

## “IN THE REALM OF FORMS”

GARY KENNEDY

I am delighted to join my colleagues in the Art Department at the Mansfield campus of The Ohio State University — Professors Kate Shannon and John Thrasher — in curating an exhibition in the Conard gallery, especially since they allowed me to have all the fun of choosing pieces for the show, and none of the labor of communications, shipping, installation, etc. Together we chose artworks that caught our fancy, without any precise rules for what constitutes “mathematical art.” Seeing the works of art assembled together, what themes can we discern? This is an attempt to group the works in the show by mathematical themes, based on my own interpretations. The artists may perhaps have different views.

### 1. KEY EXAMPLES

There is a long tradition in mathematics of key examples, which may range from useful illustrations of a theory to essential parts of it. One of the latter type is Klein’s

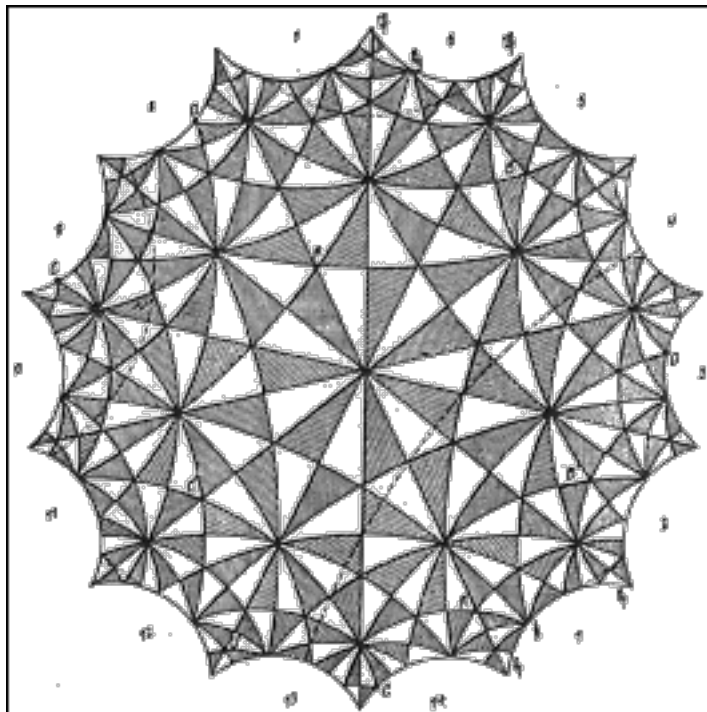


FIGURE 1. Klein’s Riemann surface of genus 3

Riemann surface of genus 3 with 168 automorphisms, the simplest surface realizing Hurwitz's upper bound of  $84(g - 1)$  for the number of automorphisms. Figure 1 shows the illustration<sup>1</sup> from Klein's original paper [3]. (An English translation appears in [4].) Only part of the beautiful symmetry of this object is visible to the uninitiated viewer, namely the seven-fold rotational symmetry about the central point. One has to imagine that the illustration is printed on some extremely flexible material, so that its fourteen edges can be glued in pairs; the resulting surface will have 168 symmetries. Indeed, one can pick any pair of black triangles on the surface, and there will be a symmetry of the surface taking one triangle to the other. Conan Chadbourne's "Contorted Plan of Obscure Regularity" offers us another view of Klein's surface, with the gluings indicated by interlaced arcs. The delicate tinting and shading create a calm and elegant setting.

Saul Schleimer and Henry Segerman's sculpture "Conformal Chmutov," custom-made for this exhibition, depicts a Chmutov surface, which is named for its discoverer: Professor Sergei Chmutov of the Ohio State University [2]. Chmutov's surfaces can be thought of as 3-dimensional versions of the Lissajous curves which appear on an oscilloscope, such as the curve  $(2x^2 - 1)^2 + (4y^3 - 3y)^2 = 1$  shown in Figure 2. The curve is remarkable for the following reason: although its equation is merely of degree 6, it has 7 nodes (the points where the curve crosses over itself). Similarly, Chmutov's surfaces are remarkable for having many cone points; for certain degrees, they are the surfaces of specified degree having (as far as is known) the largest number of cone points. An example is shown in Figure 3, whose equation is

$$-8(x^4 + y^4 + z^4) + 8(x^2 + y^2 + z^2) = 4;$$

the equation has degree 4, and the surface has 12 cone points. Now it must be confessed that the sculpture of Schleimer and Segerman is not really a Chmutov surface, and for a very good reason: the actual Chmutov surface would fall apart at its cone points. Thus they have created the surface with equation

$$-8(x^4 + y^4 + z^4) + 8(x^2 + y^2 + z^2) = 3.24114;$$

the change in the parameter on the left has the effect of replacing each cone point by a small tube.

## 2. TECHNOLOGIES

Several pieces in the show were created by 3D printing, including those by Schleimer and Segerman, Brittany Ransom, and Mark Stock.

Ransom has reproduced the shapes of arctic sea ice through a multi-step process, beginning with photographs taken on a cell phone. The photographs were stitched together and processed to create a 3-dimensional scan. This virtual 3D model was then sliced, and the resulting slices were cut out using a laser. Assembled together, they create a "ghosted form" that evokes the floating ice.

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<sup>1</sup>copied from [www.umpa.ens-lyon.fr/JME/Vol11Num1/artCBavard/domfond.gif](http://www.umpa.ens-lyon.fr/JME/Vol11Num1/artCBavard/domfond.gif)

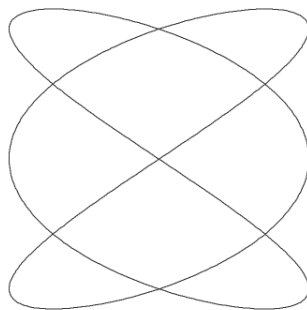


FIGURE 2. A Lissajous curve

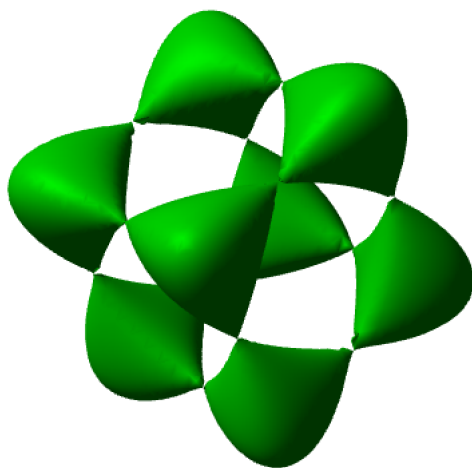


FIGURE 3. Chmutov surface of degree 4

Stock has created intriguing models of quaternion Julia sets. The usual notion of Julia sets uses the complex numbers: numbers of the form  $z = x + iy$ , where  $i$  is the square root of  $-1$ . Iterating some quadratic function  $z \mapsto z^2 + az + b$ , one observes which initial values fail to “escape to infinity”; these points constitute the Julia set. Quaternion Julia sets are determined by the same recipe, except that now one uses quaternions  $w + ix + jy + kz$ .

Another technological innovation is the use of computer programming, which is especially apt when one is creating a form with a recursive structure, such as a fractal: the idea of the form is reflected in the recursive structure of the program which creates it. Robert Fathauer’s ceramic “Four-Fold Development” provides a stunning example. The first five generations of a fractal curve are assembled as horizontal cross-sections to create an intricate sculpture reminiscent of a coral reef.

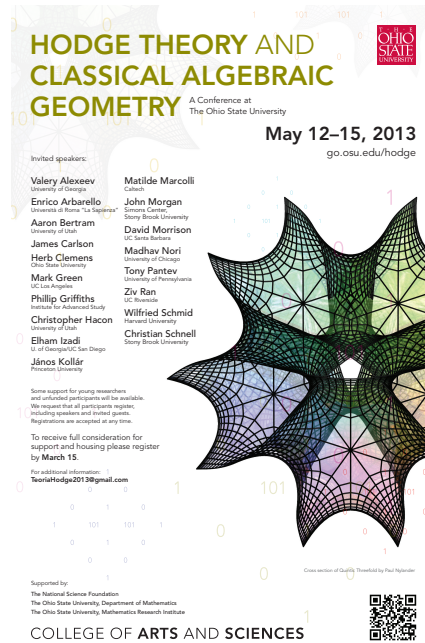


FIGURE 4. Hodge Theory conference poster

### 3. SYMMETRY

Fathauer’s piece also exhibits symmetry in 3-dimensional space. To this, Paul Nylander adds the element of time, as in his video illustrating how a dodecahedron must be “puffed out” so that — when its faces are appropriately glued — it can be given a hyperbolic structure. His videos of Boy’s surface and of a Calabi-Yau threefold are again “key examples.” The former shows a real projective plane immersed in three-dimensional space, i.e., placed in three-dimensional space so that, although there are self-collisions, there is no need to crimp the surface at any point. [1] A still frame from his Calabi-Yau video was used in the poster advertising a conference in Hodge Theory at The Ohio State University in May 2013; see Figure 4. The quintic threefold is the simplest example of a Calabi-Yau space, which can embody the tightly-wrapped hidden dimensions required by contemporary String Theory in theoretical physics.

Lynn Bodner’s eye-popping “Eleven and Nine Star” is the result of setting herself the task of creating a plausible reconstruction of an historic Islamic design using only Euclidean compass-and-straightedge constructions. The design is an example of a wallpaper pattern, meaning a design having translational symmetry in two independent directions. This design also has reflection symmetries. One might think that the design has additional symmetries, namely rotation symmetries centered at the centers of the eleven- and nine-pointed stars, but mathematical analysis shows that rotations of these orders are impossible in a wallpaper pattern; the rotation of the individual stars cannot be extended to the entire design.

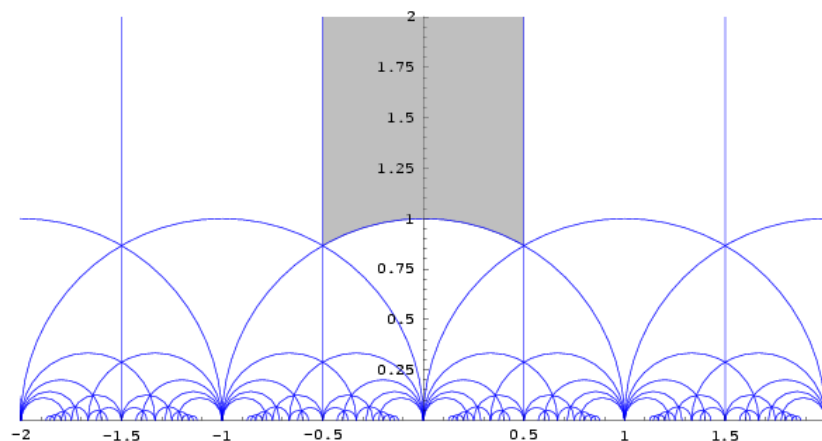


FIGURE 5. Fundamental domains in the upper-half-plane model of the hyperbolic plane

Taken together, Frank Farris’s visually stunning pieces for this exhibit cover the gamut of two-dimensional symmetry. “A  $p4g$  from a Mountain Gentian” is, mathematically speaking, a wallpaper design. With “Imaginary Planets: A Polyhedral Sampler,” he shows us all possible types of finite symmetry patterns on a sphere. In both of these pieces, some of the mathematical elements are subtle and easily overlooked: the colors in the mountain gentian piece are derived not by simple repetition of a selected portion of the source image, but by a more elaborate computation which singles out individual pixels, while the background of the polyhedral sampler reflects relations between their symmetry groups. Looking at “Peaches to the Edge of the Universe,” one may not even be aware of all the symmetry, since here it is of a more abstract kind, based on the negatively-curved geometry of hyperbolic space via the upper-half-plane model of Poincaré; Figure 5 shows the underlying pattern of “fundamental domains.”<sup>2</sup>

#### 4. COMPLETE COLLECTIONS

At many places in mathematics, one wants a complete list of all the possible objects of a specified type. A spectacular example of this is the classification of all possible finite simple groups, a project first envisioned in the middle of the last century and completed over the ensuing decades, through a series of hundreds of papers; its details are still being simplified and reorganized. At a smaller scale, the making of complete lists and classifications is almost a daily task. Mathematical art often reflects the impulse to list, categorize, or classify; it may do this playfully or obsessively.

The artist statement of James Mai is explicit about this: he says that in each piece he exhibits a complete set of geometric forms. “The forms in each set are at once similar, in that all forms share the same geometric features, and different, in that each form is a unique arrangement of those features. As important, each form-set is both complete,

<sup>2</sup>copied from [https://en.wikipedia.org/wiki/Modular\\_group](https://en.wikipedia.org/wiki/Modular_group)

in that all [possibilities] are present.” In “Dozen Suns” he presents all possible ways of connecting the vertices of a regular octagon so as to make two polygons; there are five ways to do so that result in a triangle plus a pentagon, and a further five resulting in two quadrilaterals. Background shading is used to group the 12 possibilities by symmetry type.

Margaret Kepner’s “Octet-Variations in Blue” systematically arranges the 30 ways of shading a subset of eight equal-sized sectors of a circle, where two such shadings are considered equivalent if they differ by a rotation or reflection. The 30 possibilities are furthermore arranged in a cunning way, so that a  $180^\circ$  rotation takes the design into itself, with colors interchanged.<sup>3</sup>

## 5. SELF-REFERENCE

Self-reference play a role in several of the artworks in our exhibition. For example, in David Reimann’s pieces “Reptiling Escher” and “Kepler Constellation” the medium refers obliquely to the subject matter. A portrait of the Dutch artist M.C. Escher is created by varying the weight of a reptilian figure similar to that found in Escher’s own work, such as the lithographic print<sup>4</sup> shown in Figure 6. In a similar vein, Reimann creates a portrait of the German mathematician and astronomer Johannes Kepler by varying the intensity of light shining through astroidal figures.

In other works, the mathematical subject matter is itself self-referential: this is the essential idea of a fractal, as exhibited in Fathauer’s ceramic sculpture or in the delicately beaded Sierpinski tetrahedron of Gwen Fisher. Fathauer’s sculpture is arbitrarily cut off at a certain height, but one can imagine it growing further by applying the recursive rule which generates it for another step. Fisher’s tetrahedron is assembled out of four tetrahedra, each of which is in turn an assemblage of four tetrahedra, etc. Since the beads are of a finite size, this process soon terminates, but in mathematics we can imagine it continued to ever smaller scales.

## 6. “THE WRONG KIND OF SUBJECT MATTER”

In a perceptive essay on learning mathematics (part of his book [5]), Stephen Pinker asks:

How can people use their Stone Age minds to wield high-tech mathematical instruments? The first way is to set mental modules to work on objects other than the ones they were designed for. Ordinarily, lines and shapes are analyzed by imagery and other components of our spatial sense, and heaps of things are analyzed by our number faculty. But to accomplish [Saunders] Mac Lane’s ideal of disentangling the generic from

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<sup>3</sup>The artist points out that I have been careless here. The first 11 shadings and the last 11 have this property, but for others the symmetry is broken. Indeed, it would be impossible to arrange all 30 in the way I have described, since some of the individual shadings themselves have this color-interchange symmetry.

<sup>4</sup>copied from [https://en.wikipedia.org/wiki/Reptiles\\_\(M.\\_C.\\_Escher\)](https://en.wikipedia.org/wiki/Reptiles_(M._C._Escher))



FIGURE 6. M.C. Escher's "Reptiles"

the parochial (for example, disentangling the generic concept of quantity from the parochial concept of the number of rocks in a heap), people might have to apply their sense of number to an entity that, at first, feels like the wrong kind of subject matter.

Creativity, whether in mathematics or art, is often a matter of applying a concept or tool to the “wrong kind of subject matter.” Guang Zhu uses parametric curves to create woodblock prints and video displays. She says:

I romanticize mathematical equations. To me, they seem to live through stories and places. I work with formulas and code to create geometrical simulations. I study those moving forms to create art and to research the history and application of parametric equations. I wonder whether we can feel mathematics as synesthetic experiences through its abstract motions. Can this aesthetic evoke metaphysical understandings of art objects?

Hector Rodriguez applies the theory of orthogonal decomposition in linear algebra, specifically the Gram-Schmidt procedure, to a pair of *films noirs*: Godard's *Alphaville* and Deren & Duchamp's *Witch's Cradle*. While orthogonal decomposition is generally regarded as a means of extracting useful information from a noisy source, I think that here it has the opposite effect: it heightens the brooding and menacing qualities of these films.

Conversely, it may seem that a subject is being addressed with the “wrong tools” or “wrong materials.” Leslie Berns's “Magic Garden Series, III” provides a lovely example.

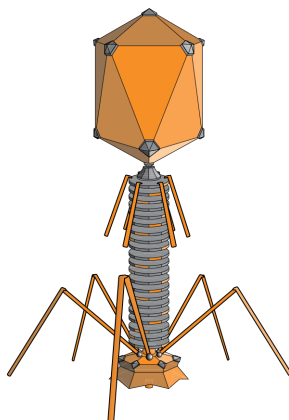


FIGURE 7. A bacteriophage

Using the medium of throw-away paper cups, she creates models of symmetry with a gentle and contemplative feel. Her piece in our show is relatively small; she has created much larger assemblages of cups, and sometimes floats them on bodies of water. David Reimann's polyhedral sculpture "Inconceivable Symmetries" is more startling. It amuses us twice: first by its Dadaesque choice of inappropriate materials, and then by its tongue-in-cheek title.

At first glance, traditional handicrafts such as needlework and beading may seem odd choices for the realization of mathematical ideas, and yet they have unique qualities which make them very suitable. Using beaded beads, "a cluster of smaller beads, woven together with a needle and thread to form a sculpture with one or more holes running through its center," Gwen Fisher creates elegant models of fractals, hyperbolic tilings, and polyhedra: cuboctahedron, truncated icosahedron, etc. In a particularly striking piece, she has assembled a realistic model of a bacteriophage; see Figure 7 for an encyclopedia illustration.<sup>5</sup>

Knitting and crocheting create flexible structures capable of bending and curving; indeed curvature can be built into them as they are created rather than imposed afterwards. A great deal of traditional needlework is concerned with how to add or leave out stitches so as to make the object conform to a curved torso. Thus expert needleworkers understand perfectly well how to create surfaces with either positive or negative curvature. Daina Taimiņa's book *Crocheting Adventures with Hyperbolic Planes* [6] explains how to create a surface with constant negative curvature, one on which non-Euclidean geometry applies. Starting with a small disk on this surface, as its radius increases its area grows much faster than in Euclidean geometry: the familiar formula  $A = \pi r^2$  needs to be replaced by  $A = \pi(\cosh r - 1)$ , so that the growth in area is essentially exponential.

To understand this, think about the process of crocheting a flat doily: one adds successive circles of stitches, increasing the number of stitches by a fixed number, e.g.,

<sup>5</sup>copied from <https://en.wikipedia.org/wiki/Bacteriophage>



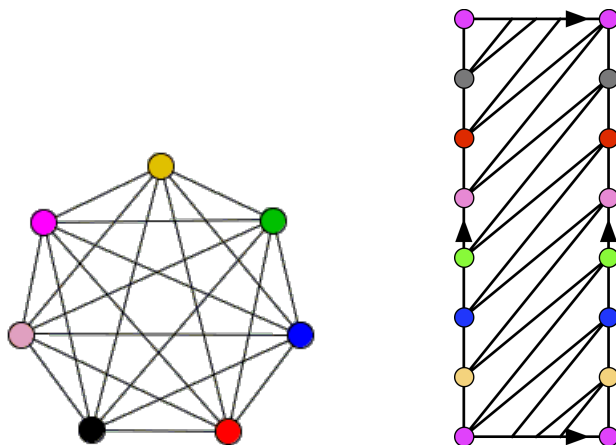


FIGURE 8. The complete graph  $K_7$ , in the plane and on a torus

using 6, 9, 12, 15, 18 stitches in successive circles. If one begins to add fewer stitches, then the piece will begin to curve like a sphere. Going the other way, if one adds more stitches at an approximately exponential rate (always increasing the number of stitches by the same factor, insofar as possible) then the piece will begin to acquire negative curvature. Thus Taimiņa’s piece is a misshapen topological disk whose circumference is far larger than it ought to be, from the viewpoint of Euclidean geometry. The surface grows as if it wants to fill space, and has an uncanny organic appearance, perhaps like fungi or coral.

Sarah-Marie Belcastro’s mathematical models show how perfectly suited these needlework media are for illuminating concepts in topology. Our show includes her models of the Klein bottle, knots and graphs embedded on surfaces, as well as links and braids. Her website <http://www.toroidalsnark.net/mathknit.html> provides a gateway to artworks, exhibitions, and seminars by a dedicated coterie of mathematical artists in needlework and fiber.

One of her knitted pieces illustrates how the complete graph  $K_7$  can be imbedded on a torus. On the left of Figure 8, we see that if we try to draw this graph in the plane, then there are many places where edges are forced to cross.<sup>6</sup> On the right we see how to draw the same graph on a torus so that there are no crossings. In this picture one must imagine that the top and bottom edges have been glued, as well as the left and right edges, so that there are just seven vertices. In Bel Castro’s knitted model, these gluings have been carried out.

Robert Bosch has found a unique way to use the Traveling Salesman Problem of discrete mathematics to create highly original artworks. The TSP is the problem of connecting a specified set of vertices in the plane by a polygonal path in an efficient way, striving to minimize the total length of the path. What Bosch discovered was that he could control the density of the path, regarded as a monochrome image. In the

<sup>6</sup>copied from <http://www.cut-the-knot.org/Curriculum/Geometry/Polyhedra/K7.gif>

resulting artworks, the image consists of one continuous loop with no self-intersections, a so-called Jordan curve, and yet it is instantly recognizable as the Mona Lisa, the nearly-touching fingers of God and Adam, or Warhol's soup can. Since the image is a Jordan curve, it divides the plane into two pieces, interior and exterior. Bosch's piece in our show is a diptych: on the left is the Jordan curve; on the right is its interior.

Our exhibition also features the witty commentary of Tiberiu Chelcea in a selection from his "Trigonopoetry" series, which undercuts the intentions of mathematical exposition. By painting over most of the words on pages of trigonometry textbooks, he has masked them out, leaving just gnomic utterances. Are they pieces of advice about life or hijinks? Maybe they are telling us that it's time for recess.

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OHIO STATE UNIVERSITY AT MANSFIELD, 1680 UNIVERSITY DRIVE, MANSFIELD, OHIO 44906, USA

*Email address:* `kennedy@math.ohio-state.edu`