

# Spherical varieties and tropical geometry

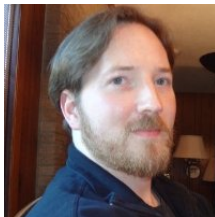
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2014

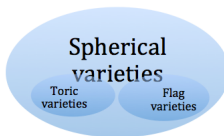
# Advertisement

- This is intended as a prospectus of research, and I'm looking for students who want to work on this project.
- The project is related to the recent work of Jason Miller, including his 2014 dissertation.

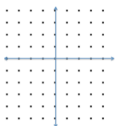


- Lecture slides are available at [u.osu.edu/kennedy.28](http://u.osu.edu/kennedy.28).

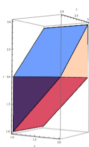
- The theme: to study aspects of algebraic geometry through combinatorial and piecewise linear methods.



Cones



Fans



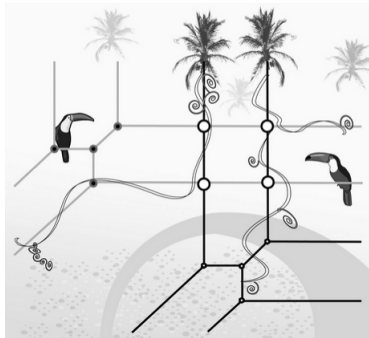
Convex bodies



Tropical varieties

(image credits given later)

- The goal: To find notions akin to those of *tropical geometry* in the wider world of *spherical varieties*.



(image by Cowdery and Challas, featured in June 2009 Mathematics Magazine)

# The plan for the lectures

- 1 Advertisement
- 2 Toric varieties
- 3 Flag varieties
- 4 Newton-Okounkov bodies
- 5 Spherical varieties
- 6 Jason Miller's research program
- 7 Tropical geometry
- 8 Tropical geometry in the spherical world

- A *toric variety*  $X$  is an irreducible algebraic variety containing an open subset which is an *algebraic torus*  $T = (\mathbf{C}^*)^n$ , such that the action of  $(\mathbf{C}^*)^n$  on itself extends to an action on  $X$ .
- In other words, there is an effective action of  $T$  on  $X$ , and it has a dense orbit which is a copy of  $T$ .

## Example

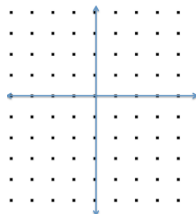
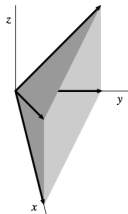
Let  $X = \{(w, x, y, z) : wz = xy\}$ , with torus  $T = \{(w, x, y, \frac{xy}{w}) : \text{all coordinates nonzero}\} \cong (\mathbf{C}^*)^3$ . There are 10 orbits, including  $\{(0, 0, 0, 0)\}$ .

## Example

$X = \mathbf{P}^1 \times \mathbf{P}^1$ , the self-product of the projective line, where each  $\mathbf{P}^1$  is regarded as  $\mathbf{C}^*$  plus the origin plus the point at infinity. Here  $T = \mathbf{C}^* \times \mathbf{C}^*$ , and there are 9 orbits.

# Cones and fans

- The study of toric varieties in affine space leads one to *cones*.
- The study of general toric varieties leads one to *fans*.

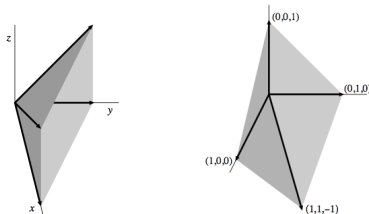


(Figure on left from Cox, Little, Schenk)



# Cones and fans

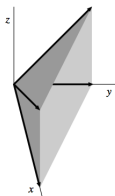
- Here is a cone  $\sigma$  in  $\mathbf{R}^3$  and its *dual cone*  $\sigma^\vee$ . (Cox, Little, Schenk)



- Create an algebra with one generator for each ray of the dual cone:  $x$  corresponds to  $(1, 0, 0)$ , etc. It's the algebra  $\frac{\mathbf{C}[w,x,y,z]}{wz-xy}$ , the ring of functions on the variety  $wz = xy$  in  $\mathbf{C}^4$ .

# Cones and fans

- Note that  $\sigma$  has four faces, four edges, and a vertex. Each of these corresponds to a  $T$ -orbit.



- Considering closures of orbits, the correspondence is inclusion-reversing: the vertex corresponds to the dense orbit, and  $\sigma$  itself corresponds to the one-point orbit.

# Cones and fans

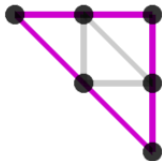
- Here is the *fan* specifying  $\mathbf{P}^1 \times \mathbf{P}^1$ .



- There are two cones of largest dimension, corresponding to the four single-point orbits.

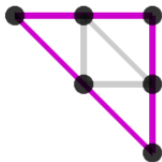
# Cones and fans

- For a toric variety in projective space, one can also naturally associate a *moment polytope*. Here's a polytope associated to  $\mathbf{P}^2$ .



- This polytope records both the variety  $\mathbf{P}^2$  and how it's embedded in projective space, via a certain *line bundle*.

- This polytope specifies that  $\mathbf{P}^2$  is embedded in  $\mathbf{P}^5$  using all quadratic monomials:  $[x, y, z] \mapsto [x^2, xy, y^2, xz, yz, z^2]$ .



- One can obtain the fan for  $\mathbf{P}^2$  as the *normal fan* of this polytope.

- Let  $G$  be a semisimple algebraic group (e.g.,  $SL_n$ ). A subgroup  $P$  is called *parabolic* if the homogeneous space  $X = G/P$  is compact, and  $X$  is called a *flag variety*.
- A *Borel subgroup*  $B$  is a maximal closed and connected solvable subgroup.
- The group  $G$  acts on  $X$  by left multiplication. If one restricts to the action by  $B$ , one finds that there is still a dense orbit.

# The complete flag variety

- For a specific example, let  $G = SL_n$  and let  $P$  and  $B$  consist of all upper triangular matrices.
- In this case the points of  $G/B$  are *complete flags* of vector subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1}$$

of  $\mathbf{C}^n$ . We call  $G/B$  a *complete flag variety*.

- Fix one complete flag  $F$ . Roughly speaking, a *Schubert variety* parametrizes those flags which meet  $F$  in a specified way, e.g., we could require that the first element of the flag meets the first element of  $F$ . The recipes for calculating intersections of Schubert varieties are called *Schubert calculus*.

## Example

How many lines in projective 3-space meet four specified lines?

*Answer: 2*



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# Two Schuberts



[www.carus-verlag.com](http://www.carus-verlag.com)

Franz Schubert (1797–1828)



<http://people.su.se>

Hermann Schubert (1848–1911)

# Convex bodies for flag varieties

- We can associate a convex body to the complete flag variety
  - by a general theory of convex bodies associated to line bundles on varieties, or
  - by a construction adapted to the geometry of the flag variety.

- The general theory was developed by Okounkov, Kaveh-Khovanskii, and Lazarsfeld-Mustață.
- The situation: Consider a very ample line bundle  $L$  on a smooth projective variety  $X$  of dimension  $d$ .

- The basic idea: Choose a flag of smooth subvarieties

$$Y_1 \supset Y_2 \cdots \supset Y_d,$$

where  $\text{codim}(Y_i) = i$ .

- To each section  $s$  of the line bundle, associate a sequence of nonnegative integers

$$(v_1, v_2, \dots, v_d).$$

- Here  $v_1$  is the order of vanishing of  $s$  along  $Y_1$ . Remove  $v_1 Y_1$  from  $s$  and restrict to  $Y_1$ ; let  $v_2$  be the order of vanishing of  $s$  along  $Y_2$ . Iterate this construction.

- Let  $v(L)$  be the resulting set of points.
- The *Newton-Okounkov body* is

$$\Delta(X, L) = \text{closed convex hull} \left( \bigcup_{m \geq 1} \frac{1}{m} v(L^{\otimes m}) \right).$$

# N-O bodies for flag varieties

- To apply this idea to the complete flag variety, it's natural to let the flag of subvarieties consist of Schubert varieties.

$$Y_1 \supset Y_2 \cdots \supset Y_d,$$

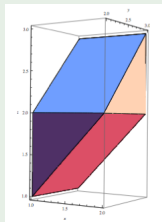
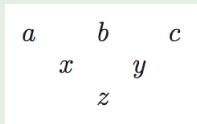
In this case the convex body is a polytope.

- One can do the same thing for any  $G/B$ .
- Kaveh showed that the construction can be interpreted via work of Littelman and others in which they construct a natural basis for any  $G$ -module, using a notion of “paths” and “strings.” For this reason we call  $\Delta(G/B, L)$  a *string body*.

## Example

The *Gelfand-Zetlin polytope* in 3 dimensions

$$a \leq x \leq b, \quad b \leq y \leq c, \quad x \leq z \leq y$$



(Figures from Kiritchenko, Smirnov, Timorin)



# The KST correspondence

- Kiritchenko, Smirnov, Timorin — *Schubert calculus and Gelfand-Zetlin polytopes* (2011)
- They describe a way to associate a set of faces to each Schubert variety, so that intersection of faces faithfully reflects the intersection theory on the complete flag variety.
- I.e., they show how to carry out Schubert calculus via combinatorics on faces of the Gelfand-Zetlin polytope.

- A complex algebraic variety is a *spherical variety* if it's acted upon by a reductive group  $G$  and there is a dense orbit under the action of a Borel subgroup  $B$ .
  - *Reductive groups* include semisimple groups (e.g.,  $SL_n$ , symplectic groups, orthogonal groups), tori  $(\mathbf{C}^*)^n$ , and general linear groups.
  - A *Borel subgroup* is a maximal closed and connected solvable subgroup.
- We have seen these two examples:
  - A toric variety — here  $G = B = T$ , an algebraic torus, and  $X$  is some variety containing an open dense copy of  $T$ .
  - The complete flag variety — here  $G = SL_n$ ,  $B$  is its Borel subgroup, and  $X$  is the homogeneous space  $G/B$ .

# Comparing toric varieties and complete flag varieties

- For a toric, the moment polytope and the Newton-Okounkov body coincide.
- For  $G/B$ , if one tries to imitate the definition of the moment polytope, one discovers that it's trivial.
- This suggests that they are two extreme cases.

# Three polytopes

- Indeed, for a general spherical variety  $X$  and a very ample line bundle  $L$ , one can construct both sorts of polytopes (they will be polytopes), and there is a map

$$\Delta(X, L) \rightarrow \Delta^{\text{mom}}(X, L).$$

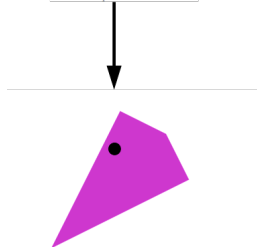
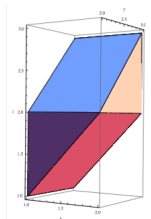
- Furthermore each fiber is an instance of a string polytope for  $G/B$ .

- Conics in the projective plane are parametrized by  $SL_3/N(SO_3)$ .
- A classical completion is the space of *complete conics* .

# Complete conics

- Here the moment polytope is a quadrilateral, and the string polytopes are 3-dimensional Gelfand-Zetlin polytopes (of varying sizes).
- Thus the Newton-Okounkov polytope is 5-dimensional.

(top of figure taken from Kiritchenko, Smirnov, Timorin)



- The volume of the Newton-Okounkov polytope is

$$\frac{1}{5!}(m^5 + 10m^4n + 40m^3n^2 + 40m^2n^3 + 10mn^4 + n^5)$$

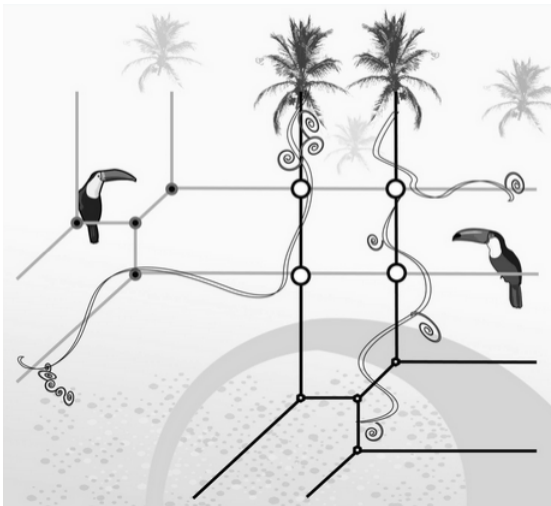
where  $(m, n)$  is a pair of numbers specifying the line bundle.

- To get the line bundle corresponding to the condition “the conic is tangent to a specified conic,” take  $m = n = 2$ . Then the volume is  $\frac{3264}{5!}$ . The numerator tells you the number of conics tangent to five specified conics.
  - First computed by Chasles in 1864.

- In his 2014 Ohio State dissertation *Okounkov bodies of Borel orbit closures in wonderful group compactifications*, Jason Miller combined aspects of the Newton-Okounkov body theory and the KST correspondence.
- Given the Newton-Okounkov polytope associated to a certain type of spherical variety, he showed how to set up a correspondence between  $B$ -orbit closures and sets of faces of the polytope. Like the KST correspondence, this correspondence reflects the intersection theory on the spherical variety, and it's packed with other information as well.



# Tropical geometry



(image by Cowdery and Challas, featured in June 2009 Mathematics Magazine)

# Tropical geometry in the spherical world

