Rotational Virtual Links, Parity Bracket Polynomials and Quantum Link Invariants

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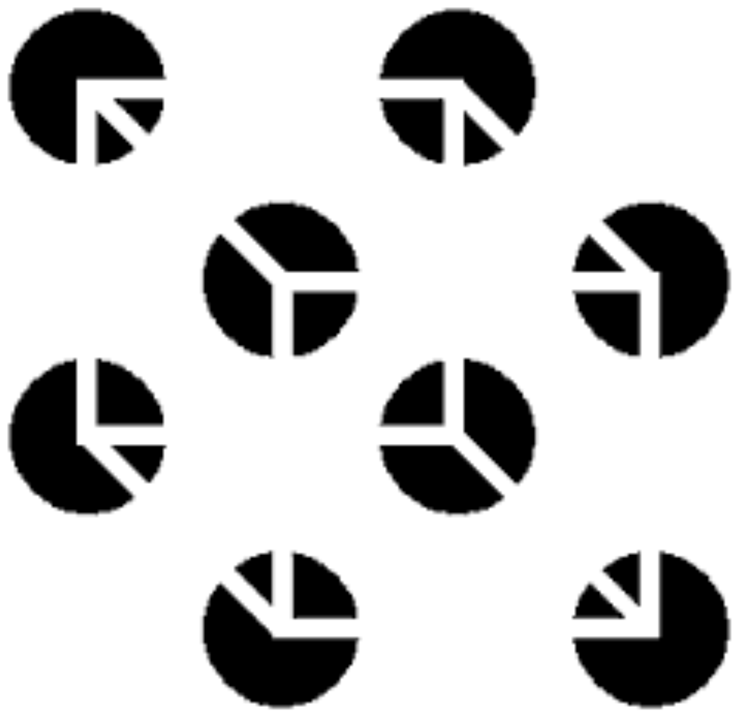

ROTATIONAL VIRTUAL KNOTS AND QUANTUM LINK INVARIANTS

LOUIS H. KAUFFMAN

Abstract. This paper studies rotational virtual knot theory and its relationship with quantum link invariants. Every quantum link invariant for classical knots and links extends to an invariant of rotational virtual knots and links. We give examples of non-trivial rotational virtual links that are undetectable by quantum invariants.

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Virtual Knot Moves

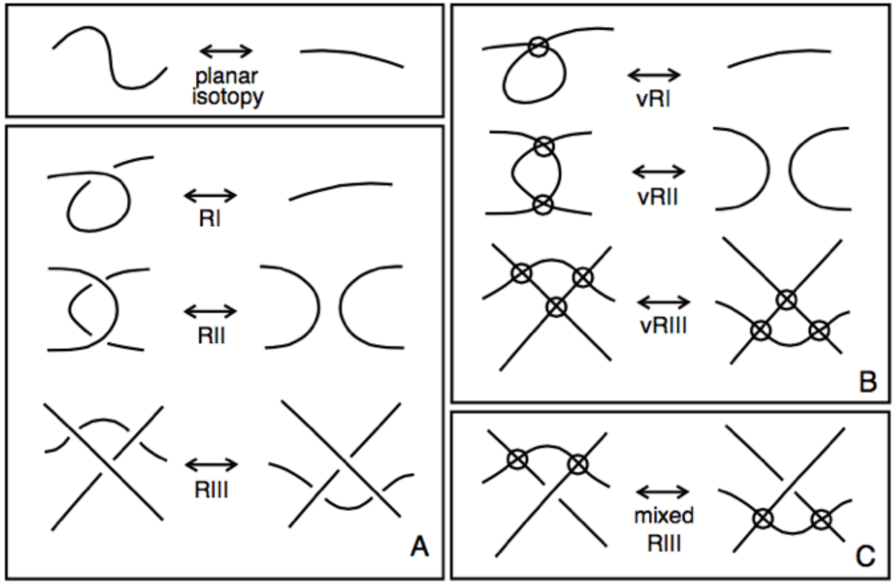


FIGURE 1. **Moves**

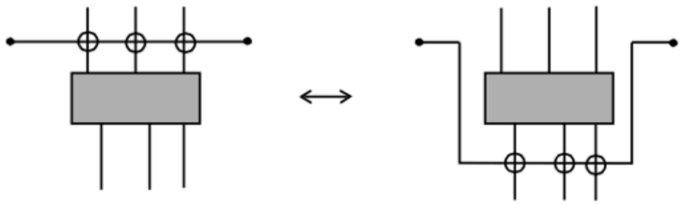
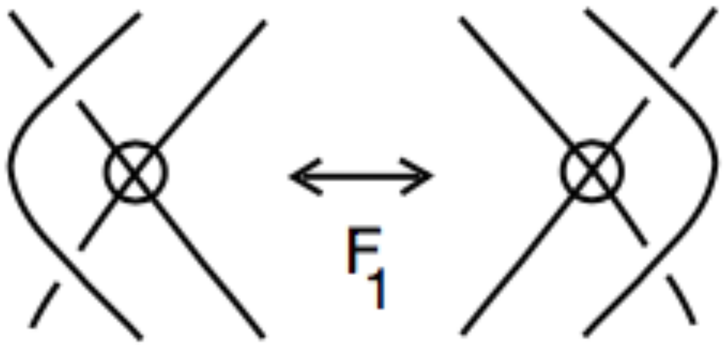
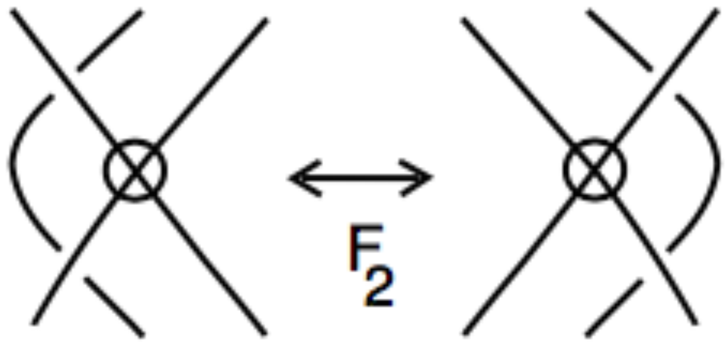


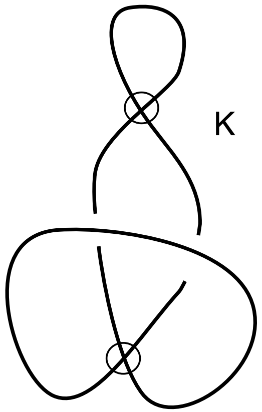
FIGURE 2. **Detour Move**

Forbidden Moves

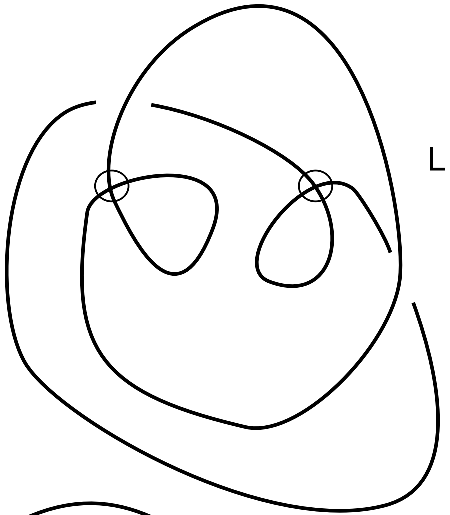




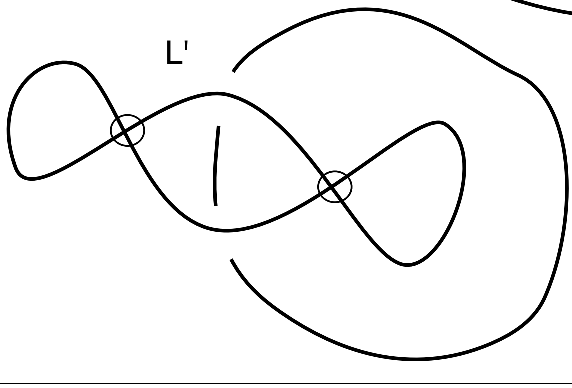
Rotational Virtual Knot Theory
=
VKT with Detour Move restricted to regular homotopy in the plane or on the two sphere.



K



L



L'

All quantum link invariants extend to invariants of rotational virtual links.

Rotational Virtuals are the CORRECT DOMAIN for studying quantum link invariants.

In *rotational virtual knot theory* (introduced in [18]) the detour move is restricted to regular homotopy of plane curves. This means that the virtual curl of Figure 1 can not be directly simplified, but two opposite virtual curls can be created or destroyed by using the Whitney Trick of Figure 9. Another way to put this is to say that rotational virtual knot theory is virtual knot theory without the first virtual move (thus one does not allow the addition or deletion of a virtual curl).

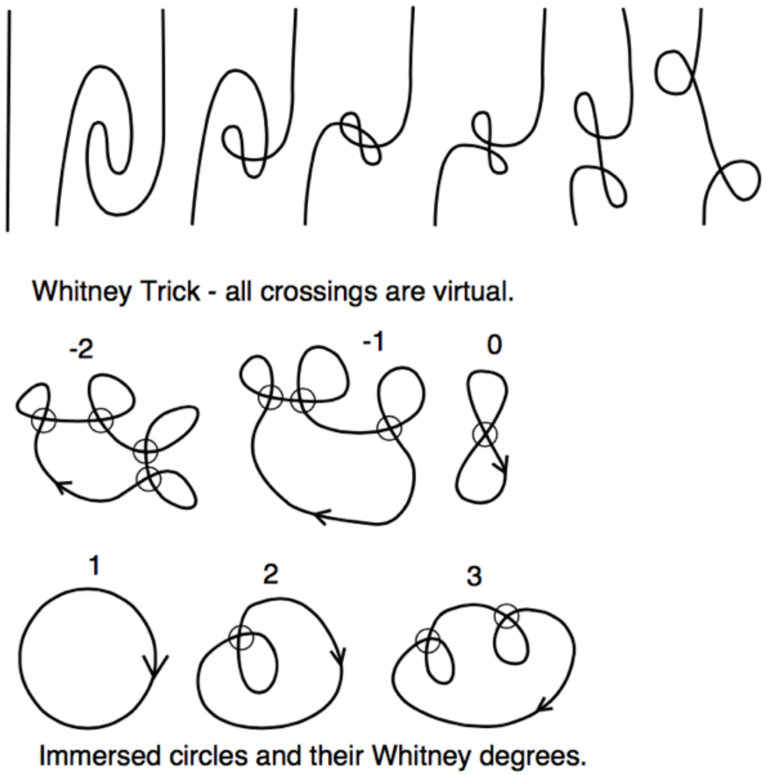


FIGURE 9. **Whitney Trick and Whitney Degrees**

Rotational Virtual Knot Cobordism

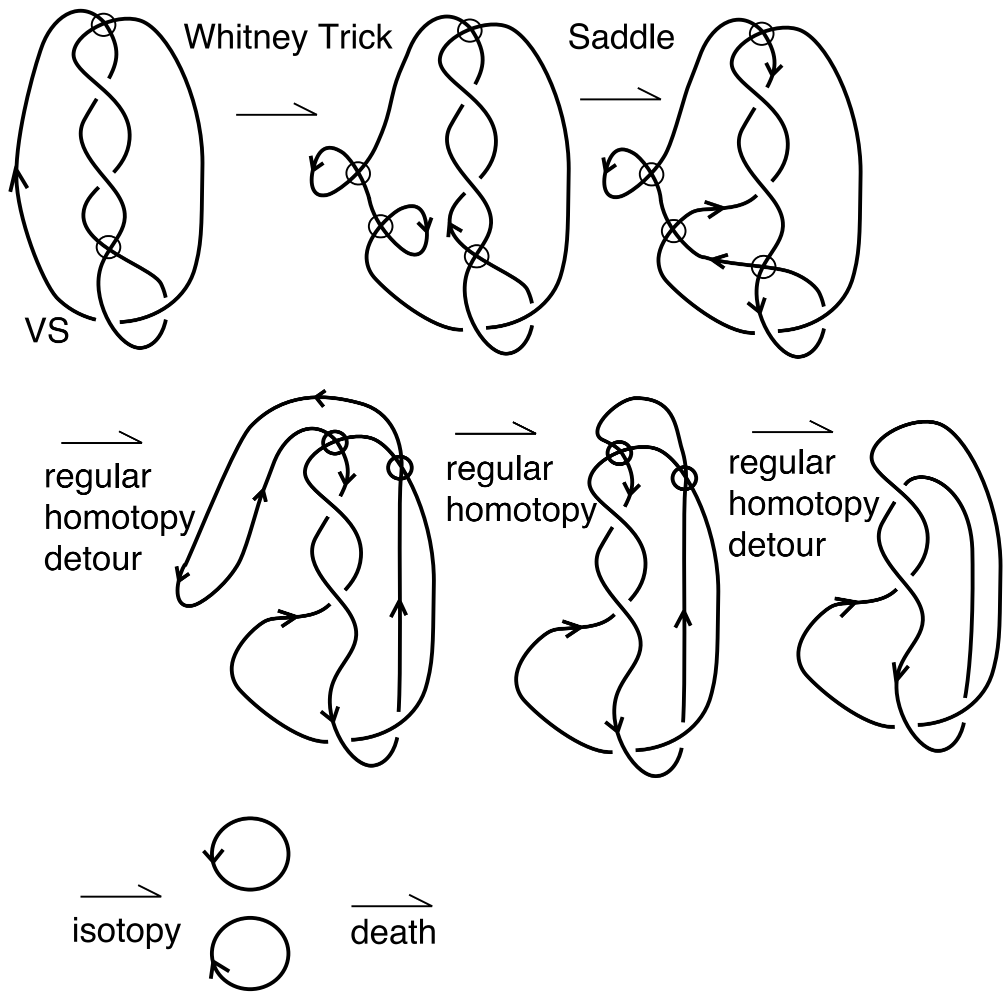
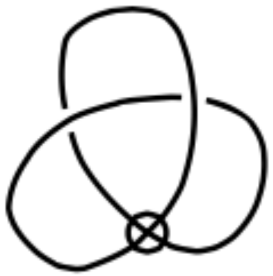
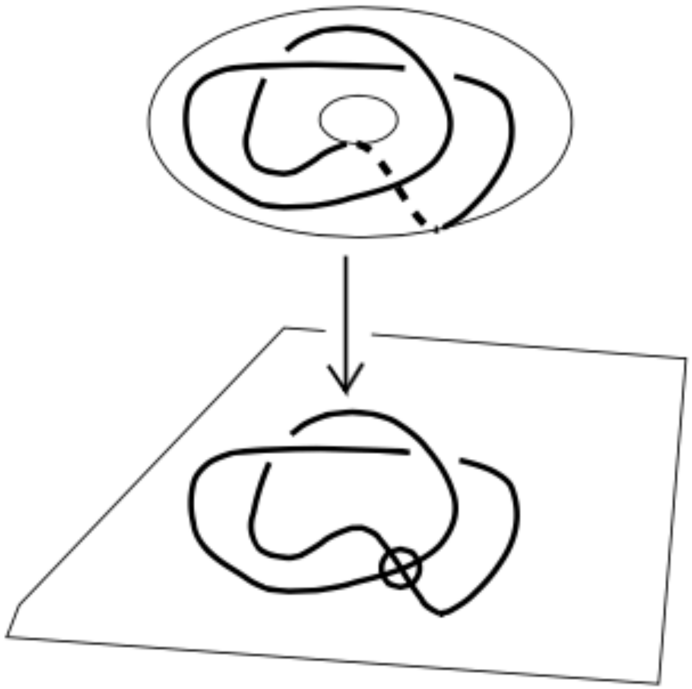
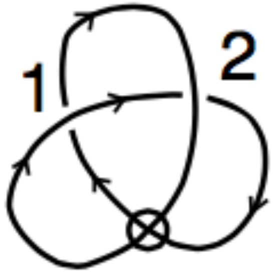


Figure 28: **The Virtual Stevedore is Rotationally Slice**

Abstract Diagrams, Surfaces and Gauss Codes

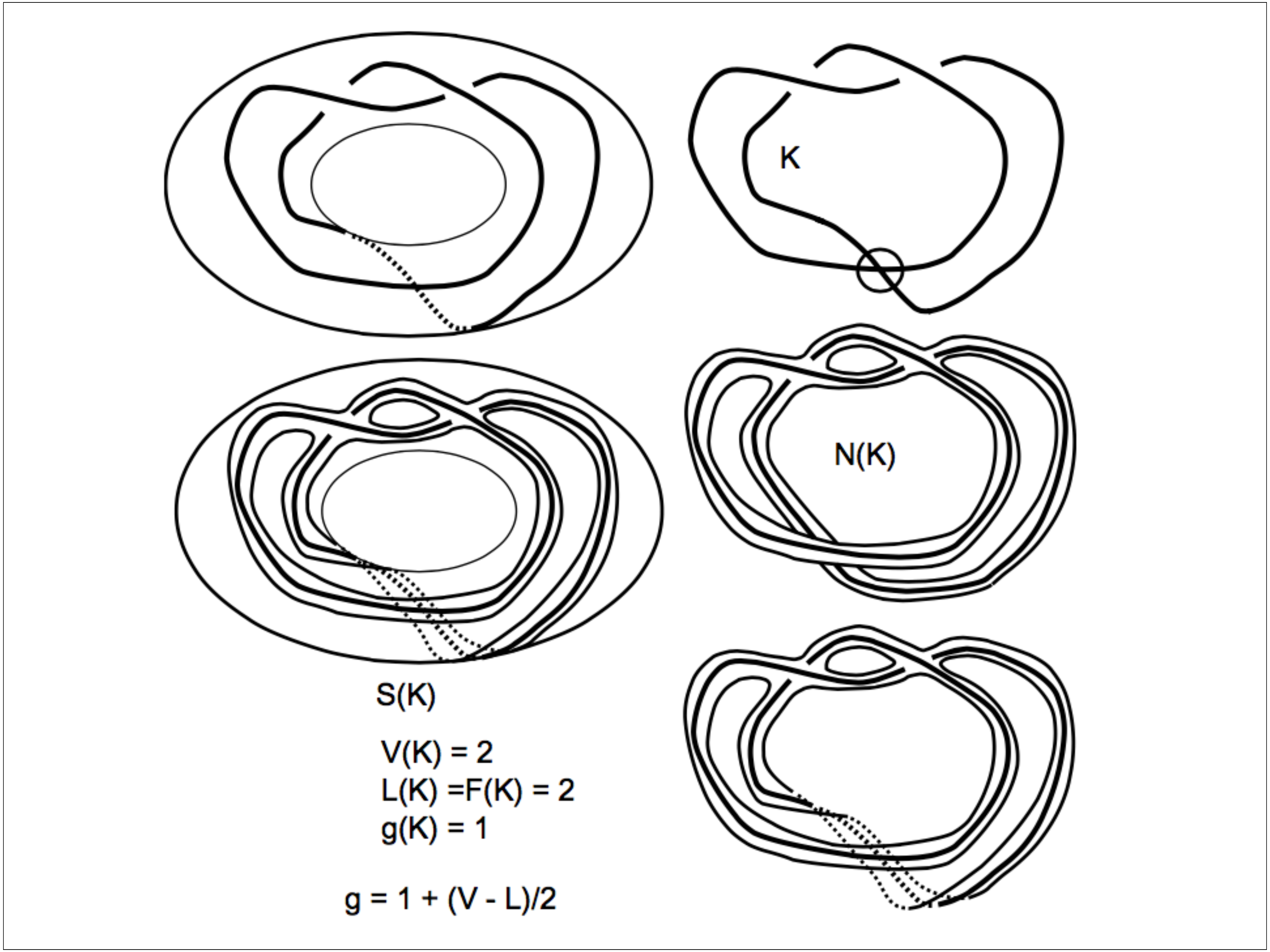






1212
Bare Gauss Code

O1+U2+U1+O2+
Full Gauss Code


K
N(K)
S(K)
V(K) = 2
L(K) =F(K) = 2
g(K) = 1
g = 1 + (V - L)/2

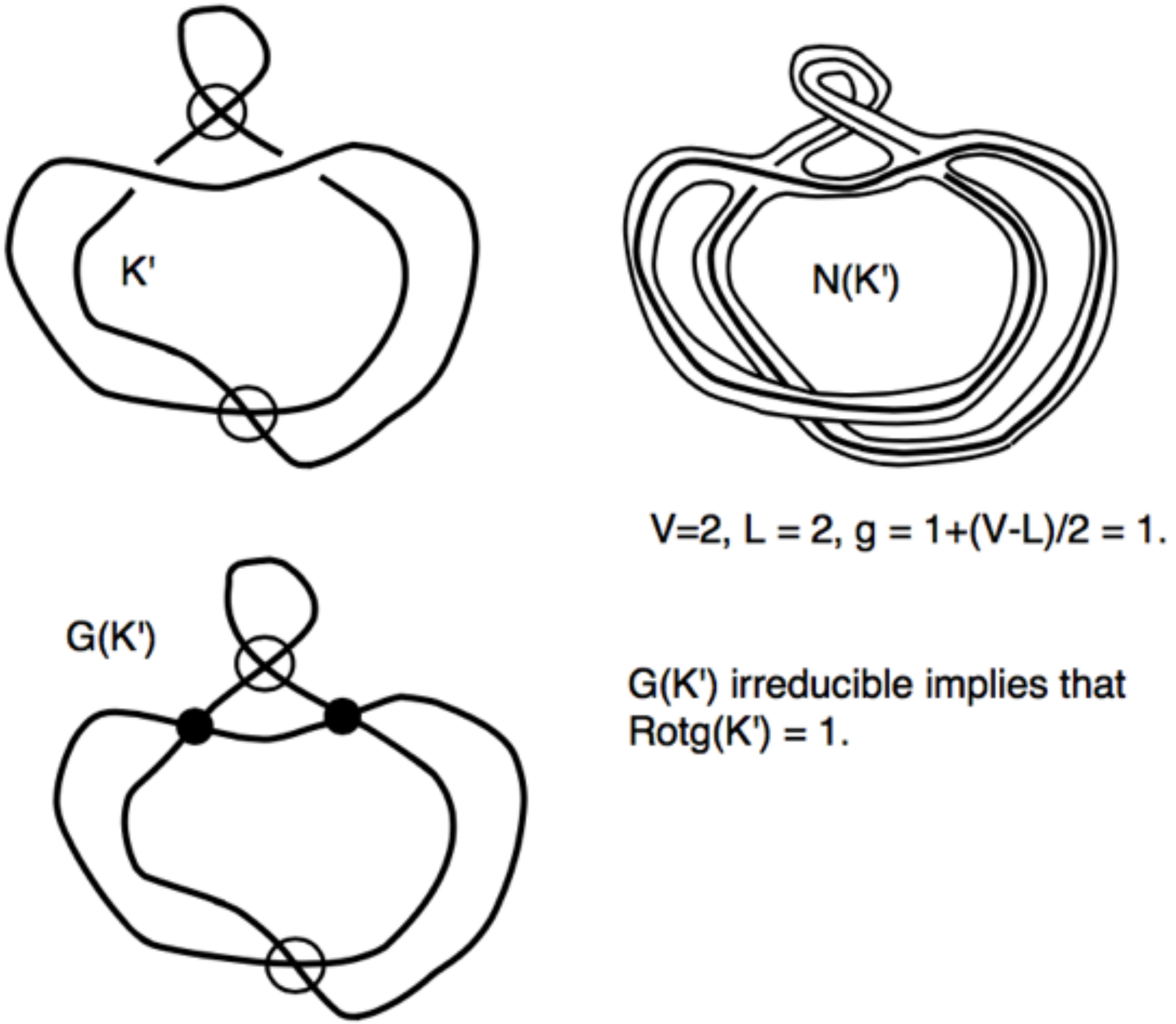
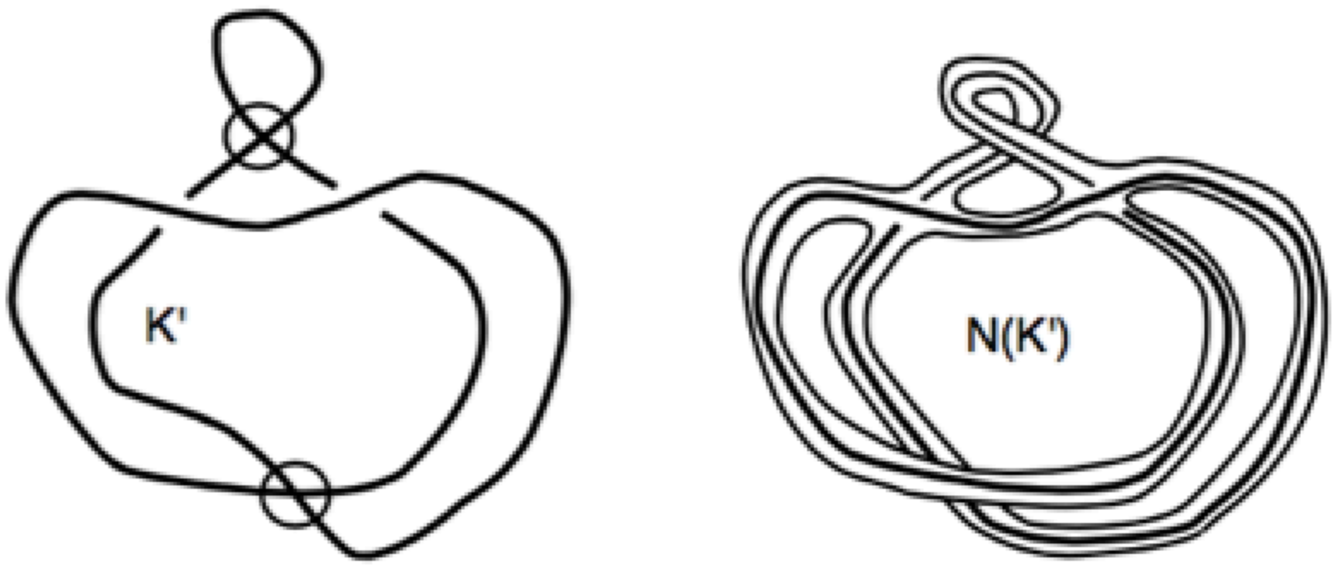


FIGURE 8. **Genus of Rotational Virtual Diagrams**



Rotational virtual knot theory is a theory of knots in oriented ribbon surfaces (abstract link diagrams) with twisting allowed in the bands of the surface. This twisting is indexed by the virtual crossing structure in the rotational diagram.

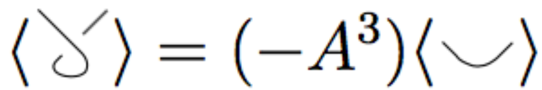
Bracket Polynomial for Rotational Virtuals

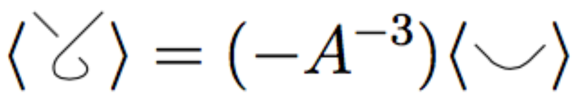
Keep track of the regular homotopy class of the state loops.

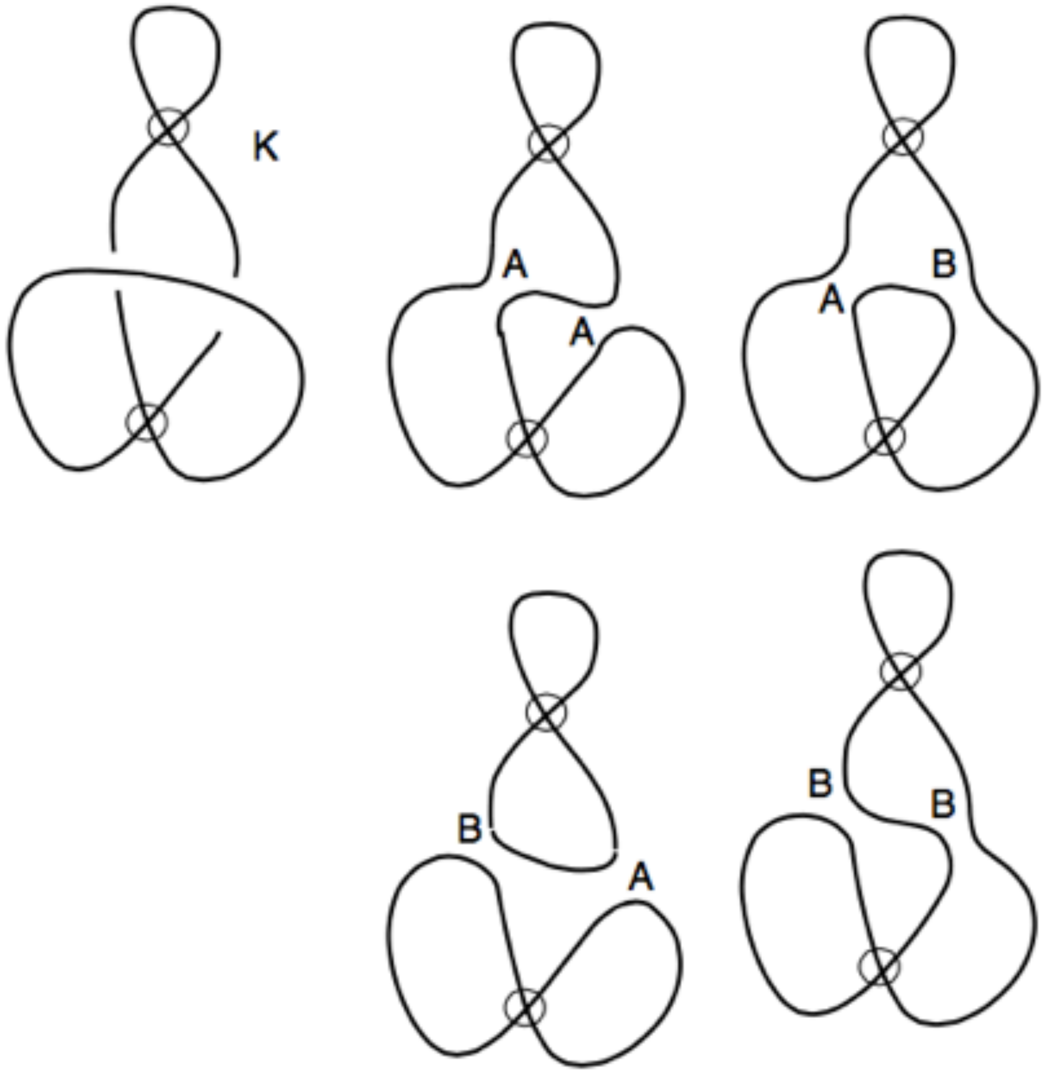
$$\langle K \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$

$B = A^{-1}$

$d = -A^2 - A^{-2}$







$[K] = (AA + BB + AB)d + AB\,[$  $]$

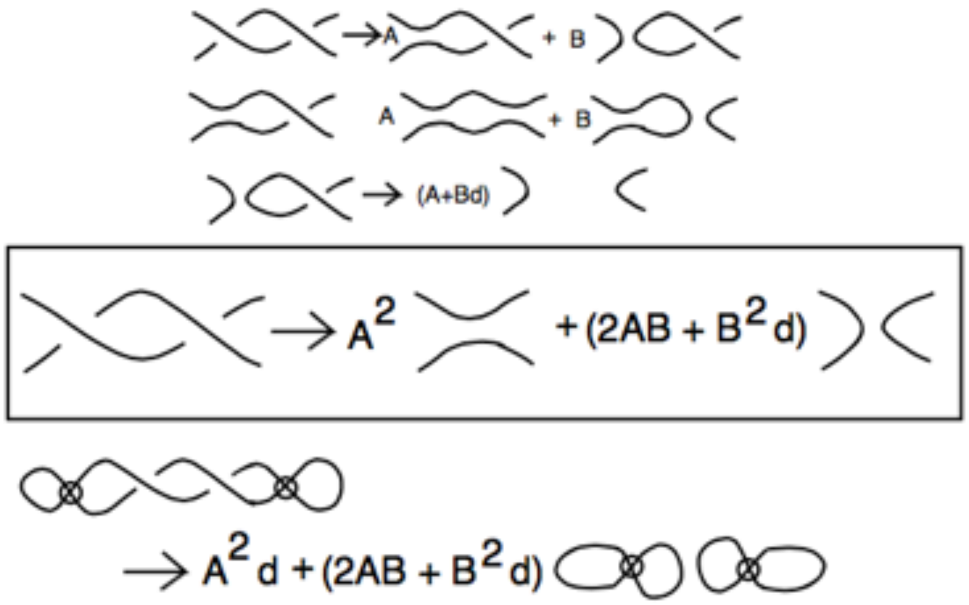
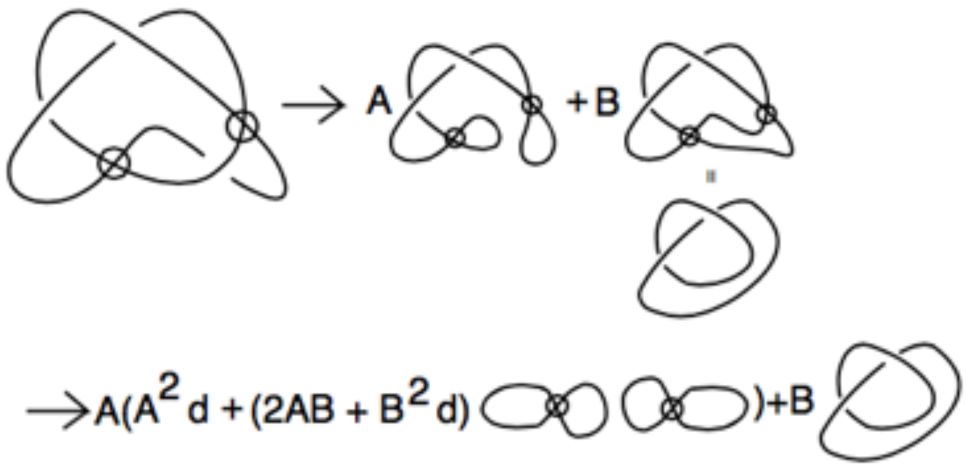
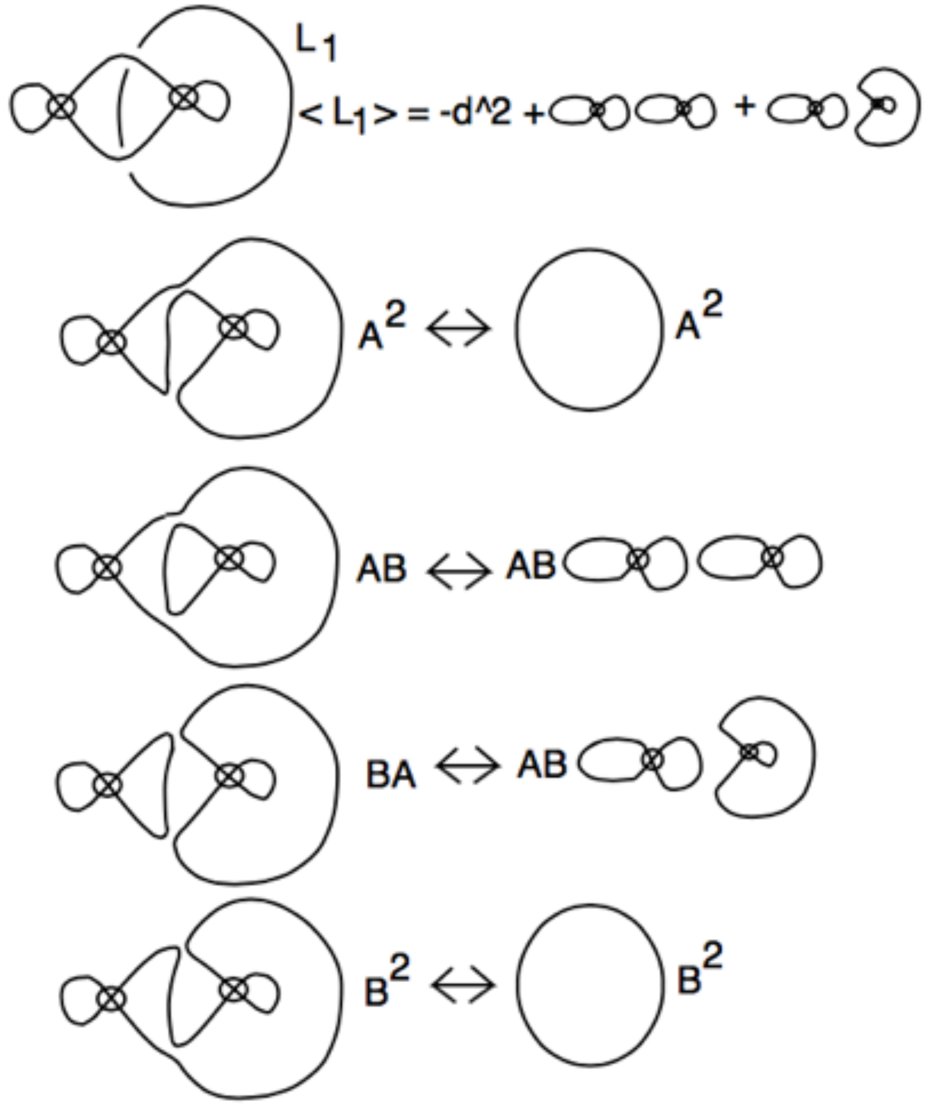


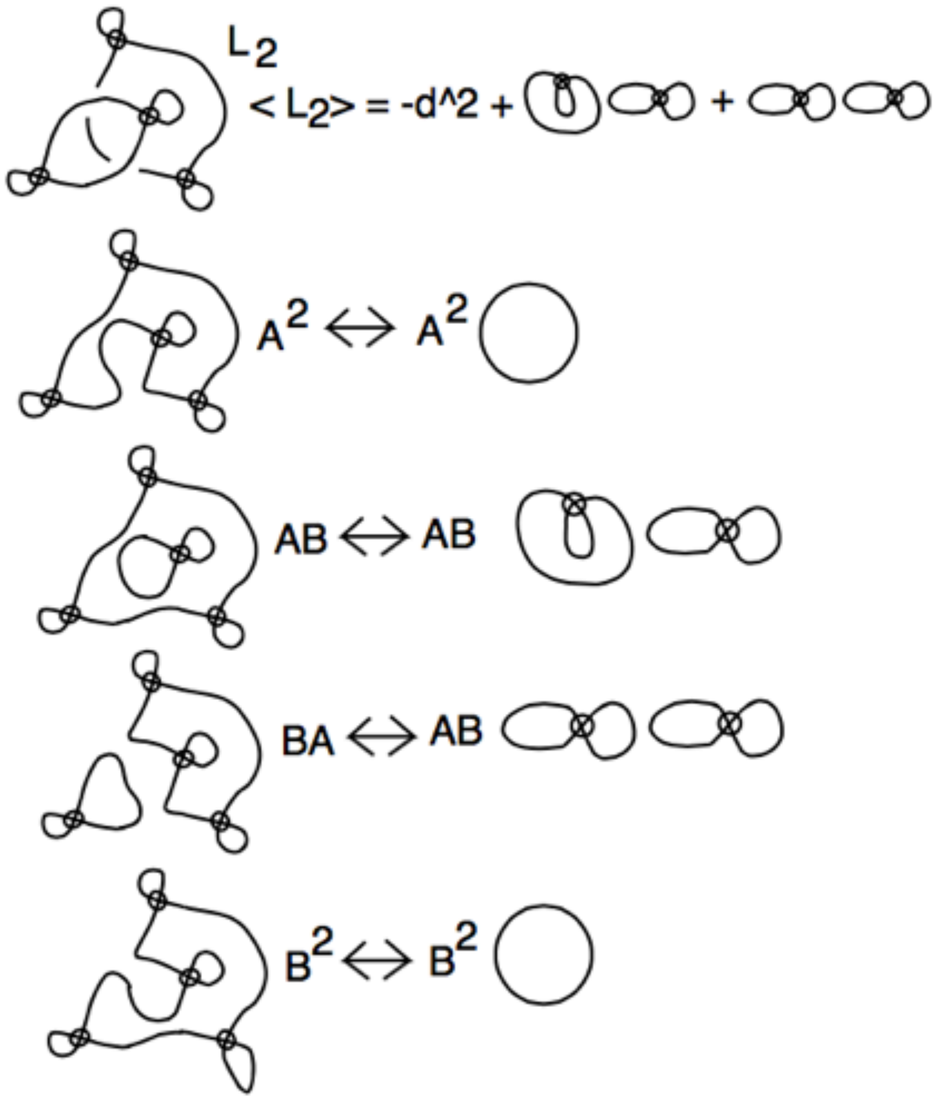
FIGURE 11. **Expanding a Tangle**





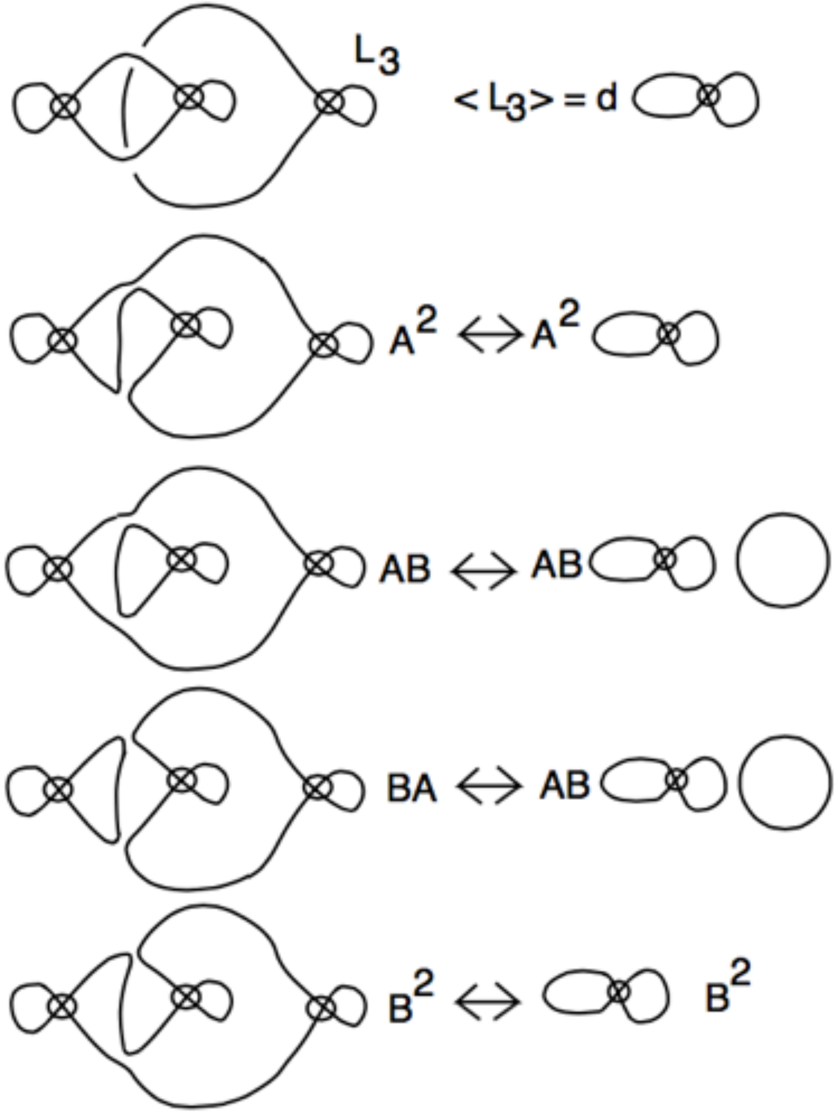
L1 is detected by bracket.

Evaluation of Rotational Bracket on a Link L_1



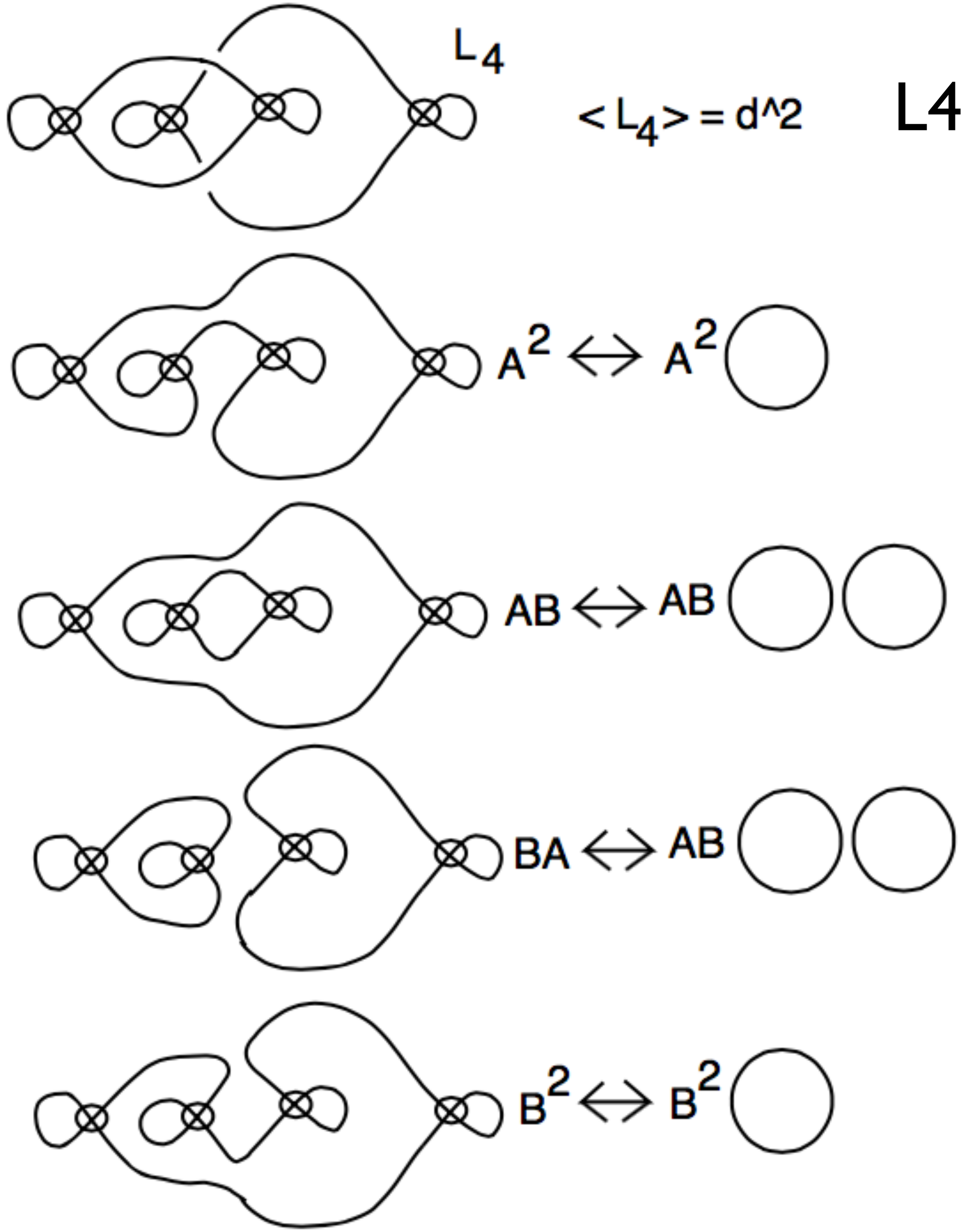
L2 is detected by bracket.

Evaluation of Rotational Bracket on a Link L_2



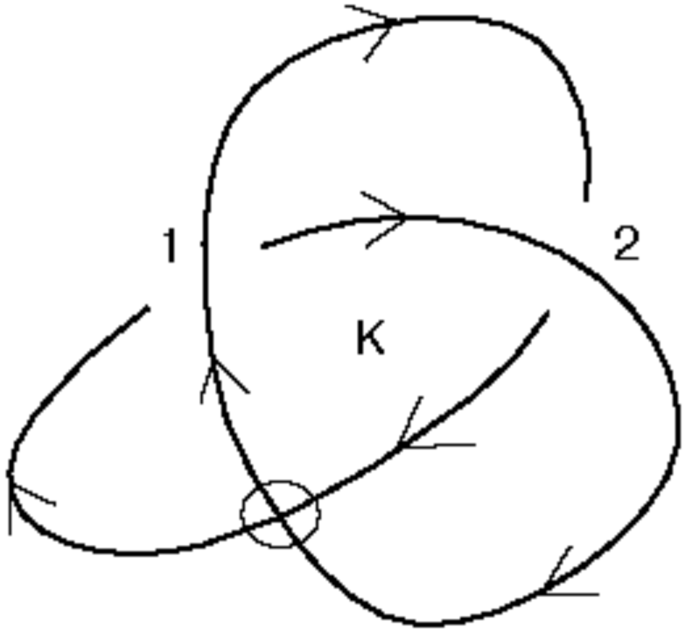
L3 not detected by bracket.

Evaluation of Rotational Bracket on a Link L_3



L4 not detected by bracket.

Parity



Bare Gauss Code
1212

Crossings 1 and 2 are odd.

A crossing is odd if it flanks an odd number of symbols in the Gauss code.

The odd writhe of K, J(K).

J(K) = Sum of signs of the odd crossings of K.

Here J(K) = -2.

Facts: J(K) is an invariant of virtual isotopy.
J(K) = 0 is K is classical.
J(Mirror Image of K) = -J(K).

Hence this example is not classical and is not isotopic to its mirror image.

The Manturov Parity Bracket Extends to Rotational Virtual Knots and Links

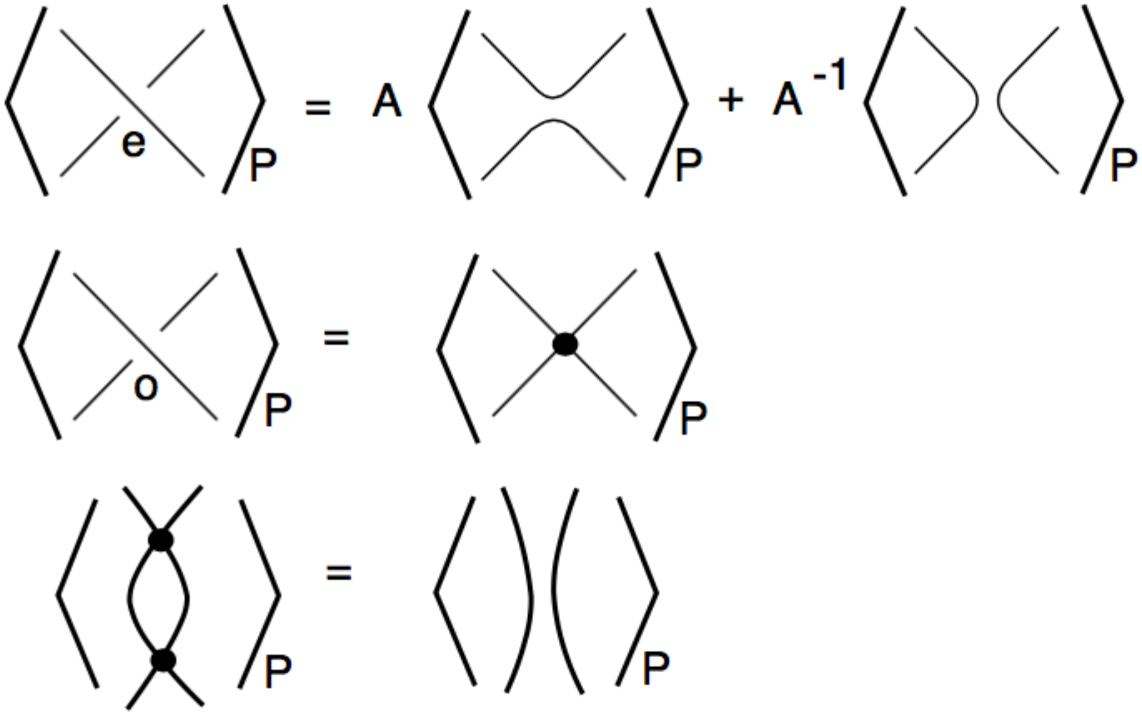
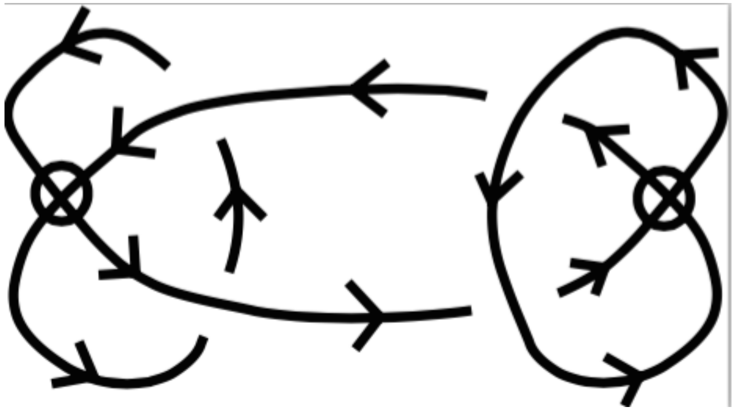
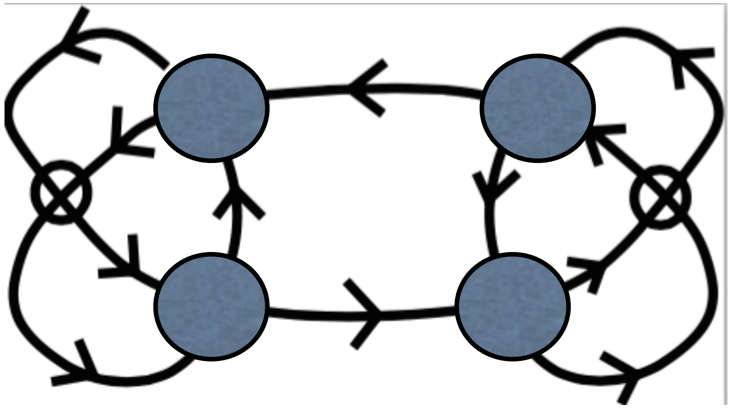


FIGURE 20. **Parity Bracket Expansion**



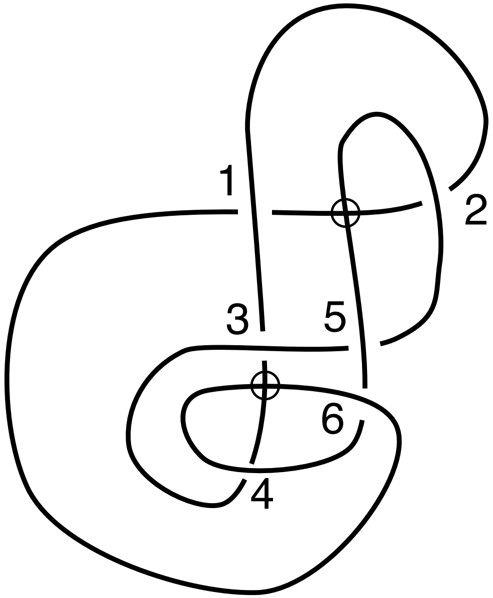
Kishino Diagram

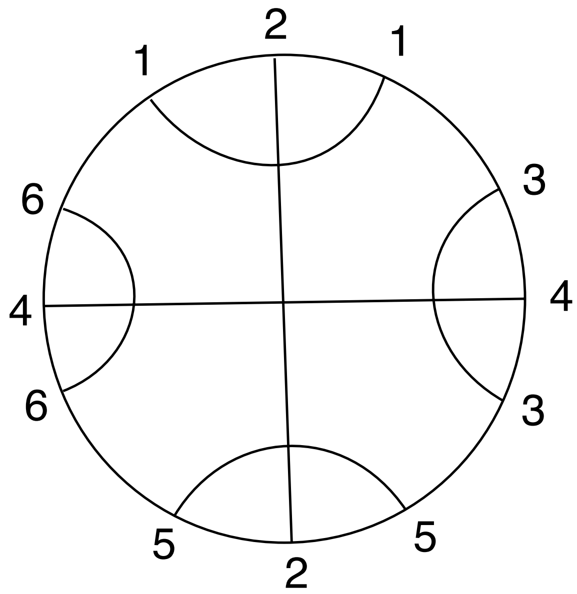
(all odd crossings)

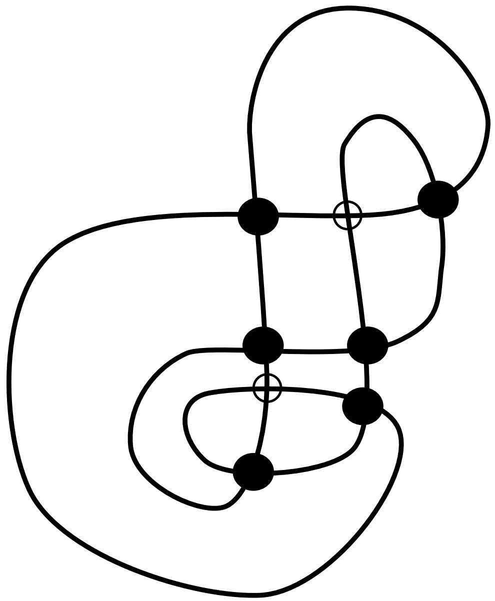


This picuture is the proof that the Kishino diagram represents a non-trivial virtual knot.

Determining Genus for Odd Knots







All classical nodes are odd.
Graph is irreducible.
One parity bracket state.
Genus g = 2.

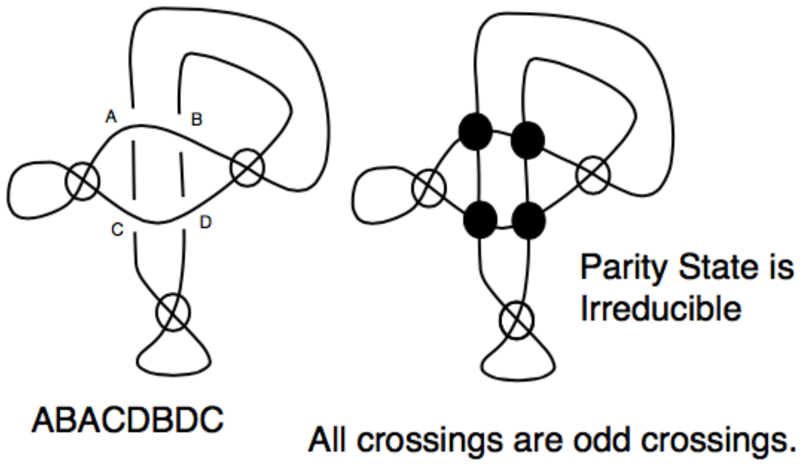
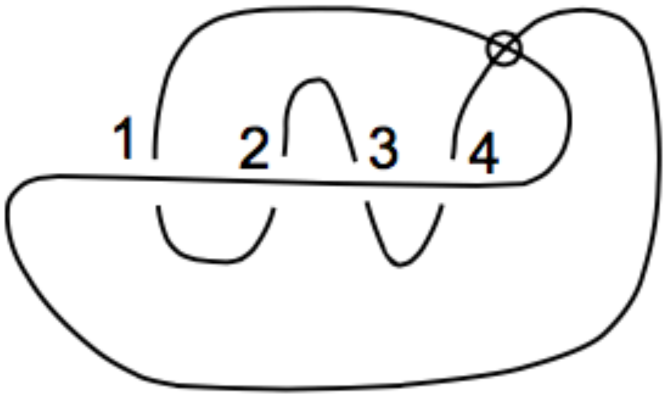


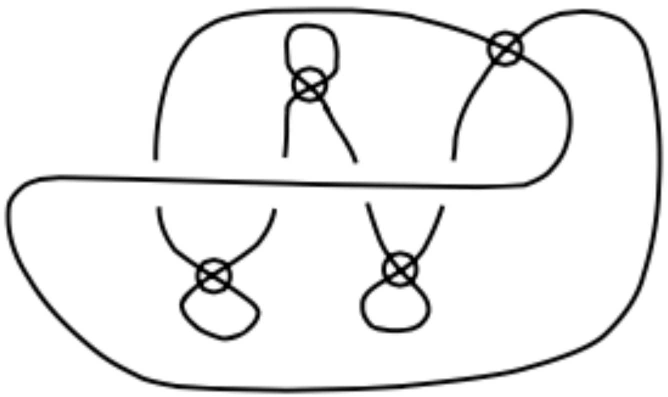
FIGURE 19. **Single State Detection of a Rotational Virtual Knot**

Irreducibility Theorem.
For every virtual diagram K with all odd crossings, there exists a decoration by virtual curls to form a new diagram L where the graph obtained by replacing all odd crossings in L by nodes is irreducible, proving that L is non-trivial.

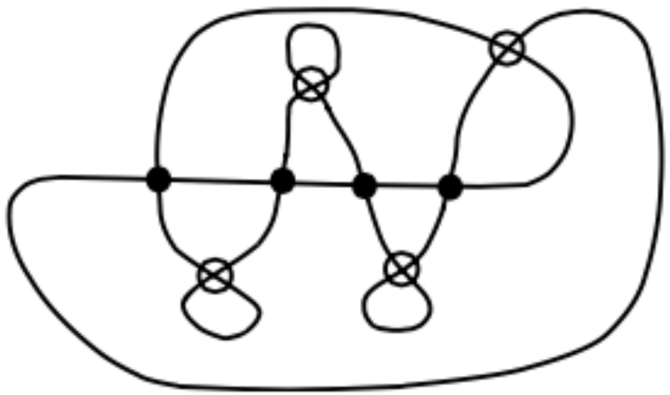


D 12341234

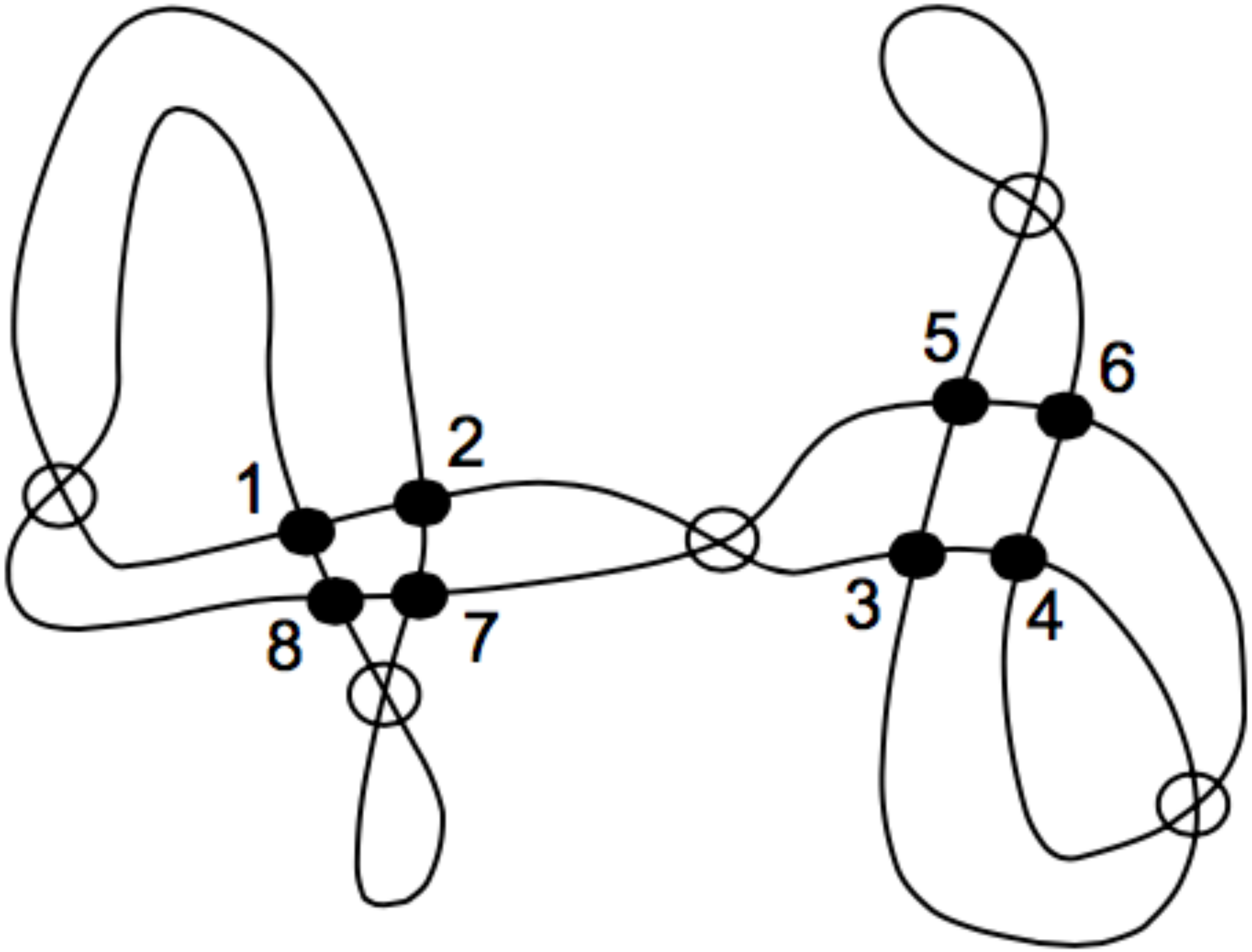
All odd crossings.



D' Knotted
Rotational



G(D')
Irreducible



12343564657818 72

All odd crossings.

Genus Two

The Parity Bracket can be extended to links. This requires an extension of the notion of a crossing that is selected for nodification. Kauffman and Kaestner do this by making all crossings between link components selected. Then a wider choice of reduction relations is needed.

Kaestner, Aaron M.; Kauffman, Louis H. Parity, skein polynomials and categorification. *J. Knot Theory Ramifications* 21 (2012), no. 13, 1240011, 56 pp.

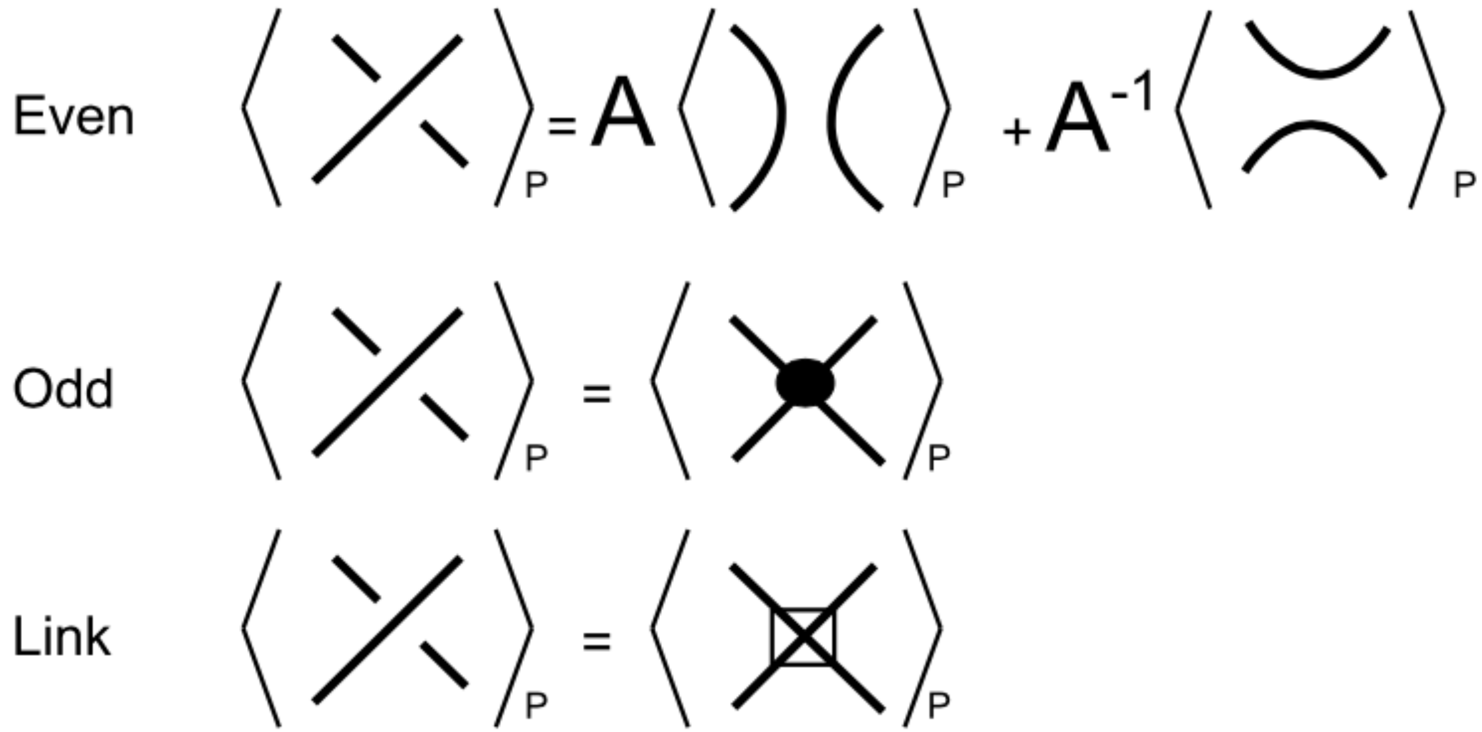


Figure 33: Link Parity Bracket Polynomial Smoothing Relations

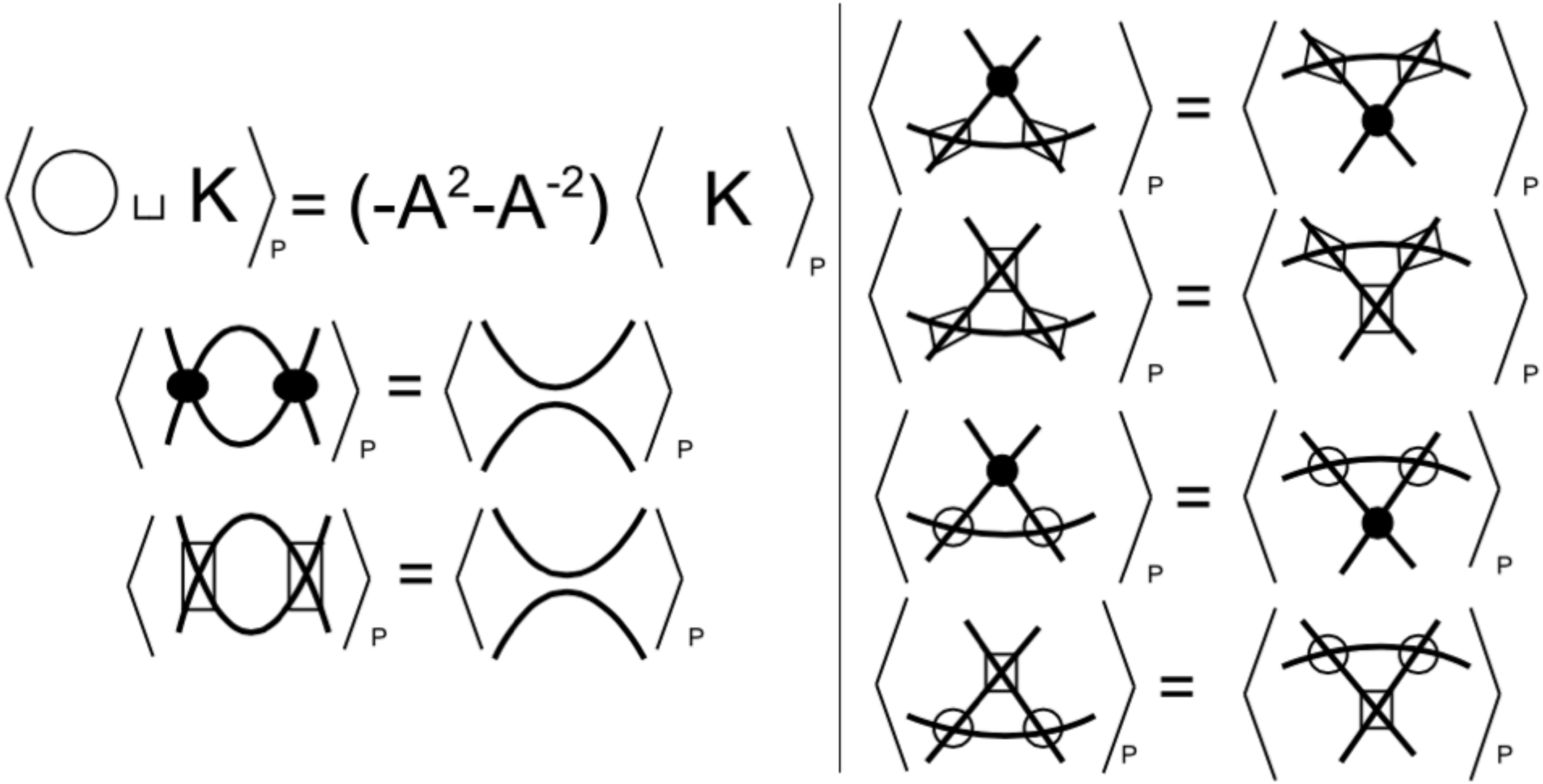
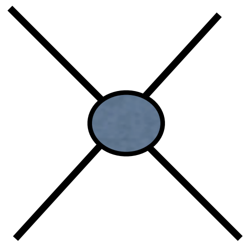


Figure 34: Link Parity Bracket Polynomial Reduction Relations

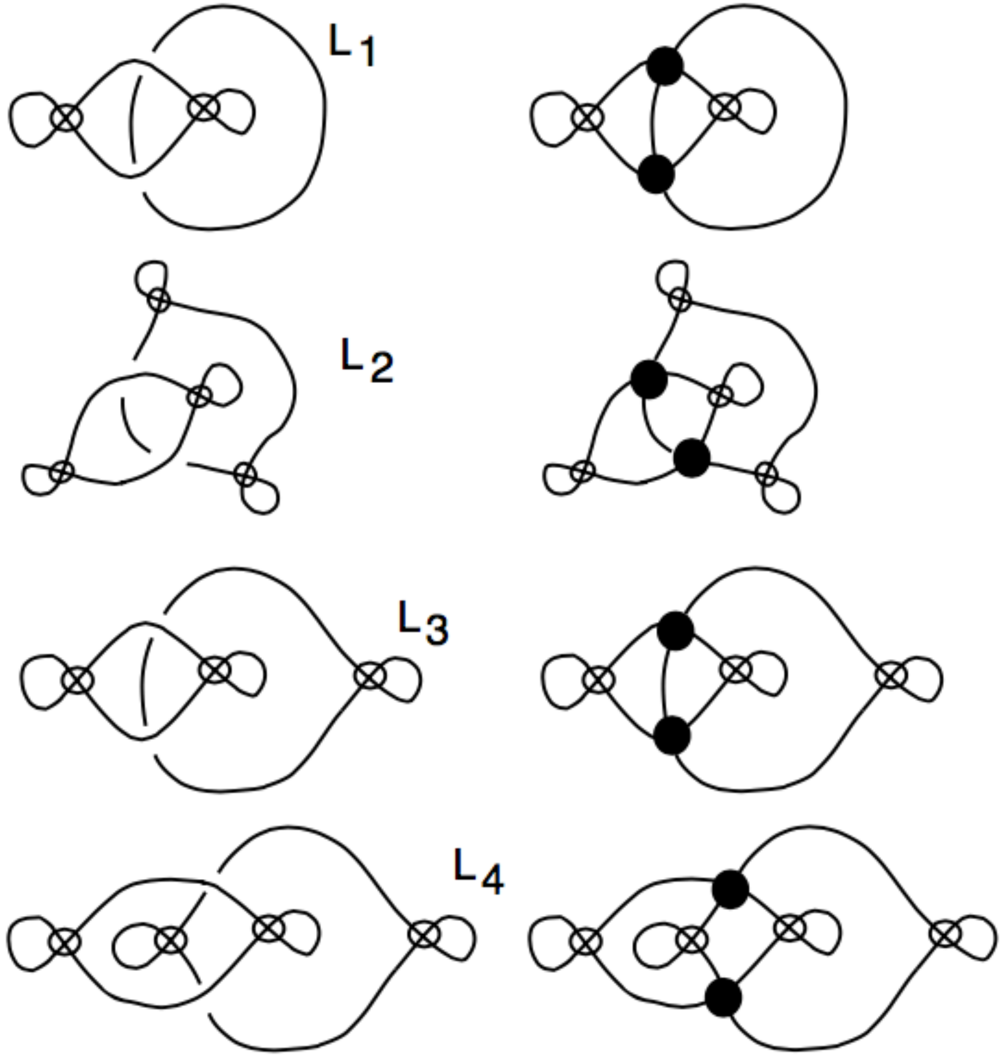
In the next slide, I have used a regular graphical node



at link crossings because there are no self-crossings in these diagrams.

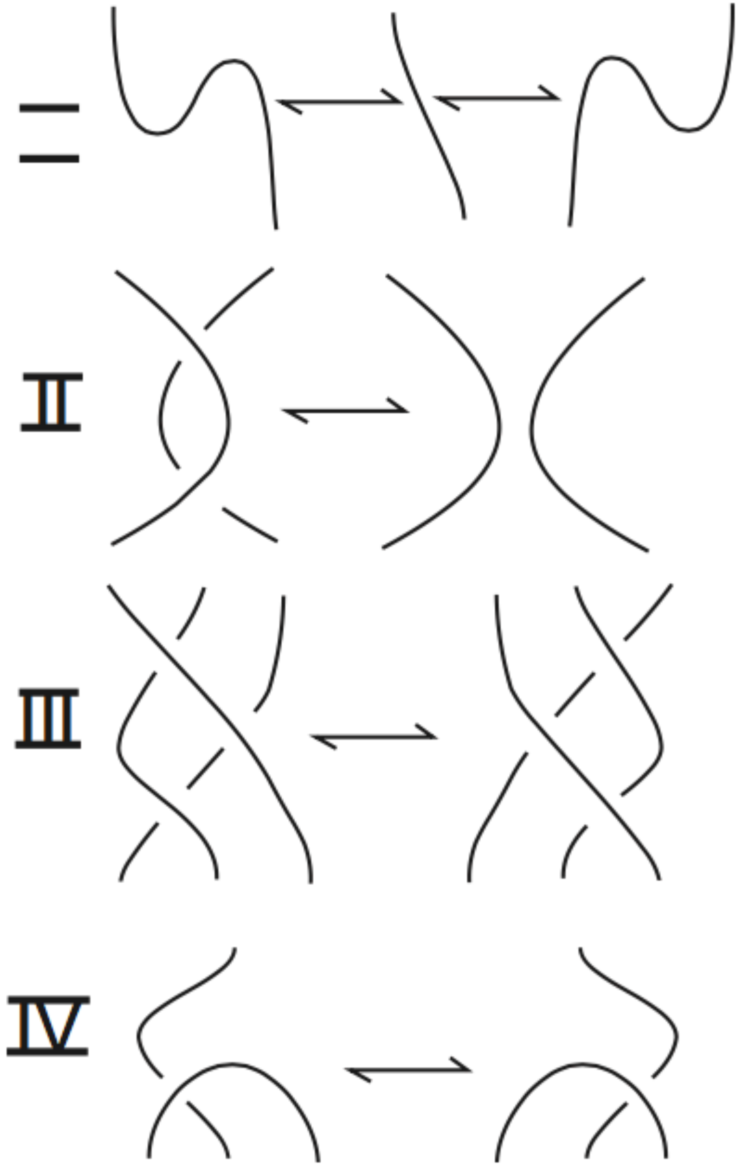
The reader can verify that the diagrams are irreducible under the conditions for the parity link bracket polynomial.

All Non-Trivial, All Pairwise Distinct Via Parity Bracket for Links (Kaestner - Kauffman)



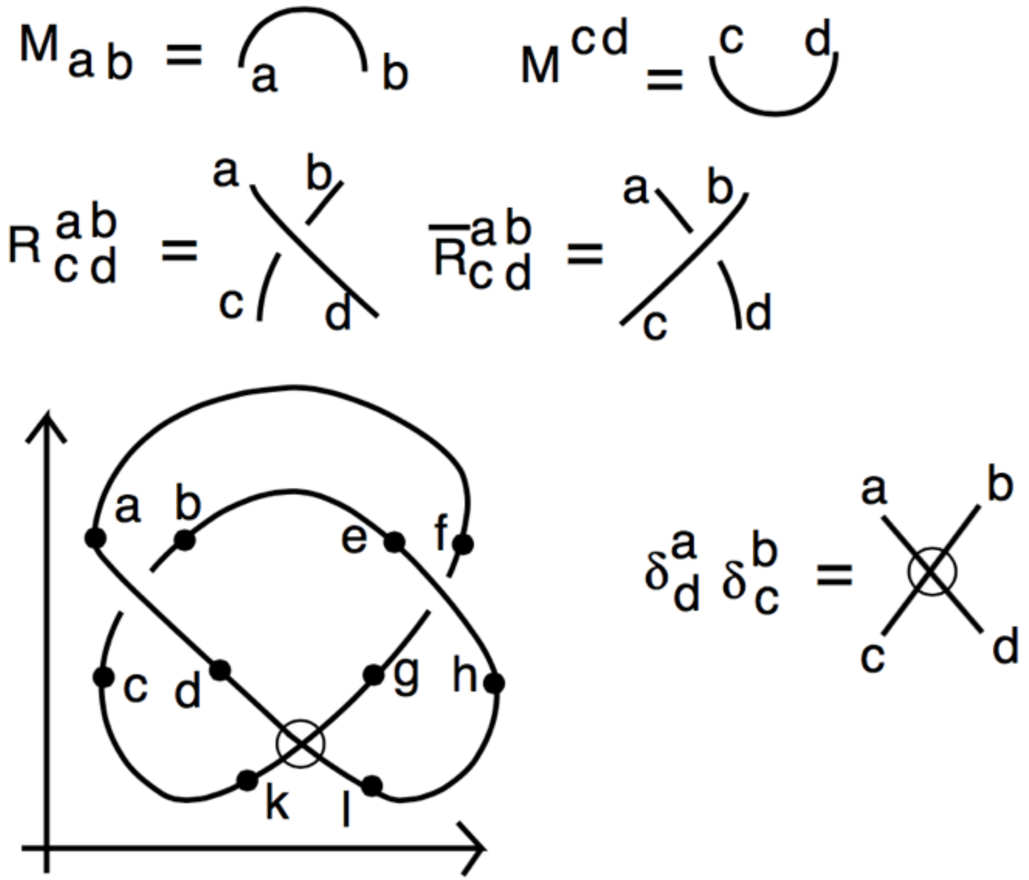
Combinatorial Topology (parity) can show that many
rotational virtual links
undectectable by the bracket polyomial are
non-trivial and distinct.

What about Quantum Invariants?



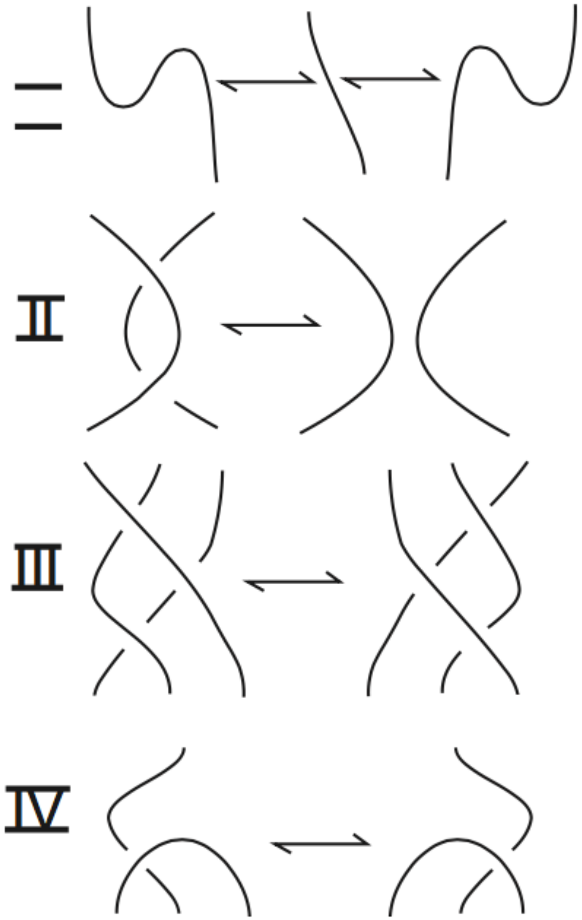
Regular isotopy with respect to a vertical direction.

Tangle
Category



$$Z_K = M_{af} M_{be} M^{ck} M^{lh} R^{ab}_{cd} R^{ef}_{gh} \delta^{d}_{l} \delta^{g}_{k}$$

FIGURE 25. **Quantum Link Invariants**



$$V^{ab}_{cd} = \delta^a_d \delta^b_c.$$

$$0. \quad M^{ai} M_{ib} = \delta^a_b$$

$$II. \quad R^{ab}_{ij} \overline{R^{ij}_{cd}} = \delta^a_c \delta^b_d$$

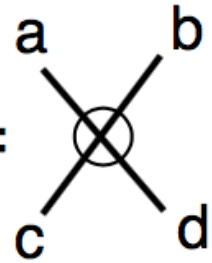
$$III. \quad R^{ab}_{ij} R^{jc}_{kf} R^{ik}_{de} = R^{bc}_{ij} R^{ai}_{dk} R^{kj}_{ef}$$

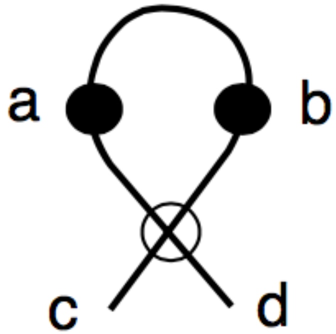
$$IV. \quad R^{ai}_{bc} M_{id} = M_{bi} \overline{R^{ia}_{cd}}$$

$$IV'. \quad \overline{R^{ai}_{bc}} M_{id} = M_{bi} R^{ia}_{cd}$$

Regular isotopy with respect to a vertical direction.

$M_{ab} =$ 

$\delta^{a}_{d}\,\delta^{b}_{c} =$ 

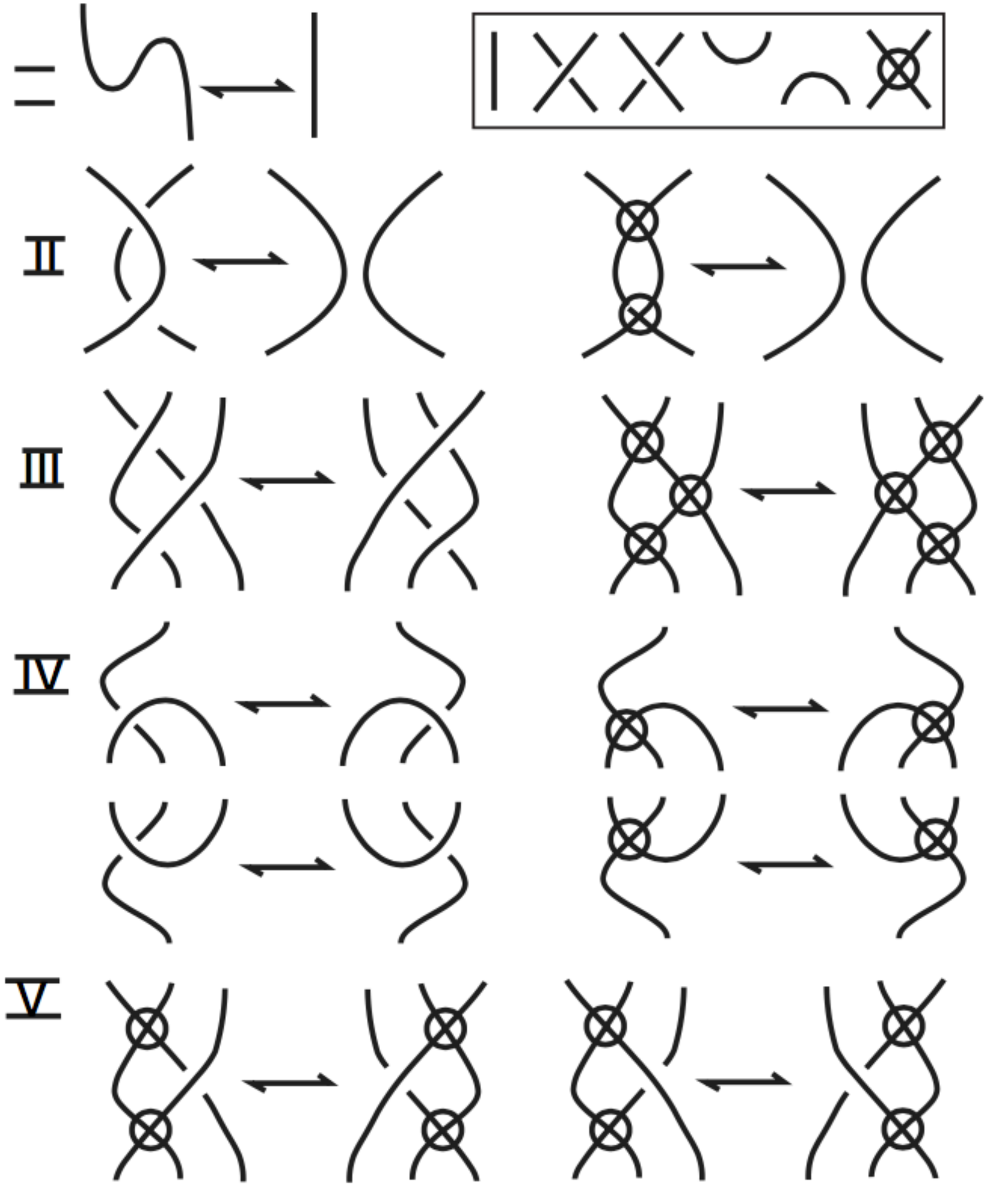


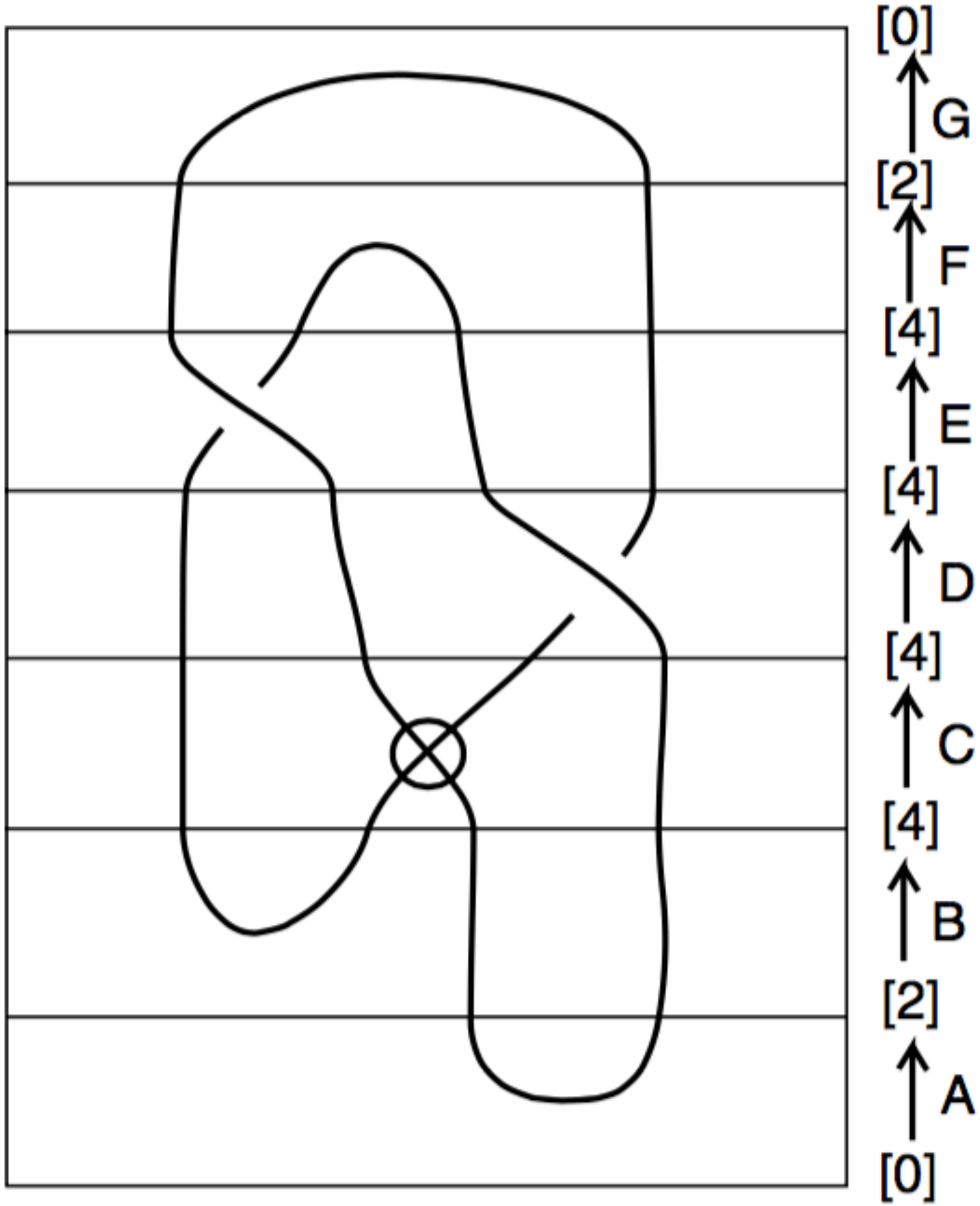
$= M_{ab}\,\delta^{a}_{d}\,\delta^{b}_{c} = M_{dc} =$ 

FIGURE 26. **Quantum Virtual Curl**

In general, quantum invariants see the presence of virtual curls.

Virtual Rotational Tangle Category




[0]
G
[2]
F
[4]
E
[4]
D
[4]
C
[4]
B
[2]
A
[0]

Quantum Link Invariants via the Category of a Quantum Algebra

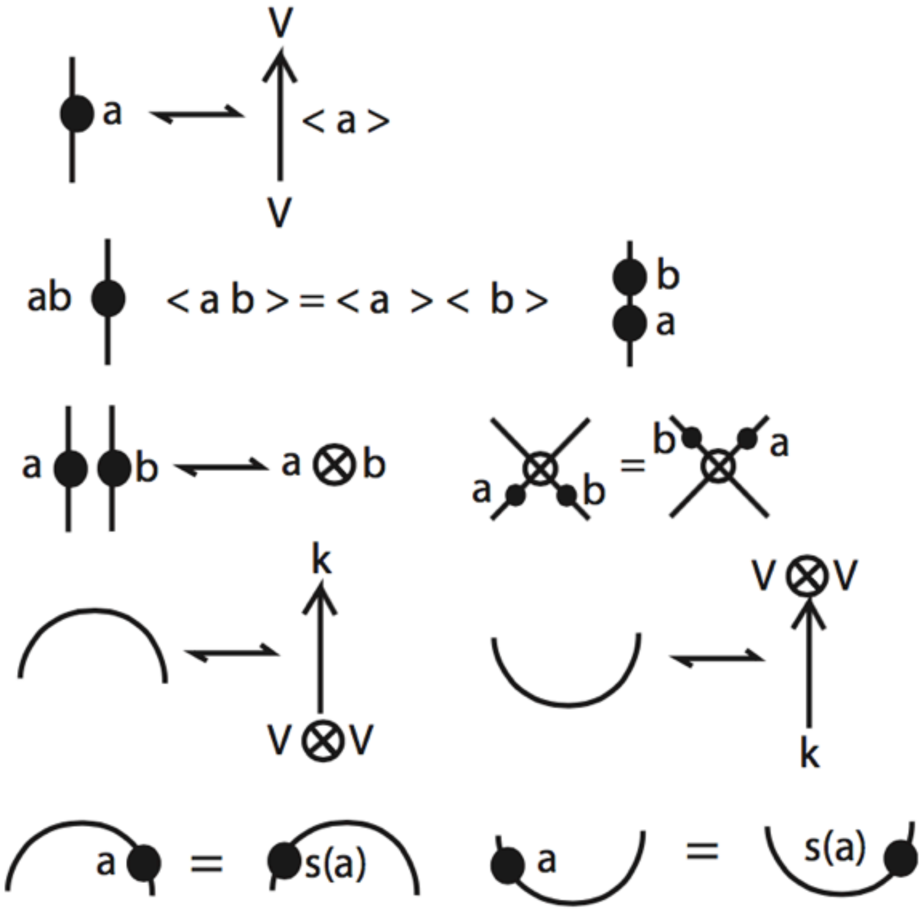
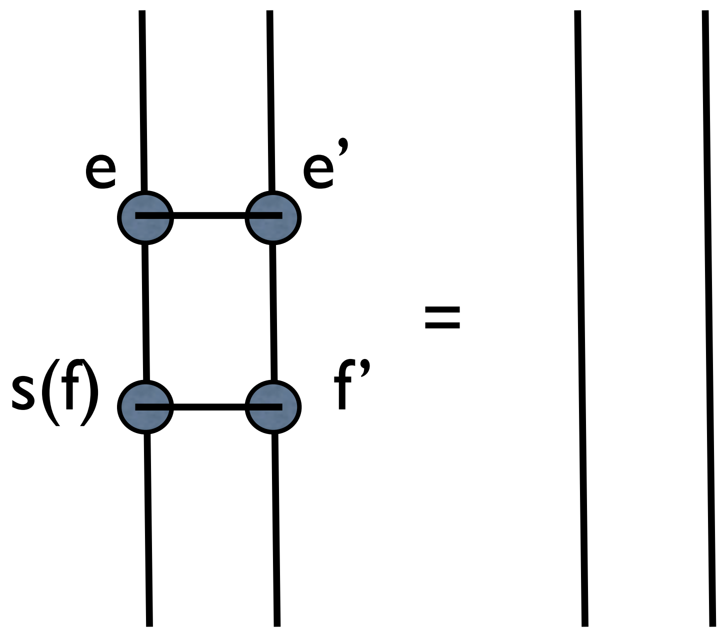
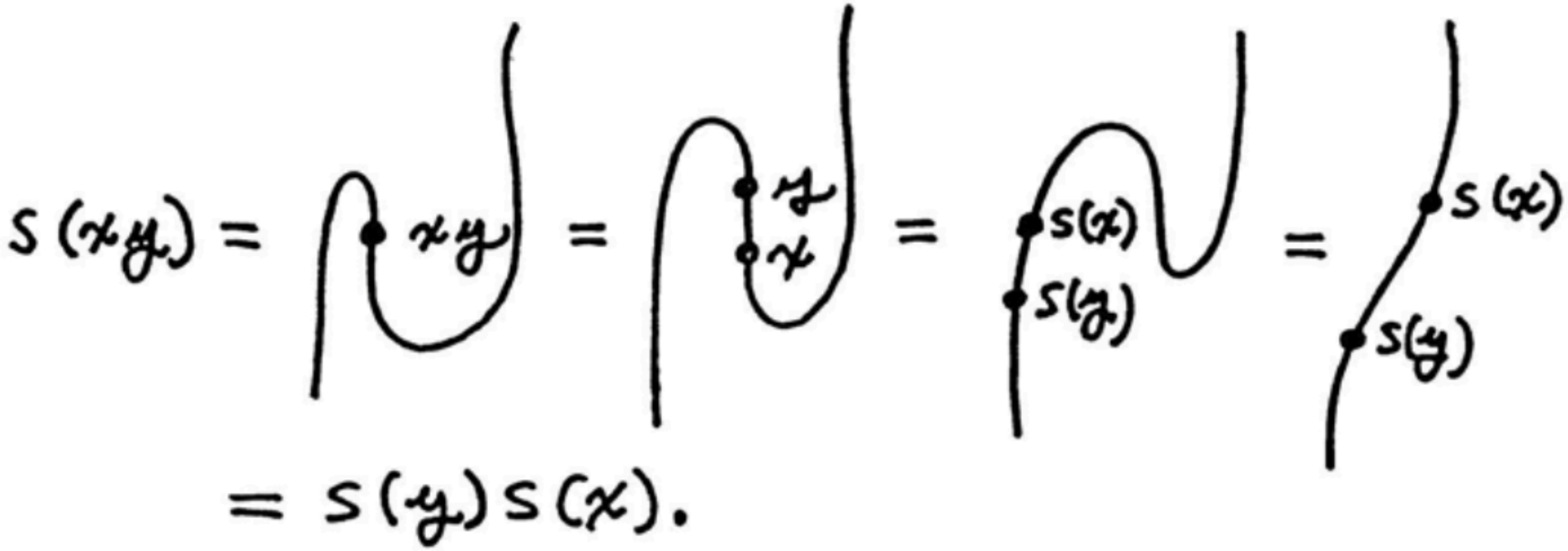


FIGURE 37. Morphisms in $Cat(A)$

A quantum algebra has an antipode and a solution to the algebraic Yang-Baxter equation.



Algebraic Yang-Baxter
Element and Its Inverse



The antipode is an antimorphism of algebras.

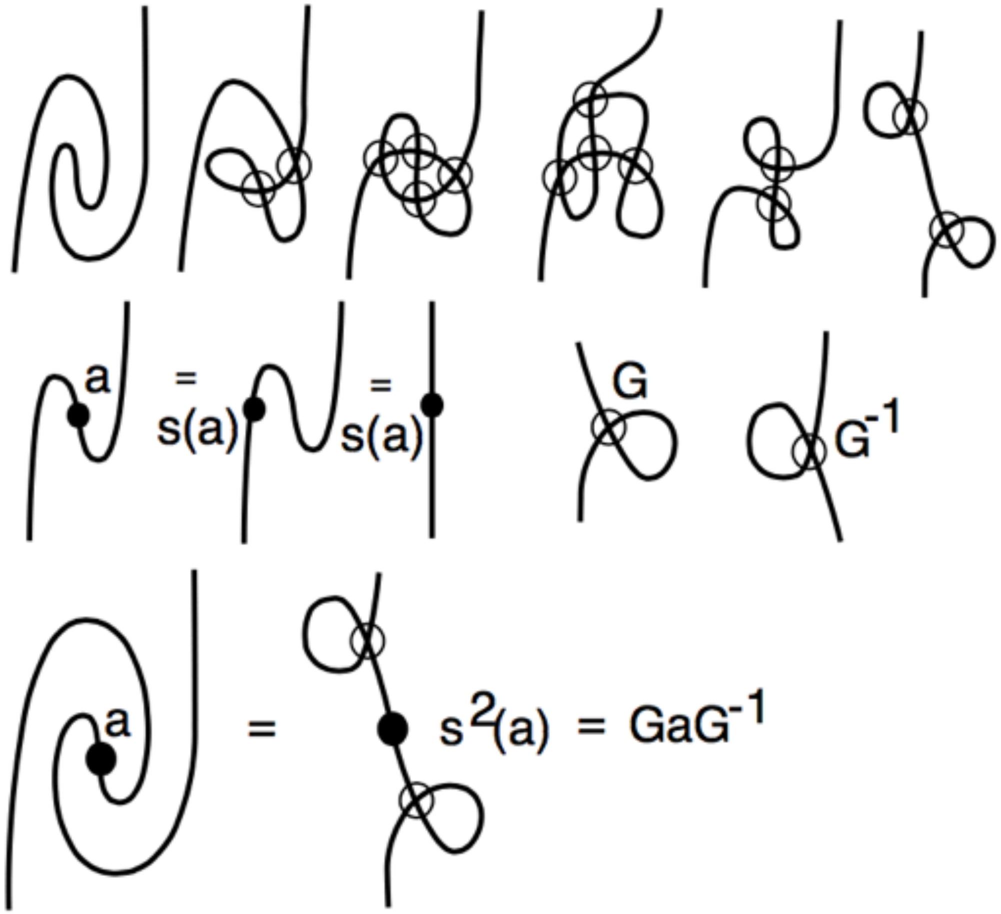
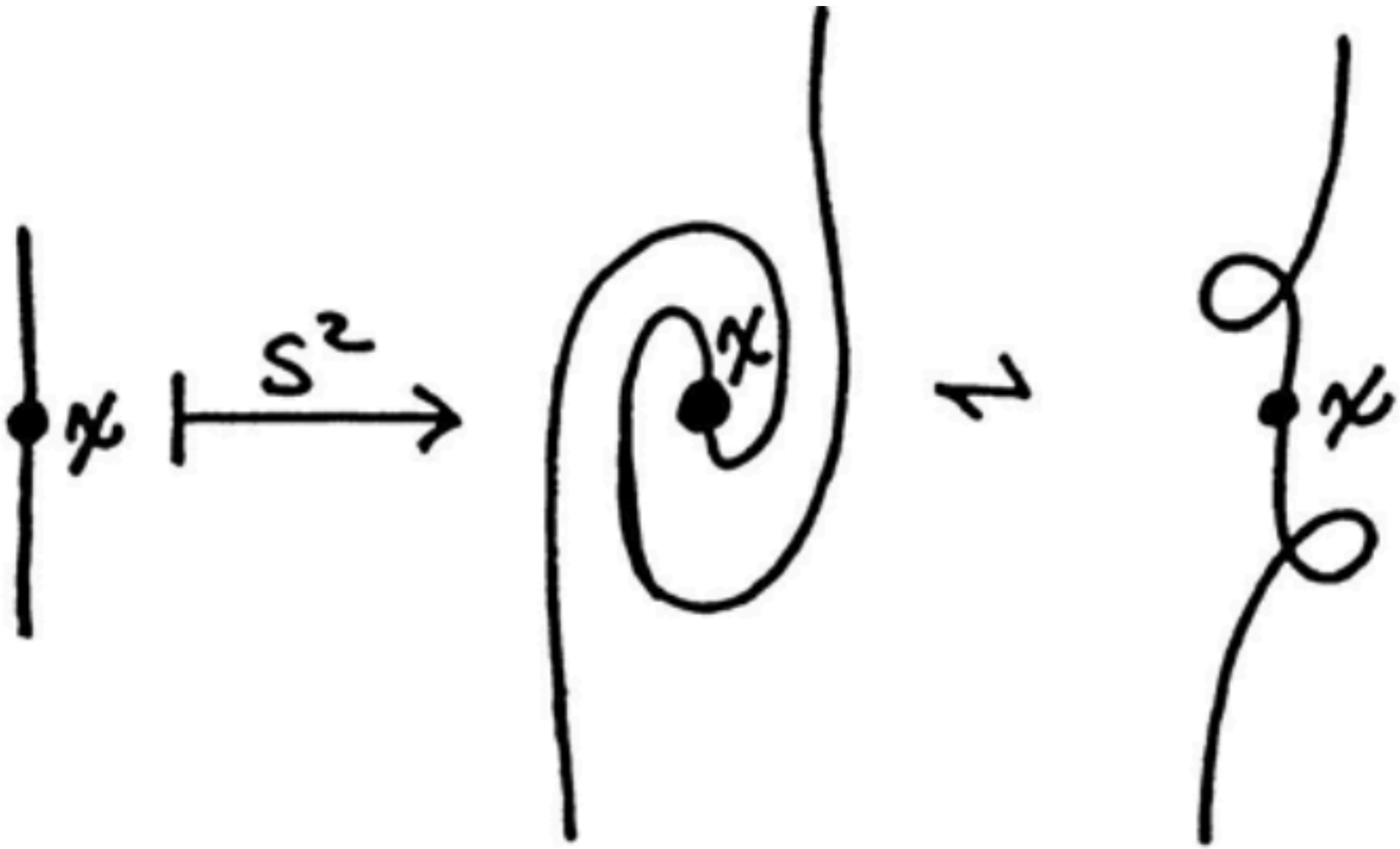


FIGURE 38. Diagrammatics of the antipode



x
S^2
x
x

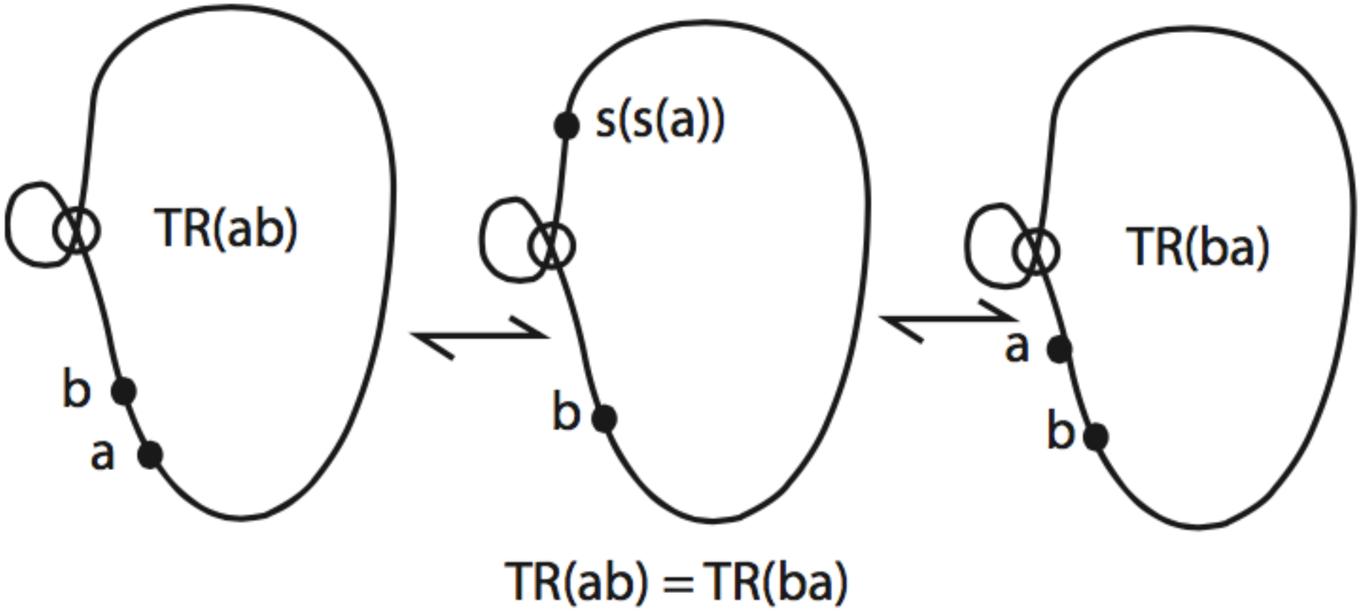


FIGURE 39. Formal trace

Basic Functor

$$F: \text{Rotational Tangle Category} \longrightarrow \text{Category of Quantum Alg}$$

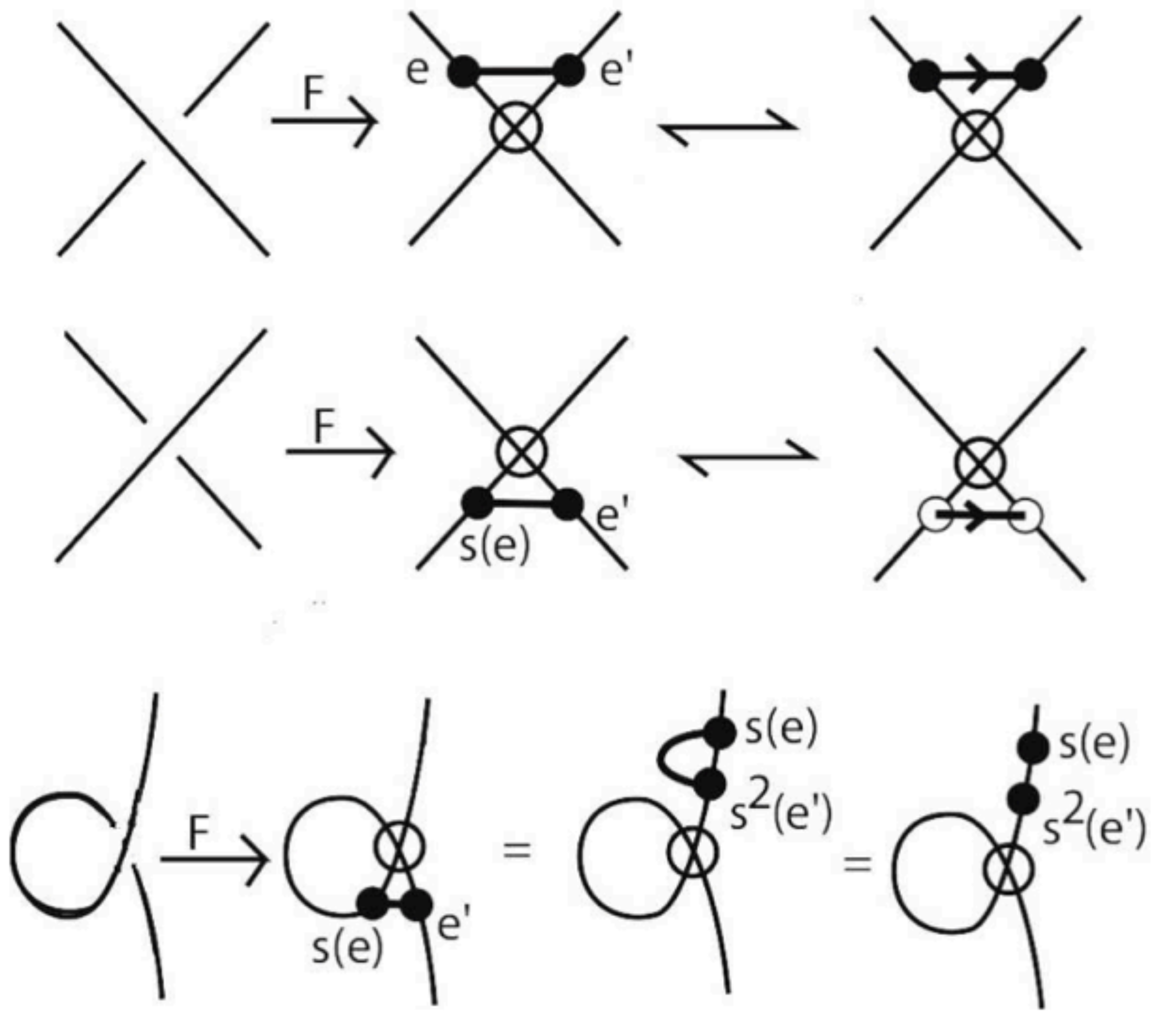


FIGURE 40. The functor $F : VTC \longrightarrow Cat(A)$

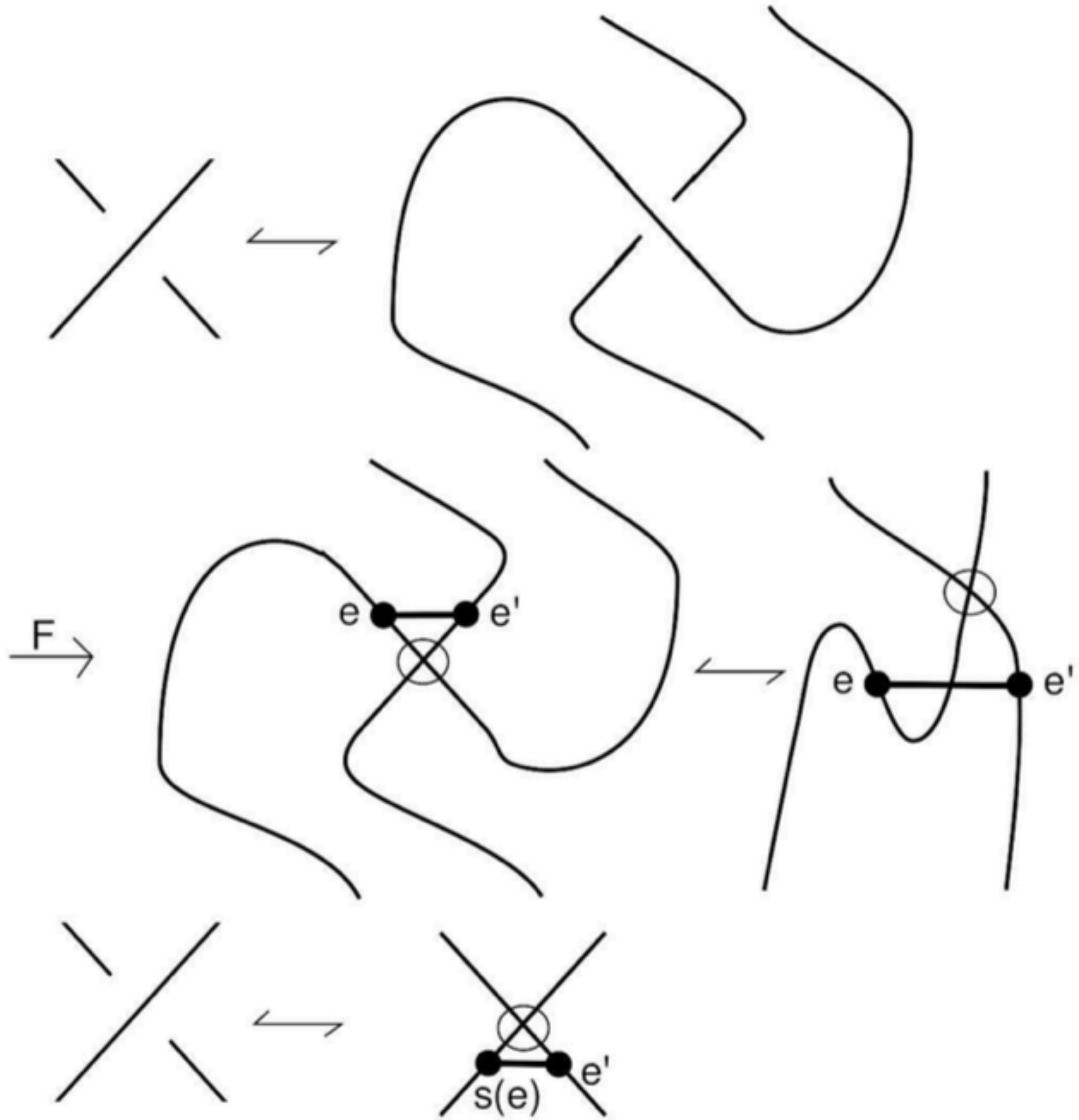
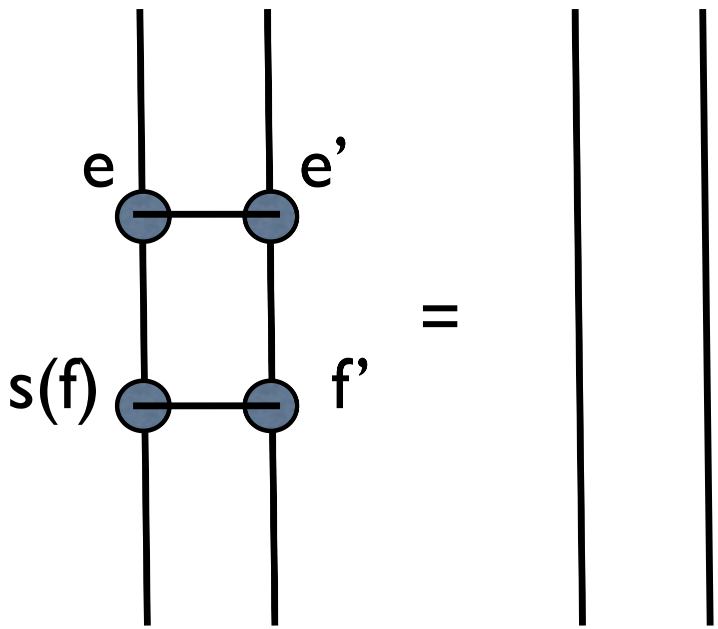


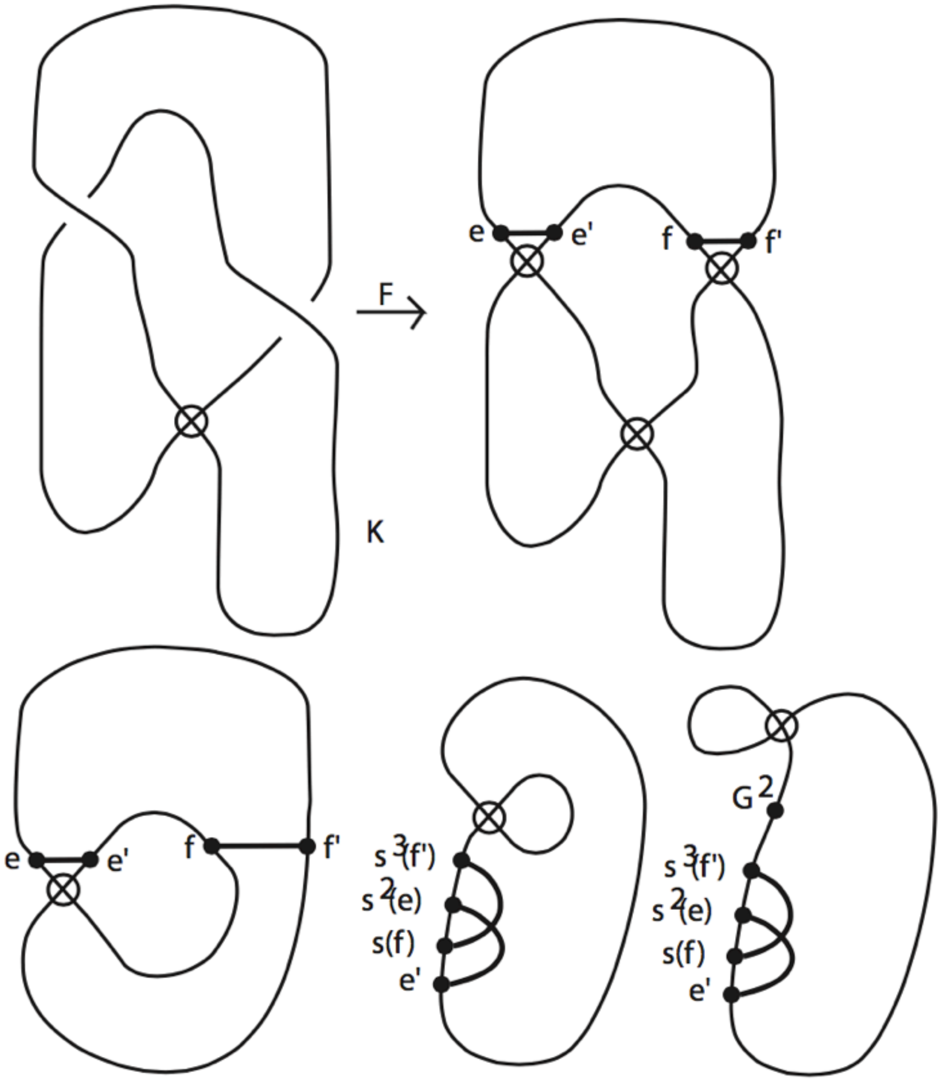
FIGURE 41. Inverse and antipode



Algebraic Yang-Baxter
Element and Its Inverse

The structure of
the inverse of
the Yang-Baxter
element is implied
by the structure of
the category of the
quantum algebra.

Functor Applied to the Virtual Trefoil



F(K) = TR[e' s(f) s(s(e)) s(s(s(f'))) GG]

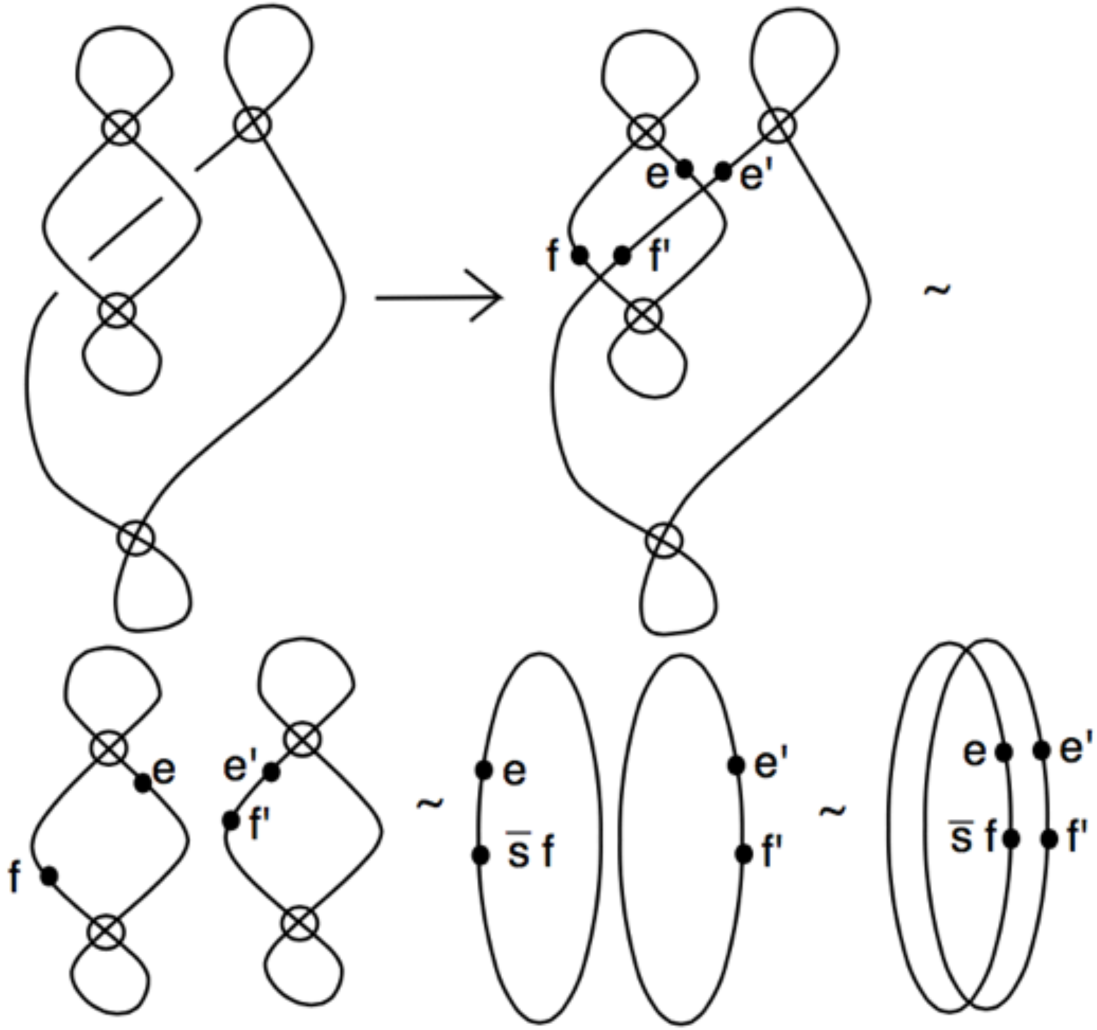
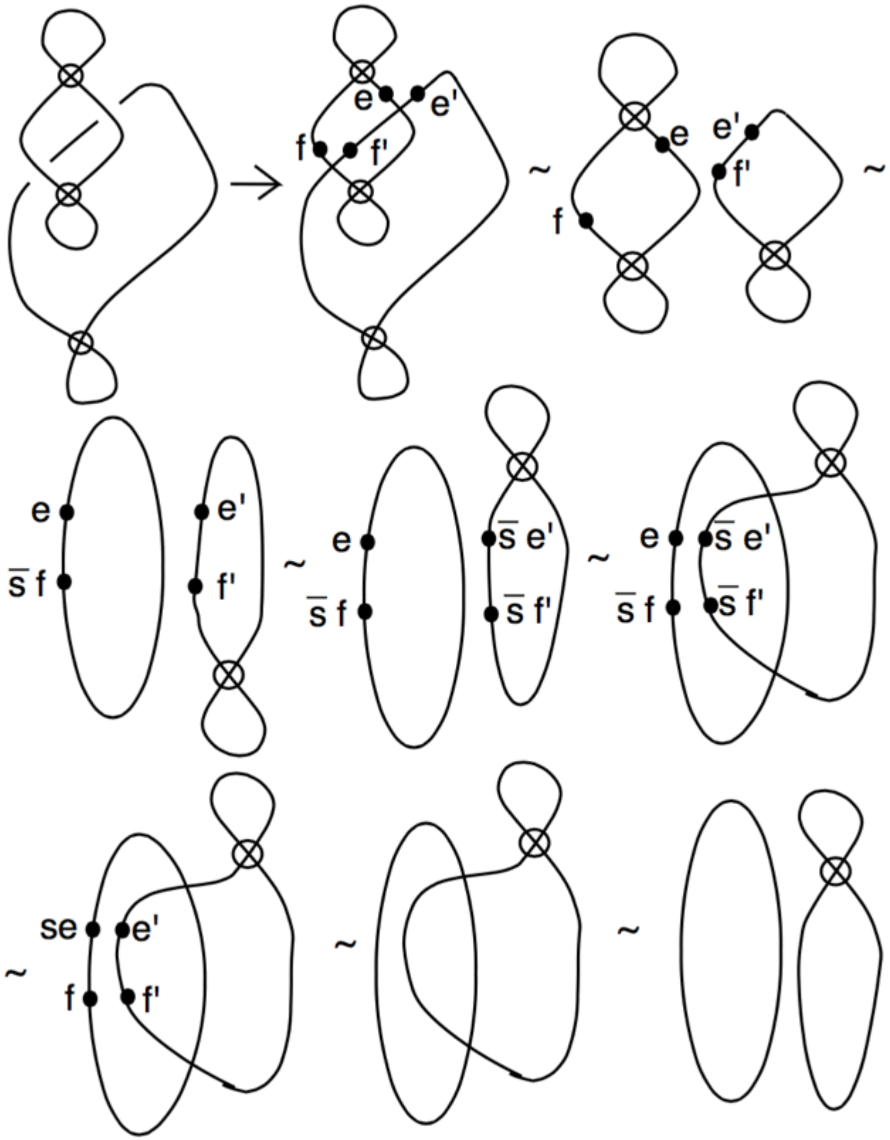
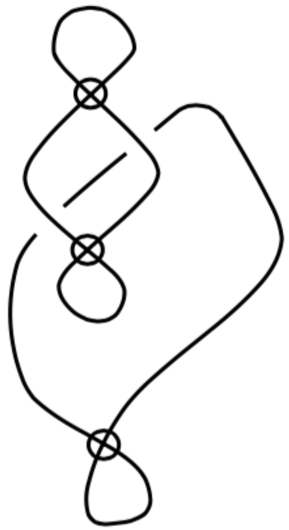


FIGURE 43. First Rotational Link - Non-Trivial Invariants



A Non-Trivial Rotational Link with Trivial Quantum Invariants.



Thus we find simple examples of non-trivial virtual rotational links that are not detectable by any quantum algebraic invariants.

This leads to many new questions and the prospect of using rotational virtual links as a testing category for the strength and properties of quantum invariants.