Recall:[1]

1. Some facts:
   (a) For a differentiable function f, $\text{curl}(\nabla f) = 0$
   (b) If $\vec{F}$ is conservative, then $\text{curl}\vec{F} = 0$.
   (c) If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ and $\text{curl}\vec{F} = 0$, then $\vec{F}$ is a conservative vector field.
   (d) If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$, then
   $$\text{div}(\text{curl}\vec{F}) = 0$$
2. The **curl** of a vector field $F = P\vec{i} + Q\vec{j} + R\vec{k}$ is given as
$$\text{curl}\vec{F} = \nabla \times \vec{F} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} =$$
$$= \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\vec{i} + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\vec{j} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\vec{k}$$
3. The **divergence** of a vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is given as
$$\text{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$$
4. (**Surface Integrals of a vector field**) If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal $\vec{n}$, then the surface integral of $\vec{F}$ over S is
$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_S \vec{F} \cdot \vec{n} dS$$
This integral (on the left side of the equality) is also called the *flux* of $\vec{F}$ across S. Remember that *flux* means the rate at which $\vec{F}$ flows through S.

[1] Author: Andrew Castillo

5. (**Surface Integrals of a vector field continued**) If we parameterize S by $\vec{r}(u,v)$ with parameter domain D, then the unit normal $\vec{n}$ is given by $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ and the surface integral of a vector field F over a surface S is then given by

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

THIS IS THE FORMULA YOU WANT TO USE TO COMPUTE SURFACE INTEGRALS OF VECTOR FIELDS.

6.

Theorem 1. *(**Stokes' Theorem**) Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_S curl\vec{F} \cdot d\vec{S}$$

7.

Theorem 2. *(**Divergence Theorem**) Let D be a simple solid region and let S be the boundary surface of D, given with positive outward orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains D. Then*

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int\int_D div\vec{F} dV$$

1 Stokes' and Divergence Theorem Examples:

Problem 1. Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y, z) = (x + y^2)\vec{i} + (y + z^2)\vec{j} + (z + x^2)\vec{k}$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

Problem 2. Use Stokes' Theorem to evaluate

$$\int\int_S \text{curl}\vec{F} \cdot d\vec{S}$$

where $\vec{F}(x, y, z) = y\vec{i} - x\vec{j} + yx^3\vec{k}$ and S is the part of the part of the sphere $x^2 + y^2 + z^2 = 4$ for $z > 0$, with upwards orientation. Assume that C is the circle $x^2 + y^2 = 4$ (in $\mathbb{R}^3$), with counterclockwise orientation.

Problem 3. Verify that Stokes' Theorem is true for the vector field $\vec{F}(x, y, z) = y^2\vec{i} + x\vec{j} + z^2\vec{k}$ and S is the part of the paraboloid $z = x^2 + y^2$ that lies strictly below the plane $z = 1$, oriented upwards. Furthermore, C is the curve in which the the surface of the paraboloid $z = x^2 + y^2$ intersects the plane $z = 1$, oriented counterclockwise.

Problem 4. Use Stoke's Theorem to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y, z) = e^{-x}\vec{i} + e^{x}\vec{j} + e^{z}\vec{k}$. Here, C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant, and is oriented counterclockwise as viewed from above.

Problem 5. Verify that the Divergence theorem is true for the vector field $\vec{F}(x, y, z) = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and S is the part of the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 1$.

Problem 6. Use Divergence theorem to evaluate

$$\int\int_S \vec{F} \cdot d\vec{S}$$

where $\vec{F}(x,y,z) = x^2\vec{i} + xy\vec{j} + z\vec{k}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy plane.

Problem 7. Use the Divergence Theorem to evaluate

$$\int\int_S \vec{F} \cdot d\vec{S}$$

where $\vec{F}(x,y,z) = z\tan^{-1}(y^2)\vec{i} + z^3\ln(x^2+1)\vec{j} + z\vec{k}$ and S is the part of the cone $z = 2 - \sqrt{x^2+y^2}$ that lies above the plane $z = 1$, oriented upwards.

2 Vector Identities:

Problem 8. Prove the following identity

$$\int\int_S \text{curl}\vec{F} \cdot d\vec{S} = 0$$

assuming S satisfies the conditions of Divergence Theorem.

Problem 9. Suppose that S and C satisfies the condition of Stokes' Theorem and f and g are smooth functions. Use Stokes' Theorem to prove the following identity.

$$\int_C (f\nabla g) \cdot d\vec{r} = \int\int_S (\nabla f \times \nabla g) \cdot d\vec{S}$$

Problem 10. Assume that S and D satisfy the conditions of Divergence Theorem $\vec{F}$ is a smooth funtion such that $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$. Use the Divergence Theorem to prove the following identity:

$$\mathrm{Vol}(D) = \frac{1}{3}\int\int_S \vec{F} \cdot d\vec{S}$$

Problem 11. Assume that S and E satisfy the conditions of Divergence Theorem and both f and g are smooth functions. Use the Divergence Theorem to prove the following identity:

$$\int\int_S (f\nabla g)\cdot\vec{n}dS = \int\int\int_D (f\nabla^2 g + \nabla f\cdot\nabla g)dV$$

where ∇^2 is the differential operator (called the **Laplacian**)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$