Progress in Adaptive Electrical Capacitance Tomography

Dissertation

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By

Zeeshan, B.E., M.Sc., M.S.

Graduate Program in Department of Electrical Engineering

The Ohio State University

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Dissertation Committee:

Professor Fernando Teixeira, Advisor
Professor Kubilay Sertel
Professor Bradley D. Clymer
Abstract

Electrical Capacitance Tomography (ECT) is a sensing modality to find dielectric permittivity distribution from the boundary electrode capacitance measurements. ECT find applications in various different industrial applications due to its low cost, non invasive and non intrusive nature. ECT is a nonlinear problem, where imaging resolution is limited by the soft-field nature and limited number of measurements due to minimum electrode size constraints to provide a given SNR for capacitance measurement system. In order to increase the number of measurement without degrading SNR, adaptive electrical capacitance tomography (AECT) and its 3-D variant adaptive electrical capacitance volume tomography (AECVT) is introduced. The proposed strategy is based on the manipulation of synthetic electrodes comprised of a set of smaller physical electrodes (segments) enabled by AECT/AECVT hardware. AECT increases the number of measurements however, the increased amount of correlation between different measurements makes the inverse problem (image reconstruction) more ill-conditioned. Spatially adaptive reconstruction techniques (SART) are introduced that take advantage of the Laplacian nature of the interrogating field in AECT/ECT by utilizing synthetic electrodes based on different segment partition sizes while reconstructing in different portions of the region of interest (RoI). The reconstruction can be sequentially performed from the peripheral region of the RoI, where the achievable resolution is higher to the center region. SART also makes
it possible to use different AECT sensing mode together to achieve higher spatial and radiometric resolution as well as to ameliorate some the inherent non linear artifacts of ECT sensing.

The simultaneous activation of multiple electrodes can be advantageous to manipulate the resulting field distributions and capture additional spatial information for imaging purposes. However, conventional methods for sensitivity map computation in ECT are not adequate in the presence of multielectrode activation because the mutual coupling between electrodes is not properly accounted for. This coupling becomes especially critical in adaptive electrical capacitance volume tomography (AECVT) sensors, where signals from many small electrode segments are combined into synthetic electrodes. A more general approach is presented for sensitivity map computation in AECVT.
This work is dedicated to my beloved wife Qurat-ul-Ain for all her love and support
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Vita

March 6th, 1978 .......................... Born - Karachi, Pakistan

2001 ................................. B.E. Electrical Engineering
                             National University of Science and
                             Technology, Pakistan

2003 ................................. M.Sc. Systems Engineering
                             Pakistan Institute of Engineering &
                             Applied Sciences, Pakistan

2003-2011 ............................. Senior Engineer
                             Pakistan Atomic Energy Commission
                             Islamabad, Pakistan

2013 ................................. M.Sc. Electrical Engineering
                             New Mexico State University, USA

2015 ................................. M.S. Electrical Engineering
                             The Ohio State University

2013-present .......................... Graduate Research Associate,
                             The Ohio State University.

Publications

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Fields of Study

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Major Field: Electrical Engineering

Studies in:

- Electromagnetics  Prof. Fernando Teixeira
- Signal Processing  Prof. Bradly Clymer
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Chapter 1: Introduction

Electrical capacitance tomography (ECT) is an electrical sensing modality like microwave tomography (MWT), electrical resistance tomography (ERT) and magnetic induction tomography (MIT). Conventionally tomography is defined as a technique/method to acquire distribution of some physical property from an array of measurements made by sensors at the periphery of process or object under investigation. The sensor can be of different kind having acoustic, electrical, optical or radiation (X-ray or \( \gamma \) rays) sources.

ECT is used to find cross sectional as well as volumetric permittivity distribution from inter-electrode capacitance measurements [50, 56]. The array of sensing electrodes are mounted outside of the dielectric process to be monitored, therefore the technique is non invasive and non intrusive. ECT is usually performed in frequency range of 1kHz-10MHz [23, 66] whereas microwave tomography utilize the frequency range of 300MHz-30GHz [19, 20] to determine both permittivity and conductivity distribution from scattered field [10, 17, 18, 59, 95], similarly the EIT [14, 32, 35, 64, 69] and MIT [22, 34, 44, 58, 70] seek to determine conductivity and permeability contrast respectively. ECT is quite benign as it does not produce harmful effects such as
heating produced by microwave tomography or ionization in X-ray tomography, furthermore since sensing electrodes are not in physical contact with process, therefore it is suitable for medical and industrial applications.

ECT suffer from low spatial resolution, whereas X-ray, MRI, optical and microwave tomography can achieve very high spatial resolution as well as contrast detection capability. ECT on the other hand has the advantages of fast imaging rate that is required to capture transient phenomena such as flows and combustion flames. ECT finds applications in various different industrial processes such as fluidized beds, oil/gas two phase flows [5, 37, 86], combustion flames in engine cylinders [84]. It is also employed for non destructive testing [16], measuring mass and level of cryogenic fluids [60, 61] in space-borne applications as well as moisture detection in space shuttle thermal protection system (TPS) tiles [61, 96]. ECT is still under constant development, the first ECT system was developed by the US Department of Energy, Morgantown in 1970s [40, 89] and UMIST in 1980s [37].

1.1 Introduction and Motivation

ECT finds the 2D tomograms of the dielectric distribution, whereas electrical capacitance volume tomography (ECVT) [46, 74, 75, 83] determine the 3D volumetric permittivity distribution in the sensing domain. The references [1, 93] provides a good overview of the current state of ECT sensor technology.

ECT sensing electrodes respond differently to the permittivity distribution in the sensing domain. That slight difference in measured capacitance between electrode is used to reconstruct the dielectric distribution in the sensing domain. ECT systems consist of three components: sensing electrodes that are mounted around the non
conducting process to be monitored, data acquisition system that measure the capacitance and process the data for digital processing and third component is image reconstruction algorithm that compute permittivity distribution from the measured capacitance. The system block diagram is shown in Figure 1.1.

The frequency used for the measurement of capacitance range from 1kHz-10MHz [53,92]. Sensor dimensions are much smaller then the wavelength at those frequencies therefore the sensing can be considered as quasi-static phenomenon.

The ECT sensing field is governed by the scale invariant Laplace equation, therefore ECT finds applications in wide variety of applications having different physical dimensions ranging from a few tens of microns to a meters [82].
Minimum electrode size constraint the number of electrodes that can be mounted around the periphery of the sensing domain. The ECT sensor having \( n \) electrodes has \( \frac{n(n-1)}{2} \) independent measurements e.g. for 12 electrode ECT sensor shown Figure 1.2 has 66 independent measurements. The sensing domain on the other hand is discretized into pixel, such that no pixels are much larger than available measurements. The ECT has a three major difficulties limiting the achievable resolution.

1. The relationship between the permittivity and the capacitance is nonlinear, i.e. electric field is distorted by the dielectric object in the sensing domain, this is also refereed to as the soft field effect. The term softfield is used in contrast with hard field tomography like X-ray where interrogating field does not depend on the medium.
2. The second issue with the ECT is that number of independent measurement are limited, therefore problem is under-determined.

3. ECT problem is ill posed and ill conditioned, therefore reconstruction is prone to noise and error.

This work focus on sensor design and image reconstruction techniques to ameliorates these difficulties. This dissertation presents novel ECT sensor design and image reconstruction methods to achieve high resolution.

1.2 Contributions

The research is focused on ECT/ECVT sensor design using adaptive schemes that increases the available measurements and development of imaging algorithms catered for the adaptive methods to achieve higher resolution images. The following results are discussed in this dissertation:

1. AECT and AECVT sensor design and simulation.

2. Enhancing spatial resolution using adaptive sensor.

3. Identifying optimal sensor excitation methods for a given region of interest (ROI) in the sensing zone.

4. Space adaptive imaging techniques (SART) are described to fully exploit the AECT.

5. Sensitivity computation method for multi-electrode excitation are proposed, this allows adaptive electrodes to have a voltage distribution.
1.3 Organization of Dissertation

This dissertation comprises of seven chapters. Following introduction in Chapter 1, Chapter 2 discuss forward problem in Electrical Capacitance Tomography (ECT) and Electrical Capacitance Volume Tomography (ECVT). It introduces the sensitivity based sensor model and presents the results for the analytical formulation to find the electric potential and field in ECT sensor. The analytical solution is also used to find sensitivity map for ECT sensor.

ECT/ECVT is an inverse problem where the objective is to determine dielectric permittivity distribution $\epsilon(x, y)$ from the inter-electrode capacitance measurements. The inverse problem is both under determined and ill-posed, therefore solution requires use of regularization method. Chapter 3 presents the review of image reconstruction methods and address some of the issues related to reconstruction artifacts in ECT/ECVT.

The conventional approach to compute sensitivity is not applicable in case when multiple electrodes other than sensing electrode are also excited. Chapter 4, introduces novel method for sensitivity map computation for multi-electrode excitation. The multi-electrode excitation is introduced to have diversity in the sensitivity.

Adaptive electrical capacitance tomography (AECT) is introduced in Chapter 5 of the thesis as a means to ameliorate the undeterminacy of the ECT problem. The reconstruction results are provided for ECT and ECVT sensor configurations to provide imaging resolution improvements.

Chapter 6 presents the spatially adaptive reconstruction techniques (SART) that utilizes the additional degrees of freedom made available by AECT sensor. The
reconstruction allows the full exploitation of the adaptive sensor and introduces the novel methods for domain selective reconstruction.

Chapter 7 presents the conclusions and recommendations for future research in ECT/ECVT.
Chapter 2: ECT Sensor Model and Forward Problem

The forward problem in ECT consists of evaluating capacitance between electrode in response to a given dielectric spatial distribution in the sensing domain. The physical model of the sensor relates how a particular permittivity distribution would effect the boundary electrode measurements. This chapter presents linearized sensor model starting from fundamental Maxwell’s equation. The numerical and analytical results for sensor model are presented. The sensor model is utilized in the dissertation for image reconstruction.

2.1 ECT Background

Electrical Capacitance tomography (ECT) is an imaging technique that is used to image industrial process having dielectric materials for example multiphase flow such as gas-oil mixture and solid-gas mixture. The multi-phase flow occurs in industrial processes such as oil pipelines, fluidized beds, flame visualization [37] etc. This imaging modality provides the advantages of fast imaging rate that is required to capture transient phenomena so that they may be modeled or controlled. ECT technique besides being fast has the advantages such as being non invasive and non intrusive [93], since the sensor is not in physical contact with the process. ECT is also employed for non destructive testing as it does not disturb the object or process to be imaged.
Typically ECT Sensor consists of several electrodes that are mounted around the pipe or process to be sensed. The sensor are usually mounted around the pipes in circular fashion, in general ECT sensor electrodes can be designed in any shape in order to conform to the sensing domain or process [7, 8]. The measuring electrodes are enclosed in a metallic shield so that the external field disturbance do not induce any undesirable voltages and/or charges on the measurement electrode and hence introduce noise and degrade the measurement [1]. Electrical capacitance tomography consists of two problems, Forward problem consists of finding the mutual capacitance given a dielectric distribution [72, 73]. whereas the computation of permittivity from capacitance is know as inverse problem.

In order to measure the inter-electrode capacitance single electrode excitation method is used. In this measurement scheme one of the electrode is excited and the rest of the electrodes are held at ground potential and capacitance is measured between the two electrodes. If an ECT sensor consists of \( n \) electrodes there are \( \frac{n(n-1)}{2} \) distinct capacitance measurements are possible.

### 2.2 ECT Sensor Forward Model

ECT sensor physical model is derived in this section. The single electrode is excited with a potential \( V_i \) and the induced charge \( Q_j \) is measured on the other electrodes. In order to find the charge on any electrode in ECT sensor basic electromagnetic model is employed. The Maxwell’s equations describing electrostatic phenomenon are

\[
\nabla \cdot \vec{D} = \rho \tag{2.1}
\]
Where $\vec{D}$ is the electric flux density measured in the $C/m^2$, $\rho$ is the free charge density. $\vec{E}$ is the electric field intensity measured in $V/m$, $\vec{B}$ is the magnetic field density is measured in tesla. Since the sensing domain in ECT is filled with loss less dielectric media with $\sigma = 0$, therefore $J_c = 0$ in equation (2.4). Since ECT problem is an electrostatic phenomenon, therefore the terms involving the time derivative do not play a role.

Dielectric medium in the sensing domain is considered to be linear and isotropic, the constitutive relation for the medium are

$$\vec{D} = \epsilon \vec{E}$$  
$$\vec{B} = \mu \vec{H}$$  
$$\vec{J} = \sigma \vec{E}$$

where $\epsilon$ is the permittivity, $\mu$ is permeability and $\sigma$ is the conductivity of the medium. ECT sensing finds the permittivity distribution $\epsilon(x,y)$ of a non conducting
(σ = 0) medium in the sensing domain from multielectrode capacitance measurements.

ECT system consisting of 12 electrodes is shown in Figure 2.1, the electrodes are arranged in a circular fashion inside the metallic outer shield. The potential distribution φ(x, y) in the sensing domain, is related to the electric field by (2.8)

\[ \vec{E} = -\nabla \phi(x, y) \]  \hspace{1cm} (2.8)

using Gauss’s law given by equation (2.1) and equation (2.8) we have,

\[ \nabla \cdot \epsilon(x, y) \nabla \phi(x, y) = 0 \]  \hspace{1cm} (2.9)
Potential distribution in the sensing domain can be found using poisson equation (2.9), subject to the dirichlet boundary conditions on the electrodes $\phi(\Gamma_k) = \phi_k$, where $\phi_k$ is the potential on the $kth$ electrode in the ECT sensor.

ECT employ single electrode excitation scheme, where one electrode is excited at a time and potential distribution is found in the sensing domain. The closed form analytical solution of (2.9) is only possible for a limited number of cases having symmetrical permittivity distribution in the sensing domain. Numerical method such as finite elements methods (FEM) [62] or finite difference method (FDM) [88] are employed to determine the potential and electric field in the sensing domain. The simulation performed in this dissertation utilizes COMSOL Multiphysics, which is an FEM based solver.

The capacitance $C_{ij}$ between source electrode $i$ and detector $j$ is given by (6.1), where $Q$ is the charge on detector electrode and $V_{i,j}$ is the potential difference between source and detector electrodes

$$C_{ij} = \frac{Q}{V_{i,j}}$$ (2.10)

The charge on any electrode can be found by finding the electric field by equation (2.8) and using the Gauss law

$$Q = \oint_{\Gamma_j} \epsilon \vec{E} \cdot d\Gamma$$ (2.11)

$$C_{i,j} = -\frac{1}{V_{i,j}} \iint_{\Gamma_j} \epsilon(x, y) \nabla \phi(x, y) \cdot d\Gamma$$ (2.12)

The capacitance of 12 electrode ECT sensor when sensor is completely empty ($\epsilon_r = 1$) and fully filled with a medium having $\epsilon_r = 3$ is given in Figure 2.2. The peak values
shows the capacitance between adjacent electrodes, furthermore if separation between electrodes increases capacitance value decreases.

Figure 2.2: Capacitance of 12 electrode ECT sensor

2.3 Linear Sensitivity Model

The relation between the mutual capacitance $C = [C_{1,2}, C_{1,2}, \cdots, C_{n-1,n}]$ between electrodes, and the permittivity distribution $\epsilon(x, y)$ in the sensing domain given by (2.12) is non linear.

$$C = F(\epsilon(x, y))$$ (2.13)

The inverse problem is to find the permittivity distribution $\epsilon(x, y)$ from capacitance $C$ in (2.13) is non unique because of existence of annihilators (null space) [65, 77]. The vector of non linear functional $F = [f_1(\epsilon), f_2(\epsilon), \cdots, f_M(\epsilon)]$ describe permittivity to capacitance map.
In order to find discrete linear model, sensing domain is partitioned into small pixel as shown in Figure 2.3, the sensing zone is discretized into $40 \times 40$ pixels, however only 1184 pixels are in the sensing zone. The permittivity in each pixel is assumed to be homogenous. Pixel permittivities are mapped to changes in capacitances by first-order Born approximation [79,102] of equation (2.13), this relation can be expressed in linear form as [51] shown in 2.14.

$$\Delta C = \frac{dF}{d\varepsilon} \cdot \Delta \varepsilon + O((\Delta \varepsilon)^2) = J \cdot \Delta \varepsilon + O((\Delta \varepsilon)^2)$$  \hspace{1cm} (2.14)

neglecting the second order terms $O((\Delta \varepsilon)^2$, the relation (2.14) can be approximated as (2.15).

$$\Delta C = J \cdot \Delta \varepsilon$$  \hspace{1cm} (2.15)
\begin{equation}
\mathbf{J} = \begin{pmatrix}
\frac{\partial f_1}{\partial \epsilon_1} & \frac{\partial f_1}{\partial \epsilon_2} & \cdots & \frac{\partial f_1}{\partial \epsilon_N} \\
\frac{\partial f_2}{\partial \epsilon_1} & \frac{\partial f_2}{\partial \epsilon_2} & \cdots & \frac{\partial f_2}{\partial \epsilon_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_M}{\partial \epsilon_1} & \frac{\partial f_M}{\partial \epsilon_2} & \cdots & \frac{\partial f_M}{\partial \epsilon_N}
\end{pmatrix}
\end{equation}

Here $\Delta \mathbf{C}$ is $M \times 1$ vector where $M$ is the number of capacitance measurements, for $n$ electrode ECT sensor there are $M = n(n-1)/2$ measurements. The number of pixels in the sensing domain are $N_p$, typically, $M \ll N_p$, e.g. for 12 electrode ECT sensor $M = 66$ and the number pixel shown in Figure 2.3 are 1184. $\mathbf{J}$ is the $M \times N_p$. The Jacobian matrix is given in (2.16) whose elements maps the change in dielectric permittivity in $k$th pixel to change in $m$th capacitance.

\begin{equation}
\mathbf{J}_{m,k} = \frac{\partial C_m}{\partial \epsilon_k}
\end{equation}

The change in capacitance is related to change in permittivity in sensing domain by linear relation (2.15). It is however evident from Figure 2.2 that dynamic range of each measurement is different. In order to find a consistent solution to the linearized inverse problem it has been found advantageous to normalize each measurement from 0 to 1 [43]. The normalization is performed by filling the entire sensor with high and low permittivity material respectively and measuring the capacitance $C^{\text{high}}$ and $C^{\text{low}}$ for each measurement channel. The calibration performed against the materials of known permittivity are then used to normalize any measurement $C_{i,j}$ as shown in (2.18) [57, 87].
\[ C_{i,j}^{\text{norm}} = \frac{C_{i,j} - C_{i,j}^{\text{low}}}{C_{i,j}^{\text{high}} - C_{i,j}^{\text{low}}} \] (2.18)

The sensor linear model \( \mathbf{J} \) is consequently normalized as indicated in relation (2.18) [57, 87].

\[ S_{i,j} = \frac{J_{i,j}}{\sum_{k=1}^{N_p} J_{i,j}(k)} \] (2.19)

The normalized linearized sensor model is shown in (2.20), where the capacitance vector \( \mathbf{c} \) with all normalized capacitances relates pixel permittivity vector \( \mathbf{g} \) by the sensitivity matrix \( \mathbf{S} \) [43, 54].

\[ \mathbf{c} = \mathbf{S} \cdot \mathbf{g} \] (2.20)

where the sensitivity at a pixel location between two electrodes is obtained by first computing the electric potential \( \phi_i \) produced by exciting electrode \( i \) with unit voltage and keeping all other electrodes grounded, and repeating the procedure for all \( i \). The following integral is then evaluated over each pixel area \( \sigma(p) \) in the sensing domain:

\[ [S]_{(ij),p} = -\int \int_{\sigma(p)} \nabla \phi_i(x,y) \cdot \nabla \phi_j(x,y) \, d\sigma, \] (2.21)

The sensitivity map for 12 electrode ECT sensor are shown in Figure 2.4. Sensitivity computation for ECVT is provided in Appendix A.

### 2.4 Analytical Modeling of ECT Sensor

The sensitivity computation requires finding the field distribution in the sensor. It is often time required to validate the numerically computed results with the close
Figure 2.4: Sensitivity map for 12 electrode ECT sensor
form solution [47]. In order to find electric field and potential analytically ECT sensor is modeled as the circular disk with some potential distribution at the boundary.

The problem is concerned with determining the potential distribution $V(r, \phi)$ inside circular disk of radius $a$, when excited at the boundary by a potential distribution $V(a, \phi)$.

![Figure 2.5: Potential in the disk](image)

The potential is governed by the Laplace equation subject to the boundary condition $V(a, \phi)$.

There is no $z$ dependance in this case.

$$\nabla^2 V = 0 \quad (2.22)$$

$$\nabla^2 V = \frac{\partial^2 V(r, \phi)}{\partial r^2} + \frac{1}{r} \frac{\partial V(r, \phi)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V(r, \phi)}{\partial \phi^2} = 0 \quad (2.23)$$
applying the separation of variable technique, assumeing that the potential is a function of $R(r)$ and $\Psi(\phi)$

$$V(r, \phi) = R(r)\Psi(\phi)$$ (2.24)

$$\frac{r}{R} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) + \frac{1}{\Psi} \frac{\partial^2 \Psi}{\partial \phi^2} = 0$$ (2.25)

since both variables are now separated, therefore

$$\frac{r}{R} \frac{\partial}{\partial r} \left( r \frac{\partial R}{\partial r} \right) = k^2$$ (2.26)

$$\frac{1}{\Psi} \frac{\partial^2 \Psi}{\partial \phi^2} = -k^2$$ (2.27)

Solution to the $\Psi$ is periodic in $\phi$, therefore we have

$$\Psi_n = A_n \cos(n\phi) + B_n \sin(n\phi)$$ (2.28)

similarly for $R$

$$R(r) = C_1 r^n + C_2 r^{-n}$$ (2.29)

Since the potential is to be found inside the disk, therefore we consider only the $r^n$ term.

$$V_n(r, \phi) = [A_n \cos(n\phi) + B_n \sin(n\phi)] r^n$$ (2.30)

now the total potential can be expressed as
\[ V(r, \phi) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\phi) + B_n \sin(n\phi)] r^n \] (2.31)

now at the boundary at \( r = a \) we have \( V(a, \phi) \), therefore it can be deduced

\[ V(a, \phi) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\phi) + B_n \sin(n\phi)] a^n \] (2.32)

The coefficients now can be found from the orthogonality condition in the Fourier series expansion

\[
A_0 = \frac{1}{2\pi} \int_{0}^{2\pi} V(a, \phi) d\phi \] (2.33)

\[
A_n = \frac{1}{a^n \pi} \int_{0}^{2\pi} V(a, \phi) \cos(n\phi) d\phi \] (2.34)

\[
B_n = \frac{1}{a^n \pi} \int_{0}^{2\pi} V(a, \phi) \sin(n\phi) d\phi \] (2.35)

2.4.1 Potential and Electric Field due to a single Electrode Excitation

The boundary potential distribution is given by

The Fourier coefficients from previous section are computed to find the series solution of the Laplace equation for the given boundary condition.

\[
A_0 = \frac{1}{2\pi} \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} 1 d\phi = \frac{1}{12} \] (2.36)

\[
A_n = \frac{1}{a^n \pi} \int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \cos(n\phi) d\phi = \frac{2 \sin\left(\frac{n\pi}{12}\right)}{n \pi a^n} \] (2.37)

The \( B_n \) coefficient is 0, due to the symmetry of the problem now potential can be found anywhere in the disk as
Figure 2.6: Potential in the disk

\[ V(r, \phi) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{2\sin\left(\frac{n\pi}{12}\right)}{n\pi a_n} r^n \cos(n\phi) \] (2.38)

The coefficients are plotted and mean square error is computed to reconstruct the boundary potential. The Gibbs phenomenon can be observed since we are taking only finite number of terms and there is sharp discontinuity in potential.

Electric field is computed by the gradient of the potential and is plotted for given boundary conditions inside the disk.

\[ \vec{E} = -\nabla V(r, \phi) = -\frac{\partial V(r, \phi)}{\partial r} a_r - \frac{1}{r} \frac{\partial V(r, \phi)}{\partial \phi} a_\phi \] (2.39)

\[ \vec{E}_n(r, \phi) = \frac{2r^{n-1}\sin\left(\frac{n\pi}{12}\right)}{\pi} \left[-\cos(n\phi) a_r + \sin(n\phi) a_\phi\right] \] (2.40)

\[ \vec{E}(r, \phi) = \sum_{n=1}^{\infty} E_n(r, \phi) \] (2.41)
Figure 2.7: Fourier coefficients

Figure 2.8: Potential distribution at the electrode
2.4.2 Analytical Sensitivity Map of ECT Sensor

Sensitivity is computed by considering ECT sensor consisting of 12 Electrodes. The problem has circular symmetry therefore the electric field computed for the first electrode is rotated to find the field produced by all sensor considering all other sensor at the ground potential. The sensitivity is computed by the relation (2.21). The sensitivity distribution is found by considering infinitesimal gap between electrodes. In case of 12 Electrode sensor there are 66 sensitivity maps. Since the Sensitivity pattern is having symmetry only 6 maps are shown and the rest can be obtained by rotational transformations. The sensitivity maps so computed can be used for single plane ECVT sensor also assuming no variation in the axial direction, therefor it cannot give the cross plane coupling for ECVT volume imaging. Appendix B provide the analytical procedure for finding the electric field and potential for ECVT sensor.
Figure 2.10: Electric field in the circular ECT sensor

Figure 2.11: Sensitivity of ECT sensor
Chapter 3: Review of Image Reconstruction Methods

The inverse problem in electrical capacitance tomography (ECT) is concerned with finding permittivity distribution from the mutual-capacitance measurements. The computation of dielectric permittivity in the sensing domain is known as image reconstruction.

The challenges in image reconstruction arise because ECT is inherently a nonlinear problem where interrogating electric field is dependent on the dielectric material present in the sensing domain. It is for this reason known as soft-field tomography as opposed to hard field tomography where material present in the sensing domain do not cause changes in the source field. The second difficulty with ECT is limited number of measurements due to constraint on the electrode size. While the forward problem in ECT is well posed as it gives unique set of measurement to a given dielectric distribution however, the inverse problem is ill-posed and ill conditioned.

The term ill-posed was first used by Hadamard [3, 24] who defined an inverse problem to be well posed if following conditions are met:

1. The problem have a solution.

2. The solution must be unique.

3. The solution must continuously depend on the measured data.
If a problem violate any of the above conditions, it is referred to as ill-posed. In case of ECT ill-posedness is due to the non linearity and Laplacian nature of the field. Ill-posedness can cause small changes in measured capacitance leading to large changes in permittivity, thereby violating the Hadamard stabiltiy condition. The change in capacitance $C = F(\epsilon)$ is not detectable [78], if permittivity is perturb a little $F(\epsilon+\delta)$ therefore inverse mapping yield unbounded results from the RiemannLebesgue lemma [26,27].

Image reconstruction in ECT make use of sensitivity based linearized model. Computationally discrete problem does not yield unbounded results therefore a discrete problem is categorized as ill-posed if:

1. The singular values decays to zeros without any gap.

2. The condition number is very large.

The singular value spectrum and condition number would be discussed in detail in this chapter.

There are a number of techniques developed for ECT image reconstruction, broadly they are classified into iterative and non-iterative or direct method, references [13,15,93,97] are suggested for an exhaustive review. This chapter provide a brief overview of the image reconstruction methods that are employed in this dissertation.

3.1 Direct Methods

The direct methods are also known as single step methods to solve the inverse problem. The direct methods are usually faster and computationally less expansive as compared to the iterative methods. Therefore direct methods are suitable for
real time reconstruction, however they yield low spatial resolution and are good for qualitative image analysis because of averaging or low pass filtering effect.

### 3.1.1 Linear Back Projection

ECT linearized model \( S \in \mathbb{R}^{M \times N} \) maps pixel permittivity \( g \in \mathbb{R}^{N} \) to the normalized capacitance \( c \in \mathbb{R}^{M} \) as shown in (3.1). Since \( M \ll N \), therefore inverse \( S^{-1} \) does not exist.

\[
c_{M \times 1} = S_{M \times N} \cdot g_{N \times 1} \tag{3.1}
\]

Linear back projection is based on \( S^T \in \mathbb{R}^{N \times M} \) as a mapping from capacitance vector space to permittivity vector space indicated in (3.2). It is however indicated here that \( \hat{g} \) is an approximate solution of (3.1).

\[
\hat{g} = S^T \cdot c \tag{3.2}
\]

### 3.1.2 Singular Value Decomposition

The sensitivity matrix \( S \) in ECT provides the information regarding the detectability of any change in the sensing domain. The sensitivity matrix is dependent on the particular measurement strategy such as source/detector electrode numbers and the sensor electrode spatial arrangement. Singular value decomposition(SVD) is a method of data analysis that is used to find the principal components of a data set. It is a powerful methods for the analysis of the ill conditioned problem and provide how much information can be recovered from a particular data set. SVD decomposes the sensitivity matrix into set of orthonormal vectors and corresponding singular values \( \sigma_i \).
\[ S = U \Sigma V^T \]  \hspace{1cm} (3.3)

Here \( U_{M \times M} \) and \( V_{N \times N} \) are unitary matrices whose columns \( u_i \) and \( v_i \) span \( \mathbb{R}^M \) and \( \mathbb{R}^N \) respectively. Whereas the \( \Sigma_{M \times N} \) is diagonal matrix of singular values of \( S \). The singular values are sorted in the order of importance i.e \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \geq ... \geq \sigma_r \) are arranged such that much of the information is contained in the singular values having leading indices [3,27,76]. The plot of the singular value is referred to as singular value spectrum. SVD decomposition in equation (3.3) can also be expressed as:

\[ S = \sum_{i=1}^{r} \sigma_i u_i v_i \]  \hspace{1cm} (3.4)

The \( u_i \) and \( v_i \) are the eigenvectors of \( SS^T \) and \( S^T S \) respectively.

\[ SS^T u_k = \sigma_k^2 u_k \]  \hspace{1cm} (3.5)

\[ S^T S v_k = \sigma_k^2 v_k \]  \hspace{1cm} (3.6)

Condition number \( \kappa(S) \) quantify the ill conditioning of the sensitivity matrix \( S \). It is defined as the ratio of the maximum singular value to the minimum non-zero singular value.

\[ \kappa(S) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \]  \hspace{1cm} (3.7)

The SVD can be used for computing the generalized inverse or pseudoinverse \( S^\dagger \) [3,76], thereby permittivity \( g^\dagger \) can be estimated from relation (3.9).
\[ S^i = V \Sigma^{-1} U^T \]  

(3.8)

\[ g^i = V \Sigma^{-1} U^T c = \sum_{i=1}^{n} \frac{u_i^T c}{\sigma_i} v_i \]  

(3.9)

Figure 3.1: Singular value spectrum

The singular values spectrum for the ECT sensor having 6, 8 and 12 electrodes is plotted in Figure 3.1. The singular values spectrum is spread over several order of magnitudes. The small singular values can cause solution in (3.9) to be unstable. In order to avoid unstable solution discrete picard condition (DPC) \([3, 26, 29]\) is introduced. The DPC condition states that \( |u_i^T c| \) should decay to zero faster then
corresponding singular value $\sigma_i$. The DPC is assured by plotting the $|u_i^T c|$ vs singular value $\sigma_i$ across the entire spectrum, known as the picard plot.

Since picard condition is not satisfied by all singular values hence the solution is stabalized by introducing a truncation or regularization parameter $\phi$ in equation (3.9) as indicated in modified relation (3.10).

$$g^\dagger = \sum_{i=1}^{n} \varphi_i \frac{u_i^T c}{\sigma_i} v_i \quad (3.10)$$

Two methods used to regularize the solution are, truncated singular value decomposition (TSVD) method and the Tikhonov method [28]. In TSVD the filtering factor has a sharp cutoff that discard contributions from smaller singular values.

$$\varphi_i = \begin{cases} 1, & i = 1, \ldots, k \\ 0, & i = k, \ldots, n \end{cases} \quad (3.11)$$

### 3.1.3 Tikhonov Regularization

Tikhonov method is a regularization approach to solve ill-posed inverse problem. The least square solution to the linear problem $c = Sg$ is given by (3.12).

$$\hat{g} = (S^T S)^{-1} S^T c \quad (3.12)$$

Since the inverse $(S^T S)^{-1}$ does not exist therefore diagonal loading [33] is introduced to regularize the solution in (3.13), where $\alpha > 0$ is regularization parameter and $I$ is identity matrix. The reconstructed image quality is strongly depended on the choice of $\alpha$.

$$\hat{g} = (S^T S + \alpha I)^{-1} S^T c \quad (3.13)$$
In general Tikhonov regularization replaces the ill posed problem by a nearby well posed problem. The least square minimization functional can be expressed as (3.14).

\[ C(g) = \min ||c - S\hat{g}||^2 + \alpha^2 ||\hat{g}||^2 \]  

(3.14)

In terms of Tikhonov method, the effect on SVD decomposition is that singular value are filtered [25]. The filtering factor is found by minimization of cost function (3.14) which is a compromise between residual capacitance \( ||c - S\hat{g}||^2 \) and estimated solution \( ||\hat{g}||^2 \) determines \( \alpha \), the regularization parameter.

\[ \varphi_i = \frac{\sigma_i^2}{\sigma_i^2 + \alpha^2} \]  

(3.15)

The regularization parameter or truncation value for TSVD can be found using a number of techniques such as [3,26,27]:

1. Discrepancy Principle

2. Generalized Cross Validation (GCV)

3. \( L \) curve Method

### 3.2 Iterative Methods

ECT is a nonlinear, ill-posed and undetermined problem. The single step or direct solution is not accurate. Inverse problem, therefore is solved iteratively. These methods make use of current permittivity estimate \( \hat{g} \) to update the image based on residual error \( r_k = c - S\hat{g} \) between calculated and measured capacitance. The iterative process is carried out until the discrepancy is acceptable. Since inverse problem is not unique additional constraint are imposed to regularize the solution.
The two methods employed in this dissertation are iterative Landweber methods and algebraic reconstruction based methods (ART).

### 3.2.1 Landweber Method

Landweber iterative algorithm is a variation of steepest descent method \([39,45,89]\). It is based on the minimization of the residual error given by:

\[
e(g) = \frac{1}{2} (c - S\hat{g})^T(c - S\hat{g})
\]  

(3.16)

The steepest descent method can be formulated as shown in 3.17).

\[
\hat{g}_{k+1} = \hat{g}_k - \alpha_k \nabla e(g)
\]  

(3.17)

The relaxation parameter \(\alpha_k\) can be chosen fixed for all iteration. The fixed step size \(\alpha\) is chosen based on the largest singular value \(\lambda_{\text{max}}\) of \(S\) \([39,91]\), with \(\alpha \leq 2/\lambda_{\text{max}}\).

The gradient is computed as indicated in 3.18, the landweber iteration are then shown in 3.19.

\[
\nabla e(g) = S^T(Sg - c)
\]  

(3.18)

\[
\hat{g}_{k+1} = \hat{g}_k - \alpha_k S^T(Sg - c)
\]  

(3.19)

The solution is projected to a convex set after each iteration in landweber algorithm in order to suppress numerical artifacts in the form of negative values. The projection operator is used to impose non-negativity and peak value constraint is:
\[ P([g_k]) = \begin{cases} 
0 & \text{for } g_k < 0 \\
g_k & \text{for } 0 \leq g_k \leq 1 \\
1 & \text{for } g_k > 1 
\end{cases} \quad \text{(3.20)} \]

### 3.2.2 Algebraic Reconstruction Methods

Algebraic reconstruction method (ART) or Kaczmarz method [3, 31, 42] is a row action method, where a single row \( s_i \) of the sensitivity matrix \( S \) is utilized to update the permittivity estimate. The \( kth \) iteration only utilizes only single measurement to update \( \hat{g}_k \) as indicated in 3.14, \( \alpha_k \) is relaxation parameter.

\[
\hat{g}_k = \hat{g}_{k-1} - \alpha_k \frac{(c_k - s_k \hat{g}_{k-1})}{s_k^T s_k} s_k 
\]  

(3.21)

Since ART uses only one measurement at a time, therefore it is more susceptible to noise. In order to overcome the disadvantages associate with the convergence of ART simultaneous iterative reconstruction technique (SIRT) [2, 6] is introduced. The method is called simultaneous as all the measurements are used in the same iteration and can be written as indicated in 3.22.

\[
\hat{g}_{k+1} = \hat{g}_k + \alpha_k T S^T M(c - S \hat{g}_k) 
\]  

(3.22)

here \( M \) and \( T \) are symmetric positive definite matrices whose values depends on the particular optimization used. The relaxation parameter \( \alpha_k \) is bounded by the spectral radius \( \alpha_k \leq 2/\rho(T S^T M S) \) [41].
Chapter 4: Multi electrode Excitation and Sensitivity Computation

Electrical Capacitance Tomography (ECT) is predicated on the knowledge of sensitivity maps produced by the electric fields between capacitance electrodes that blanket the imaging domain. The simultaneous activation of multiple electrodes can be advantageous to manipulate the resulting field distribution in the imaging domain and capture additional information during ECT measurement acquisition. However, conventional methods for sensitivity map computation in ECT are not accurate for the case of multi-electrode activation because the mutual coupling between electrodes is not properly accounted for. This becomes especially critical in electrical capacitance volume tomography (ECVT) systems. This chapter presents a more general approach for sensitivity map computation in ECT/ECVT based on the reaction of the fields produced by a posteriori (induced) charge distributions rather than from enforcing a Dirichlet-type boundary condition. The approach is illustrated in this chapter for both ECT and ECVT configurations.

4.1 ECT Electrodes

The functionality of capacitive plates used to date in ECT sensors can be broadly categorized into two types [1]. The first type of plates refers to the plates used for the
capacitance measurements itself, known as sensing electrodes. The second type are guard electrode plates, which have been called as driven guards or axial guards [1,93] as shown in Fig. 4.1.

![ECT sensor with guards](image)

Figure 4.1: ECT sensor with guards

The purpose of the axial guards is strictly to reduce fringing field effects when the plate size is small. In order to have higher axial resolution, it is desirable to keep the axial length of the plates small; however, reducing such axial length inevitable increases the fringing field effect, which in its turn degrades the resolution. In order to ameliorate the fringing field affect on the axial resolution, driven guards at the both ends of the sensing electrodes are used.
It has also been mentioned in the literature that use of guard plates would perturb the capacitance measurement in some way [1] but we stress here that this is not the case: the capacitance between every two plates in a ECT system is a function only of the geometry and of the electrical properties in the domain to be sensed and not of the relative excitation. 3D effects on the sensitivity by the driven guards in ECT was investigated by using different guard electrode configurations [36] and error in the 2D approximation of 3D sensor was discussed.

It should also be noted that a scheme of parallel electrode excitation by driven guards was investigated in the prior ECT literature [90]. This scheme excites electrodes other than sensing electrode with the aim to produce a higher field and thereby higher sensitivity in the central region of the sensing domain. It was found that method would instead only enhance sensitivity very near to adjacent electrodes and that sensitivity is not enhanced and is even worsened in the central region [90]. From such results, the conclusion was made that with the use of driven guards there is no possibility of having more than \( \frac{n(n-1)}{2} \) independent capacitance measurements [90,93] in an ECT sensor with \( n \) electrodes.

In this chapter the concept of active guards is applied with two specific objectives in mind. The first objective is to explicitly control the charge distribution of the sensing guards, thereby controlling the sensitivity map produced by the latter and enabling independent set of measurements to be attained with a single pair of sensing plates. The second objective of the active guards is to explicitly mitigate the high field between the adjacent plates by gradually tapering the voltage distribution on the (active) plates adjacent and nearby the sensing plate(s), thereby increasing the useful dynamic range of the capacitance measurements. These two objectives can
be sought after in tandem and are empowered by the recently-patented AECVT (adaptive electrical capacitance volume tomography) capabilities.

It is stressed here that the important difference of the presently proposed active guards and the prior axial/driven guard concepts is the intended functionality. Active guards are designed to control the spatial distribution of the electric charge density on the sensing plates to increase the number of independent sensitivity maps achievable and provide a sensitivity maps devoid of high field singularities between adjacent plates. Finally, we note that in a same ECT sensor, a given capacitance plate can act as sensing electrode for one measurement acquisition and as an active guard electrode in another measurement acquisition.

4.2 Sensitivity Computation and Multielectrode Excitation

ECT seeks to determine the dielectric distribution inside the imaging domain from a set of (mutual) capacitance measurements obtained by means of electrode plates mounted at the boundary [1]. ECT can provide imaging of two-dimensional (2-D) domains, i.e. the cross-section of a vessel, or three-dimensional (3-D) domains. In the latter case, it is more appropriately referred to as electrical capacitance volume tomography (ECVT) [83]. In order to estimate the permittivity distribution, the sensing domain is discretized into pixels in 2-D or voxels in 3-D comprising the permittivity vector \( \mathbf{g} \) represented as a function of position. Under a linear approximation, the change in permittivity values is related to a change on the measured capacitances through \( \mathbf{c} = \mathbf{Sg} \), where \( \mathbf{S} \) is the sensitivity map (matrix) or simply, the sensitivity. Conventionally, the sensitivity at a given voxel \( n \) associated with any two electrode
pair \(i, j\) is computed as \[83\]

\[ S_{ij}[n] = -\frac{1}{V_i V_j} \int_{\nu[n]} \nabla \varphi_i \cdot \nabla \varphi_j \, d\nu \quad (4.1) \]

where \(\varphi_i\) is the potential distribution inside the imaging domain produced by a voltage \(V_i\) applied on the electrode \(i\) \textit{with all other electrodes grounded}, and likewise for \(\varphi_j\). The above integral is performed over each voxel \(\nu[n]\). If the voxel size is sufficiently small, this integral can be approximated using the average field values inside each voxel. Relation (4.1) can be approached from various perspectives such as the Born approximation, or the Reaction Theorem [67,68] through which it can be interpreted as the reaction of the field produced at a given spatial point due to voltage sources at the \(i\) and \(j\) electrodes. However, eq. (4.1) assumes independent potential distributions and is not directly applicable to configurations in which multiple (i.e. simultaneous) active electrodes in addition to the source electrode are present (i.e., when the assumption of remaining electrodes being at ground potential is not true). This is because the charge distribution on a given source electrode is perturbed in the presence of other active electrodes; hence, the consequent electric field and sensitivity map is modified as well\(^1\). The use of multiple active electrodes is highly desirable in ECVT to, among other reasons, mitigate fringing effects and reduce field singularities near adjacent electrodes. Otherwise, the latter limit the peak excitation voltages and hence the signal-to-noise ratio (SNR).

\(^1\)On the other hand, it should be pointed out that the \textit{mutual capacitance} themselves remain constant irrespective of electrode excitations, as the capacitances depend on the \textit{geometry} only.
4.3 Formulation

We consider an electrical capacitance volume tomography (ECVT) system. An ECVT system with $N$ electrodes with voltage excitations $V_1, V_2, \ldots, V_N$ would cause (total) charges $Q_1, Q_2, \ldots, Q_N$ to be induced on the respective electrodes. The electrode charges can be expressed as a linear response to the the electrode potentials though the following generic relation

$$Q_i = \sum_j C_{ij} V_j$$  \hspace{1cm} (4.2)

where $C_{ij}$ is the capacitance coefficient matrix. Note that these capacitance coefficients are not the (measured) mutual electrode-pair capacitances, see [71]. The surface charge density on any electrode is in general spatially non-uniform and its distribution can be approximately computed by, for example, discretizing each of the electrodes into sufficiently small segments with uniform charge distribution. In what follows, we parametrize such segmentation by $\zeta_i, i = 1, \ldots, M$, where $M$ is the total number of such small segments in a given plate\(^2\). The charge distribution on the source $\rho^k_s(\zeta_i)$ and receiver $\rho^k_d(\zeta_i)$ electrodes due to a given multi-electrode excitation pattern $V^k = [V^k_1, V^k_2, \ldots, V^k_N]^T$ can be found from a generalization of (4.2), i.e.,

$$\rho^k_s(\zeta_i) = \sum_j [\mathcal{R}_s]_{ij} V^k_j$$  \hspace{1cm} (4.3)

$$\rho^k_d(\zeta_i) = \sum_j [\mathcal{R}_d]_{ij} V^k_j$$  \hspace{1cm} (4.4)

The presence of other active electrodes perturbs the charge distributions $\rho_s(\zeta_i)$ and $\rho_d(\zeta_i)$, through the coupling encoded by reaction matrices $[\mathcal{R}_s]$ and $[\mathcal{R}_d]$. To account

\(^2\)As a practical matter, if ECVT electrodes identical in shape, then the same type of parametrization can be used for all electrodes. This is assumed here for simplicity.
for such perturbation on the charge distributions at the two electrodes forming a given capacitive pair caused by a multi-electrode excitation (mutual coupling effect), the former charge distributions are first determined from (4.3) and (4.4) as \textit{induced sources} and then used as \textit{impressed sources} to compute the consequent electric fields inside the imaging domain [11, 21].

4.4 ECVT Sensitivity under Multi-Electrode Excitation

In this Section, we first verify that the application of driven electrode guards in a multi-electrode excitation pattern for ECT indeed reduces field “singularities” in the region between the source electrode and the adjacent ones (i.e., guards), thereby allowing for an increase in the field strength deeper into the sensing domain. In addition, ECT sensitivity maps based on different multi-electrode excitation patterns are computed to illustrate the effecting of different sensitivity maps between source/receiver electrodes pairs by, e.g., a modification on the excitation pattern of the guard electrodes. This is in contrast to conventional ECT setting with single-electrode excitation, where the number of different sensitivity maps is limited to \(n(n - 1)/2\).

The presence of large gradients in the electric potential near to the source electrode edges gives rise to a highly localized sensitivity as illustrated in Fig. 4.2(a). This hampers the imaging of regions near the center of the imaging domain. This field singularity can be reduced by exciting guard electrodes using a tapered voltage distribution. The ensuing sensitivity can then be computed using the impressed charge distribution approach outlined above. This is referred here as an “active control guard” (ACG) strategy, and is illustrated in Fig. 4.2(b), where the source
electrode numbered as 1 as well as the electrodes numbered as 2 and 8 are excited by 1 Volt, and the electrodes numbered as 3 and 7 are excited by 0.5 Volts. The entire (two-dimensional) sensing domain is discretized by $128 \times 128$ pixels in Fig. 4.2.

![Electric field magnitude [V/m] distribution in the 2-D ECT domain under (a) conventional excitation and (b) multi-electrode excitation. See text for details.](image)

We next consider a adaptive electrical capacitance volume tomography (AECVT) [55] configuration, where each acquisition is performed using synthetic electrodes that combine the signals from a set of small segments around the three-dimensional cylindrical domain, as shown in Fig. 4.3. In practice, a minimum number of segments is chosen to form a synthetic electrode as dictated by the required SNR. Simulations are performed to compute AECVT sensitivity maps with and without active control guards.

In the conventional (i.e. single-electrode excitation) AECVT configuration, the electric field at a chosen point near the source electrode edges was computed as 81.1
(a) Perspective view of the electric field magnitude distribution on the cross-section of an AECVT sensor (including the outer dielectric wall between the electrodes and the inner cylindrical imaging domain) without (L) and with (R) active guards. For ease of comparison, the distributions are clipped to the same magnitude interval as indicated by the colorbars. Highest and lowest values are also indicated.

(b) Cross-sectional view of the electric field magnitude distribution inside the inner cylindrical domain (outer cylindrical dielectric wall not included) without (L) and with (R) active guards.

Figure 4.3: Electric field magnitude distribution in a AECVT sensor configuration.
V/m, whereas in the AECVT configuration with active guards, the field value at the same point decreased to 34.8 V/m. In addition, the lowest field magnitude in the sensing domain was found to be about 5.5 V/m in the latter configuration, whereas in the conventional configuration, the lowest field magnitude was 1.1 V/m. This indicates that, by using active guards, the ratio of the field value at the center to the field value near the source electrode can be considerably increased, enabling larger voltage signals to be used in the source electrode and hence better SNR to capture small capacitance variations due to perturbations near the center of the domain. Similar conclusions can be drawn from the electric field magnitude distributions depicted in Fig. 4.3.

Fig. 4.4 shows the induced charge distributions on the twelve segments comprising the synthetic source electrode plate when using three different excitations: a conventional one with source electrode at 1 V and all others grounded shown in Fig. 4.4(a), a symmetric multi-electrode excitation shown in Fig. 4.4(b), and an asymmetric multi-electrode excitation shown in Fig. 4.4(c). The respective induced charge density distributions are plotted on the right. It is evident that the resulting charge density is less uniform in the conventional AECVT configuration of Fig. 4.4(a) than in the AECVT configuration with active guard electrodes of Fig. 4.4(b). The higher charge concentration near the edges of the synthetic electrode in the conventional configuration is associated to the presence of a field singularity there. Fig. 4.4(c) illustrates that, when active guards are excited at only one side of the source electrode, an asymmetry in the charge density is induced, as expected. Fig. 4.5 shows the sensitivity maps produced by these three configurations. The possibility of producing correlated

\[^{3}\text{Note that mitigation of field singularities reduces the chances of electric arc discharges due to electrostatic buildup.}\]
(a) AECVT without Active Guards

(b) AECVT with Active Guards

(c) AECVT with Half Active Guards

Figure 4.4: AECVT charge density with and without active guards
but not identical sensitivity maps between fixed electrode pairs that is enabled by
different guard electrode activations can be exploited for imaging purposes.

![Figure 4.5: Sensitivity maps for different multi-electrode excitations.](image)

(a) Sensitivity map without active guards.

(b) Sensitivity map with symmetric active guards.

(c) Sensitivity map with asymmetric active guards.

Figure 4.5: Sensitivity maps for different multi-electrode excitations.
4.5 Conclusions

This chapter presented the computation of sensitivity maps in AECVT configurations employing multi-electrode excitations based on induced charge densities computed a posteriori. This approach fully accounts for the mutual-coupling arising from simultaneous and arbitrary multi-electrode excitations, including active control guards. We also illustrated the advantages of employing active guards to (1) suppress field singularities due to large gradients between adjacent electrodes thus enabling better SNR for capturing capacitance variations due to perturbations near the center of the domain, as well as to (2) provide additional degrees of freedom for manipulating spatial sensitivity maps and hence acquiring more spatial information from a fixed set of (boundary) electrodes.
Chapter 5: Adaptive Electrical Capacitance Tomography

A new measurement acquisition strategy is introduced to increase image resolution in adaptive electrical capacitance tomography (AECT) and adaptive electrical capacitance volume tomography (AECVT). The proposed strategy is based on the manipulation of synthetic electrodes comprised of a set of smaller physical electrodes (segments) enabled by AECT/AECVT hardware. The synthetic electrodes are sequentially activated with partial overlap of constituent segments to provide a fine-stepped axial and/or azimuthal electronic scan along the entire sensor. Consequently, an increased number of independent capacitance measurements are made available. Reconstruction results are shown to illustrate the enhancement on resolution versus conventional electrical capacitance volume tomography.

5.1 Introduction and Motivation

Recent advances on ECT technology have included the development of electrical capacitance volume tomography (ECVT) [46, 80, 81, 85] and adaptive (AECVT) versions thereof [55, 100, 101]. Mathematically, the ECVT problem involves first computing the sensitivity map (forward problem), i.e. determining the influence of small (voxel) permittivity perturbations on the inter-electrode capacitances, for every voxel in the RoI and every electrode pair. The relation between the mutual capacitance
$C_{i,j}$ between electrodes $i$ and $j$, and the permittivity distribution $\epsilon(\vec{r})$ inside the RoI can be expressed as

$$C_{i,j} = -\frac{1}{V_{i,j}} \int \int_{S} \epsilon(\vec{r}) \nabla \phi(\vec{r}) \cdot \hat{n} \ dS,$$  

(5.1)

where $S$ is the electrode area, $\hat{n}$ is the unit vector normal to it, $\phi(\vec{r})$ is the electric potential, and $V_{i,j}$ is impressed voltage difference between electrodes $i$ and $j$ with $i,j = 1,\ldots,N_e$, $i \neq j$ and $N_e$ being the number of electrodes. The total number of distinct electrode pairs (mutual capacitance measurements) is $N_p = N_e(N_e - 1)/2$.

Relation (6.1) is an integral equation for $\epsilon(\vec{r})$, where the (measured) mutual capacitance $C_{i,j}$ is related to $\epsilon(\vec{r})$ in a nonlinear fashion due to the dependence of $\phi(\vec{r})$ on $\epsilon(\vec{r})$. By discretizing the RoI into small voxels, changes on individual voxel permittivities can be mapped to changes on the measured capacitances. Under a first-order Born approximation, this relation can be expressed in linear form as [51]

$$c = S \cdot g$$  

(5.2)

where $c$ is a column vector representing the set of mutual capacitances (measurement vector) and $g$ is a column vector describing each voxel permittivity (image vector). The elements $S_{pq}$ of the sensitivity matrix $S$ write as [54]

$$S_{pq} = -\int \int_{V_q} \nabla \varphi_i(\vec{r}) \cdot \nabla \varphi_j(\vec{r}) \ dV,$$  

(5.3)

where $\varphi_i(\vec{r})$ is the potential due to a unit voltage excitation on electrode $i$ with all other electrodes at ground potential, and the indices $i,j$ above comprise an electrode pair, indexed by $p = (i-1)N_e + j$, with $i < j$. The integral is carried out over each voxel volume $V_q$, $q = 1,\ldots,N_q$ with $N_q$ being the number of voxels comprising the RoI.
A major challenge for ECT, as well as for ECVT, is the relatively low spatial resolution because of the ‘soft-field’ nature of the interrogating electric field. Although resolution can benefit from better imaging reconstruction algorithms, diminishing returns are expected without concomitant advances on sensor hardware capabilities and data acquisition strategies. For adequate resolution, the RoI should be discretized by a sufficiently large number of voxels, implying $N_q \gg N_p$. In other words, the dimension of the vector $g$ in (6.2) is much larger than that of $c$ and the problem becomes severely underdetermined. Although this does not preclude solving for $g$ given $c$ since regularization or similar strategies can be utilized, it negatively impacts the achievable image resolution. To mitigate the underdeterminacy of the problem, one could attempt to increase $N_p$ by employing smaller electrodes; however, this solution is not satisfactory because it decreases the signal-to-noise ratio (SNR) of the capacitance signal, which is proportional to the electrode area. A number of alternative approaches have been developed for ECT and ECVT to increase the number of independent measurements without penalizing the SNR, including mechanical rotation and combination of pre-existing electrode responses [63, 94]. More recently, AECVT has been introduced to enable the electronic combination of a large number of small-sized electrode segments to form synthetic electrodes [55, 100], while maintaining a given minimal area requirement.

In this work, we describe a new capacitance acquisition strategy, enabled by adaptive sensors, based on the sequential activation of synthetic electrodes having partial overlap of constituent segments and providing a fine-stepped electronic scan along the axial or azimuthal direction of the sensor. This strategy significantly increases the number of independent capacitance measurements. Image reconstruction results
based on the Landweber algorithm are included to illustrate the resolution enhancement of the proposed strategy versus conventional ECT/ECT.

5.2 AECVT-enabled electronic scanning

We first consider an ECVT sensor with a total of 24 electrodes. The electrodes are arranged into 4 planes (along axis) and each plane has 6 electrodes (along azimuth), as shown in the drawing at the top left portion of Figure 5.1. The electrodes are mounted around the dielectric wall surrounding the RoI with relative permittivity equal to 3.0. The AECVT sensor has similar geometry to the ECVT sensor except that the area corresponding to each original electrode is now segmented into a set of 9 equally-sized and equally-spaced small electrodes, as depicted in Figure 5.1. This yields a AECVT sensor comprising a total of 216 electrode segments. The inner diameter of the ECVT sensor is 5 cm and the outer dielectric wall (vessel) thickness is 0.5 cm.

As noted before, for acquisition purposes such individual electrode segments are combined to form larger synthetic electrodes as to meet the minimum SNR requirements. In principle, the segments comprising a given synthetic electrode can be selected arbitrarily, with the optimal selection of segments depending on the geometry of the RoI and the desired sensitivity patterns. In particular, individual electrode segments can be combined to form $3 \times 1$ synthetic electrodes of rectangular shapes, as illustrated in Figure 5.2, and further combined to form square-shaped $3 \times 3$ synthetic electrodes by combining three adjacent $3 \times 1$ segments along the axial direction. A similar arrangement can be obtained by combining segments into $1 \times 3$ synthetic electrodes and further combining three adjacent segments along the azimuthal direction.
Figure 5.1: AECVT sensor configuration with $6 \times 4 \times 9 = 216$ segments. Each segment can be excited independently and their signal combined, thus enabling synthetic (reconfigurable) electrodes. The top-left inset illustrates a conventional ECVT sensor with $6 \times 4 = 24$ electrodes.

into $3 \times 3$ synthetic electrode segments, as illustrated in Figure 5.3. These synthetic electrodes can be activate sequentially to scan the entire RoI. In particular, by allowing a partial overlap of synthetic electrode area, axial and azimuthal scannings based on $3 \times 3$ synthetic electrodes can be done in a fine-stepped manner to provide a larger number of independent mutual capacitance measurements. This is illustrated in Figure 5.2 by means of synthetic electrodes indicated as 01, 02, and 03. The azimuthal acquisition mode scans the entire AECVT sensor in such a way that each successive synthetic electrode is offset by an equivalent $3 \times 1$ electrode, so as to produce $18 \times 3$ distinct synthetic $3 \times 3$ electrodes along the azimuthal direction in each plane. Since there are four planes, this corresponds to 72 synthetic electrodes overall.

The axial acquisition mode on the other hand scans the entire AECVT sensor so as to yield 10 synthetic $3 \times 3$ electrodes along each axial column. Since there are
Figure 5.2: Schematics of an AECVT sensor with axial synthetic electrode scanning where each synthetic electrode is formed by $3 \times 3$ segments. During scanning, different synthetic electrodes are formed by a shift of 1/3 of the electrode length along the axial direction, as exemplified by the synthetic electrodes labeled as 01, 02, and 03 in this figure. This means that different synthetic electrodes can have overlapped areas. This allows for a finer scanning of the RoI and higher achievable resolution. Note that a given mutual capacitance measurement involving two synthetic electrodes does not admit overlapping electrode pairs, of course.

6 columns, this corresponds to a total of 60 synthetic electrodes. The azimuth and axial scans can be applied together to further increase the number of capacitance pair combinations being created. The resulting combined-mode sensor employing $3 \times 3$ synthetic plates yields 180 synthetic electrodes. Note that, since any two adjacent synthetic electrodes with overlapped segments cannot be combined into an electrode pair for the purpose of measuring the mutual capacitance, there are 2412 possible capacitance pair combinations (as opposed to 2556, if the mutual capacitance from overlapped pairs could be measured) in the azimuth case with 72 synthetic electrodes, 1668 possible capacitance pair combinations (as opposed to 1770) in the axial case with 60 synthetic electrodes, and 14220 possible capacitance pair combinations (as
opposed to 16110) in the combined case. Table 1 summarizes the total number of synthetic electrodes and mutual capacitance measurements in each case.

It should be stressed that the measurements enabled by the set of all such $3 \times 3$ synthetic electrodes is actually larger than the set of acquisitions from either a mechanical rotation (azimuthal case) or a mechanical translation (axial case) of the ECVT sensor in steps of a $1/3$ fraction of the original ECVT electrode size. This is because the scanning of the ‘sender’ and ‘receiver’ electrodes is done independently, whereas in a mechanical rotation, both are necessarily moved together. More importantly, the electronic scanning proposed here does not incur any of the practical disadvantages associated with a physical movement of sensor parts such as higher power consumption, limitations on speed, and the need for high mechanical precision and reliability.
Table 5.1: Sensor Configurations

<table>
<thead>
<tr>
<th>Sensor</th>
<th>No. of Electrodes</th>
<th>Measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECVT</td>
<td>24</td>
<td>276</td>
</tr>
<tr>
<td>axial-scan AECVT</td>
<td>60</td>
<td>1668</td>
</tr>
<tr>
<td>azimuth-scan AECVT</td>
<td>72</td>
<td>2412</td>
</tr>
<tr>
<td>full-scan AECVT</td>
<td>180</td>
<td>14220</td>
</tr>
</tbody>
</table>

5.3 Image Reconstruction

Although the adaptive sensing scheme increase the number of independent measurements, the ill-conditioning of the problem is exacerbated and so it is important to tailor image reconstruction algorithms to the new measurement data set to be able to extract higher resolution. The sensitivity matrix $S$ in ECVT provides the information regarding the ability to detect voxel permittivity perturbations in the RoI. The singular value decomposition (SVD) can be used to analyze this problem by finding the principal components of the sensor model. The SVD decomposes the sensitivity matrix as

$$S = U\Sigma V^T$$  \hspace{1cm} (5.4)

where $U$ and $V$ are $N_p \times N_p$ and $N_q \times N_q$, respectively, unitary matrices with column vectors spanning $R^{N_p}$ and $R^{N_q}$ respectively, and $\Sigma$ is a diagonal matrix of singular values. The singular values in $\Sigma$ can be arranged in magnitude order, i.e. $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and so on [3,27,76,98].

Figure 5.4 and Table 5.2 show the SVD distribution of the sensitivity matrices for AECVT sensors with different number of synthetic electrodes, as well as of the sensitivity matrix for the ECVT sensor with 24 electrodes. In each case, the singular
Figure 5.4: Spectral plot of the singular value distributions of the sensitivity matrix $S$ for conventional ECVT with 24 electrodes, and AECVT with axial, azimuthal and full scanning with 60, 72 and 180 synthetic electrodes respectively.

values are normalized by the largest one. The condition number of the sensitivity matrix $\kappa(S)$, which is a measure of the amount of ill-conditioning of the problem, is related to the ratio between the maximum and minimum singular values. From this results presented in Figure 5.4 and Table 5.2, it is observed that an increase on the number of measurements produces an increase on $\kappa(S)$. Consequently, the reconstruction becomes very sensitive to measurement errors since the relative measurement error in the capacitance $c + \delta c$ and permittivity $g + \delta g$ vectors are related through [3,27].

$$\frac{\|\delta g\|}{\|g\|} \leq \kappa(S)\frac{\|\delta c\|}{\|c\|}$$  \hspace{1cm} (5.5)
Even though the measurement error $\delta c$ can be reduced by averaging a larger number of measurements in time, this averaging would however increase the acquisition time and hamper real-time imaging. The increase in the ill-conditioning is attributed to the fact that although added measurements enable a fine-stepped scanning, they also produce a higher degree of correlation on the measured capacitances. Even though an increase in the number of synthetic electrodes exacerbate the ill-conditioning of the problem, it important to note that useful information is still added to the measurement data set. This is evident by examining Figure 5.4 again, and also the cumulative distributions shown in Figure 5.5 and Table 5.3: the number of singular values with relative magnitude above a given threshold, say $10^{-3}$ in Figure 5.4, is considerably larger (and increases with the number of synthetic electrodes used) in the AECVT case than in the ECVT case. This means that, if the ill-conditioning is properly dealt with, an increase in the number of electrodes has indeed the potential to increase resolution. An effective way to combat ill-conditioning is to apply regularization techniques to retain only the contribution from the most dominant singular values. This will be discussed in more detail below.

<table>
<thead>
<tr>
<th>Sensor</th>
<th>No of Electrodes</th>
<th>Condition Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional</td>
<td>24</td>
<td>$9.034 \times 10^4$</td>
</tr>
<tr>
<td>Axial 3 $\times$ 3</td>
<td>60</td>
<td>$1.753 \times 10^8$</td>
</tr>
<tr>
<td>Azimuthal 3 $\times$ 3</td>
<td>72</td>
<td>$5.492 \times 10^{14}$</td>
</tr>
<tr>
<td>Full-scan 3 $\times$ 3</td>
<td>180</td>
<td>$6.065 \times 10^{16}$</td>
</tr>
</tbody>
</table>
Figure 5.5: Percentage Cumulative contribution of Singular values of the sensitivity matrix $S$ for conventional ECVT with 24 electrodes, and AECVT with axial, and azimuthal scanning with 60, and 72 synthetic electrodes respectively.

<table>
<thead>
<tr>
<th>Sensor</th>
<th>No of Measurements</th>
<th>95%</th>
<th>98%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional</td>
<td>276</td>
<td>119</td>
<td>155</td>
<td>178</td>
</tr>
<tr>
<td>Axial $3 \times 3$</td>
<td>1668</td>
<td>289</td>
<td>434</td>
<td>550</td>
</tr>
<tr>
<td>Azimuthal $3 \times 3$</td>
<td>2412</td>
<td>338</td>
<td>505</td>
<td>639</td>
</tr>
</tbody>
</table>
5.3.1 Reconstruction Algorithm

We next illustrate the increase in resolution enabled by the proposed electronic scanning though a series of examples. Forward simulations are performed using COMSOL™ Multiphysics finite-element solver. This forward solver obtains the set of mutual capacitance between each electrode pairs using a mesh density that is kept high enough to ensure the accuracy of the solution (as verified through a convergence analysis). To illustrate the relative improvement in resolution, the image reconstruction is performed based on the steepest descent based iterative Landweber algorithm with linear back projection (LBP) initialization. The iterations on the Landweber algorithm are governed by the equation below:

\[ g_{n+1} = g_n - \alpha S^T (Sg_n - c). \]  

(5.6)

where the step size \( \alpha \) is chosen based on the largest eigenvalue \( \lambda_{max} \) of \( S \) [39,91], with \( \alpha \leq 2/\lambda_{max} \).

To combat the ill-conditioning noted above of the adaptive sensor, we adopt a regularization strategy based on a truncated SVD, whereby singular values below a given threshold are discarded [3,28]. Among the various alternatives to determine the threshold parameter [3,26,27], we adopt a two-step strategy. We first use the percentage cumulative contribution rate of singular values. A sharper decay in singular values indicate that the relative contribution of the further (truncated) values becomes less significant. We select the 99% cumulative criterion as tabulated in Table 5.3. Then, the generalized solution for truncated singular values in terms of orthonormal vectors is provided in the relation (5.7) below

\[ g^\dagger = \sum_{i=1}^{k} \frac{u_i^T \cdot c}{\lambda_i} v_i \]  

(5.7)

58
where $\mathbf{u}_i$ and $\mathbf{v}_i$ are the columns of the matrices $\mathbf{U}$ and $\mathbf{V}$. As a second step, the discrete Picard condition (DPC) [27] is applied. The DPC is a stability condition which specifies that $|\mathbf{u}_i^T \mathbf{c}|$ should decay to zero faster then corresponding singular value $\lambda_i$. After regularization, the truncated sensitivity matrix is denoted as $\mathbf{S}_k = \mathbf{U}_k \Sigma_k \mathbf{V}_k^T$, where $\Sigma_k$ is the truncated matrix with the set of $k$ remaining singular values. This truncated matrix is used instead in (6.6).

The simulations that follow also incorporate additive white Gaussian noise such that the noisy signal $\mathbf{c}^* = \mathbf{c} + \mathbf{n}$ has 60 dB of SNR. Note that the residual error $|\mathbf{c} - \mathbf{S}_k \mathbf{g}_k|$ cannot be employed as the stopping criterion in the Landweber iterations since regularization introduce a bias, therefore an empirically-determined value of 200 iterations is used in all simulations.

5.3.2 AECVT Scanning example

Figure 5.6: Arc-shaped dielectric object located centered at middle of the RoI in the axial direction and near the periphery along the RoI cross-section.
Figure 5.7: Comparison of reconstructed images of an arc-shaped dielectric object placed at different positions near the periphery of the RoI: (a) Axially-scanned AECVT sensor. (b) Conventional ECVT sensor. (c) Corresponding test-object locations in the RoI.
Figure 5.8: Two small dielectric spherical objects placed near to each other, with subtended angle of $45^\circ$ as measured from the axis of the RoI. The two objects are successively moved towards the center of the RoI (i.e., along the horizontal direction in the top-left drawing), with fixed lateral separation (and decreasing subtended angle).

Figure 5.6 depicts the test case of an object with the shape of an arc (section of a torus) positioned in the middle of the RoI along the axial direction and near the periphery of the RoI cross-section. Figure 5.7 shows Landweber-based reconstruction results for this object employing an AECVT sensor with electronic scanning along the azimuthal direction comprising 60 synthetic electrodes and a conventional ECVT sensor with 24 electrodes. The reconstruction is done while the test object is moved along azimuthally to different positions near the periphery of the RoI, as shown in Figure 5.7 (c). As evident by comparing Figure 5.7 (b) and (c), the AECVT reconstruction shows better fidelity. It also shows also better image stability as the test object is repositioned inside the RoI. The instability of the conventional ECVT result
is caused by the more abrupt change of the object response between two successive acquisitions along the azimuth, bring forth by the field singularities present in the small gap between the activated electrodes and the grounded ones next to it.

Figure 5.9: AECT synthetic electrode scanning by combining 3 segments

5.3.3 Resolving two small objects: AECT case study

We next show the reconstruction results for two small cylindrical objects placed near to each other with subtended angle of 45° as measured from the axis of the RoI. This configuration is depicted in Figure 5.8. The two objects are simultaneously moved from the periphery to the center of the RoI while maintaining a fixed lateral separation [99]. This process is repeated assuming a smaller initial subtended angle between the objects of 35°. The AECT sensor used in this case has 36 electrode segments.
Figure 5.10: Image reconstruction results for two nearby spherical dielectric objects, placed at different locations inside the RoI with initial subtended angle equal to 45° as depicted in Figure 5.8. (a-e) Results using conventional ECT with 12 fixed electrodes (no electronic scanning). (f-j) AECT results with synthetic electrode scanning by combining 3 segments. (k-o) AECT results with synthetic electrode scanning by combining 4 segments.

The electrodes segments are combined into synthetic electrodes by combining 3 or 4 segments as shown in Figure 6.2. The conventional 36 electrode ECT sensor provides 630 measurements, whereas the AECT sensors combining 3 and 4 electrodes provide 558 and 522 measurements, respectively. The results are also compared with a conventional 12-electrode ECT sensor with 66 measurements. Figure 5.10 and Figure 5.11 show the reconstruction results when the initial subtended angle is equal...
Figure 5.11: Same as Figure 5.10 but with the two objects with initial subtended angle equal to $35^0$ to $45^0$ and $35^0$, respectively. The results in Figure 5.10 indicate that better resolution is obtained using AECT scan than by using conventional ECT.

While AECT is able to resolve the two objects in the $45^0$ case for all positions, including at the center of the RoI, ECT has difficulty doing so. The case shown in Figure 5.11 is more challenging because of the smaller separation, but the AECT result based on 3 segments is able to extend the region where the two objects can be well-resolved. As expected, there is a gradual degradation in resolution for all cases as the objects are moved to the center. This phenomenon is an inevitable consequence of
the smoothing nature of the quasi-static electric field in the RoI, governed by Laplace equation.

5.4 Experimental Results

We now consider experimental results involving a sensor mounted around a plastic pipe with outer diameter 4.5" and inner diameter 4.25" as shown in Figure 5.12 and 5.14. For simplicity, we consider a single-plane sensor with 18 electrode segments and consequently restrict ourselves azimuthal scanning. This provides a proof-of-concept in the ECT context as well.

Figure 5.12: Experimental configuration showing three successive positions of the two dielectric test objects.
The dynamic range of a given channel is obtained by taking the difference $\Delta C$ between the capacitance for an empty RoI and that for a RoI completely filled by a known material (in this case, dry rice), averaged over 2000 samples. The noise strength is measured by taking the variance $\sigma$ of the latter capacitance values. The experiment is performed while keeping the SNR, taken as $20 \log(\Delta C/\sigma)$, above 80 dB for all channels. The measurements are performed at 250kHz with frame rate of 13 frames per second.

Adaptive configurations are explored by scanning the sensor with 2 and 3 segments combined together as shown in Figure 5.12. The resolution is assessed using...
the test configuration outlined in Figure 5.12 and 5.13. The results are compared against those of a conventional ECT sensor with 12 electrodes as shown in Figure 5.14 (b), which provides 66 independent measurements. In all cases, the electrodes have a length of 2” along the axial direction. The total number of measurements using the sensor with 18 segments shown in Figure 5.14 (a) is 117 when 3 segments are combined and 135 when 2 segments are combined. It is clear that the dimension of each synthetic electrode for the 3-segment AECT configuration is comparable to a 6-electrode ECT sensor, whereas that for the 2-segment AECT configuration is
comparable to a 9-electrode conventional ECT sensor. Hence, the adaptive configurations utilize (synthetic) electrodes of larger areas as compared to the conventional 12-electrode ECT sensor, while still providing a higher number of measurements.

Figure 5.15: AECT sensor with synthetic electrodes comprised of 3 segments: measured capacitances for an object at position 3 in Figure 5.12.

Measured capacitances from both adaptive sensors for a test object at position 1-3 are plotted in Figure 5.15 and Figure 5.16. The measured capacitances from a conventional 12-electrode ECT sensor for an object at same position are shown in Figure 5.17. The image reconstruction is performed using iterative Landweber algorithm and 200 iterations. It can be observed from Figure 5.18 that the adaptive sensors, and in particular the one based on synthetic electrodes comprised of 2 segments, yield better resolution, especially away from the periphery of the RoI in this case. It is
expected that the resolution can be further improved by employing smaller segments (with a corresponding increase on the number of synthetic electrode combinations). Consideration of a larger number of segments require additional hardware resources and is beyond of the scope of this paper as better tailored image reconstruction methods also need to be developed to fully harness the capabilities of AECT/AECVT.

### 5.5 Conclusion

A new measurement acquisition strategy enabled by AECVT sensors was introduced whereby overlapped synthetic electrodes comprised of electrode segments of small size are employed to scan the RoI in a finite-stepped fashion along the axial and
azimuthal directions. This proposed strategy yields an increased number of measurement acquisitions while maintaining a minimum electrode area size for all capacitance measurement acquisitions, and hence without compromising the signal-to-noise-ratio. Three-dimensional reconstruction results based on the Landweber algorithm show enhanced resolution and stability in the imaging of dielectric objects compared to conventional sensors. The increase in the resolution enabled by a finite-stepped electronic scanning can be understood from the fact that the spatial sensitivity of a capacitive sensor electrode—based on quasi-static or Laplacian fields—to dielectric perturbations on any voxel inside the RoI depends on the lateral distance of the electrode to said voxel [12]. By employing a fine-stepping, such distance can be more
Figure 5.18: Image reconstruction results based on the Landweber algorithm for two test objects at the three successive positions indicated in Figure 5.12 for: (a)-(c) Conventional 12-electrode ECT sensor, (d)-(f) 18-segment AECT sensor with synthetic electrodes comprised of 3 segments, and (g)-(i) 18-segment AECT sensor with synthetic electrodes comprised of 2 segments.

finely varied thus increasing spatial resolution. In contrast to mechanical scanning strategies introduced before, the proposed electronic scanning does not require any
movement of parts and retains the high-degree of reliability and high-speed of acquisition of conventional ECVT.
Chapter 6: Spatially Adaptive Reconstruction Methods

Image reconstruction techniques for electrical capacitance tomography (ECT) are based on the measurement of set of mutual capacitances between fixed electrode pairs placed at the boundary of the region of interest (RoI). Adaptive ECT (AECT) enriches the available measurement set and the amount of acquired information by utilizing a sensor with much smaller electrode segments and electronically combining them into synthetic electrodes to measure mutual capacitances. However, the increased amount of correlation between different measurements in AECT makes the inverse problem (image reconstruction) more ill-conditioned. As a result, conventional image reconstruction strategies used for ECT need to be modified to fully exploit the potential of AECT. This work introduce space-adaptive image reconstruction algorithms that take advantage of the Laplacian nature of the interrogating field in AECT/ECT by utilizing synthetic electrodes based on different segment partition sizes while reconstructing different portions of the RoI. The reconstruction can be sequentially performed from the peripheral region of the RoI (where the achievable resolution is higher) to the center region (where the achievable resolution is lower). In addition to providing a robust reconstruction that mitigates ill-conditioning, the proposed reconstruction techniques caters to the AECT sensor capability for selective image reconstruction in RoI subdomains.
6.1 Introduction

The increase in the number of electrodes is limited by the signal-to-noise (SNR) level in the measurement as the latter is proportional to the electrode area. To ameliorate the under-determinacy of the reconstruction problem and increase the resolution capabilities, adaptive electrical capacitance tomography (AECT) can be used [55, 100] whereby the sensor is comprised of many small segments that can be electronically combined together to form synthetic electrodes while meeting the minimum area requirement set by a given SNR. The entire sensor can be reconfigured as desired; for example, to provide extra degrees of freedom in the sensitivity map and/or to yield electronic scanning capabilities as illustrated in Figure 6.2, where an azimuth scan is realized with synthetic plates made of multiple segments. However, the added degree of correlation between measurements in AECT makes the inverse problem (image reconstruction) more ill-conditioned. As a result, conventional ECT reconstruction techniques need to be modified to fully exploit the potential of AECT.

This work introduce space-adaptive image reconstruction techniques that cater to the Laplacian nature of the interrogating field in AECT/ECT and utilize different segment electrode partitions while reconstructing different portions of the RoI. The reconstruction can be sequentially performed starting from the peripheral region of the RoI adjacent to the electrodes (where the ultimate achievable resolution is higher) to the center of the RoI (where the achievable resolution is lower). In addition to providing a more robust reconstruction algorithms that mitigates the ill-conditioning problem, the proposed reconstruction techniques also allow multiple AECT sensor modes to be combined for image reconstruction of selected portions the RoI.
6.2 Problem Formulation

6.2.1 ECT sensitivity matrix

The inter-electrode capacitance $C_{ij}$ in ECT for a given a permittivity distribution $\epsilon(\vec{r})$ in the RoI [51,52,81] is given by

$$C_{ij} = -\frac{1}{V_{ij}} \int_{A} \epsilon(\vec{r}) \hat{n} \cdot \nabla \phi(\vec{r}) dA,$$  \hspace{1cm} (6.1)

where $\phi(\vec{r})$ is the electrical potential at a point $\vec{r}$ in the RoI, $A$ is the electrode surface, $\hat{n}$ is the unit vector normal to it, and $V_{ij}$ is the potential difference between a given electrode pair $i, j$. The relation in eq. (6.1) is nonlinear since $\phi(\vec{r})$ depends on $\epsilon(\vec{r})$. By discretizing the RoI into $P$ small pixels (or voxels), changes in each pixel permittivities can be mapped to changes on the measured capacitances assuming a uniform permittivity within each voxel. Under a Born approximation, this relation can be linearized and cast into matrix form as [51]

$$c = S \cdot g$$ \hspace{1cm} (6.2)

where $c$ is a $M \times 1$ column vector representing the set of mutual capacitances (measurement vector), $g$ is a $P \times 1$ column vector describing the set of voxel permittivities (image vector), and $S$ is the $M \times P$ sensitivity matrix [54]. The sensitivity matrix element $[S]_{(ij),p}$ corresponding to an electrode pair $ij$ and a given pixel $p$ is obtained by first computing the electric potential $\phi_i$ produced by exciting electrode $i$ with unit voltage and keeping all other electrodes grounded, and repeating the procedure for all $i$. The following integral is then evaluated over each pixel area $\sigma(p)$ (or voxel volume in ECVT) in the RoI:

$$[S]_{(ij),p} = -\int_{\sigma(p)} \nabla \phi_i(\vec{r}) \cdot \nabla \phi_j(\vec{r}) d\Omega,$$ \hspace{1cm} (6.3)
where \(d\Omega\) is an area element in 2D or volume element in 3D.

In the examples below, we employ a AECT sensor with 36 electrode segments as shown in Figure 6.1 and combine them into synthetic electrodes with 2, 3, 4, 5, or 6 segments. An example of the combination of three segments is depicted in Figure 6.2. The corresponding AECT configurations are abbreviated as AECT \(NE\), where \(N\) is used to indicate the number of electrodes being combined. An 24 electrode AECT configuration is also considered where 2, 3, or 4 segments are combined to form synthetic electrode. The total number of capacitance pair measurements for the 36- and 24-electrode AECT sensors are summarized in Table 6.1-6.2\(^4\).

\(^4\)These numbers take into account the fact that the mutual capacitance between overlapped synthetic electrodes cannot be measured.
Figure 6.2: Electronic scanning by combining 3 segments.

Table 6.1: AECT sensor configurations based on 36 segments

<table>
<thead>
<tr>
<th>Segments combined</th>
<th>Independent measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>594</td>
</tr>
<tr>
<td>3</td>
<td>558</td>
</tr>
<tr>
<td>4</td>
<td>522</td>
</tr>
<tr>
<td>5</td>
<td>486</td>
</tr>
<tr>
<td>6</td>
<td>450</td>
</tr>
</tbody>
</table>

Table 6.2: AECT sensor configurations based on 24 segments

<table>
<thead>
<tr>
<th>Segments combined</th>
<th>Independent measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>252</td>
</tr>
<tr>
<td>3</td>
<td>228</td>
</tr>
<tr>
<td>4</td>
<td>204</td>
</tr>
</tbody>
</table>
6.2.2 Ill-conditioning and SVD analysis

Singular value decomposition (SVD) is a powerful tool for the analysis of discrete ill-conditioned problems such as the ECT/AECT reconstruction. Using SVD, the $M \times P$ sensitivity matrix $S$ can be decomposed as the product $S = U\Sigma V^T$, where $U$ is a $M \times M$ unitary matrix, $V$ is a $P \times P$ unitary matrix, and $\Sigma$ is a $M \times P$ is a diagonal matrix with the singular values of the problem as its elements. The singular values are typically arranged in descending order of magnitude $\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots \sigma_{\min(M,P)}$. The plot of the singular values $\sigma_i$ is referred to as the spectral plot or spectrum in this context. The condition number of $S$ is equal to ratio $\sigma_{\max}/\sigma_{\min}$ and is a measure of the degree of ill-conditioning of the problem.

![Spectral SVD plot for a conventional ECT sensor with different number of electrodes.](image)

Figure 6.3: Spectral SVD plot for a conventional ECT sensor with different number of electrodes.
As an example, the normalized spectral plot for the conventional ECT sensors with different number of electrodes is shown in Figure 6.3. This plot indicates that as the number of electrodes increases problem becomes more ill-conditioned. In comparison, the spectral plot of the AECT sensor with 36 electrode shown in Figure 6.1 is in Figure 6.4 for different scanning modes. The spectral plot indicate that, although number of measurements have increased, the problem has become more ill-conditioned. The increase in ill-conditioning is attributed to higher correlation in sensitivity map of successive measurements. Despite the ill-conditioning challenge, it is important to stress that the amount of information provided by the AECT data is higher than the ECT data. This is reflected in the more gradual slope of the curves in Figure 6.4 versus those in Figure 6.3 and from the larger overall number of singular values above a significant threshold. The AECT sensors exhibit about 130 singular values above the $10^{-2}$ threshold, whereas ECT sensors include no more than about 80. Note that a deeper steep in the spectral plot means that successive singular value components contribute with progressively less extra information.

Although extra information is extracted by AECT, the increase in ill-conditioning makes the reconstruction task more difficult. Therefore, a specialized reconstruction algorithm need to be design that cater to the characteristics of the AECT. This is discussed next.

6.2.3 Exploiting the Laplacian nature of the ECT field

In order to develop an efficient image reconstruction method for AECT, the fundamental nature of ECT sensing field is examined here. The potential distribution inside the ECT sensor is governed by the Laplace equation, $\nabla \cdot (\epsilon \nabla \phi) = 0$. For a
given charge density distribution $\rho(\vec{r}')$, the potential is determined by the integral

$$\phi(\vec{r}) = \int_{\Omega} G(\vec{r}, \vec{r}') \rho(\vec{r}') \, d\Omega$$

(6.4)

where in 3D, the Green’s function for the Laplace operator is given by

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi |\vec{r} - \vec{r}'|}$$

(6.5)

The Laplacian field has smoothing properties as evident from (6.5). In particular, it can be easily verified that higher order spatial derivatives of $G(\vec{r}, \vec{r}')$ become progressively smaller as the distance from the observation point $\vec{r}$ to the source point $\vec{r}'$ increases. In other words, the resulting electrical field becomes smoother and its spatial bandwidth gradually decreases with distance. This means that the Laplacian field can’t convey information contained in its near field to the far field [12]. In the context of ECT, this translates to the fact that the achievable resolution is higher

Figure 6.4: Spectral SVD plot for 36-electrode AECT sensor with different synthetic electrode configurations (combined segments).
closer in portions of the RoI closer to the electrodes than far from them. This is illustrated in Figure 6.5, where the sensitivity map for an ECT sensor with nine electrodes is shown for different electrode pairs. It can be observed that electrodes that are close enough only interrogate a limited sensing zone close to the electrodes (hence with higher resolution) whereas the farthest away electrode interrogate much broader sensing area (hence with lower resolution). If a small dielectric object is present near the sensing electrodes it would effect the capacitance measurement more strongly than when the same object is located near the center of the sensing domain. The sensitivity is also function of the electrode size. For smaller electrodes, the sensitivity becomes more focused near the electrode.

In order to quantify these observations in the context of ECT, we define the cumulative sensitivity distribution (CSD) parameter as the integral of the sensitivity map over a portion of the RoI, with the integral over the whole RoI is normalized to one. For the region of integration, we define the peripheral domain as those points in the RoI which are less than a given distance \( R_p \) away from the boundary of the RoI. As illustrated in Figure 6.6, the peripheral domain thus defined corresponds to the annular region in grey color. The inner domain is indicated in white color in the same figure. We let \( R_p \) vary and compute the CSD of a given electrode pair \( i, j \) over the peripheral domain. The CSD is computed as function of \( R_p \) with \( 0 \leq R_p \leq R \), where \( R \) is the overall radius of the RoI. The resulting CSD plotted for ECT sensors with 6 and 12 electrodes is shown in Figure 6.7 and Figure 6.8, for different electrode pairs. These CSD plots indicate that for close electrodes pairs (1,2 and 1,3 for example) the sensitivity is mostly distributed in the peripheral zone whereas for far apart electrode pairs (1,4 in the 6-electrode case and 1,6 and 1,7 in the 12-electrode case,
Figure 6.5: Sensitivity plots for $S_{1,3}$ and $S_{1,d}$ (where $d$ stands for diagonally opposite electrodes) for (A)ECT sensors with different number of (segments) electrodes.
Figure 6.6: Partitioning the sensing domain

Figure 6.7: Cumulative sensitivity distribution plot for 6-electrode ECT sensor.
Figure 6.8: Cumulative sensitivity distribution plot for 12-electrode ECT sensor.

Figure 6.9: Cumulative sensitivity distribution of electrode pair $S_{1,2}$ for ECT sensors with different number of electrodes.
for example) the sensitivity is more uniformly distributed throughout the depth of the RoI. The CSD for the different AECT sensor configurations is plotted in Figure 6.10 and Figure 6.11. An AECT sensor with smaller synthetic electrode scanning has many more available measurements interrogating the peripheral region that an AECT sensor with larger electrodes. Moreover, the AECT sensing depth can be increased by using larger synthetic electrode as shown in Figure 6.12 for $S_{1,2}$. A key conclusion from the above observations is that one can exploit the flexibility afforded by AECT and utilize synthetic electrodes of different sizes: *smaller ones to sense the peripheral region with higher resolution and larger ones to sense the central region of the RoI with higher sensitivity*. This strategy will be implemented in the space adaptive reconstruction algorithms discussed in the next Section.

![Figure 6.10: CSD for 36-segments AECT sensor with synthetic electrodes comprised of 2-segment(2E).](image)

Figure 6.10: CSD for 36-segments AECT sensor with synthetic electrodes comprised of 2-segment(2E).
Figure 6.11: CSD for 36-segments AECT sensor with synthetic electrodes comprised of 6-segment (6E) combining.

### 6.3 Image Reconstruction Algorithms

The CSD can be used to determine the effective distance from periphery where a given electrode pair measurement is most effected by perturbations on the dielectric distribution. Therefore, measurements are partitioned on the basis of CSD contribution to different spatial regions (extending from periphery). Image reconstruction is now performed using iterative algorithms using selective measurements. Three basic algorithms are proposed operating on the basis of a partitioning of the RoI into different regions as indicated in Figure 6.13 In all three cases, the iterative reconstruction engine utilizes either the well-known Landweber iterative reconstruction technique or the diagonally relaxed orthogonal projection (DROP), which is a block algebraic
Algorithm 1 utilizes a single acquisition, i.e. fixed synthetic electrode to achieve reconstruct in different portions of the RoI as summarized in the descriptor below. Algorithm 2 on the other hand incorporates multiple AECT sensor scanning modes for different spatial regions, as detailed in the second descriptor below. Finally, Algorithm 3 is a variant of Algorithms 1 and 2 where in a nonlinear FEM model of the sensor for imaging in the center of the sensing zone is utilized (instead of the approximate linear model.)
**Algorithm 1** Spatially adaptive reconstruction using fixed AECT sensor

**Input:** Capacitance vector $[c]_{M \times 1}$, sensitivity matrix $[S]_{M \times P}$, sensor radius $R_{\text{inner}}$, desired CSD threshold $q$ (see below, default $q = 0.99$).

**Output:** Permittivity distribution vector $[g]_{P \times 1}$.

1. Find the total number of synthetic electrodes $N$
2. for $k = 1$ to $N/2$ do
3. Compute cumulative sensitivity distribution (CSD) as a function of radial distance $R_p$ from periphery
4. end for
5. Partition the RoI into $R_1, R_2, R_3, \ldots$ subdomains according to CSD, such that $P = P_1 + P_2 + P_3$ where $P$ is the total number of pixels.
6. Find the measurements $m_1 \in M$ having (100 $\times$ $q$)% CSD in $R_1$.
7. Apply iterative solver such as Landweber or DROP to find $[g_1]_{P_1 \times 1}$ in $R_1$, $[g_1]_{P_1 \times 1} = f([S]_{m_1 \times P_1}, [c]_{m_1 \times 1})$.
8. Find measurements $m_2 \in M$ with (100 $\times$ $q$)% CSD in $R_1 + R_2$.
9. Let $m_2 = m - m_1$.
10. Proceed with image reconstruction using $g_1$ as initial value for iterative solver to compute $[g_1 + g_2]_{P \times 1}$ in the $R_1 + R_2$ partition, $[g_1 + g_2]_{P \times 1} = f([S]_{m_2 \times P}, [c]_{m_2 \times 1})$.

**Algorithm 2** Spatially adaptive reconstruction using multiple AECT sensor modes

**Input:** Capacitance vectors $[c_1]_{M_1 \times 1}, [c_2]_{M_2 \times 1}, [c_3]_{M_3 \times 1}, \ldots$, sensitivity matrices $[S_1]_{M_1 \times P}, [S_2]_{M_2 \times P}, [S_3]_{M_3 \times P}, \ldots$, sensor radius $R_{\text{inner}}$, $q$ (see description on Algorithm 1).

**Output:** Permittivity distribution vector $[g]_{P \times 1}$.

1. Find the total number of synthetic electrodes $N$
2. for $k = 1$ to $N/2$ do
3. Compute CSD for each AECT sensing mode, e.g. $CSD_{2E}, CSD_{3E}, \ldots$
4. end for
5. Partition RoI into $R_1, R_2, R_3, \ldots$ subdomains according to CSD such that $P = P_1 + P_2 + P_3$.
6. Find the measurements $m_1 \in M_1$ having (100 $\times$ $q$)% CSD in $R_1$ using smaller electrode AECT sensor e.g. $CSD_{2E}$.
7. Apply iterative solver such as Landweber or DROP to find $[g_1]_{P_1 \times 1}$ in $R_1$, $[g_1]_{P_1 \times 1} = f([S_1]_{m_1 \times P_1}, [c_1]_{m_1 \times 1})$.
8. Find measurements $m_2, m \in M_2$ with (100 $\times$ $q$)% CSD in $R_1$ and $R_1 + R_2$ subdomains, respectively, using larger electrode AECT sensor e.g. $CSD_{3E}$.
9. Let $m_2 = m - m_1$.
10. Compute the residual capacitance vector $c_2 = c_2 - S_2 g_1$.
11. Use iterative solver such as Landweber or DROP to compute $[g_2]_{P_2 \times 1}$ in $R_2$ subdomain, $[g_2]_{P_2 \times 1} = f([S_2]_{m_2 \times P_2}, [c_2]_{m_2 \times 1})$. 


Figure 6.13: RoI partitioned into $R_1, R_2, R_3$ subdomains.

In all cases considered here, the iterations for the fixed step-size Landweber algorithm [45,91] are governed by

$$g_{k+1} = g_k - \alpha S^T(Sg_k - c).$$  \hspace{1cm} (6.6)

where the step size $\alpha$ is bounded by the largest eigenvalue $\lambda_{max}$ of $S^TS$ [39,91], with $\alpha \leq 2/\lambda_{max}$. Since this algorithm may produce numerical artifacts in the form of negative values, a projection operator is used as a non-negativity constraint:

$$P([g_k]_i) = \begin{cases} [g_k]_i & \text{for } [g_k]_i \geq 0 \\ 0 & \text{for } [g_k]_i < 0 \end{cases}$$  \hspace{1cm} (6.7)

for $i = 1, \ldots, P$. On the other hand, the ART-based DROP method can expressed in matrix form as

$$g_{k+1} = g_k + w_k AS^T W (c - Sg_k)$$  \hspace{1cm} (6.8)

where $w_i$ is a relaxation parameter, $0 < w_i < 2$, and $A$ and $W$ are diagonal matrices as defined in [9,30,31]. A detailed discussion on DROP is not the objective of this paper,
Figure 6.14: Flow chart summarizing the two-stage space-adaptive methods. \(M_1, M_2, M_3\) shows the number of measurements for different AECT modes. \(P\) is the total number of pixels in the RoI, and \(R_1, R_2\) indicated the subdomains as shown in Figure 6.13.
see further implementation details can be found [9, 30, 31]. The DROP technique is considered here for evaluating its possible advantages versus the Landweber technique, as considered in the next Section. Since both iterative Landweber and DROP have semi convergence properties, the residual error $r_k = \|c - Sg_k\|^2_2$ is used as convergence criterion.

6.4 Results and Discussion

Apart from the Laplacian nature of the interrogating field discussed in the previous Section, resolution in ECT is limited by the number of measurements, electrode size, and smoothing effects introduced by the regularization methods. When evaluating resolution in ECT, it is important to distinguish two types of resolution: spatial resolution and permittivity contrast. Spatial resolution has to do with the sharpness of the image and the ability to resolve (distinguish) two nearby dielectric objects [48]. Permittivity contrast resolution is the ability of evaluate the permittivity contrast between different objects (sometimes also referred as rationometric resolution).

6.4.1 Spatial Resolution and Electrode Size

The first set of simulations considered here evaluates the enhancement in spatial resolution by using AECT sensor. The set up consists of two small objects with same permittivity placed near the periphery of the RoI, i.e. very close to sensing electrodes. The image reconstruction is performed using two different set of measurements. The first set is based on the use of measurement pairs that provide the dominant contributions for the region up to 2 cm away from periphery ($R_1$ subdomain) and the other for the region up to 3 cm away from the periphery ($R_1 + R_2$ subdomains). The RoI radius is 5cm. The reconstruction is performed first for the $R_1$ subdomain and the
results are then used to performed reconstruction for the $R_2$ subdomain, as shown in Figure 6.15. The results in Figure 6.16 show that the two-stage reconstruction strategy based on AECT is able to better resolve the objects compared to conventional ECT results. In addition, the resolution is improved by a decrease on the synthetic electrode size, from 4 segments (AECT-4E) to 2 segments (AECT-2E), with a consequent increase on the total number of measurements, as indicated in Table 6.3. Landweber and DROP give roughly equivalent results.

![Figure 6.15](image)

Figure 6.15: RoI partitioning for adaptive reconstruction: (a) $R_1$ subdomain (in yellow) and (b) $R_1 + R_2$ subdomain (in yellow).

<table>
<thead>
<tr>
<th>AECT sensor configuration</th>
<th>$R_1$ subdomain</th>
<th>$R_1 + R_2$ subdomain</th>
</tr>
</thead>
<tbody>
<tr>
<td>2E</td>
<td>144</td>
<td>216</td>
</tr>
<tr>
<td>3E</td>
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<td>180</td>
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<td>4E</td>
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<td>144</td>
</tr>
<tr>
<td>6E</td>
<td>36</td>
<td>108</td>
</tr>
</tbody>
</table>
6.4.2 Rationometric resolution

The next set of simulations addresses rationometric resolution aspects. First, two test objects of different sizes and same permittivity are considered placed opposite in the RoI and both close to the periphery. It is well known that a conventional ECT sensor responds strongly to the larger object than to the smaller object, leading to a larger permittivity estimate for the former. In order to ameliorate this problem, an AECT sensor is utilized with an adaptive reconstruction strategy having $R_1 = 2.5$ cm and $R_2 = 4$ cm. The simulation results shown in Figure 6.17 indicate that AECT provides a more close estimate for the (same) permittivity of the large and small objects.
than the conventional ECT result. In addition, the two stage reconstruction strategy also reduces the ill-conditioning as shown in singular value spectrum presented in Figure 6.18. The convergence plot for the residual error for the second stage in Figure 6.19 indicates nearly identical convergence rate for the Landweber and DROP techniques for the AECT-6E sensor configuration.

![Figure 6.17: Conventional ECT versus two-stage AECT image reconstruction based on different (2E, 3E, 6E) synthetic electrode sizes, using Landweber (a-d), and DROP (e-h) methods for a test distribution including large and small objects near the periphery.](image)

Next, we examine two objects with same size and permittivity, one placed near the periphery and the other at the center of the RoI. In this case, due to the weaker sensitivity at the center, the permittivity of the latter object is underestimated by conventional ECT, as show in Figure 6.20. In contrast, the increase on the synthetic
Figure 6.18: Spectral plot for the two-stage reconstruction using AECT sensor based on synthetic electrodes combining six segments (6E).

Figure 6.19: Residual error plot for Landweber (LWB) and DROP methods.
electrode size enabled by AECT indicate a progressive improvement in the permittivity value estimate for the object at the center (due to the increase in the sensitivity produced by larger plates). Here, a two-stage reconstruction is used for AECT with $R_1 = 2.5 \text{ cm}$ for a RoI radius equal to 5 cm. In this case, DROP present better imaging performance than the Landweber technique. The residual error plot shown in Figure 6.21 illustrated the convergence results for DROP and Landweber techniques for second stage AECT-6E reconstruction.

![Image of reconstructed images](image)

Figure 6.20: Conventional ECT versus two-stage AECT image reconstruction based on different (3E, 4E, 6E) synthetic electrode sizes, using Landweber (a-d), and DROP (e-h) methods for identical objects near periphery and center of the sensor.

In summary, it is observed that AECT sensors can improve the performance versus ECT sensors in two main ways. First, AECT sensors enable the use of smaller synthetic electrodes to increase the image resolution particularly for objects near
the periphery of the RoI. Second, AECT sensors enable the use of larger synthetic electrodes to increase the rationometric resolution, particularly for objects near the center of the RoI. Note that since synthetic electrodes are easily reconfigurable in AECT, both smaller and larger electrodes can be used to yield these two improvements at the same time.

![Residual error plot for Landweber (LWB) and DROP methods.](image)

Figure 6.21: Residual error plot for Landweber (LWB) and DROP methods.

### 6.5 Conclusion

We have discussed advantages of using space adaptive reconstruction in conjunction with AECT versus conventional ECT. The results shows that higher spatial resolution can be achieved using smaller synthetic electrodes for reconstruction near the periphery of the RoI. This can also be explained from the Laplacian nature of the interrogating field in the RoI. Peripherical reconstruction data can be subsequently used to aid in the reconstruction of inner subdomains, in a nested fashion. The
results also show that larger synthetic electrode can provide better permittivity estimates (rationometric resolution) than conventional ECT, in particular for objects near the center of the RoI, where the sensitivity is weak. The use of space adaptation also mitigates ill-conditioning in AECT.
Chapter 7: Conclusions and Recommendations

ECT problem is inherently a nonlinear problem, where imaging resolution is limited by the number of measurements and soft-field effect. This work has focused on three distinct aspect of ECT.

1. AECT/AECVT sensor design.

2. SART Image Reconstruction


7.1 Conclusions

The conclusions regarding the contributions made in this dissertation are as follow:

7.1.1 AECVT Sensing

A new measurement acquisition strategy enabled by AECVT sensors was introduced, whereby overlapped synthetic electrodes comprised of 3 × 3 segments each are employed to scan the imaging domain in the axial and azimuthal directions. This proposed strategy yields an increased number of measurement acquisitions while maintaining a minimum electrode area size for each acquisition. Reconstruction results show enhanced resolution and stability for imaging objects compared to the
conventional ECVT sensing. The increase in the resolution provided by electronic scanning in AECVT can be understood from the fact that the voxel sensitivity of an electrode based on interrogating fields of a quasi-static or Laplacian (‘near-field’) nature depends on the distance of the electrode to said voxel [12]. By using electronic scanning, such distance can be more gradually varied without change the (synthetic) electrode plate size and thus without compromising the signal-to-noise-ratio of the measurements.

7.1.2 SART Image Reconstruction

Adaptive electrical capacitance tomography (AECT) increasing the available measurements previously limited by SNR considerations by electronic scanning, however it makes inverse problem more ill-conditioned. This work introduce spatially adaptive image reconstruction technique (SART) that takes advantage of the Laplacian nature of the electrostatic field by utilizing dominant measurements for different spatial regions of the sensor. The reconstruction is performed by partitioning the sensing domain and reconstruction is sequentially performed starting at the pixels in the annular region adjacent to the sensing electrodes and progressively reconstructing deeper inside the sensing domain. In addition to providing robust reconstruction this technique also allows multiple AECT sensor modes to be incorporated for image reconstruction in the selective sensing domain.

The proposed scheme demonstrate the advantages of using space adaptive reconstruction (SART). The results shows that higher spatial resolution can be achieved using smaller size electrode for reconstruction near periphery, whereas larger size electrode can provide better permittivity contrast. The new method also ameliorate the
non-linear artifacts peculiar to ECT reconstruction. These methods provide a robust reconstruction as condition number is reduced by many order of magnitude.

7.1.3 Multi Electrode Excitation

The computation of sensitivity maps in AECVT configurations employing multi-electrode excitations based on induced charge densities computed a posteriori was demonstrated. This approach fully accounts for the mutual-coupling arising from simultaneous and arbitrary multi-electrode excitations, including active control guards. It was also illustrated the advantages of employing active guards to:

1. suppress field singularities due to large gradients between adjacent electrodes thus enabling better SNR for capturing capacitance variations due to perturbations near the center of the domain.

2. To provide additional degrees of freedom for manipulating spatial sensitivity maps and hence acquiring more spatial information from a fixed set of (boundary) electrodes.

7.2 Recommendations

The following recommendations are made for future research in to ECT.

1. AECVT scanning be performed with arbitrary shape electrodes in order to utilize the diversity in sensitivity afforded by the AECVT.

2. Image reconstruction methods need to be develop that combining multiple AECVT sensing modes.

3. It is proposed that SART methods be extended to AECVT sensing.
4. Application of AECVT and SART methods to displacement current phase tomography (DCPT).

5. AECVT requires significant hardware resources to make additional measurement, as a future research direction it is recommended that fast and efficient hardware be develop.

6. Image Reconstruction methods that utilize voltage distribution by employing guard electrodes to provide added dimension in AECVT.
Appendix A: Electrical Capacitance Volume Tomography
Sensitivity Computation

Electrical capacitance volume tomography (ECVT) finds the volumetric distribution of permittivity from the capacitance measurements [51,100]. The electrodes are mounted in the azimuthal and axial direction as shown in Figure A.1. The cross plane measurements provide the added degree of freedom to determine volumetric distribution. The sensitivity of ECVT sensor is found in similar manner as ECT sensor, the only difference is now sensing domain is discretized into voxels [4,49,51].

\[ S_{i,j} = -\iiint_{V_q} \nabla \varphi_i(x,y,z) \cdot \nabla \varphi_j(x,y,z) dV, \quad (A.1) \]

where \( \varphi_i(x,y,z) \) is the potential due to a unit voltage excitation on electrode \( i \) with all other electrodes at ground potential, and the indices \( i, j \) above comprise an electrode pair, with \( i < j \). The integral is carried out over each voxel volume \( V_q \), \( q = 1, \ldots, N_q \) with \( N_q \) being the number of voxels comprising the ROI.

The electrodes placed in the axial direction are referred to as planes as indicated in Figure A.1. Sensitivity map for the 2 plane ECVT sensor having 6 electrodes in each plane are shown in Figure A.2. The sensitivity map interrogates different sensing zones, the cross planes electrodes provide permittivity information along axial direction.
Figure A.1: 6 electrode 2 plane ECVT sensor
Figure A.2: Sensitivity map for 6 electrode 2 plane ECVT sensor
Appendix B: ECVT Sensor Analytical Solution

This section presents the modelling of ECVT sensor, that is to find the electric potential, electric field and hence compute the sensor sensitivity. The ECVT sensor problem considered here is equivalent to finding potential distribution inside an infinite cylinder of radius $a$ due to the boundary excitation $V(a, \phi, z)$. The cylinder is shown in figure below.

![Cylinder with Boundary Excitation](image)

Figure B.1: Cylinder with Boundary Excitation
The potential inside infinite cylinder is governed by the dirichlet boundary condition and can be found

\[ V(r, \phi, z) = \frac{1}{4\pi \epsilon_0} \int G_D(r, r') \rho(r') dv' - \frac{1}{4\pi} \int \Psi(r') \frac{\partial G_D(r, r')}{\partial n'} dS' \]  (B.1)

here \( G_D(r, r') \) is the dirichlet green function for the arbitrary source at \( r' \), assuming the no volume charge density inside the cylinder \( \rho(r') = 0 \) we have

\[ V(r, \phi, z) = -\frac{1}{4\pi} \int \Psi(r') \frac{\partial G_D(r, r')}{\partial n'} dS' \]  (B.2)

The dirichlet green function \( G_D(r, r') \) has to be solved in the cylindrical coordinates. Laplace equation has to solved for the point source to find the green function.

\[ \nabla^2 V(r, \phi, z) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial V(r, \phi, z)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V(r, \phi, z)}{\partial \phi^2} + \frac{\partial^2 V(r, \phi, z)}{\partial z^2} = 0 \]  (B.3)

According to the separation of variable approach, we have

\[ V(\rho, \phi, z) = R(\rho) \Psi(\phi) Z(z) \]  (B.4)

The choice for the interior of the cylinder is given by \( e^{ikz}, e^{im\phi}, I_m(k\rho) \) since we want to have z variation as well. Now using the approach outlined in [38] the green function in the interior of the cylinder can be written as

\[ G_I(r, r') = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_m(k) I_m(k\rho) e^{ikz} e^{im\phi} dk \]  (B.5)

Similarly for the region outside the infinite cylinder we have

\[ G_{II}(r, r') = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} B_m(k) \left[ I_m(k\rho) + C_m K_m(k\rho) \right] e^{ikz} e^{im\phi} dk \]  (B.6)
now for the case of $\rho = a$ we have $G_{II}(r, r') = 0$, so we have

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} B_m(k) \left[ I_m(ka) + C_m K_m(ka) \right] e^{ikz} e^{im\phi} dk = 0 \quad (B.7)$$

$$C_m = -\frac{I_m(ka)}{K_m(ka)} \quad (B.8)$$

and for the continuity of potential at the source point $\rho = \rho'$ we have

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} A_m(k) I_m(k\rho') e^{ikz} e^{im\phi} dk = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} B_m(k) \left[ I_m(k\rho') - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho') \right] e^{ikz} e^{im\phi} dk \quad (B.9)$$

Now we can find the green function in both the region, inside or outside the infinite cylinder. Since we are interested only inside the cylinder we have

$$G_{II}(r, r') = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} B_m(k) \left[ 1 - \frac{I_m(ka) K_m(k\rho')}{I_m(k\rho') K_m(ka)} \right] I_m(k\rho) e^{ikz} e^{im\phi} dk \quad (B.10)$$

considering the source to be at the point $(\rho', \phi', z')$ we have

$$\nabla^2 G = -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z') \quad (B.11)$$

The procedure outlined in [38] is used to find the final expression for the cylindrical green function for an infinite cylinder and for the source to be located at the $(\rho', \phi', z')$.

$$G_I(r, r') = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(k\rho)}{I_m(ka)} \left[ I_m(ka) K_m(k\rho') - K_m(ka) I_m(k\rho') \right] e^{ik(z-z')} e^{im(\phi-\phi')} dk \quad (B.12)$$

The dirichlet green function can be used to find the potential distribution inside the cylinder due to the boundary excitation $V(a, \phi, z)$ now using normal as the radial direction we have
\[ V(\rho, \phi, z) = -\frac{1}{4\pi} \int_{s'} V(a, \phi, z) \frac{\partial G_D(r, r')}{\partial \rho'} dS' \quad (B.13) \]

\[ V(\rho, \phi, z) = -\frac{1}{4\pi} \int_{s'} \frac{1}{\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_m(ka)}{I_m(ka)} V(a, \phi, z) \frac{\partial}{\partial \rho'} [I_m(ka)K_m(k\rho') - K_m(ka)I_m(k\rho')] \bigg|_{\rho'=a} e^{ik(z-z')} e^{im(\phi-\phi')} dk d\phi dz \quad (B.14) \]

Since the term after evaluating the derivative in the above expression is the Wronskian, therefore have

\[ \frac{\partial}{\partial \rho'} [I_m(ka)K_m(k\rho') - K_m(ka)I_m(k\rho')] \bigg|_{\rho'=a} = \frac{k}{ka} \quad (B.15) \]

Therefore potential distribution can be found inside cylinder due to boundary excitation \( V(a, \phi, z) \) by

\[ V(\rho, \phi, z) = -\frac{1}{4\pi^2 a} \int \int \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{I_m(ka)}{I_m(ka)} V(a, \phi, z) e^{ik(z-z')} e^{im(\phi-\phi')} dk d\phi dz \quad (B.16) \]

**B.1 Potential due to a Single Electrode Excitation**

The potential due to a single electrode can be model as a cylindrical patch and can be found analytically as the the boundary condition

\[ V(a, \phi, z) = \begin{cases} 1 & \text{for } (\phi, z) \in ([-15, 15], [-d, d]) \\ 0 & \text{elsewhere} \end{cases} \quad (B.17) \]
\[ V(\rho, \phi, z) = \frac{1}{4\pi^2a} \int_{-d}^{d} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{I_m(k\rho)}{I_m(ka)} e^{ik(z-z')} e^{im(\phi-\phi')} dk ad\phi dz \]  

(B.18)

The above expression can be simplified as

\[ V(\rho, \phi, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{I_m(k\rho)}{I_m(ka)} \int_{-d}^{d} e^{ik(z-z')} dz \int_{-\pi/2}^{\pi/2} e^{im(\phi-\phi')} d\phi dk \]  

(B.19)

\[ V(\rho, \phi, z) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{I_m(k\rho)}{km I_m(ka)} e^{im\phi} e^{ikz} \sin\left(\frac{m\pi}{12}\right) \sin(kd) dk \]  

(B.20)

\[ V(\rho, \phi, z) = \begin{cases} 
\frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{e^{im\phi} \sin\left(\frac{m\pi}{12}\right)}{m \int_{-\infty}^{\infty} I_m(ka) e^{ikz} \sin(kd) dk} & m \neq 0 \\
\frac{1}{6\pi} \int_{-\infty}^{\infty} I_0(ka) e^{ikz} \sin(kd) dk & m = 0
\end{cases} \]  

(B.21)

\[ V(\rho, \phi, z) = \begin{cases} 
\frac{4}{\pi^2} \sum_{m=-1}^{\infty} \frac{\cos(m\phi) \sin\left(\frac{m\pi}{12}\right)}{m \int_{0}^{\infty} I_m(ka) \cos(kz) \sin(kd) dk} & m \neq 0 \\
\frac{1}{3\pi} \int_{0}^{\infty} I_0(ka) \cos(kz) \sin(kd) dk & m = 0
\end{cases} \]  

(B.22)

Electric field can be computed for the ECVT Electrode by \( \vec{E} = -\nabla V \) and 3D sensitivity map can also be found. These analytical solution provide a comparison with the sensitivity map computed using COMSOL Multiphysics.
Figure B.2: Potential in the XY Plane

Figure B.3: Potential in the YZ Plane
Bibliography


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