $k_x$ and $k_y$ upper-half planes, by extension, mandates placing the weights \{\ln(w_x)\} and \{\ln(w_y)\} into both exponentials.

Now we exhibit the explicit spectral-domain representation of two commonly encountered distributed source geometries, the linear (wire) and rectangular aperture antennas.

5.4.3 Linear Antennas

Consider a linear wire antenna of length $L$ centered at $r'_o = 0$ whose current distribution can be written, without loss of generality, as a superposition of harmonic current modes [15]:

$$J(r) = \hat{z}\delta(x)\delta(y)\text{rect}(z/L) \sum_{r=1}^{\infty} \left[ J'_c(2r-1) \cos \left( \frac{(2r-1)\pi z}{L} \right) + J'_s(2r) \sin \left( \frac{2\pi r z}{L} \right) \right]$$

(5.4.10)

where $J'_c$ and $J'_s$ are the complex-valued modal current amplitudes for cosinusoidal and sinusoidal spatial current variation (resp.). The unit pulse function is defined as $\text{rect}(u) = 1$ for $|u|<1/2$ and zero otherwise. It is a simple exercise to show that

$$\tilde{J}(k) = \frac{-r\hat{z}}{L} \sum_{r=1}^{\infty} \left[ \xi_r J'_c(2r-1) \cos \left( \frac{k_z z}{L} \right) \sin \left( \frac{2\pi r z}{L} \right) \right]$$

(5.4.11)

with

$$\xi_r = \begin{cases} 1, & r = 1, 5, 9, \ldots \\ -1, & r = 3, 7, 11, \ldots \\ 0, & \text{else} \end{cases}, \quad \xi'_r = \begin{cases} 1, & r = 2, 4, 6, \ldots \\ -1, & r = 4, 8, 12, \ldots \\ 0, & \text{else} \end{cases}$$

(5.4.12)

Now let the wire antenna be oriented along an arbitrary direction $\hat{a}$ relative to the $(x, y, z)$ system (i.e., $\hat{z}$ relative to the now-rotated antenna system) and with $J_c = J'_c\delta(\hat{x})\delta(\hat{y})$ and $J_s = J'_s\delta(\hat{x})\delta(\hat{y})$, where $\hat{r} = (\hat{x}, \hat{y}, \hat{z})$ and $\hat{k} = (\hat{k}_x, \hat{k}_y, \hat{k}_z)$ represent
the position and wave vectors (resp.) in the antenna’s local coordinates. To effect
the spectral-domain current’s representation in the original \((x, y, z)\) system, one can
use polar and azimuthal rotation angles \(\alpha\) and \(\beta\) (resp.), along with their respective
individual rotation matrices
\[
\bar{U}_\alpha = \begin{pmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{pmatrix}, \quad \bar{U}_\beta = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
and the composite rotation matrix \(\bar{U} = \bar{U}_\beta \cdot \bar{U}_\alpha\). Having defined these rotation
matrices, we observe that \(\dot{k}\) and \(\dot{r}\) transform as
\[
\dot{k} = \bar{U} \cdot \dot{k} \quad \text{and} \quad \dot{r} = \bar{U} \cdot \dot{r},
\]
respectively.

Now we comment on some features of the resulting spectral-domain integral solu-
tion. Using Cauchy’s Integral Theorem, the \(n\)th modal direct field residue \((n=1,2,3,4)\)
due to the source in layer \(M\) writes as\(^{108}\)
\[
\tilde{a}_{M,n} \tilde{e}_{M,n} = 2\pi i \left[ (k_z - \tilde{k}_{M,nz}) i k_0 \eta_0 \text{Adj} \left( \tilde{A} \right) \cdot \tilde{J}(\tilde{k}) e^{i k_z z} / \text{Det} \left( \tilde{A} \right) \right]_{k_z = \tilde{k}_{M,nz}} (5.4.14)
\]
for an arbitrary electric source distribution. The particular case of a vertically-
oriented electric linear antenna centered at \(r'_o = (x_o, y_o, z_o)\) writes as
\[
\tilde{a}_{M,n} \tilde{e}_{M,n} = 2\pi k_0 \eta_0 \text{Adj} \left( \tilde{A} \right) \cdot \tilde{z} e^{i k_z z - i k r'_o} (k_z - \tilde{k}_{M,nz}) \times 
\sum_{r=1}^{\infty} \left[ r \frac{\xi_r J'_r(2r - 1) \cos \left( \frac{k L}{2} \right) + i \xi'_r J'_r(2r) \sin \left( \frac{k L}{2} \right)}{L \left( k_z - \frac{r \pi}{L} \right) \left( k_z + \frac{r \pi}{L} \right)} \right]_{k_z = \tilde{k}_{M,nz}} (5.4.15)
\]
where the entire expression for \(\tilde{a}_{M,n} \tilde{e}_{M,n}\) (not just the bracketed portion) is subject
evaluation at a particular eigenvalue \(\tilde{k}_{M,nz}\). As can be observed in (5.4.14)-(5.4.15),
the distributed source spectrum manifests as a multiplicative sinc function-like “ta-
per” augmenting (and accelerating, versus \(\{k_x, k_y\}\), the decay of) the computed direct
field modal amplitudes of a Hertzian dipole (c.f. Ch. 2). Finally, we should note that
\(^{108}\)Note that \(\text{Adj}(\bar{M})\) is the adjugate (not adjoint) of matrix \(\bar{M}\) \([78]\), and \(\text{Det}(\bar{M})\) is the determinant.
despite the factor \((k_z - r\pi/L)(k_z + r\pi/L)\) in the denominator of \(\tilde{J}(k)\) in (5.4.11) and (5.4.15), the values \(k_z = \pm r\pi/L\) are not poles because of the zeros in the numerator at those same points [99].

### 5.4.4 Aperture Antennas

Now consider a rectangular aperture possessing a tangential EM field distribution on its plane that one recasts, via the equivalence theorem, as a tangentially-polarized, surface-confined, magneto-electric current distribution [15]. Assuming the aperture’s (1) principal axes are parallel with the \(x\) and \(y\) axes with principal lengths \(L_x\) and \(L_y\) (resp.) and (2) central location is \(r'_o = 0\), its spectral-domain representation readily follows from (5.4.11) upon letting \(r\) and \(q\) be the modal current indices describing current amplitude oscillation along \(x\) and \(y\) (resp.). However, unlike the linear antenna supporting a physical current that must vanish at the wire’s ends, there are no such restrictions on the aperture’s equivalent currents. Therefore, there may be both sinusoidal and cosinusoidal modal variations for each modal index \(r > 0\) and modal index \(q > 0\) in addition to a “DC” term comprising a constant current amplitude sheet. Letting \(J'_{a,p',q'}(r,q)\) stand for the (possibly complex-valued) Fourier coefficient of a current mode with constant current direction \(\hat{a} (a=x\ or\ y)\), either sinusoidal \((p'=s)\) or cosinusoidal \((p'=c)\) current variation along the \(x\) direction, and either sinusoidal \((q'=s)\) or cosinusoidal \((q'=c)\) current variation along the \(y\) direction, then
upon defining the sub-expressions

$$\zeta(\mathbf{r}) = \text{rect}\left(\frac{x}{L_x}\right) \text{rect}\left(\frac{y}{L_y}\right) \delta(z)$$  \hspace{1cm} (5.4.16)

$$\mathbf{J}_1(r, q)(\mathbf{r}) = \zeta(\mathbf{r}) \left[ \mathbf{\hat{x}} J'_{x,s,s}(r, q) \sin \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} + \mathbf{\hat{y}} J'_{y,s,s}(r, q) \sin \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} \right]$$  \hspace{1cm} (5.4.17)

$$\mathbf{J}_2(r, q)(\mathbf{r}) = \zeta(\mathbf{r}) \left[ \mathbf{\hat{x}} J'_{x,s,c}(r, q) \cos \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} + \mathbf{\hat{y}} J'_{y,s,c}(r, q) \cos \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} \right]$$  \hspace{1cm} (5.4.18)

$$\mathbf{J}_3(r, q)(\mathbf{r}) = \zeta(\mathbf{r}) \left[ \mathbf{\hat{x}} J'_{x,c,s}(r, q) \cos \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} + \mathbf{\hat{y}} J'_{y,c,s}(r, q) \cos \frac{r \pi x}{L_x} \sin \frac{q \pi y}{L_y} \right]$$  \hspace{1cm} (5.4.19)

$$\mathbf{J}_4(r, q)(\mathbf{r}) = \zeta(\mathbf{r}) \left[ \mathbf{\hat{x}} J'_{x,c,c}(r, q) \cos \frac{r \pi x}{L_x} \cos \frac{q \pi y}{L_y} + \mathbf{\hat{y}} J'_{y,c,c}(r, q) \cos \frac{r \pi x}{L_x} \cos \frac{q \pi y}{L_y} \right]$$  \hspace{1cm} (5.4.20)

one has the following expression for the equivalent aperture currents in the space domain:

$$\mathbf{J}(\mathbf{r}) = \mathbf{J}_4(0, 0)(\mathbf{r}) + \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} \sum_{p=1}^{4} \mathbf{J}_p(r, q)(\mathbf{r})$$  \hspace{1cm} (5.4.21)

with associated spectral-domain representation

$$\mathbf{\tilde{J}}(\mathbf{k}) = \mathbf{\tilde{J}}_4(0, 0)(\mathbf{k}) + \sum_{r=1}^{\infty} \sum_{q=1}^{\infty} \sum_{p=1}^{4} \mathbf{\tilde{J}}_p(r, q)(\mathbf{k})$$  \hspace{1cm} (5.4.22)

Analogous to the linear antenna, a more general aperture plane orientation can be effected using appropriate rotation matrices to represent arbitrarily-oriented rectangular aperture antennas. Further akin to the wire antenna case, we observe again the manifestation of a tapering in the field’s Fourier spectrum (except now along both $k_x$ and $k_y$), the property of the distributed field computation imparting a (deceptively simple) multiplicative factor into the computed Hertzian dipole direct field modal amplitudes, the presence of (now four) fictitious poles, and the vulnerability of numerical instability when the observation point lies within the region ($|x| < L_x/2) \cup (|y| < L_y/2)$. 155
The latter instability aspect, when one is detouring into the upper-half $k_x$ and $k_y$
planes, is mitigated in the same manner to that shown concerning linear antennas,
i.e., via placing the natural logarithm of one or both of the Laguerre-Gauss quadra-
ture weights into the exponentials prior to evaluating them, as well as only evaluating
the scattered fields. Now, however, due to the multiplication of two sinusoid-type
functions in the spectral domain one will have for each current mode functional de-
pendence (i.e., cosinusoidal along both $x$ and $y$, etc.) four exponentials into which
one places the (natural logarithm of the) quadrature weights, rather than two expo-
nentials in the case of wire antennas.

5.4.5 Validation Results: Linear Antennas

In this subsection we first show results concerning the fields radiated by an in-
finitesimally thin linear (wire) antenna radiating at $f=30$MHz in unbounded free
space. We set the wire antenna’s length at half the free space wavelength (wire
length $L = \lambda_0/2 \sim 5$m), partition free space into three fictitious layers, place the
antenna in the 5m-thick central layer, position the observation point always either
in the top or bottom layer to compute the total field, and restrict attention to an
electric current distribution with mode index $r = 1$. The radial distance between the
antenna’s center and all observation points is held fixed at $|r - r'|= 50$m, while the
adaptive integration error tolerance was set to $1.2 \times 10^{-4}$.

Figure 5.4 shows the accuracy of the electric field, radiated by a vertically-oriented
wire antenna, versus polar angle $\theta$ for a fixed azimuthal observation angle $\phi = 0^\circ$.
Note that for $|r - r'|= 50$m the sampled polar angles $\theta = 88^\circ$, $90^\circ$, and $92^\circ$ correspond
to observation points lying within the central free space layer and thus zero scattered-field result. Thus, the polar angle sweep data is shown sub-divided into two plots to remove the artificial discontinuity in the data (versus $\theta$). We see that between $\theta = 0^\circ - 76^\circ$ and $\theta = 104^\circ - 180^\circ$, one realizes an accuracy of between thirteen to fourteen digits in $E_z$. An analogous statement applies for the error in $E_x$ except at $\theta = 0^\circ$ and $\theta = 180^\circ$, where the algorithm’s computed solution (to within machine precision) and closed form solution yield answers for $E_x$ having magnitude equal to zero (hence the error’s coercion to -150dB). We notice that the accuracy degrades as the polar observation angle tends towards horizon, but the algorithm still manages to deliver results accurate to approximately four digits. This trend is qualitatively consistent with the results in Section 5.3.3, where instead the field and observation points were in the same layer. Further extensive error studies (not shown herein) were also performed to better characterize the algorithm’s performance, which consisted of all the following parameter permutations\textsuperscript{109}: $(E_{\hat{x}}, E_{\hat{y}}, E_{\hat{z}}, H_{\hat{x}}, H_{\hat{y}}) \times (\alpha = 0^\circ, \alpha = 45^\circ, \alpha = 90^\circ) \times (\theta = 45^\circ, \theta = 135^\circ) \times (\phi = 0^\circ, \phi = 45^\circ, \phi = 90^\circ, \phi = 135^\circ, \phi = 180^\circ, \phi = 225^\circ, \phi = 270^\circ, \phi = 315^\circ)$. The results in all these permutations indicated error ranging between -130dB to -140dB.

\textsuperscript{109}The dot over the field component directions denotes components expressed with respect to the antenna’s local (rotated) coordinate system.
Figure 5.4: Accuracy of $E_x$ and $E_z$, versus polar angle $\theta$, for the vertical wire antenna. Reference results computed using expressions from [15].

To gain a better understanding of the computational efficiency realized with our proposed distributed radiator simulation approach, we examined the time required to compute the total field $E_z$ radiated by a vertically-oriented, half-wavelength wire antenna ($L = \lambda_0/2$) which radiates at $f = 30$MHz, is centered at the origin in free space, and has current distribution $\mathbf{J}(\mathbf{r}) = \hat{z}\cos(\pi z/L)\text{rect}(z/L)\delta(x)\delta(y)$ (i.e., the current variation is characterized by the harmonic $r = 1$); the observation points examined were $\mathbf{r} = (0, 0, 50)$m and $\mathbf{r} = 50(\cos80^\circ, 0, \sin80^\circ)$m. In particular, we compared the time required to obtain the field solution from using the proposed spectral-domain approach versus performing space-domain radiation integral evaluation via Legendre-Gauss quadrature.\(^{110}\) Indeed, our study revealed that a factor of approximately one

\(^{110}\)That is, with the Gauss quadrature nodes being the equivalent Hertzian dipole locations on $V'$.\)
order of magnitude in acceleration can potentially be realized. Furthermore, by the sampling theorem one reasons that when $r$ significant spatial current harmonics are required to adequately capture the spatial variation of current on $V'$, for a comparable accuracy with our proposed approach one would require approximately $r \times N_{\text{avg}}$ Hertzian dipole sampling points (as compared to just $r$ evaluations with our approach).

To validate the algorithm’s capability to simulate fields radiated by wire antennas in planar-layered, anisotropic media, consider a vertically-oriented wire antenna which radiates at $f = 10\text{MHz}$, has vertical length $L = \lambda_0/2 \sim 15\text{m}$, is centered at $r_o = (0, 0, 0)$, and supports an electric current sheet with distribution $\mathbf{J}(r) = \hat{z} \cos(\pi z/L) \text{rect}(z/L) \delta(x) \delta(y)$. The wire antenna resides in the top vacuum layer $z \geq -\lambda_0/4$, the PEC ground plane\textsuperscript{112} half-space occupies the region $z \leq -(d + \lambda_0/4)$ (the longitudinal spacing between the wire antenna and ground plane is $d = 5\text{m}$), and a ground plane-coating substrate with material properties $\bar{\epsilon}_r = \bar{\mu}_r = \text{diag}[s, s, 1/s]$ ($s = 1/10$) occupies the region $-(d + \lambda_0/4) \leq z \leq -\lambda_0/4$. See Figure 5.5a.

The use of this coating layer has the special properties of being perfectly impedance-matched to free space for all plane wave incidence angles and polarizations (hence the name “isoimpedance” medium [94]), as well as (equivalently) effecting the metric expansion of space within the isoimpedance layer by a factor of $s$ [17, 16]. As a consequence, no reflections arise at the isoimpedance/vacuum interface, while plane waves

\textsuperscript{111}We found $N_{\text{avg}} \sim 10$ Hertzian dipole sampling points were required to achieve at most -100dB error in our study.

\textsuperscript{112}Akin to the image theory study done in Section 5.3.3, for the sole purpose of facilitating the present study we coerce, within the code, the $\text{TE}_z$ and $\text{TM}_z$ reflection coefficients concerning down-going fields impinging upon the ground plane.
traversing this $d$-meter thick layer exit it having accumulated a (in general complex-valued) phase commensurate with having traversed a $sd$-meter thick region of free space [16, 94]. Equivalently (for our problem), the field solution at some point in the vacuum region $\mathbf{r} = (x, y, z \geq -\lambda_0/4)$ is exactly identical to the field solution at the same $\mathbf{r}$ but in a two-layer vacuum/PEC ground environment where the semi-infinitely thick PEC ground half-space occupies instead the region $z \leq -(sd + \lambda_0/4)$ (see Figure 5.5b). Subsequently recalling image theory, we conclude that the solution (in the vacuum/PEC geometry) at $\mathbf{r} = (x, y, z \geq -\lambda_0/4)$ in turn is identical to the solution at the same $\mathbf{r}$ in a homogeneous vacuum but with two sources [15]: the original wire antenna along with the image wire source $\mathcal{J}'(\mathbf{r}) = \hat{z} \cos(\pi(z + d')/L)\text{rect}((z + d')/L)\delta(x)\delta(y)$, where $d' = \lambda_0/2 + 2sd$; see Figure 5.5c. Alternatively stated, setting $s < 1$ makes the wire antenna radiate in the region $z \geq -\lambda_0/4$ as if the ground plane were moved closer to the base of the antenna while setting $s > 1$ causes the wire antenna to radiate in the region $z \geq -\lambda_0/4$ as if the ground plane were moved further downward in the $z$ direction. Furthermore, the “scattered” field comprises the direct field distribution, within the region $z \geq -\lambda_0/4$, established by the image source $\mathcal{J}'(\mathbf{r})$. 

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Figure 5.5: $xz$ plane view of the three problem geometries leading to an identical field distribution in the region $z \geq 0$. Physical arguments grounded in Transformation Optics theory and the form invariance of Maxwell’s Equations [16, 17] lead to equivalence in the field distributions between the first two sub-figures (for $z \geq -\lambda_0/4$). On the other hand, image theory-based considerations lead to equivalence in the field distributions between the latter two sub-figures (again, for $z \geq -\lambda_0/4$). We plot the field distribution ($E_z$) on a flat observation plane, residing at $z = 10m$, occupying the region $\{ -5 \leq x \leq 5, -5 \leq y \leq 5 \}$ m (i.e., at the elevation of the dashed green line seen in the above three $xz$ plane views).
Figure 5.6: Algorithm-computed electric field $E_z$ distribution (Figure 5.6a) and relative error $10\log_{10}(|E_z - E_z^{\text{val}}|/|E_z^{\text{val}}|)$ [dB] (Figure 5.6b) in the region $\{-5 \leq x \leq 5, -5 \leq y \leq 5, z = 10\}$ [m]. $E_z^{\text{val}}$ is the closed-form, scattered-field result comprising the image wire current source’s radiated field.

5.4.6 Validation Results: Aperture Antennas

In this subsection we first exhibit validation results concerning rectangular aperture antennas radiating at $f=30\text{MHz}$ in free space. We set the aperture antenna’s dimensions along the principal directions as $(L = L_x = L_y = \lambda_0/2)$, partition free space into three fictitious layers with the antenna in the central layer of 2m thickness, set the observation point always either in the top or bottom layer to compute the direct field, restrict attention to the $r$ modal index value $r = 1$ when cosinusoidal electric current variation along the $\hat{x}$ direction is present (and likewise, set $q=1$ for cosinusoidal electric current variation along $\hat{y}$), and assume cosinusoidal current variation along the direction of current flow.
Figure 5.7 shows the accuracy of the electric field, radiated by an aperture antenna with area normal \( \mathbf{\hat{n}} = \mathbf{\hat{z}} \) and centered at \( \mathbf{r}'_0 = \mathbf{0} \), versus polar angle \( \theta \) for a fixed azimuthal observation angle \( \phi = 0^\circ \). The radial distance between the center of the aperture and all observation points is held fixed at \( |\mathbf{r} - \mathbf{r}'| = 50 \text{m} \), while the adaptive integration error tolerance was set to be \( 1.2 \times 10^{-4} \). Figures 5.7a-5.7b concern an aperture with \( \hat{x} \cos(\pi x/L) \cos(\pi y/L) \text{rect}(x/L) \text{rect}(y/L) \delta(z) \) current amplitude pattern, while the line plots of \( E_{y2}, E_{y3}, \) and \( E_{y4} \) in Figures 5.7c-5.7d concern an aperture with \( \hat{x} \cos(\pi x/L) \sin(2\pi y/L) \text{rect}(x/L) \text{rect}(y/L) \delta(z), \hat{y} \cos(\pi x/L) \cos(\pi y/L) \text{rect}(x/L) \text{rect}(y/L) \delta(z), \) and \( \hat{y} \sin(2\pi x/L) \cos(\pi y/L) \text{rect}(x/L) \text{rect}(y/L) \delta(z) \) surface current distributions (resp.).

Note that the sampled polar angle \( \theta = 90^\circ \) corresponds to an observation point lying within the central free space layer and thus yields a null scattered-field result. Thus, the polar angle sweep data (both for the first, as well as the latter three, current distribution cases) are shown sub-divided into two plots to remove the artificial discontinuity in the data (versus \( \theta \)). From Figures 5.7a-5.7b we see that for \( \theta \in [0^\circ, 78^\circ] \) and \( \theta \in [102^\circ, 180^\circ] \), one realizes an accuracy of between thirteen to fourteen digits in \( E_x \). An analogous statement applies for the error in \( E_z \) excepting at \( \theta = 0^\circ \) and \( \theta = 180^\circ \), where the algorithm’s computed solution (to within machine precision) and closed form solution yield answers for \( E_z \) having magnitude equal to zero. We notice that the accuracy degrades as the polar observation angle tends towards horizon, but the algorithm still manages to yield results accurate to approximately three to four digits. Observing the three current cases in Figures 5.7c-5.7d leads to similar conclusions for the only non-trivial electric field component \( E_y \): accuracy for \( E_{y2} \) and \( E_{y3} \) is between thirteen to fourteen digits in the polar range \( \theta \in [0^\circ, 78^\circ] \) and \( \theta \in [102^\circ, 180^\circ] \), accuracy for \( E_{y4} \) is between thirteen to fourteen digits in the polar range \( \theta \in [0^\circ, 80^\circ] \).
and $\theta \in [100^\circ, 180^\circ]$, all three accuracies degrade for observation points near to the horizon, and for $E_{y2}$ and $E_{y4}$ the accuracy results at $\theta = 0^\circ$ and $\theta = 180^\circ$ are coerced to -150dB since the computed results (to within machine precision) and validation results were of zero magnitude.

![Graphs](image)

Figure 5.7: $\epsilon(\mathcal{E})$ versus $\theta$ for the aperture antenna. Reference results computed using expressions from [15].
Akin to the wire antenna case, extensive error studies were performed to better characterize the algorithm’s performance concerning rectangular aperture sources, which consisted of all the following parameter permutations: \((E_x, E_y, E_z, H_y, H_z) \times (\alpha = 0^\circ, \alpha = 45^\circ, \alpha = 90^\circ) \times (\theta = 45^\circ, \theta = 135^\circ) \times (\phi = 0^\circ, \phi = 45^\circ, \phi = 90^\circ, \phi = 135^\circ, \phi = 180^\circ, \phi = 225^\circ, \phi = 270^\circ, \phi = 315^\circ)\). The error results in all these permutations are between -130dB to -140dB. Again, the dot over the field component directions denotes components expressed with respect to the antenna’s local (rotated) coordinate system. Furthermore, a baseline computational efficiency study was performed which is identical to the wire antenna study except we observe the electric field \(E_x\) due to the source distribution \(\mathbf{J}(\mathbf{r}) = \hat{x} \cos(\pi x/L_x) \cos(\pi y/L_y) \text{rect}(x/L_x) \text{rect}(y/L_y) \delta(z)\).

Analogous conclusions hold, except that now one realizes two orders of magnitude in solution speed acceleration due to only one sinusoidal current harmonic required to represent a current sheet otherwise requiring (on the order of) \(N_{\text{avg}} = 10 \times 10\) Hertzian dipole sampling points for comparable accuracy.

To validate the algorithm’s capability to simulate fields radiated by aperture antennas in planar-layered, anisotropic media, consider a flat, rectangular-shaped aperture antenna radiating at \(f = 30\text{MHz}\), having dimensions \(L_x = \lambda_0/2 \sim 5\text{m}\) and \(L_y = 1\text{m}\), oriented such that it is parallel to the \(xy\) plane, centered at \(\mathbf{r}_o = (0, 0, 0)\), and supporting an electric current sheet with distribution \(\mathbf{J}(\mathbf{r}) = \hat{x} \cos(\pi x/L_x) \cos(\pi y/L_y) \times \text{rect}(x/L_x) \text{rect}(y/L_y) \delta(z)\). The aperture antenna resides in the top vacuum layer \(z \geq 0\), the PEC ground plane\(^{113}\) half-space occupies the region \(z \leq -d\) (the longitudinal spacing between the aperture antenna and ground plane is \(d = 10\text{mm}\)),

\(^{113}\)Akin to the image theory study done in Section 5.3.3, for the sole purpose of facilitating the present study we coerce, within the code, the TE\(_z\) and TM\(_z\) reflection coefficients concerning down-going fields impinging upon the ground plane.
and a ground plane-coating layer with material properties $\bar{\epsilon}_r = \bar{\mu}_r = \text{diag}[s, s, 1/s]$ ($s = 10$) occupies the region $-d \leq z \leq 0$ (see Figure 5.8a). With $s = 10$, the aperture will radiate into the region $z \geq 0$ as if the aperture-ground separation were in fact $sd=100\text{mm}$ (see Figure 5.8). In particular, the “scattered” field comprises the field distribution, within the region $z \geq 0$, radiated by the image source

$$\mathbf{J}'(\mathbf{r}) = -\mathbf{x} \cos(\pi x/L_x) \cos(\pi y/L_y) \text{rect}(x/L_x) \text{rect}(y/L_y) \delta(z + 2sd).$$
Figure 5.8: $xz$ plane view of the three problem geometries leading to an identical field distribution in the region $z \geq 0$. Physical arguments grounded in Transformation Optics theory and the form invariance of Maxwell’s Equations [16, 17] lead to equivalence in the field distributions between the first two sub-figures (for $z \geq 0$). On the other hand, image theory-based considerations lead to equivalence in the field distributions between the latter two sub-figures (again, for $z \geq 0$). We plot the field distribution ($E_x$) on a flat observation plane, residing 10m above the aperture source, occupying the region $\{-10 \leq x \leq 10, -4 \leq y \leq 4\}$m parallel to the $xy$ plane (i.e., at the elevation of the dashed green line seen in the above three $xz$ plane views).
Figure 5.9: Algorithm-computed electric field $E_x$ distribution (Figure 5.9a) and relative error $10\log_{10}\left|\frac{E_x - E_{\text{val}}}{E_{\text{val}}}\right|$ [dB] (Figure 5.9b) in the region $\{-10 \leq x \leq 10, -4 \leq y \leq 4, z = 10\}$ [m]. $E_{\text{val}}$ is the closed-form scattered-field result comprising the image aperture current source’s radiated direct field.

5.5 Case Study: Marine Hydrocarbon Exploration

Next, we validate and then illustrate one application of the proposed algorithm: facilitating computation of the fields excited by transmitters operating in highly inhomogeneous and absorptive marine environments characterizing typical controlled source EM (CSEM) operational scenarios. CSEM transmitters, typically radiating in the frequency range 0.01Hz-10Hz [35], serve as active illuminators to facilitate detection and characterization of thin, highly resistive hydrocarbon-bearing formations embedded deep under the ocean, which can complement data from magnetotelluric (MT) sounding-based methods [41]. Indeed, use of an active source allows one to also exploit galvanic, in addition to inductive, generation of the scattered field that arises from “blockage”, due to a highly resistive layer (e.g., of hydrocarbons), of what
was (with the resistive layer absent) a dominantly normally-directed vector current field [44]. Figure 5.10 below describes the geometry of the problem considered; note that a hydrocarbon-bearing formation buried at 1km under the sea and having 100m thickness is a typical case study found in the related literature [41, 18].

Figure 5.10: The two contrasting environment geometries with (Figure 5.10a) and without (Figure 5.10b) the embedded hydrocarbon reservoir. The observation points, mimicking the receiver instruments, lie at the seafloor in the $xz$ plane.

For the first study, we compare our code’s results against those found in Figures 6c, 6d, 7c, and 7d of [18] involving a Hertzian electric dipole. To this end, we set the $x$-directed source’s position to be $d=30m$ above the sea floor, the sea depth $H=300m$, the sea water’s conductivity $\sigma_2 = 3.2S/m$, and the antenna’s radiation frequency as $f = 0.25Hz$. The sub-plots in Figure 5.11 refer to when the uniaxial-anisotropic hydrocarbon reservoir has in-plane conductivity $\sigma_h = 10mS/m$ and orthogonal conductivity $\sigma_v = 2.5mS/m$, while the sub-plots in Figure 5.12 refer to when $\sigma_h = 500mS/m$

\[\text{That is, } \sigma_h \text{ and } \sigma_v \text{ are the principal conductivity components corresponding to applying an electric field either parallel or perpendicular to the reservoir’s principal bedding plane, respectively [9]. The representation of the reservoir’s (the fourth layer in Figure 5.10a) conductivity tensor } \bar{\sigma}_4 \text{ can}\]

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and $\sigma_v = 125\text{mS/m}$. We examine the “in-line”, $x$-directed total electric field $E_x$ observed at the receivers positioned at the sea floor for different source-receiver separations $x - x'$. As can be seen in both Figures 5.11 and 5.12, there is very good agreement observed for all three anisotropy cases exhibited: The (trivial/reference) isotropic reservoir case (Figures 5.11a, 5.11b, 5.12a, and 5.12b), the intermediate dipping (“cross-bedding” [9]) anisotropy case $\{\alpha_4 = 30^\circ, \beta_4 = 0^\circ\}$ (c.f. footnote 114) seen in Figures 5.11c, 5.11d, 5.12c, and 5.12d, and finally the fully dipping anisotropy case $\{\alpha_4 = 90^\circ, \beta_4 = 15^\circ\}$ seen in Figures 5.11e, 5.11f, 5.12e, and 5.12f.\footnote{For brevity, we omit results from the transverse-isotropic case $\alpha_4 = 0^\circ$ since the closed-form validation results, in Figures 5.6 and 5.9, adequately demonstrate the algorithm’s performance when media with this orientation of principal material axes are present.}

then be found using the reservoir’s conductivity bedding plane polar and azimuthal orientation angles ($\alpha_4$ and $\beta_4$, resp. [9]).
Figure 5.11: Anisotropic resistive reservoir, with $\sigma_h = 10 \text{mS/m}$ and $\sigma_v = 2.5 \text{mS/m}$. Figures 5.11a, 5.11c, and 5.11e show the magnitude of the observed electric field versus $x - x'$ for the isotropic case, intermediate dipping anisotropy case $\{\alpha_4 = 30^\circ, \beta_4 = 0^\circ\}$, and fully dipping anisotropy case $\{\alpha_4 = 90^\circ, \beta_4 = 15^\circ\}$, respectively. Figures 5.11b, 5.11d, and 5.11f indicate the phase of $E_x$ in these three anisotropy cases, respectively. The curve “Pres.” is our algorithm’s result while the curve “Ref.” is the reference result from [18].
Figure 5.12: Anisotropic conductive reservoir, with $\sigma_h = 500\text{mS/m}$ and $\sigma_v = 125\text{mS/m}$. Figures 5.12a, 5.12c, and 5.12e show the magnitude of the observed electric field versus $x - x'$ for the isotropic case, intermediate dipping anisotropy case $\{\alpha_4 = 30^\circ, \beta_4 = 0^\circ\}$, and fully dipping anisotropy case $\{\alpha_4 = 90^\circ, \beta_4 = 15^\circ\}$, respectively. Figures 5.12b, 5.12d, and 5.12f indicate the phase of $E_x$ in these three anisotropy cases, respectively. The curve “Pres.” is our algorithm’s result while the curve “Ref.” is the reference result from [18].
For the second study, we position a 100m long, \( x \)-directed wire antenna \( d=50 \)m above the ocean-sediment interface while maintaining again the depth of the observation points ("receivers") at the seafloor.\[^{116}\] Furthermore, both the transmitter antenna and observation points are confined to the \( xz \) plane while \( \sigma_2 \sim 3.33 \)S/m and the isotropic reservoir has conductivity \( \sigma_4 = 10 \)mS/m. Since the field strength can vary significantly over the transmitter-receiver separation distances (taken with respect to the wire’s center) \( x - x' \) considered herein \((1-20 \)km\) \[^{[41]}\], we plot the magnitude and phase of the ratio of the scattered fields received in the two geometries considered in Figures 5.10a and 5.10b: For example, in the case of \( E_s^x \), we observe the phase and magnitude of the received scattered field ratio \( E_{xxr} = E_{x1r}^s/E_{x2r}^s \), where \( E_{x1r}^s \) and \( E_{x2r}^s \) are the scattered fields observed at a particular receiver in the geometries described by Figures 5.10a and 5.10b (resp.). As a result, a measurement’s responsiveness to the hydrocarbon formation’s presence is indicated by the extent of phase deviation from \( 0^\circ \) (in the phase plots) and the extent of magnitude swing from \( 0 \)dB (in the magnitude plots), as observed in Figures 5.13-5.15.

Figures 5.13, 5.14, and 5.15 illustrate the phase and magnitude of the scattered field ratios concerning \( E_s^x, H_s^x, \) and \( E_s^z \) (resp.), both for the shallow water \((H=100 \)m\) and deep water \((H=500 \)m\) cases.\[^{117}\] From these Figures, we notice that for both sea water depth scenarios the electric field ratios \( E_s^x \) and \( E_s^z \) (but particularly \( E_s^z \), which corresponds to a pure Transverse-Magnetic to \( z \) mode [TM\(_z\)] \[^{[77, 45]}\]) exhibit strong responsiveness to the presence of the deeply buried hydrocarbon bed. On the

\[^{116}\]The length, orientation, and depth above the seafloor of the transmitter antenna, as well as the receiver positions, lead to a case study qualitatively following, and is primarily inspired from, the CSEM field campaign reported in [42].

\[^{117}\]In the top-left corner of Figure 5.14c, note the vortex-like behavior of the phase. The seemingly solid vertical black strip corresponds to closely spaced (black) contour lines that, upon zooming in at high resolution, do in fact illustrate the locally rapid variation of phase.
other hand, in shallow water $H^x_x$ provides little useful information, as can be seen by its relatively poor response to the presence of the resistive hydrocarbon formation compared to the electric field measurements. However, upon increasing the water depth to 500m, both the phase and magnitude of $H^x_x$ show a very high response to the presence of the hydrocarbon bed. By contrast, the phase and magnitude of $H^x_x$ (not shown here), corresponding to a pure Transverse-Electric to $z$ mode (TE$_z$) [77, 45], fails to yield significant responsiveness to the resistive formation even when the water depth is increased to 500m. These results qualitatively corroborate prior studies indicating that the sea-air interface can significantly dampen instrument sensitivity to deeply buried hydrocarbon reservoirs [37, 100]. However, the sensitivity reduction effect is strongly dependent on the field type (electric versus magnetic) and component ($x, y, z$), with the dampening effect much more pronounced in measurements derived from the TE$_z$ modes as compared to the TM$_z$ modes [37, 100].
Figure 5.13: Figures 5.13a and 5.13c denote the phase (degrees) of $E_{xr}$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$), while Figures 5.13b and 5.13d denote the magnitude [dB] of $E_{xr}$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$).
Figure 5.14: Figures 5.14a and 5.14c denote the phase (degrees) of $H_{xr}^s$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$), while Figures 5.14b and 5.14d denote the magnitude [dB] of $H_{xr}^s$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$).
Figure 5.15: Figures 5.15a and 5.15c denote the phase (degrees) of $E_{sr}^s$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$), while Figures 5.15b and 5.15d denote the magnitude [dB] of $E_{sr}^s$ when the transmitter operates (resp.) in either shallow water ($H=100m$) or deep water ($H=500m$).
5.6 Concluding Remarks

We have introduced and validated numerical algorithms performing two functions widely applicable to myriad applications ranging from radar, antenna, and microwave circuit modeling to aperture synthesis and geophysical prospecting. First we discussed how to extract the scattered field radiated by sources, both Hertzian and distributed, embedded within planar-stratified environments with generally anisotropic and lossy media based on the spectral-domain/Fourier modal synthesis technique. Some of the key features of the extraction algorithm are: (1) Numerical robustness with respect to large material, source, and observer parameter variations, (2) non-reliance on space domain tensor Green’s functions, and (3) no added computational burden versus computing the total field (c.f. Ch. 2). Second, we discussed how to circumvent the tedious and oftentimes computationally expensive task of evaluating the radiation integral in translation-variant environments via calculating the radiation integral directly in the Fourier domain itself. Beyond exhibiting applicability to general sources admitting a closed form Fourier domain representation, this algorithm also enjoys the benefit (as compared to Hertzian dipole-based sampling) of further acceleration due to efficiently sampling and simulating the dominant spatial amplitude distribution of many commonly encountered source current profiles. To validate the algorithms, numerical results were compared against closed-form field solutions in both free space and layered, anisotropic medium radiation scenarios. To illustrate the applicability of the algorithm in geophysical prospecting, we applied it to the modeling of CSEM-based sensing of sub-oceanic hydrocarbon deposits using active wire antenna transmitters.
6.1 Introduction

6.1.1 Chapter Summary and Contributions

We derive the key expressions to robustly address the eigenfunction expansion-based analysis of electromagnetic (EM) fields produced by current sources within planar non-birefringent anisotropic medium (NBAM) layers. In NBAM, the highly symmetric permeability and permittivity tensors can induce directionally-dependent, but polarization independent, propagation properties supporting “degenerate” characteristic polarizations, i.e. four linearly-independent eigenvectors associated with only two (rather than four) unique, non-defective eigenvalues. We first explain problems that can arise when the source(s) specifically reside within NBAM planar layers when using canonical field expressions. To remedy these problems, we exhibit alternative spectral-domain field expressions, immune to such problems, that form the foundation for a robust eigenfunction expansion-based analysis of time-harmonic EM radiation and scattering within such type of planar-layered media. Numerical results demonstrate the high accuracy and stability achievable using this algorithm.118

118NOTE: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.
6.1.2 Background

Environments with (locally) planar-layered profiles are encountered in diverse applications such as geophysical exploration, ground penetrating radar, conformal antenna design, and so on [61] (also, c.f. Chs. 2 and 4). To facilitate electromagnetic (EM) radiation analysis in such environments, eigenfunction (plane wave) expansions (PWE) have long been used because of their relative computational efficiency versus brute-force numerical methods such as finite difference and finite element methods. Moreover, PWE can accommodate linear, but otherwise arbitrary anisotropic layers characterized by arbitrary (diagonalizable) $3 \times 3$ material tensors (c.f. Ch. 2). This proves useful when rigorously modeling planar media simultaneously exhibiting both electrical and magnetic anisotropy, such as (i) isoimpedance beam-shifting devices and (to facilitate proximal antenna placement) ground-plane-coating slabs systematically designed via transformation optics (T.O.) techniques [101] (also, c.f. Ch. 4), (ii) more practically realizable (albeit not necessarily isoimpedance) approximations to T.O.-inspired media such as metamaterial-based thin, wide-angle, and polarization-robust absorbers to facilitate (for example) radar cross section control [102], as well as (iii) numerous other media such as certain types of liquid crystals, elastic media subject to small deformations, and superconductors at high temperatures [103]. These named, amongst other, modeling scenarios share in common the potential presence of a particular class of anisotropic media in which the magnetic permeability ($\bar{\mu}_r$) and electric permittivity ($\bar{\epsilon}_r$) tensor properties are “matched” to each other and hence together define media supporting four “degenerate” plane wave eigenfunctions that, while possessing four linearly independent field polarization states (eigenvectors) as usual, share only two unique (albeit, critically still, non-defective) eigenvalues [77].
Alternatively stated, propagation characteristics within such media are still (in general) dependent on propagation direction but *independent* of polarization, eliminating “double refraction” (“birefringence”) effects [77, 103]. Hence our proposed moniker “Non-Birefringent Anisotropic Medium” (NBAM), rather than the “pseudo-isotropic” moniker [103].

From an analytical standpoint, said PWE constitute spectral integrals exactly quantifying the radiated fields (c.f. Ch. 4). Except for some very simple cases however, these expansions must almost always be evaluated by means of numerical quadratures or cubatures, whose robust computation (with respect to varying source and layer properties) is far from trivial and requires careful choice of appropriate quadrature rules, complex-plane integration contours, etc. to mitigate discretization and truncation errors as well as accelerate convergence [71] (also, c.f. Chs. 2 and 4). In addition to such considerations of primarily *numerical* character, a distinct problem occurs, due to said eigenvalue degeneracy, when sources radiate within NBAM layers. Indeed, this case requires proper analytical “pre-treatment” of the fundamental spectral-domain field expressions to avoid two sources of “breakdown”: (i) Numerically unstable calculations (namely, divisions by zero) during the computation chain, as well as (ii) Corruption of the correct form of the eigenfunctions, viz. \( z \exp[i k_z z] \) instead of the proper form \( \exp[i k_z z] \), the former resulting from a naive, “blanket” application of Cauchy’s integral theorem to the canonical field expressions [104, 105].

To this end, we first show the key results detailing the degenerate “direct” (i.e., homogeneous medium) radiated fields in the “principal material basis” (PMB) representation with respect to which the material tensors are assumed simultaneously
diagonalized by an orthogonal basis \([101]\). Subsequently, we transform these PMB expressions to the Cartesian basis (the PWE’s employed basis). Finally, we employ a robust, numerically-stable NBAM polarization decomposition scheme to obtain the Cartesian-basis direct field polarization amplitudes. The two-dimensional (2-D) Fourier integral-based PWE algorithm, resulting from implanting these derived field expressions into an otherwise highly robust PWE algorithm (c.f. Ch. 4), comprises this paper’s central contribution.

6.2 Problem Statement

We assume the \(e^{-i\omega t}\) convention in what follows. Within a homogeneous medium of material properties \(\{\bar{\epsilon}_r, \bar{\mu}_r\}\), the electric field \(\varepsilon(r)\) radiated by electric (\(J\)) and (equivalent) magnetic (\(M\)) current sources satisfies

\[
\tilde{A}(\cdot) = \nabla \times \bar{\mu}_r^{-1} \cdot \nabla \times (\cdot) - k_0^2 \bar{\epsilon}_r \cdot (\cdot) \quad (6.2.1)
\]

\[
\tilde{A}(\varepsilon) = i k_0 \eta_0 J - \nabla \times \bar{\mu}_r^{-1} \cdot \mathcal{M} \quad (6.2.2)
\]

\(^{119}\)Note: The material tensor eigenvectors \(\{\hat{v}_1, \hat{v}_2, \hat{v}_3\}\) are not to be confused with the field polarization eigenvectors.

\(^{120}\)K \(k_0 = \omega \sqrt{\mu_0 \epsilon_0}, \mu_0, \eta_0 = \sqrt{\mu_0 / \epsilon_0}, \bar{\epsilon}_r, \) and \(\bar{\mu}_r\) are the vacuum wave number, vacuum permittivity, vacuum permeability, vacuum plane wave impedance, NBAM relative permittivity tensor, and NBAM relative permeability tensor, respectively. An infinitesimal point/Hertzian dipole current resides at \(r' = (x', y', z')\), the observation point resides at \(r = (x, y, z)\), \(\Delta r = r - r' = (\Delta x, \Delta y, \Delta z)\), \(u(\cdot)\) denotes the Heaviside step function, and \(k = (k_x, k_y, k_z)\) denotes the wave vector. Furthermore, \(\tilde{\tau}_r = \bar{\mu}_r^{-1}\) and \(d_0 = k_0^2 \epsilon_{zz} (\tau_{xy} \tau_{yx} - \tau_{xx} \tau_{yy})\), where \(\gamma_{ts} = \hat{t} \cdot \hat{\gamma}_r \cdot \hat{s} \) (\(\gamma = \tau, \epsilon; t, s = x, y, z\)). All derivations are performed for the electric field, but duality in Maxwell’s Equations makes immediate the magnetic field solution. Finally, a tilde over variables denotes they are Fourier/wave-number domain quantities.
and can be expressed via a 3-D Fourier integral over the field’s plane wave constituents 
\{\tilde{E}(k)e^{ikr}\}^{121}:

\[
\tilde{A}^{-1} = \text{Adj}\left(\tilde{A}\right) / \text{Det}\left(\tilde{A}\right) \tag{6.2.3}
\]

\[
\tilde{E}(k) = \tilde{A}^{-1} \cdot \left[ ik_0 \eta_0 \tilde{J} - \tilde{\nabla} \times \tilde{\mu}^{-1} \cdot \tilde{M} \right] \tag{6.2.4}
\]

\[
\mathcal{E}(r) = \left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{E}(k) e^{ikr} \, dk_z \, dk_x \, dk_y \tag{6.2.5}
\]

where, anticipating planar layering along \(z\), the \(k_z\) spectral integral is “analytically” evaluated for every \((k_x, k_y)\) doublet manifest in the (typically numerically) evaluated outer 2-D Fourier integral. That is to say, by “analytically” evaluated we mean that the general (symbolic) closed-form solution of the \(k_z\) integral for arbitrary \((k_x, k_y)\) doublet, obtained by equivalently viewing the \(k_z\) real-axis integral as a contour integral evaluated using Jordan’s Lemma and residue calculus, is well-known and can be numerically evaluated at the \((k_x, k_y)\) doublets \([45]\) (also, c.f. Ch. 2). In particular, analytical evaluation of the \(k_z\) integral yields the “direct” field \(\mathcal{E}^d(r)\) (c.f. Ch. 2):

\[
\mathcal{E}^d(r) = \frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(\Delta z) \sum_{n=1}^{2} \tilde{a}_n^d \tilde{e}_n e^{i\tilde{k}_n z \Delta z} + u(-\Delta z) \sum_{n=3}^{4} \tilde{a}_n^d \tilde{e}_n e^{i\tilde{k}_n z \Delta z} \left[ e^{i(k_x \Delta x + i k_y \Delta y) \, dk_x \, dk_y} \right. \tag{6.2.6}
\]

where \(\tilde{a}_n^d(k_x, k_y)\) is the (source dependent) direct field amplitude of the \(n\)th polarization, while \(\tilde{e}_n(k_x, k_y)\) and \(\tilde{k}_n(z, k_y)\) are (resp.) the electric field eigenvector (i.e., polarization state) and eigenvalue of the \(n\)th mode \((n = 1, 2, 3, 4)\) (c.f. Ch. 2).

\(^{121}\text{Adj}(\cdot)\) and \(\text{Det}(\cdot)\) denote the adjugate and determinant of said argument, respectively. \(\text{Det}(\tilde{A}) = d_0(k_z - \tilde{k}_{1z})(k_z - \tilde{k}_{2z})(k_z - \tilde{k}_{3z})(k_z - \tilde{k}_{4z})\), where \(\{\tilde{k}_{nz}\}\) are the eigenvalues (i.e., longitudinal \([z]\) propagation constants).
Modes labeled with \( n = 1, 2 \) correspond to up-going polarizations, and similarly for down-going modes \( (n = 3, 4) \).

The problem with the canonical numerical implementation of this residue calculus approach lies in its tacit assumption of non-degeneracy (distinctness) in the eigenvalues \( \{\tilde{k}_{1z}, \tilde{k}_{2z}, \tilde{k}_{3z}, \tilde{k}_{4z}\} \), which does not hold for NBAM media. As an illustration of the polarization-independent dispersion behavior of NBAM, consider the dispersion relations of a uniaxial-anisotropic medium slab \( \{\bar{\epsilon}_r = \text{Diag} [a, a, b], \bar{\mu}_r = \text{Diag} [c, c, d]\} \) \( (k^2_p = k^2_x + k^2_y) \) \cite{77, 45}: \( \tilde{k}_{1z} = [k^2_0 ac - (c/d)k_y^2]^{1/2}, \tilde{k}_{2z} = [k^2_0 ac - (a/b)k_y^2]^{1/2}, \tilde{k}_{3z} = -\tilde{k}_{1z}, \text{ and } \tilde{k}_{4z} = -\tilde{k}_{2z} \). Setting \( \{a = \bar{y}^2 c, b = \bar{y}^2 d\} \) \( (\bar{y} \text{ is an arbitrary, non-zero multiplicative constant}) \) renders \( \tilde{k}_z^+ = \tilde{k}_{1z} = \tilde{k}_{2z} \) and \( \tilde{k}_z^- = \tilde{k}_{3z} = \tilde{k}_{4z} \), demonstrating the plane wave propagation direction dependent, but polarization independent, dispersion characteristics of uniaxial NBAM \cite{103}. This conclusion applies also for more general uniaxial NBAM material tensors possessing PMB rotated with respect to the Cartesian basis \cite{103}. Similarly, for biaxial NBAM with PMB-expressed tensors \( \{\bar{\mu}^{\text{pmb}}_r = \text{Diag} [a, b, c], \bar{\epsilon}^{\text{pmb}}_r = \bar{y}^2 \bar{\mu}^{\text{pmb}}_r\} \), the polarization-independent dispersion relations are:

\[
\tilde{k}_{1z}^{\text{pmb}} = [(\bar{y}k_0)^2 ab - (a/c)k_y^2 - (b/c)k_y^2]^{1/2}
\]

(6.2.7)

\[
\tilde{k}_{3z}^{\text{pmb}} = -[(\bar{y}k_0)^2 ab - (a/c)k_y^2 - (b/c)k_y^2]^{1/2}
\]

(6.2.8)

with \( \tilde{k}_z^+ = \tilde{k}_{2z}^{\text{pmb}} = \tilde{k}_{1z}^{\text{pmb}} \) and \( \tilde{k}_z^- = \tilde{k}_{4z}^{\text{pmb}} = \tilde{k}_{3z}^{\text{pmb}} \).

Now, the two-fold degenerate eigenvalue \( \tilde{k}_z^+ \) has associated with it two linearly independent field polarizations describing up-going plane waves \cite{103}; this holds likewise for the two down-going polarizations with common eigenvalue \( k_z^- \). Mathematically

\[\text{Please see [45] and Ch. 2 for other relevant layered-medium expressions.}\]
speaking, the eigenvalues \( \{ \tilde{k}_z^+, \tilde{k}_z^- \} \) are each twice-repeating (i.e., algebraic multiplicity of two) but have associated with each of them two linearly independent eigenvectors (i.e., geometric multiplicity of two), making them non-defective and rendering the four NBAM polarization states suitable as a local EM field basis within NBAM layers [104]. Despite the existence of four linearly independent eigenvectors, it is worthwhile to further exhibit the key results of the systematic analytical treatment of the two fictitious double-poles of \( \tilde{A}^{-1} \) to render numerical PWE-based EM field evaluation robust to the two said sources of “breakdown”; this treatment is performed in the next section.

Let us first make two preliminary remarks, however. First, assume that the source-containing layer is a biaxial NBAM with \( \tilde{\mu}_p^\text{pmb} = \text{Diag} [a, b, c] \) and \( \tilde{\epsilon}_p^\text{pmb} = \tilde{\mu}_p^\text{pmb} \). Second, the orthogonal matrix \( \tilde{\mathbf{U}} = [\tilde{\mathbf{v}}_1 \ \tilde{\mathbf{v}}_2 \ \tilde{\mathbf{v}}_3] \) transforms vectors between the PMB and Cartesian basis. For example, the relationship between the \( n \)th PMB eigenmode wave vector \( \mathbf{k}_n^\text{pmb} = (k_{nx}^\text{pmb}, k_{ny}^\text{pmb}, \tilde{k}_{nz}^\text{pmb}) \) and the (assumed available) \( n \)th Cartesian-basis wave vector \( \mathbf{k}_n = (k_x, k_y, \tilde{k}_{nz}) \) writes as \( \mathbf{k}_n^\text{pmb} = \tilde{\mathbf{U}}^{-1} \cdot \mathbf{k}_n \).

### 6.3 Direct Electric Field Radiated within NBAM

The (Cartesian basis) Fourier domain representation of the electric field, radiated in a homogeneous NBAM, writes as \( \tilde{\mathbf{E}} = -\tilde{\mathbf{A}}^{-1} \cdot \tilde{\nabla} \times \tilde{\mathbf{u}}_r^{-1} \cdot \tilde{\mathbf{M}} \) for a (equivalent) magnetic current source or \( \tilde{\mathbf{E}} = ik_0\eta_0\tilde{\mathbf{A}}^{-1} \cdot \tilde{\mathbf{J}} \) for an electric current source. These two equations, moreover, hold equally when re-represented in the NBAM’s PMB (i.e., adding “pmb” superscript to all quantities), which is what we will employ. Indeed,
the components \( \{ A_{m w} \} \) \((m, w = 1, 2, 3)\) of \( \tilde{A}^{-1, \text{pmb}} (\cdot) \) write as \( A_{m w} = A_{w m}, \) and \( \tilde{k} = k_{\text{pmb}} / k_0 \):

\[
\begin{align*}
\tilde{B} &= -cy^2 k_0^2 (k_z^2 - \left[ aby^2 - (a/c)k_y^2 - (b/c)k_z^2 \right]) \\
A_{11} &= (k_x^2 - bcy^2) / \tilde{B}, \quad A_{12} = \tilde{k}_x \tilde{k}_y / \tilde{B} \\
A_{13} &= \tilde{k}_x \tilde{k}_z / \tilde{B}, \quad A_{22} = (k_y^2 - acy^2) / \tilde{B} \\
A_{23} &= \tilde{k}_y \tilde{k}_z / \tilde{B}, \quad A_{33} = (k_z^2 - aby^2) / \tilde{B}
\end{align*}
\]

while the components of \( -\tilde{A}^{-1, \text{pmb}} \cdot \tilde{\nabla}_{\text{pmb}} \times \tilde{\mu}_r^{-1, \text{pmb}} (\cdot) \{ \tilde{A}_{m w} \} \) write as \( \{ \tilde{A}_{m w} = -\tilde{A}_{w m} \) :

\[
\begin{align*}
\tilde{B}' &= \tilde{B} / (y^2), \quad \tilde{A}_{12} = -i\tilde{k}_z / \tilde{B}' \\
\tilde{A}_{13} &= i\tilde{k}_y \tilde{k}_z / \tilde{B}', \quad \tilde{A}_{23} = -i\tilde{k}_x / \tilde{B}'
\end{align*}
\]

The expressions within Eqns. (6.3.1)-(6.3.4) describe the electric field from an electric current source while the expressions within Eqns. (6.3.5)-(6.3.6) describe the electric field from an (equivalent) magnetic current source. Duality in Maxwell’s Equations makes immediate the magnetic field results.

Next the PMB electric field \( \mathbf{E}_{\text{pmb}}(k_x, k_y; z, z') \), after re-expressing Eqns. (6.3.1)-(6.3.6) in terms of \( \{ k_x, k_y, k_z \} \) to identify the \( k_z \) (rather than \( k_{\text{pmb}}^z \) ) eigenvalues \( \{ \tilde{k}_n \} \) (using the relation \( k = \tilde{U} \cdot k_{\text{pmb}} \)) as well as “analytically” performing the \( k_z \) contour integral, can be decomposed into a linear combination of the degenerate up-going modes \( \{ \tilde{e}_{1, \text{pmb}} \}, \tilde{e}_{2, \text{pmb}} \} \) (for \( z > z' \)) or down-going modes \( \{ \tilde{e}_{3, \text{pmb}}, \tilde{e}_{4, \text{pmb}} \} \) (for \( z < z' \)).

For an electric source, we have for \( \tilde{e}_{\pm, \text{pmb}} \):

\[
\pm 2\pi i \left[ ik_0 \eta_0 \left( k_z - \tilde{k}_z^\pm \right) \tilde{A}^{-1, \text{pmb}} \cdot \tilde{J}_{\text{pmb}} \right] \bigg|_{k_z = \tilde{k}_z^\pm} (6.3.7)
\]

\[124\]When \( z = z' \), assuming the source does not lie exactly at a planar material interface, one can write the direct fields as a linear combination of either the up-going or down-going modes since both combinations lead to identical field results (save at \( r' \) on the plane \( z = z' \) [45] (also, c.f. Ch. 2).
and similarly for a (equivalent) magnetic source upon replacing $ik_0\eta_0\vec{A}^{-1,\text{pmb}} \cdot \vec{J}^{\text{pmb}}$ with $-\vec{A}^{-1,\text{pmb}} \cdot \vec{\nabla}^{\text{pmb}} \times \vec{\mu}_r^{-1,\text{pmb}} \cdot \vec{M}^{\text{pmb}}$ in Eqn. (6.3.7). Next, the degenerate PMB modal electric fields are re-expressed in the Cartesian basis ($\vec{e}_n = \vec{U} \cdot \vec{e}_n^{\pm,\text{pmb}}$) from which the Cartesian-basis direct field modal amplitudes $\{\tilde{a}_{d1}, \tilde{a}_{d2}, \tilde{a}_{d3}, \tilde{a}_{d4}\}$ can be robustly extracted using the polarization decomposition method proposed previously for sources radiating within isotropic layers (c.f. Ch. 2):

$$
\begin{bmatrix}
\tilde{a}_{d1} \\
\tilde{a}_{d2} 
\end{bmatrix} = \begin{bmatrix}
\tilde{e}_{x1} & \tilde{e}_{x2} \\
\tilde{e}_{y1} & \tilde{e}_{y2} 
\end{bmatrix}^{-1} \begin{bmatrix}
\tilde{e}_x^+ \\
\tilde{e}_y^+ 
\end{bmatrix},
\begin{bmatrix}
\tilde{a}_{d3} \\
\tilde{a}_{d4} 
\end{bmatrix} = \begin{bmatrix}
\tilde{e}_{x3} & \tilde{e}_{x4} \\
\tilde{e}_{y3} & \tilde{e}_{y4} 
\end{bmatrix}^{-1} \begin{bmatrix}
\tilde{e}_x^- \\
\tilde{e}_y^- 
\end{bmatrix}
$$

(6.3.8)

where $\{\tilde{e}_{xn}, \tilde{e}_{ym}, \tilde{e}_{zn}\}$ are the $x$, $y$, and $z$ components of the (cartesian basis) NBAM’s $n$th electric field eigenvector $\vec{e}_n$. Moreover, if the above-inverted matrices are suspected (with respect to, say, the euclidean matrix norm measure) of being ill-conditioned, one can always utilize instead say the $y$ and $z$, or alternatively the $x$ and $z$, components of the field eigenvectors (c.f. Ch. 2). Indeed, this decomposition procedure is well-defined due to the non-defective nature of the eigenvalues, and hence linear independence between the four NBAM field eigenvectors $\{\vec{e}_n\} [103]$.

### 6.4 Results

Now we exhibit some illustrative results demonstrating the developed algorithm’s performance. We investigate both the electric field $E_z$ radiated by a vertical (i.e., $z$-directed) Hertzian electric current dipole (VED), as well as the magnetic field $H_z$ radiated by a $z$-directed Hertzian (equivalent) magnetic current dipole (VMD); both sources radiate at $f = 2$MHz. In both scenarios, the source resides at depth $z' = 0\text{m}$ within a three-layer NBAM, occupying the region $-1 \leq z \leq 1 \text{ [m]}$, of material properties $\bar{\epsilon}_r = \bar{\mu}_r = \text{Diag}[10, 10, 1/10], \text{Diag}[5, 5, 1/5]$, and $\text{Diag}[2, 2, 1/2]$ within the regions $-1 < z < -1/4 \text{ [m]}, -1/4 < z < 1/4 \text{ [m]},$ and $1/4 < z < 1 \text{ [m]}$ (resp.);
see Fig. 6.1. The top layer \((z \geq 1\text{m})\) is vacuum \((\epsilon_{r1} = \mu_{r1} = 1)\) while the bottom layer \((z \leq -1\text{m})\) is a perfect electric conductor (PEC); note that this layered-medium configuration was specifically chosen to facilitate comparison with closed-form solutions through invocation of T.O. and EM Image theory (c.f. Ch. 5). Indeed the EM field solution within \(z \geq -1\text{m}\), for our five-layered configuration involving a VED source, can be shown identical to the closed-form field result of two VED’s (located at depths \(z = -1.75\text{m}\) and \(z = -19.25\text{m}\)) of identical orientation to the original VED and radiating in homogeneous, unbounded vacuum. Note that within the NBAM, an added step to compute the closed-form result must be taken, appropriately mapping the observation points within the NBAM to vacuum observation points by viewing a \(d\)-meter thick NBAM layer \(\bar{\epsilon}_r = \bar{\mu}_r = \text{Diag}\left[n, n, 1/n\right]\) as equivalent to a \(nd\)-meter thick vacuum layer. Similarly, the VMD problem can be shown identical to two VMD’s (located at depths \(z = -1.75\text{m}\) and \(z = -19.25\text{m}\)) radiating in homogeneous, unbounded vacuum; in this scenario however, image theory prescribes that the \(z = -1.75\text{m}\) VMD possess identical orientation to the original VMD, but that the \(z = -19.25\text{m}\) VMD possess opposite orientation.\(^{125}\)

Observing Figs. 6.2c-6.2d, we note the relative errors in both the electric field \((\delta_e)\) and magnetic field \((\delta_h)\) are very low,\(^{126}\) approaching in most of the observation plane near the limits of floating point double precision-related numerical noise (approximately -150 in [dB] scale); for reference, Figs. 6.2a-6.2b are the computed field amplitudes of the VED and VMD (i.e., lying within the central NBAM layer) must be scaled by a factor of 1/5 (relative to the vacuum sources) to facilitate field comparisons. Moreover the normal field components \(\{\mathcal{E}_z, \mathcal{H}_z\}\), within the NBAM layer with properties \(\bar{\epsilon}_r = \bar{\mu}_r = \text{Diag}\left[n, n, 1/n\right]\), are also scaled (artificially, for both visual display and error computation purposes) by \(1/n\) to account for their discontinuity across material interfaces.

\(^{125}\)Let \(E_c\) be the computed electric field, and let \(E_v\) be the closed-form reference solution. Then \(\delta_e = |E_c - E_v|/|E_v|\) (likewise for \(\delta_h\)).
distributions themselves from our algorithm. This is consistent with our having set
an adaptive relative integration error tolerance of $1.2 \times 10^{-14}$. We do observe however that the error noticeably increases (for fixed observer/source radial separation) as the observation angle tends closer to “horizon” (i.e., source depth $z'$ and observer depth $z$ coinciding). The error variation trend versus angle has been observed before (c.f. Ch. 5) even when the source resided in non-NBAM media, and hence the increased error versus observation angle is not likely due to instabilities in the presented NBAM-robust algorithm. We conjecture rather that the increasing error (versus observation angle) arises due to commensurately increasing numerical cancellation that can only be partially offset by a (computer resource limited) finite extent of $hp$ integration refinement performed using finite precision arithmetic. This numerical cancellation, we remark, is well known to be predominantly induced by integrand oscillation, which worsens as the observation angle tends to horizon (c.f. Chs. 2 and 5). One remedy is to use a constant-phase path [61], but a robust remedy for 2-D integrals (needed for generally anisotropic media) remains an open question. Moreover, this path would change as one varies the outer integration variable. Finally we emphasize that given the design of our particular implementation, which always first computes the direct electric field and then (if need be) computes the magnetic field using ancillary relations [45][Ch. 2], we have in fact tested the soundness of both Eqns. (6.3.1)-(6.3.4) (VED scenario) and Eqns. (6.3.5)-(6.3.6) (VMD scenario).

\footnote{Namely, cancellation from radiation field contributions arising from numerical integration along contour sub-sections symmetrically located about the imaginary $k_x$ and $k_y$ axes. By contrast, our algorithm robustly ensures (irrespective of observation angle) that the evanescent field contribution introduces little numerical cancellation-induced error and rapid convergence (c.f. Ch. 4).}
Figure 6.1: Vertically-oriented Hertzian dipole current source within a three-layer NBAM. The purple (air) and blue (NBAM) regions form the plane on which the fields are observed in Fig. 6.2. The parameter $n$ equals ten, five, and two within the regions $-1 < z < -1/4$ [m], $-1/4 < z < 1/4$ [m], and $1/4 < z < 1$ [m], respectively.

6.5 Conclusion

We addressed a fundamental origination of breakdown in the spectral-domain-based (PWE) evaluation of EM fields radiated by sources embedded within NBAM planar slabs, leading to a robust formulation that can accurately compute EM fields despite the modal degeneracy, induced by said NBAM, that would ordinarily lead to numerical instabilities and/or corruption of the functional form of the plane wave...
Figure 6.2: (a) $E_z$ radiated by a VED. (b) $H_z$ radiated by a VMD. (c) Relative error: $E_z$. (d) Relative error: $H_z$.

eigenfunctions. Indeed this instability arises due to eigenvalues that, while non-defective, have an algebraic multiplicity equal to two rather than one. The remedy is to apply a proper (analytical) “pre-treatment” of the spectral-domain tensor operators prior to polarization amplitude extraction, resulting in robust analysis of EM fields in arbitrary anisotropic planar-layered media. Results validated the high accuracy of numerical computations based on this analytical pre-treatment.
Chapter 7: Full-Wave Algorithm to Model Effects of Bedding Slopes on the Response of Subsurface Electromagnetic Geophysical Sensors near Unconformities

7.1 Introduction

7.1.1 Chapter Summary and Contributions

We propose a full-wave pseudo-analytical numerical electromagnetic (EM) algorithm to model subsurface induction sensors, traversing planar-layered geological formations of arbitrary EM material anisotropy and loss, which are used, for example, in the exploration of hydrocarbon reserves. Unlike past pseudo-analytical planar-layered modeling algorithms that impose parallelism between the formation’s bed junctions however, our method involves judicious employment of Transformation Optics techniques to address challenges related to modeling relative slope (i.e., tilting) between said junctions (including arbitrary azimuth orientation of each junction). The algorithm exhibits this flexibility, both with respect to loss and anisotropy in the formation layers as well as junction tilting, via employing special planar slabs that coat each “flattened” (i.e., originally tilted) planar interface, locally redirecting the incident wave within the coating slabs to cause wave fronts to interact with the flattened interfaces as if they were still tilted with a specific, user-defined orientation.
Moreover, since the coating layers are homogeneous rather than exhibiting continuous material variation, a minimal number of these layers must be inserted and hence reduces added simulation time and computational expense. As said coating layers are not reflectionless however, they do induce artificial field scattering that corrupts legitimate field signatures due to the (effective) interface tilting. Numerical results, for two half-spaces separated by a tilted interface, quantify error trends versus effective interface tilting, material properties, transmitter/receiver spacing, sensor position, coating slab thickness, and transmitter and receiver orientation, helping understand the spurious scattering’s effect on reliable (effective) tilting this algorithm can model. Under the effective tilting constraints suggested by the results of said error study, we finally exhibit responses of sensors traversing three-layered media, where we vary the anisotropy, loss, and relative tilting of the formations and explore the sensitivity of the sensor’s complex-valued measurements to both the magnitude of effective relative interface tilting (polar rotation) as well as azimuthal orientation of the effectively tilted interfaces.128

7.1.2 Background

A natural question arises as to which numerical technique is best suited to modeling more complex geometries involving tilted layers. In principle, one could resort to brute-force techniques such as finite difference and finite element methods [21, 90, 54, 106]. The potential for low-frequency instability (e.g., when modeling geophysical sensors operating down to the magnetotelluric frequency range [fraction of a Hertz]),

128NOTE: Unless otherwise stated, all conventions, abbreviations, and notation within this chapter are self-contained.
high computational cost (unacceptable, especially when many repeated forward solutions are required to solve the inverse problem), and accuracy limitations due to mesh truncation issues (say, via perfectly matched layers or other approaches [17, 107]) associated with the lack of transverse symmetry in the tilted-layer domain [108], render these numerical methods less suitable for developing fast forward-modeler engines for tilted-layer problems. Another potential approach involves asymptotic solutions which traces the progress of incident rays and their specular reflections within subsurface formations [109]. However, this approach is limited to sufficiently high-frequencies and hence unsuitable for modeling low-frequency sensors operating in zones where highly resistive and highly conductive (not to mention anisotropic) layers may coexist [77]. Yet a third possible approach, called the “Tilt Operator” method, which assumes lossless media and negligible EM near-fields to avoid spurious exponential field growths (arising from violation of “primitive” causality [i.e., cause preceding effect], which is inherent in this method), is another possibility [110, 111]. Akin to the other mentioned high-frequency approach however [109], the Tilt Operator method is not appropriate for our more general class of problems with respect to sensor and geological formation characteristics.

This chapter is organized as follows. In Section 7.2 we overview the 2-D plane wave expansion algorithm, derive the material blueprints for the planar “interface-flattening” coating slabs, and show how to systematically incorporate these into the computational model. Sections 7.3.1-7.3.2 show the error analysis to quantify how the accuracy of the results varies with effective interface tilting, material profile, transmitter/receiver spacing, sensor position, coating slab thickness, and complex-valued measurement component (both its real and imaginary parts). In Section 7.4 we apply the
algorithm to predicting EM multi-component induction tool responses when the tool traverses (effectively) tilted formation beds for different interface tilt orientations as well as central bed anisotropic conductivity profiles. The formation anisotropies studied will span the full gamut: All the way from isotropic to ("Transverse-Isotropic") non-deviated uniaxial, ("cross-bedded") deviated uniaxial, and full biaxial anisotropy. We adopt the $\exp(-i\omega t)$ convention, as well as assume all EM media are spatially non-dispersive, time-invariant, and are representable by diagonalizable anisotropic 3 $\times$ 3 material tensors.\textsuperscript{129}

\section{7.2 Formulation}

\subsection{7.2.1 Background: Electromagnetic Plane Wave Eigenfunction Expansions}

In deriving the planar multi-layered medium eigenfunction expansion expressions, first assume a homogeneous formation whose dielectric (i.e., excluding conductivity), relative magnetic permeability, and electric conductivity constitutive anisotropic material tensors write as $\bar{\epsilon}_r$, $\bar{\mu}_r$, and $\bar{\sigma}$. Specifically, the assumed material tensors are those of the layer (i.e., in the anticipated multi-layered case), labeled $M$, within which the transmitter resides. Maxwell’s Equations in the frequency domain, upon impressing causative electric current $\mathbf{J}(\mathbf{r})$ and/or (equivalent) magnetic current $\mathbf{M}(\mathbf{r})$, yields the electric field vector wave equation (duality in Maxwell’s Equations yields \textsuperscript{129}Diagonalizability of the material tensors, which physically corresponds to a medium having a well-defined response for any direction of applied electric and magnetic field, is required for completeness of the plane wave basis. All naturally-occurring media, as well as the introduced interface-flattening slabs, are characterized by diagonalizable material tensors.\textsuperscript{195}}
the magnetic field vector wave equation) [45] (also, c.f. Ch. 2):\(^{130}\)
\[
\bar{A}(\cdot) = \nabla \times \bar{\mu}_r^{-1} \cdot \nabla - k_0^2 (\varepsilon_r + i \sigma / \omega) \cdot \bar{A}(\mathcal{E}) = i k_0 \eta_0 \mathcal{J} - \nabla \times \bar{\mu}_r^{-1} \cdot \mathcal{M} \tag{7.2.1}
\]

Now define (c.f. Ch. 2) the three-dimensional spatial Fourier Transform (FT) pair for some generic vector field \(L\) (e.g., the magnetic field or current source vector):
\[
\tilde{L}(k) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} L(r) e^{-i k \cdot r} \, dx \, dy \, dz,
\]
\[
L(r) = \left( \frac{1}{2\pi} \right)^3 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{L}(k) e^{i k \cdot r} \, dk_x \, dk_y \, dk_z \tag{7.2.2}
\]

where \(r = (x, y, z)\) is the position vector and \(k = (k_x, k_y, k_z)\) is the wave vector. Expanding the left and right hand sides, of the second equation in Eqn. (7.2.1), in their respective wave number domain 3-D integral representations and matching the Fourier-domain integrands on both sides, one can multiply the inverse of \(\tilde{A}\) (the FT of \(A\)) to the left of both integrands. Admitting a single Hertzian/infinitesimal-point transmitter current source located at \(r' = (x', y', z')\), and denoting the receiver location \(r\), one can then procure the “direct” (i.e., homogeneous medium) radiated time-harmonic electric field \(\mathcal{E}_d(r)\) (c.f. Ch. 2). Indeed, performing “analytically” (i.e., via contour integration and residue calculus techniques) the \(k_z\) integral leads to the following expression:
\[
\frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \left[ u(z - z') \sum_{n=1}^{2} \tilde{a}^{D}_{M,n} \tilde{e}_{M,n} e^{i k_{M,n} \Delta z} + u(z' - z) \sum_{n=3}^{4} \tilde{a}^{D}_{M,n} \tilde{e}_{M,n} e^{i k_{M,n} \Delta z} \right] \times e^{i k_x \Delta x + i k_y \Delta y} \, dk_x \, dk_y \tag{7.2.3}
\]

where \(\Delta x = x - x', \Delta y = y - y', \Delta z = z - z', u(\cdot)\) denotes the Heaviside step function, and \(\{\tilde{e}_{p,n}, \tilde{k}_{p,n}, \tilde{a}^{D}_{p,n}\}\) stand for the electric field polarization state vector, \(\varepsilon_0, c, \mu_0 = 1/(\varepsilon_0 c^2)\) represent vacuum electric permittivity, vacuum speed of light, and vacuum magnetic permeability, respectively. Furthermore, \(\omega = 2\pi f\) is the angular temporal radiation frequency, \(k_0 = \omega / c\) is the vacuum wave number, and \(\eta_0 = \sqrt{\mu_0 / \varepsilon_0}\) is the intrinsic vacuum plane wave impedance [15, 45].
longitudinal wave number component, and direct field polarization amplitude of the
pth formation bed’s nth plane wave polarization \((1 \leq p \leq N, 1 \leq n \leq 4)\), respectively.

Now introducing additional formation beds will induce a modification, via reflection
and transmission mechanisms interfering with the direct field, to the total observed
electric field. We mathematically codify this interference phenomenon by deriving
the (transmitter layer \([M]\) and receiver layer \([L]\)-dependent) time-harmonic scattered
electric field \(\mathbf{E}_s(\mathbf{r})\) (c.f. Ch. 2):

\[
\frac{i}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[(1 - \delta_{L,N}) \sum_{n=1}^{2} \tilde{a}_{L,n}^s \tilde{e}_{L,n} e^{ik_{L,n} z} + (1 - \delta_{L,1}) \sum_{n=3}^{4} \tilde{a}_{L,n}^s \tilde{e}_{L,n} e^{ik_{L,n} z}\right] \times e^{ik_x \Delta x + ik_y \Delta y} \, dk_x \, dk_y \tag{7.2.4}
\]

where \(\tilde{a}_{p,n}^s\) is the scattered field polarization amplitude of the nth polarization in layer
\(p\), and \(\delta_{P1,P2}\) denotes the Kronecker Delta function.

### 7.2.2 Tilted Layer Modeling

Admit an \(N\)-layer medium where the \(m\)th planar interface \((m=1,2,\ldots,N-1)\) is
characterized as follows. First, its upward-pointing area normal vector \(\mathbf{\hat{z}}'_m\) is rotated
by polar angle \(-90^\circ \leq \alpha'_m \leq 90^\circ\) relative to the \(z\) axis, and azimuth angle \(0^\circ \leq \beta'_m \leq 180^\circ\) relative to the \(x\) axis. Second, the \(m\)th interface’s “depth” \(z'_m\) is defined
at the Cartesian coordinate system’s transverse origin \((x,y) = (0,0)\). See Figure 7.1
for a schematic illustration of the environment geometry’s parametrization.

\(\text{Albeit as becomes apparent below, this “polar” angle corresponds to rotation in the direction opposite to that ascribed to the spherical coordinate system.}\)
Figure 7.1: Figure 7.1a shows the original problem with tilted planar interfaces in an $N$-layer geological formation possessing the EM material tensors $\{\bar{\epsilon}_p, \bar{\mu}_p, \bar{\sigma}_p\}$. Figure 7.1b shows the transformed, equivalent problem obtained through employing special “interface-flattening” media (c.f. Eqn. (7.2.7)) that coat the underside ($\{\bar{\epsilon}_{m+1}', \bar{\mu}_{m+1}', \bar{\sigma}_{m+1}'\}$) and over-side ($\{\bar{\epsilon}_m'', \bar{\mu}_m'', \bar{\sigma}_m''\}$) of the $m$th interface. $d$ represents the thickness of each T.O. slab in meters. For simplicity of illustration, all interfaces here are tilted within the $xz$ plane (i.e., interface-tilting azimuth orientation angles $\{\beta_m'\} = 0^\circ$).
To make the \( m \)th planar interface parallel to the \( xy \) plane yet retain its tilted-interface scattering characteristics, as the first of two steps we abstractly define, within the two slab regions \((z_m' - d) \leq z < z_m' \) and \( z_m' \leq z < [z_m' + d]\) bounding the \( m \)th interface, a “coordinate stretching” transformation. Namely this transformation relates Cartesian coordinates \((x, y, z)\), which parametrize the coordinate mesh of standard “flat space”, to new oblique coordinates \((\bar{x}, \bar{y}, \bar{z})\) that parametrize an imaginary “deformed” space whose coordinate mesh deformation systematically induces in turn a well-defined distortion of the EM wave amplitude profile within said slab regions \([56, 101, 16]\).\(^{132}\)

\[
\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = z + a_m x + b_m y
\]  

(7.2.5)

where \( a_m = -\tan \alpha_m' \cos \beta_m' \) and \( b_m = -\tan \alpha_m' \sin \beta_m' \). Indeed this coordinate transform will cause wave fronts to interact with the \( m \)th flattened interface as if said interface were geometrically defined by the equation \( z = z_m' - a_m x - b_m y \) rather than \( z = z_m' \). As the second step in the interface-flattening procedure, we invoke a “duality” (not to be confused with duality between the Ampere and Faraday Laws) between spatial coordinate transformations and doubly-anisotropic EM media which “implement” the effects, of an effectively deformed spatial coordinate mesh (and

\(^{132}\)We remark in passing upon a strong similarity between the coordinate transform shown in Eqn. (7.2.5) versus the “refractor” and “beam shifter” coordinate transforms prescribed elsewhere [112, 101]. However, while the beam shifter transform (and the equivalent anisotropic medium it describes [101]) is perfectly reflectionless due to continuously transitioning the coordinate transformation back to the ambient medium (e.g., free space), our coordinate transformation is inherently \textit{discontinuous}. Indeed, note in Eqn. (7.2.5) that the mapping \( \bar{z} \), which depends on \( x \) and \( y \) in addition to \( z \), can not be made to continuously transition back to the (identity) coordinate transform \( \bar{z}(z) = z \) implicitly present within the ambient medium. Alternatively stated, our defined anisotropic coating slabs have the exact same material properties as the beam shifter, but our slabs border the ambient medium at planes that, though parallel to each other, are orthogonal relative to the junction planes between the beam shifter and its ambient medium. See other references for the importance of the junction surface’s orientation in ensuring a medium perfectly impedance-matched to the ambient medium [113, 114].
hence effectively deformed spatial metric tensor), on EM waves propagating through flat space (see references deriving this “duality” [16, 56, 57, 17, 58]). Following one of two common, equivalent conventions [16, 17] leading seamlessly from coordinate transformation to equivalent anisotropic material properties, by defining the Jacobian coordinate transformation tensor [16]:

\[
\bar{\Lambda}_m = \begin{bmatrix}
\frac{\partial \bar{x}}{\partial x} & \frac{\partial \bar{y}}{\partial x} & \frac{\partial \bar{z}}{\partial x} \\
\frac{\partial \bar{x}}{\partial y} & \frac{\partial \bar{y}}{\partial y} & \frac{\partial \bar{z}}{\partial y} \\
\frac{\partial \bar{x}}{\partial z} & \frac{\partial \bar{y}}{\partial z} & \frac{\partial \bar{z}}{\partial z}
\end{bmatrix} = \begin{bmatrix} 1 & 0 & a_m \\
0 & 1 & b_m \\
0 & 0 & 1 \end{bmatrix}
\] (7.2.6)

within the region \((z'_m - d) \leq z < z'_m\) one has the interface-flattening material tensors \(\{\bar{\gamma}'_{m+1}\}\) in place of the original formation’s material parameters \(\{\bar{\gamma}_m\}\) within layer \(p = m + 1\) (\(\gamma = \epsilon, \mu, \sigma\)). Similarly, within the region \(z'_m \leq z < (z'_m + d)\) one has the interface-flattening material tensors \(\{\bar{\gamma}''_m\}\) in place of the original formation’s material parameters \(\{\bar{\gamma}_m\}\) within layer \(m\). How are the interface-flattening material tensors defined though? Quite simply, in fact, and this definition holds regardless of the original formation layer’s anisotropy and loss (“T” superscript denotes non-Hermitian transpose) [16]:

\[
\bar{\gamma}'_{m+1} = \bar{\Lambda}_m^T \cdot \bar{\gamma}_{m+1} \cdot \bar{\Lambda}_m, \quad \bar{\gamma}''_m = \bar{\Lambda}_m^T \cdot \bar{\gamma}_m \cdot \bar{\Lambda}_m
\] (7.2.7)

Note that if the \(m\)th interface lacks effective tilt then \(\bar{\Lambda}_m\) reduces to the identity matrix, which in turn leads to the two interface-flattening media reducing to the media of the respective formation layers from which they were derived using Eqn. (7.2.7): \(\bar{\gamma}'_{m+1} = \bar{\gamma}_{m+1}\) and \(\bar{\gamma}''_m = \bar{\gamma}_m\) (as expected!). Now the new material profile,
characterized by *parallel* planar interfaces, appears for a simple three-layer, two-interface geometry as:

\[ \gamma_1, (z_1' + d) \leq z < \infty \] (7.2.8)

\[ \gamma_1'', z_1' \leq z < (z_1' + d) \] (7.2.9)

\[ \gamma_2', (z_1' - d) \leq z < z_1' \] (7.2.10)

\[ \gamma_2, (z_2' + d) \leq z < (z_2' - d) \] (7.2.11)

\[ \gamma_2'', z_2' \leq z < (z_2' + d) \] (7.2.12)

\[ \gamma_3', (z_2' - d) \leq z < z_2' \] (7.2.13)

\[ \gamma_3, -\infty < z < (z_2' - d) \] (7.2.14)

with an analogous material profile resultant for \( N > 3 \) layers.

Before proceeding, we note that the coating slabs are spatially homogeneous in the employed Cartesian coordinate system, and hence we require only one planar layer to represent each coating slab, and that too to represent each slab’s spatial material profile exactly. This homogeneity characteristic is important from a computational efficiency standpoint, as it means that for each “flattened” interface only two coating layers’ EM eigenfunctions, Fresnel reflection and transmission matrices, etc. need to be computed (c.f. Ch. 2). Second, as we are fundamentally approximating the transverse translation-variant geometry as a transverse translation-invariant one, modeling spurious scattering from the “apexes” and more complex intersection junctions of the tilted beds is out of the question. As we concern ourselves with subsurface geophysical media, which typically present inherent conductivity (typically on the order of at least \( 10^{-3} \text{S/m} \) to \( 2 \text{S/m} \) [9]), scattering from these intersections should typically be negligible so long as the sensors are not in the immediate neighborhood
of said intersections (rarely the case, for the small tilting explored herein). Third, the interface-flattening slabs are not impedance-matched to the respective ambient medium layers into which they are inserted. Indeed, one of the objectives of this paper is to quantify the impedance mismatch of the artificial slabs, which we will find practically constrains (i.e., for a given desired computation accuracy) the amount of interface tilt that can be modeled.

There are two further advisories worth mentioning. First, we recommend adaptively (i.e., depending on the transmitter and receiver positions) reducing the thickness $d$ of coating layer(s), within which receiver(s) and/or transmitter(s) may reside depending on their depth, just enough so that the receivers and transmitters are located once more within the formation layers. Why this recommendation? Although pseudo-analytical techniques are available to compute fields when the receiver and/or transmitter are located in such layers (c.f. Ch. 6), the main reason is to eliminate spurious discontinuities of the normal ($z$ in our case) electric and magnetic field components manifest when the source (or, as can be anticipated from EM reciprocity, the receiver) traverse a boundary separating a true formation layer and a coating slab [17]. Second, the thickness $d$ of the coating slabs must also be adjusted to ensure the coating layer just beneath the $m$th interface does not cross over into the coating layer just above the $(m + 1)$st interface. We account for these two points within our numerical results below.
7.3 Error Analysis

7.3.1 Overview

To briefly recap: The proposed method relies upon insertion of specially-prescribed, doubly-anisotropic material slabs above and below each (effectively) tilted original interface to manipulate wave-fronts such that they interact with the coated interfaces as if they were tilted. This technique allows one, in principle, to unequivocally and independently prescribe the arbitrary, effective polar and azimuthal tilting orientation of each interface. Moreover, in the limit of vanishingly small effective polar tilt for some interface (and irrespective of effective azimuth tilt), the material properties of the slab just above (below) this interface continuously transitions back to the material properties of the ambient medium just above (below) said interface. However, as the material properties of each coating slab are (for non-zero effective tilt) not perfectly impedance matched to the respective ambient medium into which it is inserted, spurious scattering will result that coherently interferes with the true field scattered from the effectively tilted interface.

The spurious scattering corrupts the true responses (both co-polarized and cross-polarized ones) arising from tilt and hence needs to be quantified if we are to understand the practical limitations of the proposed tilted-layer algorithm, which is inherently perturbational in nature with respect to the range of (effective) polar tilt that can be reliably modeled. It is also important to understand the error trends not just versus effective polar tilt, but also for different geophysical media (anisotropy and loss), transmitter/receiver spacing, sensor positions (relative to the interface), transmitter and receiver orientation, and thickness $d$ of the coating slabs. To simplify the geometry and admit a closed-form reference solution, we examine only a
two-layered medium with an interface effectively tilted within the \(xz\) plane by \(\alpha = \alpha_1'\) degrees. Both the reference \((H_{wq}^{\text{Ref}})\) and algorithm/transformed-domain \((H_{wq}^{\text{TQ}})\) field values are computed to fourteen Digits of Precision (DoP), with the relative error between their respective field values (either real or imaginary part) denoted \(\epsilon\) \((-\log_{10}\epsilon\) denotes DoP agreement): Either \(\epsilon = |\text{Re}[H_{wq}^{\text{TQ}}] - \text{Re}[H_{wq}^{\text{Ref}}]|/|\text{Re}[H_{wq}^{\text{Ref}}]|\) or \(\epsilon = |\text{Im}[H_{wq}^{\text{TQ}}] - \text{Im}[H_{wq}^{\text{Ref}}]|/|\text{Im}[H_{wq}^{\text{Ref}}]|\), where \(\text{Re}[H_{wq}] = H'_{wq}\) and \(\text{Im}[H_{wq}] = H''_{wq}\) denote the real and imaginary parts (resp.) of the \(q\)-oriented magnetic field component observed at the receiver due to the \(w\)-directed magnetic dipole transmitter \(H_{wq}\). Errors below \(10^{-14}\) were artificially coerced in post-processing to \(10^{-14}\).

In order to compute the reference field solution in this two-layer scenario, we first denote the transmitter-receiving spacing \(L_s\), transmitter depth \(z'\), and receiver depth \(z = z' + L_s\) in the “transformed” domain with a flat, parallel (to the \(xy\) plane) interface (residing at \(z_1' = 0\)) with two coating slabs.\(^{133}\) In the equivalent domain, involving again a flat interface \((z_1' = 0)\) but with a rotated sensor, the new transmitter position \((x_T', y_T' = 0, z_T')\) writes as \(z_T' = z' \cos \alpha\) and \(x_T' = z' \sin \alpha\), the new receiver position \((x_T, y_T = 0, z_T)\) writes as \(z_T = (z' + L_s) \cos \alpha\) and \(x_T = (z' + L_s) \sin \alpha\), the transmitting dipole orientations are physically rotated by \(-\alpha\) degrees in the \(xz\) plane, and the receiver antennas are also physically rotated by \(-\alpha\) degrees in the \(xz\) plane.\(^{134}\)

Additionally, for the latter two material profiles (described, and denoted M3 and M4, below) involving anisotropic media, we rotate the anisotropic material tensors by \(-\alpha\) degrees (isotropic tensors are invariant under rotation, by definition). Throughout this error study, when computing the reference and transformed-domain solutions the

\(^{133}\)Note: For all results in this paper, we assume a vertically-oriented sensor in the transformed domain.

\(^{134}\)Equivalently, the observed fields at the receiver location are now observed relative to a cartesian coordinate system rotated by \(-\alpha\) degrees.
transmitter radiation frequency is held fixed at $f=100$kHz. Furthermore, both the reference and transformed-domain solutions are computed using the same numerical code, albeit with interface tilt set to zero for the reference solutions.

We show relative error $\log_{10} [H'_{wq}]$ and $\log_{10} [H''_{wq}]$ ($w, q = x, y, z$) versus the effective polar tilt angle $\alpha < 0$ (degrees) and azimuth angle (fixed at $\beta' = 0^\circ$). To better illustrate error trends (e.g., linear or quadratic) versus $\alpha$, we plot this Log-scale error on the vertical axis and $\log_{10} |\alpha|$ on the horizontal axis. To show error variations versus material profile and transmitter-receiver spacing, in each plot we exhibit relative errors for two different sensor spacings $L_s (=400\text{mm ["S1"] and =1m ["S2"]})$ and four different conductivity profiles (each having two layers, with $\epsilon_r = \mu_r = 1$, prior to inserting the two coating slabs):

1. $\sigma_1 = 1\text{mS/m, } \sigma_2 = 2\text{mS/m (["M1"]- Both layers isotropic, 2:1 conductivity contrast)}$

2. $\sigma_1 = 1\text{mS/m, } \sigma_2 = 20\text{mS/m (["M2"]- Both layers isotropic, 20:1 conductivity contrast)}$

3. $\sigma_1 = 1\text{mS/m, } \sigma_h = 5\text{mS/m, } \sigma_v = 1\text{mS/m, } \alpha_2 = 60^\circ, \beta_2 = 0^\circ$ (["M3"]- Deviated uniaxial bottom layer)

4. $\sigma_1 = 1\text{mS/m, } \sigma_x = 5\text{mS/m, } \sigma_y = 2.5\text{mS/m, } \sigma_z = 1\text{mS/m, } \alpha_2 = \beta_2 = 0^\circ$ (["M4"]- Non-deviated biaxial bottom layer)

where the anisotropic tensor components $\{\sigma_h, \sigma_v, \sigma_x, \sigma_y, \sigma_z\}$ and tensor deviation angles $\{\alpha_2, \beta_2\}$ define the principal conductivity values and orientation (resp.) of the conductivity tensor relative to the cartesian axes (discussed in Section 7.4 and [9]).
The curve labeled “SXMY” refers to the sensor spacing denoted above in square brackets by [“SX”] (X=1,2) and the material scenario denoted above in square brackets by [“MY”] (Y=1,2,3,4). Fixing the flattened interface’s depth at \( z_1' = 0 \), for two coating slab thicknesses (\( d = 2\text{mm }[\text{“d1”}] \) and \( d = 200\text{mm }[\text{“d2”}] \)) we examine three mid-point sensor depths in the transformed domain: \( D = 2\text{m} \) (transmitter and receiver in top layer [“O1”]), \( D = 0\text{m} \) (transmitter in bottom layer, receiver in top layer [“O2”]), and \( D = -2\text{m} \) (transmitter and receiver in bottom layer [“O3”]). For the real and imaginary part of each field component we examine six sensor-location/d permutations, which are denoted on each page with plot labels (“Scenario:O1d1”, etc.). The transmitter depth (\( z' \)) and receiver depth (\( z \)) are computed using the sensor spacing \( L_s \) and mid-point depth \( D \): \( z = D + L_s/2 \) and \( z' = D - L_s/2 \). The scenario parameters are summarized below in Tables 7.1-7.2. In these two tables, call the transmitter layer number \( M \), transmitter depth (in meters) \( z' \), receiver layer number \( L \), receiver depth (in meters) \( z \), and coating slab thickness (in millimeters) \( d \); the top layer is layer number one, while the bottom layer is layer number two. The sensor spacing \( L_s \) is given in meters, while the principal components of layer two’s conductivity tensor \( \bar{\sigma}_2 \) are given in milli-Siemens per meter [mS/m]; material axis orientation angles \( \{\alpha_2, \beta_2\} \), provided with the principal conductivity values (when relevant), are given in degrees.

\[
\text{Table 7.1: Definitions of Plot Title Abbreviations}
\]

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Table 7.2: Definitions of Curve Label Abbreviations (σ₁ = 1mS/m)

<table>
<thead>
<tr>
<th>Abbrev.</th>
<th>Ls</th>
<th>Layer 2 Aniso.</th>
<th>σ₁₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1M1</td>
<td>0.4</td>
<td>Isotropic</td>
<td>σ₂=2</td>
</tr>
<tr>
<td>S1M2</td>
<td>0.4</td>
<td>Isotropic</td>
<td>σ₂=20</td>
</tr>
<tr>
<td>S1M3</td>
<td>0.4</td>
<td>Uniaxial</td>
<td>σ₁₂=5, σᵥ₂=1, α₂=60°, β₂=0°</td>
</tr>
<tr>
<td>S1M4</td>
<td>0.4</td>
<td>Biaxial</td>
<td>σₓ₂=5, σᵧ₂=2.5, σ₂=1, α₂=β₂=0°</td>
</tr>
<tr>
<td>S2M1</td>
<td>1</td>
<td>Isotropic</td>
<td>σ₂=2</td>
</tr>
<tr>
<td>S2M2</td>
<td>1</td>
<td>Isotropic</td>
<td>σ₂=20</td>
</tr>
<tr>
<td>S2M3</td>
<td>1</td>
<td>Uniaxial</td>
<td>σ₁₂=5, σᵥ₂=1, α₂=60°, β₂=0°</td>
</tr>
<tr>
<td>S2M4</td>
<td>1</td>
<td>Biaxial</td>
<td>σₓ₂=5, σᵧ₂=2.5, σ₂=1, α₂=β₂=0°</td>
</tr>
</tbody>
</table>

A final remark before proceeding with error analysis, concerning our choices of examined d: One could in principle make the slabs arbitrarily thin or thick, which we have not tried (we only examined, in this error study, d=0.2m and d=2mm). The algorithm’s present design does not dictate a specific “optimal” value of d, unfortunately. What we can say however is that making d comparable to the local wavelength, on either side of the interface, is highly undesirable for at least two reasons. First, since the coating slabs are in fact reflective they can confine spurious guided-wave modes that may significantly corrupt computed sensor responses. Second, as indicated in Section 7.2.2, the thicker d is one must adaptively reduce the thicknesses of slab(s) intersecting each other, as well as slab(s) containing transmitters or receivers, to eliminate artificial discontinuities in the normal field components. This adaptive thickness reduction, which becomes increasingly frequent for finer spatial sampling (i.e., versus sensor depth) of the sensor response profiles, introduces yet another level
of arbitrariness into the algorithm. Namely, the desirable minimum space maintained
between the receiver and coating layer/ambient medium interfaces (and likewise for
the transmitter), as well as between any two ambient medium/coating slab junctions.
Although both examined values of \(d\) in this error study would not necessitate their
adaptive thinning, due to the three specifically examined sensor locations, the thicker
\(d = 0.2m\) slabs \textit{would} require adaptive thinning in the Section 7.4 results due to the
sensor’s depth being varied (0.1m sampling period) throughout the studied three-layer
formation profiles.

For the cross-polarized field plots, we only show errors for \(H_{xz}\) and \(H_{zx}\) since the
other cross-polarized field components \((H_{xy}, H_{yx}, H_{yz}, H_{zy})\) had zero magnitude to
within numerical noise. Elaborating: Our chosen threshold for numerical noise is that
either \(|H_{wq}^{TO}| \leq 10^{-12}\) and/or \(|H_{wq}^{Ref}| \leq 10^{-12}\) (-12 on \log_{10} scale), which is based on
the adaptive integration tolerance \((1.2 \times 10^{-14})\) and an educated guess \((10^2)\) of the
maximum magnitude of field components that would experience cancellation during
evaluation of the oscillatory Fourier double-integral (Eqns. (7.2.3)-(7.2.4)).\(^{135}\) In
passing, we mention that to suppress numerical cancellation-based noise due to using
finite-precision arithmetic, we “symmetrically” integrate. That is to say, we integrate
along the integration contour sub-sections \((a, b)\) and \((-b, -a)\), add these results, then
integrate along contour sub-sections \((b, c)\) and \((-c, -b)\), add these results and update
the accumulated contour integral result, and so on.

\(^{135}\)Although rigorous justification for the threshold \(10^{-12}\) is lacking, the conclusion of negligible
\(\{H_{xy}, H_{yx}, H_{yz}, H_{zy}\}\) is physically reasonable. Indeed one can verify (using Eqns. (7.2.6)-(7.2.7))
that T.O. media, used to effect modeling of \(xz\) plane tilting, only perturbs the ambient medium’s
response in the \(xz\) plane (but not \(y\) direction). Hence one can reason the T.O. media should not
induce spurious scattering of \(\{H_{xy}, H_{yx}, H_{yz}, H_{zy}\}\) responses if they were absent without the T.O.
slabs.
7.3.2 Results and Discussion

Figures 7.2-7.11 display errors (respectively) for the following field components: Re[\mathcal{H}_{xx}], \text{Im}[\mathcal{H}_{xx}], \text{Re}[\mathcal{H}_{yy}], \text{Im}[\mathcal{H}_{yy}], \text{Re}[\mathcal{H}_{zz}], \text{Im}[\mathcal{H}_{zz}], \text{Re}[\mathcal{H}_{xz}], \text{Im}[\mathcal{H}_{xz}], \text{Re}[\mathcal{H}_{zx}], \text{and Im}[\mathcal{H}_{zx}]. Let \Delta \epsilon denote the rate of change of error (Log_{10} \epsilon) versus Log_{10}|\alpha|: A slope of two (one) indicates quadratic (linear) error variation versus effective tilt.

We remark that in some plots, one or more material/tool-spacing scenario curves show strange “dips” in the error levels (Figs. 7.2a-7.2b, 7.5a, 7.6d, 7.6f, and 7.8a). Given how small the error dips are (typically \leq 1 \text{ DoP}), we ascribe the dips to a combination of machine-dependent computation and problem geometry-dependent numerical cancellation (effective tilt, material profile, \textit{d}, and sensor characteristics). There are also some “kinks” in the error behavior, at very small tilt, that can be observed in one or more material/tool-spacing scenario curves within Figures 7.2c, 7.2e-7.2f, 7.3e, 7.4c-7.4f, 7.5a, and 7.6c-7.6f. Despite these two sporadically-occurring characteristics in the error curves, the error trends are well preserved, and it is this we summarize in the following observations:

1. A two order of magnitude increase in \textit{d} (from 2mm [“d1” plots] to 200mm [“d2” plots]) produces low error variation (typically 0-1 DoP error increase).

2. Error is typically 1-3 DoP higher when the transmitter and receiver are in different layers ("O2" plots), versus when they are in the same layer ("O1" and "O3" plots). By contrast, the error is approximately equal if the transmitter and receiver are both either above or below the interface.

3. Transmitter-receiver spacing ("S1" vs. "S2" curves) has little effect on error levels (typically 0-1 DoP difference).
4. The M3 scenario curves typically show greatest error (versus M1, M2, and M4 curves) in the co-polarized results, but the lowest error in the cross-polarized plots. There does not appear to be any obvious, systematic trend in error variation between the M1, M2, and M4 cases across the studied sensor/environment parameter permutations.

5. The M1, M2, and M4 curves show quadratic error variation in the co-polarized results (their cross-polarized errors are catastrophically high), while the M3 curves predominantly show instead linear error variation for both co-polarized and cross-polarized results. For Figures 7.3c-7.3d, 7.5c-7.5d, and 7.7c-7.7d the M3 curves, interestingly, show quadratic error too however.

6. The cross-pol response errors not only are much higher than their co-pol response error counterparts, but the errors vary more versus \(d, L_s\), and sensor position \(D\) too.
Figure 7.2: Relative error in computing $H'_{xx} = \text{Re}[H_{xx}]$. 

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Figure 7.3: Relative error in computing $H''_{xx} = \text{Im}[H_{xx}].$
Figure 7.4: Relative error in computing $H'_{yy} = \text{Re}[H_{yy}]$. 

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Figure 7.5: Relative error in computing $H''_{yy} = \text{Im}[H_{yy}]$. 

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Figure 7.6: Relative error in computing $H'_{zz} = \text{Re}[H_{zz}]$. 
Figure 7.7: Relative error in computing $H''_{zz} = \text{Im}[H_{zz}]$. 

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Figure 7.8: Relative error in computing $H'_{xz} = \text{Re}[H_{xz}]$.
Figure 7.9: Relative error in computing $H''_{xz} = \text{Im}[H_{xz}]$. 

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Figure 7.10: Relative error in computing $H_{zx}'=\text{Re}[H_{zx}]$. 

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Figure 7.11: Relative error in computing $H''_{zx} = \text{Im}[H_{zx}]$. 

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7.4 Application to Triaxial Induction Sensor Responses

Now we perform case studies, to illustrate the algorithm’s flexibility in modeling media of diverse anisotropy and loss, involving twenty eight variations of a three-layered medium (seven tilt orientations, four central bed conductivity tensors), where the two interfaces exhibit (effective) relative tilting and are each flattened by two coating slabs \( d=2 \text{mm} \) thick (i.e., one coating slab immediately above, and one coating slab immediately below, each flattened interface). Figure 7.12 depicts the well-logging scenario simulated: A vertically-oriented triaxial induction tool [8, 13], that operates at \( f=100 \text{kHz} \), is transverse-centered at \((x, y) = (0, 0)\), possesses three (co-located) electrically small loop antenna transmitters (modeled as unit-amplitude Hertzian [equivalent] magnetic current dipoles) directed along \( x \) \((M^T_x)\), \( y \) \((M^T_y)\), and \( z \) \((M^T_z)\), and possesses three (co-located) loop antenna receivers \( \{M^R_x, M^R_y, M^R_z\} \) situated \( L_s=1.016 \text{m} \) (forty inches) above the transmitters.\(^{136}\)

All four material scenarios share the following common geological formation features prior to inserting the interface-flattening slabs: Top interface depth \( z'_1 = 2 \text{m} \) and bottom interface depth \( z'_2 = -2 \text{m} \),\(^{137}\) all three layers possess isotropic relative dielectric constant (i.e., excluding conductivity) \( \epsilon_{r1} = \epsilon_{r2} = \epsilon_{r3} = 1 \) and isotropic relative magnetic permeability \( \mu_{r1} = \mu_{r2} = \mu_{r3} = 1 \), layer one (top layer) possesses isotropic electric conductivity \( \sigma_1 = 50 \text{mS/m} \), layer three (bottom layer) possesses isotropic electric conductivity \( \sigma_3 = 20 \text{mS/m} \), the polar tilt angles of the two interfaces are equal to \( \alpha'_2 = -\alpha'_1 = \alpha' \geq 0 \), and the azimuth tilting orientation of the

\(^{136}\)Note 1: “T” superscript here denotes “transmitter”, \textit{not} non-Hermitian transpose.

\(^{137}\)For the equivalent problem with flattened interfaces, this choice of \( z'_1 \) and \( z'_2 \) results in the coating layer just above the bottom interface, and coating layer just below the top interface, having their respective coating layer/central formation layer junctions spaced 3.996m apart.
two interfaces are equal to $\beta_1' = \beta_2' = \beta'$. In other words, the relative tilting between the two interfaces is $2\alpha'$ degrees while we choose to tilt both interfaces with identical azimuth orientation (more generally, our algorithm allows for each interface to exhibit arbitrary azimuth orientation).

In Figures 7.13-7.16 below, the labels T1, T2, etc. in the legend represent induction log signature curves corresponding to different interface tilting scenarios, namely: \{\alpha' = 0^\circ, \beta' = 0^\circ\} (T1, solid black curve), \{\alpha' = 1^\circ, \beta' = 0^\circ\} (T2, solid blue), \{\alpha' = 1^\circ, \beta' = 45^\circ\} (T3, solid green), \{\alpha' = 1^\circ, \beta' = 90^\circ\} (T4, solid red), \{\alpha' = 3^\circ, \beta' = 0^\circ\} (T5, dash-dot blue), \{\alpha' = 3^\circ, \beta' = 45^\circ\} (T6, dash-dot green), \{\alpha' = 3^\circ, \beta' = 90^\circ\} (T7, dash-dot red). We chose the azimuth tilt orientations $\beta' = 0^\circ$ and $\beta' = 90^\circ$ specifically due to our having examined already, in the previous section, the error of co-polarized fields components when they are oriented either within or orthogonal to the plane of interface tilting. Moreover for a given $\alpha'$, the $\beta' = 45^\circ$ curves exhibit results intermediate to the $\beta' = 0^\circ$ and $\beta' = 90^\circ$ curves, suggesting confidence in the presented results.

(Material) Scenario 1: The highly resistive central layer (mimicking a hydrocarbon-bearing reservoir) possesses isotropic electric conductivity $\sigma_2 = 5\text{mS/m}$. Scenario 2: The central layer’s conductivity is characterized by a non-deviated uniaxial conductivity tensor $\bar{\sigma}_2 = \sigma_h^2 (\hat{x}\hat{x} + \hat{y}\hat{y}) + \sigma_v^2 \hat{z}\hat{z}$ with $\sigma_h^2 = 5\text{mS/m}$ and $\sigma_v^2 = 1\text{mS/m}$. Scenario 3: Same as Scenario 2, except the symmetric, non-diagonal conductivity tensor writes as $[9]$ 

$$\bar{\sigma}_2 = \begin{bmatrix}
\sigma_{xx2} & \sigma_{xy2} & \sigma_{xz2} \\
\sigma_{xy2} & \sigma_{yy2} & \sigma_{yz2} \\
\sigma_{xz2} & \sigma_{yz2} & \sigma_{zz2}
\end{bmatrix} \quad (7.4.1)$$
\[ \sigma_{xx2} = \sigma_{h2} + (\sigma_{v2} - \sigma_{h2})(\sin \alpha_2 \cos \beta_2)^2 \quad (7.4.2) \]
\[ \sigma_{xy2} = (\sigma_{v2} - \sigma_{h2})(\sin \alpha_2)^2 \sin \beta_2 \cos \beta_2 \quad (7.4.3) \]
\[ \sigma_{xz2} = (\sigma_{v2} - \sigma_{h2}) \sin \alpha_2 \cos \alpha_2 \cos \beta_2 \quad (7.4.4) \]
\[ \sigma_{yy2} = \sigma_{h2} + (\sigma_{v2} - \sigma_{h2})(\sin \alpha_2 \sin \beta_2)^2 \quad (7.4.5) \]
\[ \sigma_{yz2} = (\sigma_{v2} - \sigma_{h2}) \sin \alpha_2 \cos \alpha_2 \sin \beta_2 \quad (7.4.6) \]
\[ \sigma_{zz2} = \sigma_{v2} - (\sigma_{v2} - \sigma_{h2})(\sin \alpha_2)^2 \quad (7.4.7) \]

with tensor dip (\(\alpha_2\)) and strike (\(\beta_2\)) angles equal to \(\alpha_2 = 30^\circ\) and \(\beta_2 = 0^\circ\) (compared to Scenario 2, where \(\alpha_2 = \beta_2 = 0^\circ\)).

Scenario 4: The central layer has full (albeit non-deviated) biaxial anisotropy characterized by the conductivity tensor \(\bar{\sigma}_2 = \sigma_{x2} \hat{x}\hat{x} + \sigma_{y2} \hat{y}\hat{y} + \sigma_{z2} \hat{z}\hat{z}\), where \(\{\sigma_{x2} = 5, \sigma_{y2} = 20, \sigma_{z2} = 1\}\) [mS/m]. The Scenario definitions are summarized in Table 3, while the curve abbreviation definitions are summarized below in Tables 7.3-7.4. In these tables, the principal components of layer two’s conductivity tensor \(\bar{\sigma}_2\) are given in milli-Siemens per meter [mS/m]; material axis orientation angles \(\{\alpha_2, \beta_2\}\), provided with the principal conductivity values (when relevant), are given in degrees. Polar tilt angle of the interfaces is parametrized by angle \(\alpha'\) [deg], while the two interfaces’ common azimuthal (xy plane) tilt orientation angle is given by angle \(\beta'\) [deg] (\(\beta' = \beta'_1 = \beta'_2\)).

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Layer 2 Aniso.</th>
<th>(\bar{\sigma}_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Isotropic</td>
<td>(\sigma_{2}=5)</td>
</tr>
<tr>
<td>2</td>
<td>Uniaxial</td>
<td>(\sigma_{h2}=5, \sigma_{v2}=1, \alpha_2=0^\circ, \beta_2=0^\circ)</td>
</tr>
<tr>
<td>3</td>
<td>Uniaxial</td>
<td>(\sigma_{h2}=5, \sigma_{v2}=1, \alpha_2=30^\circ, \beta_2=0^\circ)</td>
</tr>
<tr>
<td>4</td>
<td>Biaxial</td>
<td>(\sigma_{x2}=5, \sigma_{y2}=20, \sigma_{z2}=1, \alpha_2=\beta_2=0^\circ)</td>
</tr>
</tbody>
</table>

\(^{138}\)Note: These conductivity tensor dip and strike angles, completely unrelated to the interface tilting angles \(\{\alpha'_m, \beta'_m\}\), describe the tensor’s polar and azimuthal tilting (resp.) but follow a different convention [9].
Table 7.4: Definitions of Curve Label Abbreviations

<table>
<thead>
<tr>
<th>Abbrev.</th>
<th>$\alpha'$</th>
<th>$\beta'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>T2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>T3</td>
<td>1</td>
<td>45</td>
</tr>
<tr>
<td>T4</td>
<td>1</td>
<td>90</td>
</tr>
<tr>
<td>T5</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>T6</td>
<td>3</td>
<td>45</td>
</tr>
<tr>
<td>T7</td>
<td>3</td>
<td>90</td>
</tr>
</tbody>
</table>

Now we discuss the co-polarized results; note that as the cross-polarized results cannot be reliably modeled nearly as well as the co-pol fields (see previous section), we omit them. Observing Figure 7.13:

1. The real part of all three co-polarized measurements has no visibly noticeable sensitivity to interface tilting, even when the sensor is near bedding junctions (this applies likewise for Figures 7.14-7.16; hence for brevity, we omit these results). The imaginary part of these measurements *does*, by contrast, exhibit tilting sensitivity.

2. The imaginary parts’ sensitivity to tilting depends on the sensor position, with sensitivity being largest when the sensor is near the interfaces ($\pm 2m$). This is expected, since the conductive formation exponentially attenuates fields scattered and propagating away from the interfaces.

3. The 4m bed thickness and sensor frequency (100kHz) preclude observation of inter-junction coupling effects due to wave “multi-bounce” within the slab layers [45][Ch. 2].
4. At a fixed polar interface tilting (T2, T3, and T4 $[\pm 1^\circ]$ versus T5, T6, and T7 $[\pm 3^\circ]$), $\text{Im}[H_{x'x'}]$ shows no visible sensitivity to the interface’s azimuth orientation. This lack of azimuthal sensitivity is quite sensible, given the azimuthal symmetry of this $z$-transmit/$z$-receive measurement [11].

5. In contrast to $\text{Im}[H_{x'x'}]$, $\text{Im}[H_{y'y'}]$ and $\text{Im}[H_{y'y'}]$ do show azimuthal sensitivity. For fixed polar tilt, both measurements show greatest excursion (i.e., away from the black, zero-tilt curve) when the interfaces’ common azimuth tilt orientation is aligned with the transmitter and receiver orientation ($0^\circ$ [blue curves] and $90^\circ$ [red curves] for $\text{Im}[H_{x'x'}]$ and $\text{Im}[H_{y'y'}]$, respectively). On the other hand, minimal excursion occurs when the azimuth tilt orientation is orthogonal to the transmitter and receiver orientation ($90^\circ$ and $0^\circ$ for $\text{Im}[H_{x'x'}]$ and $\text{Im}[H_{y'y'}]$, respectively).

6. Tilting, irrespective of the polar orientation’s sign (e.g., $1^\circ$ versus $-1^\circ$ polar tilt), results in the responses $\text{Im}[H_{x'x'}]$ and $\text{Im}[H_{y'y'}]$ having downward excursions. Indeed for a fixed azimuth tilt (curve color), observe the sensor response near the two (effectively) oppositely-tilted interfaces for both the solid and dash-dot curves.

7. In contrast to $\text{Im}[H_{x'x'}]$ and $\text{Im}[H_{y'y'}]$ exhibiting downward excursions, $\text{Im}[H_{z'z'}]$ always shows an upward excursion irrespective of polar tilt sign.

The above observations also visibly manifest for the three remaining cases involving anisotropic media. We find, for the small tilting range explored here at least (zero to six degrees of relative tilt), that the anisotropy primarily serves to alter “baseline” (zero-tilt) sensor responses which are then perturbed by the effect of tilt.
Figure 7.12: Original geometry (Fig. 7.12a) and transformed, approximately equivalent geometry (Fig. 7.12b) employed below. For clarity of illustration, the layers are shown tilted within the $xz$ plane ($\beta' = 0^\circ$).
Figure 7.13: Imaginary part of co-polarized, complex-valued received magnetic field phasors: Material Scenario 1.

Figure 7.14: Imaginary part of co-polarized, complex-valued received magnetic field phasors: Material Scenario 2.
Figure 7.15: Imaginary part of co-polarized, complex-valued received magnetic field phasors: Material Scenario 3.

Figure 7.16: Imaginary part of co-polarized, complex-valued received magnetic field phasors: Material Scenario 4.
7.5 Conclusion

We have introduced and profiled (both quantitatively and qualitatively) a methodology, which augments robust full-wave numerical pseudo-analytical algorithms admitting formation layers of general anisotropy and loss, to incorporate the effects of planar interface tilting. Our proposed methodology, directed at extending the range of applicability of such eigenfunction expansion methods by relaxing the traditional constraint of parallelism between layers, consists of first defining a spatial coordinate transformation within a thin planar region surrounding each interface to be “flattened”. Such coordinate transformation locally distorts the EM field, effectively altering the local angle of incidence between EM waves and flat interfaces so as to mimic the presence of interfaces possessing effective, independently-defined tilt orientations. During the second stage of flattening the interfaces, the coordinate transformation is incorporated into the EM material properties of said planar regions via application of T.O. principles, which exploits the well-known “duality” between spatial coordinate transformations and equivalent material properties “implementing” these coordinate transformations in flat space.

As the proposed methodology is not limited by the loss and anisotropy properties of any layer, some combination of beds such as non-deviated sand-shale laminates, cross-bedded clean-sand depositions, formations fractured by invasive drilling processes, and simpler isotropic conductive beds can all be included along with flexibly-defined (effective) bed junction tilting with respect to deviation magnitude as well as polar and azimuth orientation. Exhibiting representative examples of the new methodology’s said flexibilities, we applied it to demonstrating the expected qualitative properties of multi-component induction tool responses when planar bedding
deviation is present. This being said, one should not confuse modeling flexibility and manifestation of expected qualitative trends with \textit{quantitative} accuracy. Indeed, although T.O. theory informs us that said interface-flattening media facilitate numerical prediction of tilted-interface effects via employing specially-designed coating slabs, the necessary \textit{orientation} of the slabs’ truncation surfaces (i.e., planes parallel to the \textit{xy} plane) predicts that spurious scattering will arise. Particularly, our numerical error analysis demonstrated that artificial scattering typically scales quadratically with effective interface tilt. We found that one could nonetheless safely model tilting effects so long as the magnitude of each interface’s polar deviation is kept small, which guided the choice of explored interface tilt ranges when examining qualitative trends of sensor responses produced by induction instruments within tilted geophysical layers.
Chapter 8: Future Work

Major tracks for future work can be summarized as follows:

Test the sub-algorithm, developed for estimating the region within the complex plane where “critical points” (branch points, branch cuts, and poles) lie, for other types of media with respect to loss and anisotropy. Although validation results were presented for layered structures involving isotropic, uniaxial, and biaxial layers, which strongly suggest hence the critical point region-estimator’s robustness for all classes of anisotropic media, all of the media studied were reciprocal, non-bianisotropic (i.e., no coupling between electric and magnetic responses), and lacking material tensor eigenvalues with negative imaginary part (i.e., no gain media, nor even Perfectly Matched Layer [PML] absorber media that possess a gain response along the longitudinal $[z]$ direction). For example, studies involving electrically gyrotropic (e.g., ionosphere subject to a DC magnetic field), magnetically gyrotropic (e.g., ferrite microwave substrates subject to a DC magnetic field bias), and/or bi-anisotropic media would be relevant when understanding the spectral-domain algorithm’s reliability and computation speed when applied to areas where said medium classes frequently arise. Moreover, to avoid false alarms, an automated means (i.e., not requiring added user input) to detect absorptive media that possessing active-medium tensor eigenvalues (e.g., PML media) would be highly desirable. We remark that the proposed contour
deformations in this thesis, which were designed under the implicit assumption of absence of gain media, as well as media simultaneously lacking magnetic and electric tensors with one or more eigenvalues possessing negative real part, can be easily adjusted if such media arise (i.e., deformed into the lower-half plane within the contour’s subsection capturing the radiation, or far-field, spectrum field contribution).

Apply path deformation, and critical point region estimation, techniques proposed herein to extend the algorithm for time-domain simulation capability. Devising an appropriate integration order that prevents undesirable critical point migration (either in the $k_x$, $k_y$, and/or complex frequency $[\omega]$ plane) would also have to be addressed (c.f. Ch. 3).

We can safely argue, at this stage, that the near-field contribution computation (integration along the semi-infinite Fourier integral “tails”) now occupies only a small fraction of the total computation time, whereas without the proposed complex-plane Gauss-Laguerre Quadrature technique, this part of the computation could easily consume well over ninety percent of the computation time (depending on the desired computation accuracy) for nearly horizontally-oriented sensors (i.e., transmitted and receive depths are, or are nearly, identical). With the far-field computation now being, consistently, the computation time “bottle-neck”, we suggest adjusting the contour deformation in the left-half complex plane. Particularly, if no double-negative or active media are present, one can deform the entire left half-plane path onto a straight (in general diagonal) path bent from the origin by angle $\tan(\Delta x/\Delta z) = \tan(\Delta y/\Delta z)$. This would further accelerate computation by (approximately) an additional factor of two (for one-dimensional Sommerfeld and Fourier-Hankel transforms), and a factor
of four computation speed acceleration for the two-dimensional plane wave (Weyl) transforms we explored within this thesis.

Implement the Fourier integral computation using the Fast Fourier Transform (FFT), as applied to a suitably-defined, real-valued integration variable (and real-axis integration path) with respect to which the deformed contours in the spectral-variable complex planes are defined. This implementation, we conjecture, would result in significant computation savings. This is particularly important for simulating EM field radiation at higher and higher frequencies, and/or performing time-domain simulations (requiring the evaluation of a triple, rather than double, spectral-domain integral). Indeed, recalling (c.f. previous paragraph) that the far-field computation is now the primary computation time “bottle-neck”, and that (c.f. Chs. 1,3) the trapezoidal detour length, along which one integrates to capture the far field contribution, is linearly proportional to the radiation frequency, computation speed of the frequency domain field, using the double integral plane wave expansion, scales \textit{quadratically} with frequency. Although not problematic for our particular application area (modeling sub-2MHz subsurface and sub-oceanic geophysics sensors), this quadratic computation time scaling poses a major practical constraint concerning the present formulation’s application to modeling microwave, millimeter, terahertz, optical, etc. wave propagation in anisotropic media (e.g., propagation through gyrotropic ionospheric media and anisotropic optical crystals). The major potential drawback of using a FFT-based implementation, however, is the lack of rigorous error-checking of the integral evaluation. Indeed at present, we utilize multi-level $hp$ refinement to check and bound\textsuperscript{139} the error of the far-field computation, and $p$ refinement to check and

\textsuperscript{139}Within constraints set by the chosen number of Gauss-Legendre Quadrature rules, and number of chosen tiers of integration path length sub-division.
bound (within the number and order of Gauss-Laguerre quadrature rules) the near-field computation error. Extensive studies should be performed, across different sensor and environment scenarios (radiation frequency, number/thickness/anisotropy/loss of layers, sensor spacing and orientation, etc.), to gauge expected numerical precision degradation.
Appendix A: Interferometric SAR Coherence Arising from the Co-Polarized Electromagnetic Interrogation of Layered, Penetrable Dielectric Media

A.1 Introduction

Wide-coverage, high-resolution DEM generation and ice pack characterization are just two of the many applications of the InSAR technique, which involves extracting interferometric information (namely, interferometric coherence’s magnitude and phase) from two or more coherent microwave observations of the same scene (c.f. Fig. 1) [115]. A non-zero perpendicular baseline $B_\perp$ between the observations renders the interferometric observables sensitive to many geophysical parameters of interest, such as terrain height and vegetation parameters, but when viewing spatially extensive targets increasing $B_\perp$ also renders increasingly independent the observations (“spatial baseline decorrelation”) [116]. The observational independence of each SAR resolution cell is quantified by the pixel’s coherence parameter $\rho$, with decreasing correlation $|\rho|$ indicating greater observational independence, speckle phase noise, and error in retrieved geophysical parameters.

Radar wave ground penetration too may exacerbate decorrelation and/or bias the coherence’s phase, affecting measurements such as terrain height [117]. Penetration
and subsurface scatter contributions from deeper layers can often be ignored when the shallower layers are highly attenuating, however for low-loss media this is not necessarily the case. Unsurprisingly then, radar penetration has historically attracted much research in relation to remote sensing of diverse terrain (snow, regolith, volcanic rock, Earthen soil, natural vegetation, etc.) and for both non-terrestrial bodies (e.g., Mercury, Mars, Venus, Europa, Earth’s moon) and the Earth alike [118, 119, 120, 121]. Indeed polarization-dependent ground penetration and subsurface scatter have often been attributed, for example, as dominant contributors in observed backscatter enhancement, elevation in the backscatter echo’s linearly-polarized component when transmitting circularly polarized (CP) waves, and increase in CP ratio [119, 120, 121]. Wave penetration moreover also has important implications in the quality of interferometric data, and must be considered when designing and deploying terrain-robust instruments. As we show, this is because highly penetrating radar waves can excite subsurface scatter mechanisms with (effectively, deeply buried) scattering phase centers, which by increasing the subsurface’s effective longitudinal extent can render unreliable the results of geophysical parameter retrieval processes employing InSAR observables [117].

Modeling effects of subsurface penetration, including their modulation from the presence of terrain topography and antenna pointing errors, on InSAR coherence would significantly extend the types of coherence trends that can be analyzed for purposes of mission planning and robust geophysical parameter retrieval [122]. Indeed past InSAR coherence models have captured (one or more of) its following contributions: surface baseline decorrelation (i.e., neglecting ground penetration), subsurface
scatter (albeit ignoring multi-bounce), terrain topography, and azimuth viewing deviation [122, 115, 118, 123, 124]. However, to the best of our knowledge a systematic incorporation of multi-bounce-related backscatter enhancement, and its effects on InSAR coherence degradation, has not been performed until recently [19, 125]. Extending upon [19], which assumed only three layers and flat terrain, we also mathematically model sloped terrain and an arbitrary number of subsurface layers.

In Sections 2 and 3 we derive, validate, and elucidate novel, key physical insights gleaned from our proposed InSAR coherence model, while Section 4 contains our concluding remarks. We adopt and suppress the \( \exp(-i\omega t) \) convention, wherein positive imaginary part of the refractive index and wave number corresponds to coherent EM wave attenuation as embedded within the subsurface’s differential Scattering Cross Section (dSCS) profile.\(^{140}\)

\(^{140}\)We neglect coherence effects due to temporal scene variation, pixel misregistration, thermal noise, atmospheric propagation, and windowing and dielectric dispersion-related SAR Point Spread Function (PSF) modification [122, 115, 116, 123]. Moreover, we assume that: (i) The scattering processes contributing to the coherence are independent, and (ii) The antennas’ angular separation is sufficiently small that if one antenna’s emitted wave interacts with a differential scattering element, then so does the other antenna’s wave (and that too identically, up to a path length difference).
Figure A.1: $\epsilon_{rm} < n_m^2 >$, and $L_{vm}$ are the average dielectric constant, refractive index variance ($n_m = n_m' + in_m''$), and refractive index correlation length characterizing layer $m$’s dielectric fluctuation statistics ($m = 1, 2, ..., N$). $\sigma^2_{hm}$ and $L_m$ are the height roughness variance and correlation length (resp.) characterizing the statistics of layer $m$’s bottom interface at depth $-d_m$. $r_p$ is the range from antenna $p$ to some arbitrarily chosen SAR image pixel’s reference location $r_0 = (0, 0, 0)$. Note: Our model allows flexible prescription of statistics (Gaussian, exponential, etc.) for each volume and rough surface. Figure/caption copied from [19].

A.2 Formulation

Observing Fig. A.1, the platform travels along direction $\hat{y}$ and sets as $\hat{x}$ the ground range direction. Using variable conventions similar to [115], two radar antennas ($A1, A2$) in the top layer view the ground scene simultaneously, with the $p$th ($p = 1, 2$) antenna viewing layer $m$ at polar angle $\theta_{Ap,m}$ and azimuth angle $\phi_{Ap,m}$. Note that $\theta_m = (\theta_{A1,m} + \theta_{A2,m})/2$, $\phi_m = (\phi_{A1,m} + \phi_{A2,m})/2$, $\delta\theta_m = \theta_{A1,m} - \theta_{A2,m}$, $\delta\phi_m = \phi_{A1,m} - \phi_{A2,m}$, and $\{|\phi_m|, |\delta\phi_m|\}$ are assumed small.

First, given terrain backscatter function $f^{vv}(r)$, layer $m$ mean wave number $k_m = (\omega/c)\sqrt{<\epsilon_{rm}>}$, and the SAR image PSF $W(r) = W(x, y, z)$ (c.f. Eqn. (A.2.3)) near
$r_0$, define that pixel’s value $s_p(\theta_{Ap,1}, \phi_{Ap,1}, r_0)$:

$$
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{vv}(r) W(r) e^{i2k_1 r} e^{i2k_{zp} x + i2k_{yy} y - i2k_{zp} z} dr
$$

(A.2.1)

Second, changing variables $(x, y, z) \to (x, y, w = z - x \tan \delta_x - y \tan \delta_y)$ and assuming a “white noise”-like scatterer distribution $< f_{vv}(r) f_{vv}(r') > = \sigma_0^{vv}(w) \delta(r - r')$ [122], define the non-normalized $vv$ coherence $d_{vv} \rho_{vv} = < s_1 s_2^* >$:

$$
e^{i2k_1 (r_1 - r_2)} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sigma_0^{vv}(w) e^{i2k_{xx} x + i2k_{yy} y + i2k_{ww} w} |W(r)|^2 dx dy dw
$$

(A.2.2)

Third, given free-space SAR range resolution $R_r$ and azimuth resolution $R_a$, and defining $c_x = \sin \theta_m - \cos \theta_m \tan \delta_x$, $c_y = \cos \theta_m \tan \delta_y/c_x$, and $c_w = \cos \theta_m/c_x$ plus layer $m$’s PSF:

$$
sinc [c_x n'_m (x - c_y y - c_w w)/R_r] \times sinc [n'_m y/R_a],
$$

(A.2.3)

perform the $x$ and $y$ integrations for the $N$-layer $< s_1 s_2^* >$ expression (note: $d_N \to +\infty$ and $d_1 = 0$):

$$
e^{i2k_1 (r_1 - r_2)} \lim_{\zeta \to 0^+} \sum_{m=2}^{N} -d_{m-1} + \zeta \int_{-d_m + \zeta}^{-d_m - \zeta} \tilde{T}_m \sigma_0^{vv}(w) e^{i2k_{ww} w} dw
$$

(A.2.4)

Finally, for some generalized antenna angular separation $(\tilde{k}_x, \tilde{k}_y, \tilde{k}_w)$ at which Eqn. (A.2.4) is evaluated, normalize this result by the corresponding Eqn. (A.2.4) result $d_{vv}$ for two co-located antennas (i.e., $\tilde{k}_x = \tilde{k}_y = \tilde{k}_w = 0$) to obtain the $vv$ coherence $\rho_{vv} = |\rho_{vv}| \exp(i\phi_{vv})$ [117, 115]. We recognize $|\rho_{vv}|$ as the coherence magnitude (“correlation”) and $\phi_{vv}$ as the coherence phase for transmitting and receiving $v$-polarized

\footnote{141} For small terrain tilt angles $\{\delta_x, \delta_y\}$, $k_{zp} \approx k_1 \sin \theta_{Ap,1} \cos \phi_{Ap,1}$, $k_{yp} \approx k_1 \sin \theta_{Ap,1} \sin \phi_{Ap,1}$, and $k_{zp} \approx k'_m \cos \theta_{Ap,m}$.

\footnote{142} $\tilde{k}_x \approx k_1 \delta \theta_1 \cos \theta_1 + k'_m \delta \theta_m \tan \delta_x \sin \theta_m$, $\tilde{k}_y \approx k_1 \delta \phi_1 \sin \theta_1 + k'_m \delta \theta_m \tan \delta_y \sin \theta_m$, and $\tilde{k}_w \approx k'_m \delta \theta_m \sin \theta_m$.

\footnote{143} $\tilde{k}_y = c_y \tilde{k}_x$, $\tilde{k}_w = \tilde{k}_w + c_w \tilde{k}_x$, $\text{Tri}(x) = 1 - |x|$, and $\tilde{T}_m = (R_r/n'_m)(R_a/n'_m)|\text{Tri}(2R_r k_x/[2\pi n'_m c_x])|\text{Tri}(2R_a k_y/[2\pi n'_m])|c_x$.}

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waves. Interpreting Eqns. (A.2.2) and (A.2.4) as spatial Fourier transforms (FT) moreover, one recognizes that increasing resolution cell (RC) volume (i.e., larger $R_r$, $R_a$, and/or ground penetration) leads to the radar observing the coherent interference from an increasing number of scatterers and hence more rapid variation (vs. $\{\delta \theta_1, \delta \phi_1\}$) of the RC’s speckle pattern [123].

Now consider the following dSCS profile for $N = 3$ layers:

$$\sigma_{vv}(w) + \sigma_{t_{12}}^{vv} \sigma_{t_{21}}^{vv} Q_2 \left[ e^{-2\tau_2 w} \sigma_{vol,2}^{vv} + e^{2\tau_2 d_2} \sigma_{b,23}^{vv} \delta(w + d_2) + e^{2\tau_2 d_2} \sigma_{t_{23}}^{vv} \sigma_{t_{32}}^{vv} e^{-2\tau_3 (w + d_2)} \sigma_{vol,3}^{vv} \right]$$

(A.2.5)

which accounts for rough surface backscatter $\{\sigma_{bb,mm'}^{vv}(w + d_m)\}$, volume backscatter $\{\sigma_{vol,m}^{vv}\}$, and multi-bounce within slab layers $\{Q_m\}$. In particular under said “white noise” assumption, the ensemble averaging process results in the following ($t_m = d_m - d_{m-1}$ and $Q_N = 0$) [45][Ch. 2]:

$$Q_m = \left[ 1 - \sigma_{r,m(m-1)}^{vv} \sigma_{r,m(m+1)}^{vv} e^{2\tau_m t_m} e^{-2i\bar{k}_w t_m} \right]^{-1} = \sum_{g=0}^{\infty} \left( \sigma_{r,m(m-1)}^{vv} \sigma_{r,m(m+1)}^{vv} e^{2\tau_m t_m} e^{-2i\bar{k}_w t_m} \right)^g$$

(A.2.6)

where the $w$ integration of the dSCS profile will impart the appropriate phase shift for the “direct” (i.e., no additional two-way round trips within the slab) surface and volume backscatter terms, while on the other hand the factor $Q_m$ has conveniently “embedded” within it the added phase shift and attenuation for each of the (infinite number of) backscatter contributions based on the number of added two-way trips.

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144 (Copied from [19]) $-\tau_m \geq 0$ is the one-way depth power attenuation coefficient in layer $m$. $\sigma_{bb,mm'}^{vv}$, $\sigma_{r,mm'}^{vv}$, and $\sigma_{t,mm'}^{vv}$ are (resp.) the backscatter, specular reflection, and specular transmission dSCS of the interface between layers $m$ and $m'$ ($m' = m + 1$ or $m - 1$) for a plane wave incident in layer $m$. Said four quantities are evaluated at the wave’s local incidence angle (LIA) $\{\theta_{Ap,m}, \phi_{Ap,m}\}$ relative to the interface. $\sigma_{vol,m}^{vv}$ is the layer $m$ volume backscatter dSCS, while $Q_m$ (Eqn. (A.2.6)) embodies multi-bounce.
each contribution makes [19] (elaborated upon below). We also require the expression
\( \bar{Q}_m \), devoid of phase shift factors (i.e., the \( \delta\theta = \delta\phi = 0^\circ \) result), when calculating \( \rho_{vv} \):
\[
\frac{1}{1 - \sigma_{r,m(m-1)}^{vw} \sigma_{r,m(m+1)}^{vw} e^{2\tau_m t_m}}.
\]

Observing Eqn. (A.2.6) and illustrated below, when the attenuation terms are small (\( |\tau_m| \ll 1 \)) multi-bounce can significantly enhance backscatter, coherence phase bias, and decorrelation via introducing strong, (effectively) deeply-buried scatterers [45][Ch. 2]. In fact when at certain LIA’s and angular separations
\[
\sigma_{r,m(m-1)}^{vw} \sigma_{r,m(m+1)}^{vw} e^{-2ikw t_m} \approx 1 \quad (A.2.7)
\]
too, we conjecture that a resonance-like condition manifests wherein the \( m \)th slab transforms (guides) the wave field (particularly, its components contributing to the InSAR observation) into a cylindrical (vs. spherical) wave that experiences only \( 1/r \) (vs. \( 1/r^2 \)) geometrical power spreading [45][Ch. 2]. This could potentially result in very strong InSAR backscatter returns originating from terrain much farther away in the \( xy \) plane (effectively deeper, however) from the radar than normally expected, dramatically increasing (effectively) the SAR resolution cell size and hence rendering virtually absolute the interferometric decorrelation even for extremely small InSAR baselines [19].

We should stress, however, that said conjectured “interferometric” resonance is distinct from standard guided mode-related resonances. Indeed, both resonances have the same physical origin (low attenuation and high dielectric contrast slabs) and are mathematically characterized by geometrical optics (GO)-like series [45][Ch. 2]. Our independent-scattering assumption however, which led to compact GO series representations \( \{Q_m\} \) for the ensemble-averaged power, stripped away from the GO series field amplitude expressions any constructive “self-interference” that can lead to
anomalously bright ground returns for any single mono-static scene observation [19]. Furthermore, while an incident wave’s $k_w$ dictates guided mode excitation [45][Ch. 2], $\bar{k}_w$ (dictating interferometric resonance excitation) depends not only on mean LIA but also the antennas’ angular separation ($\delta \theta$), with larger $|\delta \theta|$ and appropriate mean LIA required for observing interferometric resonances. Finally, the standard guided modes (starting with the fundamental modes) exhibit monotonically non-decreasing $k_w$ ($\propto \cos \theta$) corresponding to monotonically non-increasing LIA [45][Ch. 2], while $\bar{k}_w$’s dependence on mean LIA ($\propto 1/\sin \theta$) suggests a similar, albeit more involved, non-decreasing monotonic trend (i.e., versus mode dominance) as the standard modes.

How does one extend the coherence model for $N > 3$? Simply modify the dSCS profile. For example, letting $D_g = \sigma_{t,(g-1)g}^{vv} \sigma_{t,g(g-1)}^{vv} \exp(2\tau_g t_g)Q_g$, backscatter from the $m$th interface (for $m > 2$) writes as

$$\sigma_{b,m(m+1)}^{vv} \delta(z + d_m) \prod_{g=2}^{m} D_g \tag{A.2.8}$$

while the layer $m$ volume scatter expression writes as

$$\sigma_{t,(m-1)m}^{vv} \sigma_{t,m(m-1)}^{vv} e^{-2\tau_m(z+d_{m-1})} Q_m \sigma_{vol,m} \prod_{g=2}^{m-1} D_g \tag{A.2.9}$$

A.3 Validation and Discussion

Given the greater complexity of our model as compared to previous ones, we demonstrate that our model reduces to standard ones under simplifying assumptions. First consider a tilted, $N$-layered medium (albeit ignoring refraction effects) with

\footnote{The product form $\prod D_g$ is an approximation, with $D_g$ neglecting some backscatter contributions from the $g$th slab’s multi-bounce fields. Particularly, those multi-bounce contributions which on more than one occasion transmit through the slab’s lower surface, undergo scattering, and then re-enter the slab to undergo further internal reflections prior to finally being backscattered and propagated to the radar.}
a perfectly side-looking radar ($\{n_m, k_m, \theta_m, \phi_m, \delta\theta_m, \delta\phi_m\} = \{1, k, \theta, 0^\circ, \delta\theta, 0^\circ\}$); we compare Eqn. (A.2.4) with Eqns. (13)-(18) of Rogridguez and Martin’s (RM) work [122].\(^{146}\) Recalling footnote A.4, simple algebra demonstrates that $\tilde{k}_x/(2\pi c_x)$, $\tilde{k}_w$, and $\tilde{k}_y/(2\pi)$ match RM’s Eqns. (15)-(17):

\[
\tilde{k}_x/(2\pi c_x) = k'_m \delta\theta_m/(2\pi \tan [\theta_m - \delta_x]) \quad (A.3.1)
\]
\[
\tilde{k}_w = k'_m \delta\theta_m \cos \delta_x / \sin (\theta_m - \delta_x) \quad (A.3.2)
\]
\[
\tilde{k}_y/(2\pi) = \tilde{k}_w \tan \delta_y/(2\pi) \quad (A.3.3)
\]

Moreover, evaluating our $\{\tilde{k}_x, \tilde{k}_w, \tilde{k}_y\}$ expressions at $\{\delta_x = \delta_y = 0^\circ, R_a = 0, N = 2\}$ yields Eqn. (12) in [115]. Now we verify computation of Eqn. (A.2.4) for a homogeneous, non-tilted subsurface, comparing our result with Dall’s Eqn. (9) [117]:\(^{147}\)

\[
\int_{-\infty}^{0} e^{2k'_w w / \cos \theta_2} e^{2ik_w w} dw
\int_{-\infty}^{0} e^{2k'_w w / \cos \theta_2} dw = 1/(1 + ik_w \cos \theta_2/[2k'_w]) \quad (A.3.4)
\]

**A.4 Analytical Coherence Results: Phase and Magnitude in the Strong-Guidance Regime**

Let us now explore our model’s predicted volume coherence result using a more complex, three-layer dSCS profile. The magnitude (correlation), we will show, is inversely proportional to $B_\perp$ while the layer depth-normalized coherence phase bias linearly diverges versus the number of considered multi-bounce contributions in the

\(^{146}\) Interchange our and RM’s ground range and azimuth coordinates, as well as connect our $\exp(i2k[r_1-r_2])$ with RM’s $\exp(-i\Delta)$, our $\delta\theta$ with RM’s $B_\perp/r_0$ (RM’s $r_0$ equals our $r_1$), and our $d_{vv}\rho_{vv}$ with RM’s Eqn. (13).

\(^{147}\) We equate our $2\tilde{k}_w$ with Dall’s $k_z$, equate our second layer’s one-way power attenuation factor $-\tau_2 = 2k'_2 / \cos \theta_2$ with the inverse of Dall’s one-way power penetration depth, let our subsurface dSCS profile equal $\sigma_{vv}^w (w) = e^{-2\tau_2 w} \sigma_{vv}^\text{vol,2}$, and artificially suppress the multiplicative surface coherence term $\tilde{T}_m \exp(i2k_1[r_1-r_2])$. 

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limit of zero attenuation and perfectly reflecting dielectric slab walls (“guidance” or “strong-guidance” regime) [19, 125]. To facilitate manifestation of analytical results, we examine the following reduced dSCS profile:

$$\tilde{\sigma}_0^{vv}(w) = \sigma_{b,12}^{vv}\delta(w) + \sigma_{t,12}^{vv}\sigma_{t,21}^{vv}e^{2\tau_2d_2}Q_2\sigma_{b,23}^{vv}\delta(w + d_2)$$  \hfill (A.4.1)

**A.4.1 Phase Result**

Evaluating the dSCS profile’s FT \((\sigma_{b,12}^{vv} + B_0e^{-2i\tilde{k}_wd_2}Q_2)\) and normalizing this by the zero-baseline result, one can show the following approximation for coherence phase results [125]:$^{148}$

$$\arg(\rho_{vv}^{\text{vol}}) \approx \lim_{G \to \infty} \sum_{g=0}^{G} F_g[-2\tilde{k}_wd_2][1 + g]$$  \hfill (A.4.2)

Observing Eqn. (A.4.2), that multi-bounce can significantly corrupt coherence phase by introducing an unbounded series of phase shifts from backscatter contributions that underwent successively increasing numbers of round-trips within the slab. Quantitatively, the echo which underwent \(g\) round-trips contributes (albeit “weighted” by its relative intensity \(F_g\)) an interferometric phase bias of \([-2\tilde{k}_wd_2 - 2g\tilde{k}_wd_2]\); in the guidance limit, the coherence phase in fact would linearly diverge. This can be more clearly understood by viewing the factor \(Q_2\) as introducing, as commented upon earlier, virtual scattering interfaces effectively buried at successively deeper interfaces. Indeed, one can verify that replacing Eqn. (A.4.1) with the following dSCS profile

$$\sigma_{b,12}^{vv}\delta(w) + B_0\sum_{g=0}^{\infty} A_g\delta(w + d_2[g + 1])$$  \hfill (A.4.3)

$^{148}B_0 = \sigma_{t,12}^{vv}\sigma_{t,21}^{vv}e^{2\tau_2d_2}\sigma_{b,23}^{vv}, A_g = (\sigma_{r,21}\sigma_{r,23}e^{2\tau_2d_2})^g, F_g = B_0A_g/(\sigma_{b,12}^{vv} + B_0\sum_{g=0}^{G} A_g), and assume |\tilde{k}_w| \ll 1.$

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which consists of a semi-infinite number of interface “images” periodically spaced apart by $d_2$, yields the same InSAR coherence expression as $\tilde{\sigma}_{0}^{uv}(w)$. Incidentally, this equivalence motivated previously our definitions for $Q_m$ and $\bar{Q}_m$. Such a result, unambiguously connecting the slab’s guidance properties to its corruption of interferometric phase, in retrospect is expected considering the analogous (vs. $\{Q_m, \bar{Q}_m, D_m\}$) GO series representations describing EM scattering from planar-layered structures [45][Ch. 2].

Elaborating upon discussion earlier: If attenuation within the dielectric slab is high (large $\tau_2$), then subsurface backscatter contributions will exponentially decay as $\exp(2[g + 1]\tau_2 d_2)$ and negligibly corrupt the InSAR coherence. Similarly, if the slab’s dielectric contrast versus its neighboring layers is low (i.e., $\sigma_{r,21}^{uv} \sigma_{r,23}^{uv} \ll 1$), these terms additionally experience polynomial decay as $(\sigma_{r,21}^{uv} \sigma_{r,23}^{uv})^g$. On the other hand, if the slab both weakly attenuates and strongly confines EM waves (guidance), the higher order terms decay slowly, and in fact the GO series $Q_m$ may even diverge.\textsuperscript{149}

Finally, although a near-grazing LIA wave, and hence potential manifestation of quasi-guidance limit scenarios, appears unrealistic for standard InSAR systems, at least two effects can invalidate this assumption. First, terrain sloping away from the radar can induce near-grazing LIA. Second, based on resonance properties of dielectric slabs [45][Ch. 2], we conjecture that as the slab’s electrical thickness increases, waves at mean LIA’s farther from grazing LIA can both encounter quasi-guidance limit environments as well as excite interferometric resonances.

\textsuperscript{149}Practically, diffuse scattering and attenuation preclude manifestation of the guidance limit and associated series divergence.
A.4.2 Correlation Result

The correlation magnitude approximately exhibits behavior inversely proportional to the antenna angular separation $\delta \theta$ (or equivalently, to the perpendicular InSAR baseline $B_\perp$). That this relationship exists can be seen as follows. First observe, in this “guidance limit”, that $\tilde{\sigma}_0^{vv}(z)$ writes as an infinite train of Dirac delta functions (Dirac “comb”) $D_0 \sum_{g=-\infty}^{\infty} \delta(z + gd_2)$ multiplicatively truncated by a Heaviside step function $u(-z)$ followed by convolution with $\delta(z + d_2)$ (i.e., the real bottom interface and its virtual copies exist only within the region $z \leq -d_2$). Second, observe that the ground penetration-related coherence $\rho_{vv}^{\text{vol}}$ associated with $\tilde{\sigma}_0^{vv}(z)$ writes (up to normalization) as

$$\rho_{vv}^{\text{vol}} \propto \int_{-\infty}^{-d_2+z} \tilde{\sigma}_0^{vv}(z)e^{i2\bar{k}_z z} dz,$$  \hspace{0.5cm} (A.4.4)

that the FT $\mathcal{F}[f(z)g(z)]$ writes as $\mathcal{F}[f(z)] \ast \mathcal{F}[g(z)]/(2\pi)$, $\mathcal{F}[u(-z)] = \pi \delta(-2\bar{k}_z) + 1/(i2\bar{k}_z)$, and

$$\mathcal{F} \left[ \sum_{g=-\infty}^{\infty} \delta(z + gd_2) \right] = \frac{2\pi}{d_2} \sum_{g'=-\infty}^{\infty} \delta(2\bar{k}_z + g'2\pi/d_2)$$ \hspace{0.5cm} (A.4.5)

Finally, the coherence $\rho_{vv}^{\text{vol}}$ due to a semi-infinite sequence of subsurface interfaces writes as ($C' = D_0e^{-i2\bar{k}_zd_2}/[4d_2]$)

$$C' \sum_{g'=-\infty}^{\infty} \left[ \pi \delta(\bar{k}_z + g'\pi/d_2) - i/(\bar{k}_z + g'\pi/d_2) \right]$$ \hspace{0.5cm} (A.4.6)

Neglecting the “aliasing” apparent in Eqn. (A.4.6) for simplicity, extract the coherence’s $g' = 0$ term and discard its delta function dependence (the only physically meaningful procedure, considering $0 < |\delta \theta| \ll 1$) to obtain the relation

$$|\rho_{vv}^{\text{vol}}| \propto \frac{1}{|\bar{k}_z|} \propto \frac{1}{B_\perp}$$ \hspace{0.5cm} (A.4.7)
This final relation leads us indeed to the conclusion that in the guidance limit $|\rho_{vv}^{\text{vol}}|$ approximately drops off as $1/B_\perp$ and suggests yet another restriction, multiplicatively augmenting the restriction set by “surface” baseline decorrelation, on the maximum feasible InSAR baseline if appreciable ground penetration is expected.

Of course, our fairly simple analysis assumed a “worst-case” guidance scenario (completely reflective dielectric slab walls and zero slab attenuation). Nominally, we would expect a non-zero $\tau_2$ due to incoherent scatter and/or ohmic absorption, as well as non-unity $\sigma_{r,21}^{\text{vv}}, \sigma_{r,23}^{\text{vv}}$ due to the rough interfaces diffusely scattering the wave’s energy in all directions. Physically this means that the higher-order multi-bounce contributions will decay in intensity as $D_0 C_g$. This would result in the phase bias writing as a “weighted” sum with weights \( \{ F_g = D_0 C_g / \sum_{g=1}^{G} D_0 C_g \} \) [125], while $|\rho_{vv}^{\text{vol}}|$ would decay slower than $1/B_\perp$ due its writing now as Eqn. (A.4.6) convolved with the FT of the “wide-band” discrete window function $e^{2(g+1)\tau_2 d_2} (\sigma_{r,21}^{\text{vv}} \sigma_{r,23}^{\text{vv}})^g$ which is defined by these rapidly decaying weights. In other words poorer wave guidance will mask the more deeply buried virtual interfaces, allowing a more graceful degradation of coherence with increasing antenna angular separation, as expected.

### A.5 Numerical Results: Phase

Our two general observations from this numerical study are as follows. First, subsurface wave attenuation will reduce coherence phase contributions (and hence phase bias) from backscatter returns undergoing more and more two-way round trips within the central slab. The limiting (high attenuation) phase bias however depends, in a “binary”/step-wise manner, on whether the scattering top interface backscatters energy or not (or more practically, we reason, whether its backscattered energy produces
a signal above the thermal noise floor). Namely if the top interface does backscatter, then the volume coherence’s layer depth-normalized phase $\phi_{\text{Vol}}^{\text{Norm}} \to 0$ as attenuation increasingly dampens subsurface backscatter. If it does not, due (for example) to the top interface being extremely smooth, then the “direct” backscatter echo from the bottom interface (no matter how weak, i.e., if neglecting thermal noise) would dominate and lead to a phase bias proportional to the central layer’s thickness ($\phi_{\text{Norm}}^{\text{Vol}} = 1$).

Second, the guidance property of the central layer has a less intuitive effect on phase bias. This is because increasing central layer guidance (quantified by increasing GP) on the one hand allows the subsurface-transmitted fraction of the incident wave’s energy to propagate farther laterally within the central layer, and hence excite stronger higher-order backscatter returns from the lower interface. On the other hand, the higher dielectric contrast (implied by larger GP) reduces the energy transmitted into the subsurface, and hence energy available to excite subsurface backscatter. In particular, if the top interface does not backscatter power (or does so very weakly, as quantified by low SR), increasing dielectric contrast will elevate phase bias. However, as the top interface backscatters more energy (e.g., becomes rougher) one arrives at a physical scenario where the top interface’s backscattered power overwhelms the negligible energy transmitted into the central layer and subsurface-backscattered, inducing a predominantly single-surface scatter scenario (as perceived by the radar) with volume coherence tending to unity.

These two findings are indeed confirmed by the numerical results. Figure A.2 shows three sub-plots, of increasing wave attenuation, demonstrating how the central layer’s guidance behavior affects phase bias in the limiting case where the top interface
does not backscatter energy (SR=0). We observe that increasing GP increases phase bias; in fact, although not plotted here explicitly but inferred from the GP=99/100 curve behavior at low $G$ values, the phase bias linearly diverges in the “guidance limit” ($\tau \to 0$ and $GP \to 1$) when SR=0 [125]. For non-zero but small SR=1/10 (Fig. A.3), GP still increases phase bias if wave attenuation is low (Fig. A.3a) since enough energy is still transmitted into the subsurface such that subsurface backscatter dominates over top-interface backscatter. Equally important in this scenario however is the low subsurface attenuation, which allows higher-order (large $g$) subsurface backscatter terms to significantly contribute to the interferometric observation as compared to lower-order ones (rather than the higher-order terms simply being attenuated). Notice, however, that even for this weakly backscattering interface, if GP increases too much (c.f. the magenta curve of Fig. A.3a) the subsurface-transmitted energy can be so small that once more the top interface’s backscattered energy dominates the interferometric observation. As attenuation increases (c.f. Figs. A.3b-A.3c), increased guidance only reduces transmitted energy; indeed, stronger wave confinement now leads merely to a laterally-propagating wave that is rapidly attenuated, making the interferometric observation primarily dependent on the top and bottom interfaces’ “direct” backscatter echoes. Similar conclusions hold as the top interface backscatters more and more energy compared to the bottom interface (higher SR). Finally, concerning phase bias in the high-attenuation limit, we observe that for SR=0 (Fig. A.2) increasing attenuation leads to normalized phase bias of one, since the radar does not effectively “see” the top-most, backscatter-free interface. On the other hand, for
non-zero SR (Fig. A.3) we observe increasing attenuation has the effect of eliminating phase bias due to attenuating the strength of subsurface scatterer and hence its influence on the interferometric observation.

### A.6 Numerical Results: Correlation

At this point, to reduce the parameter space we assume that $\sigma_{r,21} = \sigma_{r,23}$ and $\sigma_{t,12} = \sigma_{t,21} = 1 - \sigma_{r,21}$, leading to the volume coherence depending only on the following three parameters. First, the (back)scattering ratio $\text{SR} = \sigma_{b,12}/\sigma_{b,23}$ ($\text{SR} \geq 0$), which describes the relative backscatter strength of the top versus the bottom interface. Second, the central layer’s Guidance Parameter $\text{GP} = \sigma_{r,21}/\sigma_{r,23} = \sigma_{r,21}^2$ ($0 \leq \text{GP} \leq 1$), which quantifies the fraction of the transmitted wave’s energy confined within the subsurface after the wave scatters from both the top and bottom interfaces. Third, the attenuation-depth product $|\tau d|$, which describes the fraction of the transmitted wave’s energy remaining after traveling from one interface to the other (the propagation angle being determined by the specular reflection angle). Note that SR and GP depend on a number of factors, including SAR layer-dependent viewing angle, each interface’s height roughness statistics, each interface’s mean dielectric contrast, and the radar pulse’s central frequency. $|\tau d|$ depends on layer two’s mean (complex-valued) dielectric constant, the radar pulse’s central frequency, and the subsurface-transmitted wave’s propagation angle. To facilitate our analysis, we “lump” all these dependences into the said three parameters. As we will see below however, the periodic nature of the correlation (under the low-frequency approximation, allowing neglect of volume scatter) is *invariant* to variations in these three parameters, which suggests a
parameter-robust, multi-baseline, single-polarization approach, relying upon the coherence’s magnitude rather than its highly corruption-vulnerable phase, to reliably invert subsurface topography.

With the above parameter reduction performed, the volume coherence now writes as

\[
\rho_{\text{Vol}} = \frac{SR}{\left(1 - \sqrt{GP}\right)^2 \exp[2\tau d]} + Q_2 e^{-ik_w d} + \bar{Q}_2, \tag{A.6.1}
\]

where \(Q_2 = \sum_{g=0}^{\infty} (GP \times e^{2\tau d} e^{-ik_w d})^g\), and similarly for \(\bar{Q}_2\) upon setting \(k_w = 0\). The correlation follows upon taking the magnitude of Eqn. (A.6.1).

Figure A.4 shows, for the limiting case of a backscatter-free top interface (SR=0), predicted correlation (dB scale) versus the central layer’s “interferometric” electrical thickness \(k_w d\) (Log scale) for different values of the layer’s guidance parameter GP (curve colors) and \(|\tau d|\) (sub-plot letter). Our observations are as follows. First, to illustrate the inverse relationship between correlation and \(k_w\) when tending towards the “guidance limit” of perfect wave confinement and zero attenuation (GP=1, \(\tau = 0\)) for \(0 < |k_w| \ll 1\) [19], we show plots of the correlation as well as (up to multiplicative scaling) the function \(|k_w d|^{-1}\). Although not a perfect match (with agreement worsening for increasing \(|\tau d|\)), since we use non-unity GP for stable numerical result calculations, we do observe good agreement nonetheless between the dashed blue and solid magenta curves. Second, we observe that for low attenuation, there are sharp, periodic dips in the correlation that periodically alternate with peaks of high correlation. To aid visualization of the \(2\pi\)-periodic nature (versus \(k_w d\)) of the correlation dips and peaks, at the bottom of each sub-plot we show dotted red (correlation dip) and black (correlation peak) markers at \(\log_{10} |n'\pi|\) for \(n'\) odd and even, respectively. Third, observe that increasing attenuation does not alter the correlation valley and...
peak locations, but does reduce the severity of the correlation dips since higher-order multi-bounce terms are heavily attenuated and hence contribute less and less (versus increasing attenuation) to the backscatter observation. Fourth, increasing GP worsens the correlation; this is because energy transmitted into the central layer can propagate farther within the subsurface and hence excite backscatter returns that more “similarly” (as gauged by their scattered energies) contribute to the coherence [19, 125]. Fifth, akin to varying $|\tau d|$, notice that varying GP does not alter the correlation dip and peak locations.

The second, third, and fifth observations hold likewise for Figures A.5-A.6. What we find now as SR increases, however, is the importance of recognizing (in retrospect) key implications of the connection between stronger subsurface wave energy confinement and necessarily higher dielectric contrast between the central versus the two adjoining layers. This contrast, by allowing less of the pulse’s energy to be transmitted from the top layer into the central layer to excite the infinite succession of subsurface backscatter echos, increasingly makes the terrain appear “shallow” and backscattering predominantly from the air-ground interface. If the top interface does not backscatter, then the only relevant backscatter echos (no matter how weak [if neglecting thermal noise]) come from the subsurface; hence why, for SR=0, higher GP led to worsening correlation degradation. If the top interface does backscatter however, then increasing GP and/or SR can cause the top interface’s backscatter echo to increasingly become the dominant factor in determining the volumetric coherence. Indeed, even for very small SR (Figure A.5) one observes already the beginning of a “reversal” in the trend of correlation degradation versus GP as compared to the SR=0 case (Figure A.4). The trend completely reverses by Figure A.6, with the
GP=99/100 curve exhibiting the least correlation degradation while the GP=1/100 curve shows the greatest correlation degradation. Nonetheless, varying SR also leaves the correlation dip and peak locations completely unchanged; this can be confirmed upon comparing the dip and peak locations across Figures A.4-A.6. The analytical expression for the reduced form of $\rho^{\text{Vol}}$ (c.f. Eqn. (A.6.1)) confirms this finding rigorously.

A.7 Conclusion

Summary of Work: First, we proposed a layered-medium InSAR coherence model uniformly applicable to understanding interrogation of a combination of both low and high-attenuating media, incorporating influences of the dominant subsurface scatter mechanisms and their modulation by topography. We highlighted important effects that tilted and/or strongly-guiding subsurface layers can have upon interferometric phase: Linearly growing phase bias and inverse drop-off in correlation versus perpendicular InSAR baseline length. Second, elaborated below, we performed numerical studies of the coherence’s phase and magnitude (versus system and environment parameters) in a simplified three-layer model to understand how these two measurements vary across a range of more realistic, non-limit scenarios with respect to interface dielectric contrast, volume attenuation, and backscatter cross section ratio of the top to bottom interface.

Conclusions from the numerical coherence phase study: First, wave attenuation causes phase bias to approach either zero or the subsurface slab’s thickness in a step-wise manner (versus SR), depending on the relative backscatter strengths of the slab’s top and bottom interfaces. Second, higher dielectric contrast (quantified by
the “Guidance Parameter” GP in our study) augments phase bias for a very weakly backscattering top interface. On the other hand, since higher GP also reduces energy transmitted into the central slab, if the top interface does significantly backscatter power then its relative influence on coherence (versus the bottom interface) rapidly increases versus increasing GP, and hence phase bias rapidly decreases too. We conclude that typically, subsurface scatter-related phase bias should only become significant when the top interface is a very weak backscatterer (e.g., due to this interface being very smooth) and the subsurface weakly attenuates waves, which we anticipate is a reasonable assumption for many manifestations of layered geological structures such as ice sheets, snow packs, dry soils, and hyper-arid and regolith-mantled bedrock.

Conclusions from the numerical correlation study: First, that if one expects the top layer of the terrain to be strongly backscattering and/or the subsurface to be highly attenuating, the subsurface’s “interferometric” resonances should only weakly affect the correlation and hence should be of no great concern. By contrast, if the top layer is weakly backscattering (e.g., very smooth surface) while the subsurface’s dielectric contrast (to its neighboring layers) is high and its wave attenuation is low, the interferometric resonances can severely degrade correlation. Second, the periodic spacing of the correlation peaks and dips implies, when remotely sensing electrically thick subsurface layers (i.e., large $d/\lambda_2$ and low $|\tau|$), that not only will the occurrence frequency of the correlation dips and valleys (versus $k_w$) increase, but the largest practically usable $k_w$ (i.e., baseline length), after which the first correlation valley occurs, will reduce too. Interestingly, we remark that although the “interferometric” resonances discussed here should not be confused with the standard resonances of
dielectric slabs [45][Ch. 2] (recall we assume spatially uncorrelated scattering processes), they share a common physical origin: Low dielectric slab attenuation and high dielectric contrast. Here too moreover, for interferometric resonances, the resonant $k_w$ values are not only periodic but become more closely spaced as the slab’s thickness increases [45][Ch. 2]. Mapping of subsurface interface depths, using the proposed subsurface mapping technique discussed in the next paragraph, hence requires a larger (smaller) range of swept InSAR baselines for shallower (deeper) subsurface interfaces. Our third observation, namely the invariance of the (periodic) correlation’s peak and null locations versus the product $k_wd$, suggests a multi-baseline approach to robustly extracting subsurface topography. Namely, keeping fixed the radar pulse’s central frequency and mean viewing angle of the two SAR sensors, vary the third degree of freedom with respect to which is $k_w$ is dependent (i.e., SAR angular separation, or equivalently perpendicular baseline $B_\perp$), estimate the pixel-level correlation (i.e., associated with the observation of some given terrain patch) for different SAR angular separations (e.g., starting with zero angular separation, to be conservative), and compute the angular separation extent required for the correlation to reach unity once more. If the SARs’s pulse central frequency, mean viewing angle, and angular separation are known accurately, for a subsurface interface depth $d$ (assumed fixed within the given terrain patch) one can extract $d$ according to the simple formula $|k_{w1} - k_{w0}|/(2\pi)$. In this formula, $k_{w0} = 0$ is the longitudinal interferometric wave number for zero angular separation, while $k_{w1}$ is the longitudinal interferometric wave number at the next higher angular separation producing, once more, unity correlation (i.e., $k_wd = 2\pi$).
Figure A.2: Phase Bias: Backscatter-free top interface (SR=0). Note the different y-axis scale ranges.
Figure A.3: Phase Bias: SR=1/10.
Figure A.4: Correlation: Backscatter-free top interface (SR=0).
Figure A.5: Correlation: SR=1/10.
Figure A.6: Correlation: SR=1.
Bibliography


Middle East Unconventional Gas Conference, Houston, TX, USA, 2011, pp. 1–9.


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