The Gordon-Litherland pairing for knots and links in thickened surfaces

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CVCK* Classical knots and virtual knots seminar
Ohio State University
November 23, 2020

Acknowledgements

Collaborators.
This talk is based on joint papers in preparation with *Micah Chrisman* and *Homayun Karimi*.

Support.
This research was partially supported by a grant from the *Natural Sciences and Engineering Research Council of Canada.*

Background

1. **Knot Signature.**
In *Cooper* [1982 PhD thesis], *Mandelbaum-Moishezon* [1983], and *Cimasoni-Turaev* [2007, Osaka J Math], signatures are defined for homologically trivial knots in 3-manifolds.

In *Im, Lee, Lee* [2010, JKTR] and *B-, Chrisman, Gaudreau* [2020, Indiana Univ J Math], signature-type invariants are defined on various subcategories of virtual knots and links.

Goal 1: Provide more general definitions of signature invariants for knots and links in 3-manifolds, and for virtual knots and links.

Background

2. **Gordon-Litherland Pairing.**
In Greene [2017, Duke Math J], the GL pairing is extended to $\mathbb{Z}_2$ homology 3-spheres. He used it to give a geometric characterization of alternating links (cf. *Howie* [2017, Geom Topol]), and a new proof of the Tait conjectures.

Goal 2: Extend the GL pairing to more general 3-manifolds.
Use it to characterize alternating knots and links in 3-manifolds.

Classical knot signature

There are at least three ways to define the knot signature for classical knots.

1. [Trotter, Murasugi]
Let K be a knot. Choose a Seifert surface F. The *Seifert form* Θ is given by $\Theta(\alpha, \beta) = lk(\alpha^-, \beta)$ for $\alpha, \beta \in H_1(F)$.
Any matrix V representing Θ on a basis for $H_1(F)$ is called a *Seifert matrix*. It is well-defined up to unimodular congruence.

The signature of $V + V^\mathsf{T}$ is invariant under unimodular congruence and independent of choice of F.

Definition

The knot signature is given by $\sigma(K) = \mathrm{sig}(V + V^\mathsf{T})$.

Classical knot signature

2. [Kauffman-Taylor]
View $K \subset S^3 = \partial B^4$. Push F into D^4, and let M_F be the double cover of D^4 branched along F. Then $\partial M_F = X_2$, the double cover of S^3 branched along K. Note that X_2 is a $\mathbb{Z}_2$ homology 3-sphere.

The intersection form $Q\colon H_2(M_F) \times H_2(M_F) \to \mathbb{Z}$ is non-degenerate.

Definition

The knot signature is given by $\sigma(K) = \mathrm{sig}(Q)$.

3. [Gordon-Litherland]
Let F be a spanning surface for K, not necessarily oriented.

Gordon and Litherland define a symmetric, bilinear pairing

$$\mathscr{G}_F \colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}.$$

Its quadratic form specializes to the Trotter form when F is a Seifert surface and to the Goeritz form when F is the black (or white) surface of a checkerboard coloring.

Let N be a tubular neighborhood of F, and set $\widetilde{F} = \partial N$.

Then $\widetilde{F} \to F$ is a double cover ($\widetilde{F}$ is connected iff F is not oriented).

Classical knot signature

Let $\tau \colon H_1(F) \to H_1(\widetilde{F})$ be the transfer map. If α is a simple closed curve on F, then $\tau\alpha$ is the push-off of α in both directions.

Definition

1. The Gordon-Litherland pairing $\mathscr{G}_F \colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$ is defined by setting $\mathscr{G}_F(\alpha, \beta) = lk(\tau\alpha, \beta)$.
2. The Euler number of F is given by $e(F) = -\,lk(K, K')$, where K' is a push-off of K missing F.

Remark. If F is oriented, then $\mathscr{G}_F$ coincides with the symmetrized Seifert pairing $V + V^{\mathsf{T}}$ and $e(F) = 0$.

Theorem (Gordon-Litherland (1978, Invent Math))

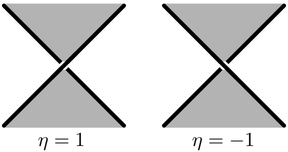
(i) $\mathscr{G}_F$ *is a symmetric bilinear pairing on* $H_1(F)$.
(ii) $\sigma(K) = \mathrm{sig}(\mathscr{G}_F) + e(F)/2.$

Checkerboard coloring and incidence number

The GL pairing leads to a simple algorithm for computing the knot signature $\sigma(K)$ from a checkerboard coloring.

Given a checkerboard coloring, let F be the spanning surface from the black regions. It is a union of disks and half-twisted bands.

Enumerate the white disks $X_0, \ldots, X_n$, they give a system of generators for $H_1(F)$. For each crossing c, set $\eta_c = \pm 1$ as below.



Goeritz matrix

For $i, j = 0, \ldots, n$, let

$$g_{ij} = \begin{cases} -\sum \eta_c & \text{if } i \neq j, \\ -\sum_{k \neq i} g_{ik} & \text{if } i = j. \end{cases}$$

The first sum is taken over all crossings c incident to both X_i and X_j.

The Goeritz matrix is given by $G = (g_{ij})_{i,j=1}^{n}$.
It represents the GL pairing $\mathscr{G}_F$ on $H_1(F)$ with basis $\partial X_1, \ldots, \partial X_n$.

Correction term

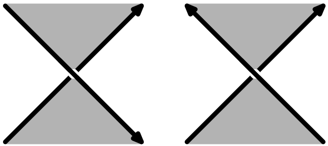
There is also a simple formula for the correction term:

$$e(F) = -2\mu(K),$$

where

$$\mu(K) = \sum_{c \text{ type II}} \eta_c.$$

Here, type is determined by:



type I
$\eta = -\varepsilon = -1$

type II
$\eta = \varepsilon = -1$

Gordon-Litherland pairing on $\Sigma \times I$

Let Σ be a compact, connected, oriented surface.

We extend the GL pairing to knots in $\Sigma \times I$ and use it to define signatures and determinants.

With more effort, the same results can be proved for links in $\Sigma \times I$.

Asymmetric linking in $\Sigma \times I$

Given disjoint knots J, K in $\Sigma \times I$, define $lk_\Sigma(J,K)$ to be the intersection of J with a 2-chain B with $\partial B = K + c$ for some 1-cycle in $\Sigma \times \{1\}$.

Then $lk_\Sigma(J,K)$ counts the number of times J goes over K with sign, where "above" is determined by the positive I direction in $\Sigma \times I$.

Gordon-Litherland pairing in $\Sigma \times I$

Let $p\colon \Sigma \times I \longrightarrow \Sigma$ denote the projection map.

Let $F \subset \Sigma \times I$ be a spanning surface for a knot $K \subset \Sigma \times I$.

Define the GL pairing $\mathscr{G}_F\colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$ by setting

$$\mathscr{G}_F(\alpha, \beta) = lk_\Sigma(\tau\alpha, \beta) - p_*[\alpha] \cdot p_*[\beta],$$

where $\tau\alpha$ is again the push-off of α in both directions and $p_*[\alpha] \cdot p_*[\beta]$ is the algebraic intersection in $H_1(\Sigma)$.

Lemma

The GL pairing $\mathscr{G}_F\colon H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$ is symmetric.

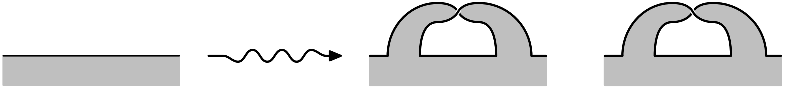
As before, $\mathrm{sig}(\mathscr{G}_F)$ can be combined with a correction term to give a signature invariant for knots in thickened surfaces.

S^*-equivalence

Definition

An S^**-equivalence* of spanning surfaces consists of:

(a) ambient isotopy,
(b) attaching (or removing) a 1-handle,
(c) attaching (or removing) a small half-twisted band.



Facts. 1. Every classical knot admits a spanning surface.
2. Any two spanning surfaces for a classical knot are S^*-equivalent.
3. Neither is true for knots in thickened surfaces.

Generalized signatures

Lemma

If F_1 and F_2 are S^*-equivalent spanning surfaces in $\Sigma \times I$, then

$$\mathrm{sig}(\mathscr{G}_{F_1}) + \tfrac{1}{2}e(F_1) = \mathrm{sig}(\mathscr{G}_{F_2}) + \tfrac{1}{2}e(F_2).$$

Note that if K' is the push-off of $K \subset \Sigma \times I$ which misses F, then $e(F) = -\, lk_\Sigma(K, K')$.

Corollary

Suppose $F \subset \Sigma \times I$ is a spanning surface for $K \subset \Sigma \times I$. Then $\sigma(K, F) = \mathrm{sig}(\mathscr{G}_F) + \frac{1}{2}e(F)$ depends only on the S^*-equivalence class of F.

Remark. If F is oriented, then $e(F) = 0$ and $\sigma(K, F) = \mathrm{sig}(\mathscr{G}_F)$ agrees with the signature of K defined using the Seifert form Θ.

Determinant and nullity

One can also use this approach to define determinant and nullity invariants by taking

$$\det(K,F) = |\det(\mathscr{G}_F)| \quad \text{and} \quad n(K,F) = \operatorname{nul}(\mathscr{G}_F).$$

Again, $|\det(\mathscr{G}_F)|$ and $\operatorname{nul}(\mathscr{G}_F)$ depend only on the S^*-equivalence class of F.

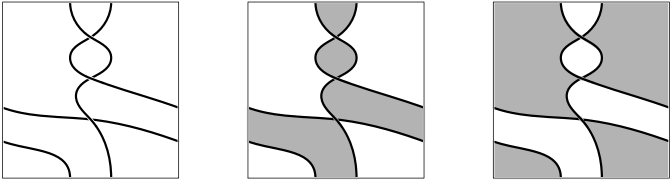
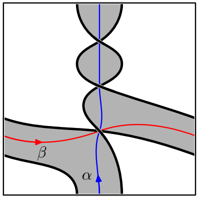
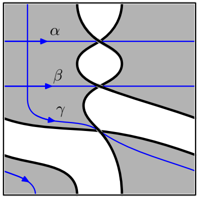


Figure: An alternating knot with dual checkerboard colorings.

Example





Let F be the black surface on left and F^* the *dual surface*.

Take basis α, β for $H_1(F)$, then $\mathscr{G}_F = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$ and $e(F) = 4$.

Thus $\sigma(K, F) = -2 + 4/2 = 0$ and $\det(K, F) = 2$.

Take basis α, β, γ for $H_1(F^*)$, then $\mathscr{G}_{F^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $e(F^*) = -2$.

Thus $\sigma(K, F^*) = 3 + -2/2 = 2$ and $\det(K, F) = 1$.

Existence of spanning surfaces $\Sigma \times I$

Fact. For classical knots, spanning surfaces always exist and are unique up to S^*-equivalence ([GL, 1978], [Yasuhara, 2014 JKTR]).

For knots in $\Sigma \times I$ with $\Sigma \neq S^2$, the situation is more complicated. Firstly, existence is not guaranteed.

Proposition

If $K \subset \Sigma \times I$ is a knot in a thickened surface, then TFAE:
(i) K is the boundary of a spanning surface $F \subset \Sigma \times I$,
(ii) the homology class $[K] = 0$ in $H_1(\Sigma \times I, \mathbb{Z}_2)$.

If either (i) or (ii) hold, then it is easy to see that K admits a diagram on Σ which is *checkerboard colorable*.

Uniqueness of spanning surfaces $\Sigma \times I$

Given a knot $K \subset \Sigma \times I$ with coloring ξ, let F be the black surface and F^* the dual surface.

Lemma

Suppose $K \subset \Sigma \times I$ is a checkerboard colorable knot and $g(\Sigma) > 0$.
(i) If F_1 and F_2 are S^*-equivalent spanning surfaces, then $[F_1] = [F_2]$ in $H_2(\Sigma, K; \mathbb{Z}_2)$.
(ii) Any spanning surface is S^*-equivalent to either F or the dual surface F^*.

Remark. F and F^* are not S^*-equivalent unless $\Sigma = S^2$. Thus, signatures, determinants, and nullities take on two possible values.

Virtual knots were introduced by Kauffman [1999, Eur J Comb] as virtual knot diagrams up to generalized Reidemeister moves.

Alternatively, virtual knots can be represented as knots in thickened surfaces up to stable equivalence Carter, Kamada, Saito [2002, JKTR].

Stabilization is the addition of a handle to Σ disjoint from K, and destablization is the removal of a handle.

A knot $K \subset \Sigma \times I$ is said to be **minimal** if it is not isotopic to one that admits a destabilization.

Kuperberg showed that for a virtual knot, any minimal representative $K \subset \Sigma \times I$ is unique up to diffeomorphism of $\Sigma \times I$.

Detecting the virtual genus

Definition

The *virtual genus* of a virtual knot is the genus $g(\Sigma)$ of a minimal representative $K \subset \Sigma \times I$.

Definition

A knot $K \subset \Sigma \times I$ is said to be *cellularly embedded* if $\Sigma \smallsetminus p(K)$ is a union of disks, where $p\colon \Sigma \times I \to \Sigma$.

Theorem

Suppose $K \subset \Sigma \times I$ is cellularly embedded and checkerboard colorable with coloring ξ. If $\det(K, F) \neq 0$ and $\det(K, F^*) \neq 0$, then K is a minimal representative for its virtual knot.

Chromatic duality

Let $F' = F\#_\tau \Sigma$ be obtained by tubing F to a parallel copy of Σ. Then F' is S^*-equivalent to the dual surface F^* with $e(F') = e(F)$.

Theorem

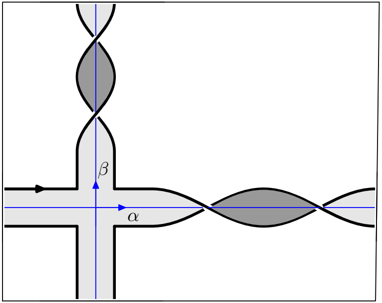
Let $F \subset \Sigma \times I$ be a spanning surface such that the map $H_1(F) \to H_1(\Sigma \times I)$ is surjective. Set $\mathscr{K} = Ker(H_1(F) \to H_1(\Sigma \times I))$. Then $\operatorname{sig}(\mathscr{G}_{F'}) = \operatorname{sig}(\mathscr{G}_F|_{\mathscr{K}})$, the restriction of $\mathscr{G}_F$ to $\mathscr{K}$.

A similar statement holds for knot determinant and nullity.

This result is useful, as it allows computation of both sets of invariants from the same surface F.

Remark. If K is cellularly embedded and checkerboard colorable, then F and its dual F^* satisfy the hypothesis of the theorem.

Example



Then $\mathscr{G}_F = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so $\sigma(K,F) = 2$ and $\det(K,F) = 4$.

Since $\mathscr{K} = 0$, it is trivial that $\sigma(K,F') = 0$ and $\det(K,F') = 1$.

Chromatic duality

Using Goeritz matrices, *Im, Lee, and Lee* defined signature, determinant, and nullity invariants for checkerboard colorable virtual knots [2010 JKTR].

Corollary

If $K \subset \Sigma \times I$ is checkerboard colored with coloring ξ with black surface F and dual surface F^*.
Then the signature from the GL pairing and the Goeritz matrices are dually equivalent. In particular,

$$\sigma(K, F) = \sigma^{ILL}_{\xi^*}(K) \qquad \textit{and} \qquad \sigma(K, F^*) = \sigma^{ILL}_{\xi}(K).$$

A similar statement holds for knot determinant and nullity.

Alternating virtual knots

Fact. Alternating virtual knots are all checkerboard colorable. A diagram is alternating iff every crossing has the same incidence number.

Convention. All crossings have incidence $\eta_c = -1$.

Theorem

If K is an alternating diagram on a surface Σ with black and white spanning surfaces B and W. Then the Gordon-Litherland pairing $\mathscr{G}_B$ and $\mathscr{G}_W$ are definite and of opposite sign.

Remark. With the above convention, $\mathscr{G}_B$ will be negative definite and $\mathscr{G}_W$ will be positive definite. Notice that $\det(K, B) \neq 0 \neq \det(K, W)$.

Corollary

If K is an alternating virtual knot diagram, then K has minimal genus.

Alternating virtual knots

Theorem

A checkerboard colorable knot K in a thickened surface $\Sigma \times I$ is alternating iff it admits positive and negative definite spanning surfaces.

Remark. This extends the results of Greene and Howie and gives a topological characterization of alternating virtual knots.

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PhD thesis, McMaster University (2018)

Thank you for your attention!