The geometric content of Tait's conjectures

Ohio State CKVK* seminar

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Historical background: Tait's conjectures, Fox's question

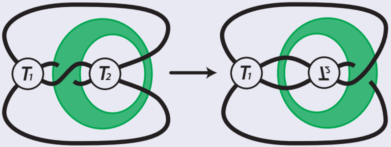
Tait's conjectures (1898)

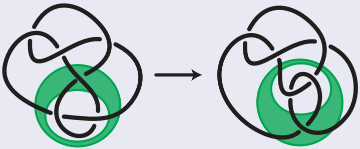
Let D and D' be reduced alternating diagrams of a prime knot L. (Prime implies $\nexists$ T_1 T_2; reduced means $\nexists$ T.) Then:

(1) *D and D'* **minimize crossings**: $|\times|_D = |\times|_{D'} = c(L)$.

(2) *D and D' have the same* **writhe**: $w(D) = w(D') = |\times|_{D'} - |\times|_{D'}$.

(3) *D and D' are related by* **flype moves**:





Question (Fox, $\sim$ 1960)

What is an alternating knot?

Tait's conjectures all remained open until the 1985 discovery of the **Jones polynomial**. Fox's question remained open until 2017.

Historical background: Proofs of Tait's conjectures

In 1987, Kauffman, Murasugi, and Thistlethwaite independently proved (1) using the Jones polynomial, whose degree span is $|\times|_D$, e.g. $V(t) = t + t^3 - t^4$. Using the knot signature $\sigma(L)$, (1) implies (2).

In 1993, Menasco-Thistlethwaite proved (3), using geometric techniques and the Jones polynomial. Note: (3) implies (2) and part of (1).

They asked if *purely geometric* proofs exist. The first came in 2017....

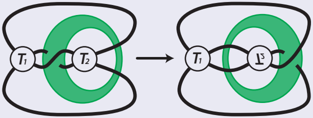
Tait's conjectures (1898)

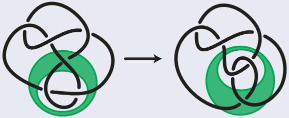
Given reduced alternating diagrams D, D' of a prime knot L:

(1) *D and D'* **minimize crossings***: $|\times|_D = |\times|_{D'} = c(L)$.*

(2) *D and D' have the same* **writhe***: $w(D) = w(D') = |\times|_{D'} - |\times|_{D'}$.*

(3) *D and D' are related by* **flype moves***:*





Historical background: geometric proofs

Question (Fox, $\sim$ 1960)

What is an alternating knot?

Theorem (Greene; Howie, 2017)

A knot $L \subset S^3$ is alternating IFF *it has spanning surfaces F_+ and F_- s.t.:*

- *Howie:* $2(\beta_1(F_+) + \beta_1(F_-)) = s(F_+) - s(F_-)$.
- *Greene: F_+ is positive-definite and F_- is negative-definite.*

Using lattice flows, Greene applied his characterization to prove:

Theorem (Greene, 2017)

Any reduced alternating diagrams D, D' of the same knot satisfy $|\times|_D = |\times|_{D'}$ and $w(D) = w(D')$.

I will describe the first *entirely geometric* proof of the flyping theorem. This also implies the theorem above. Related problems remain open.

Outline

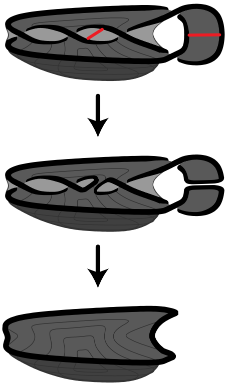
- Spanning surfaces F
  - Knot diagrams and chessboard surfaces
  - Complexity $\beta_1(F)$ and slope $s(F)$
  - Gordon-Litherland pairing $\langle \cdot, \cdot \rangle$ and signature $\sigma(F)$.
  - Greene's characterization
- (Generalized) plumbing and re-plumbing
  - Essential surfaces
  - Flyping and re-plumbing
  - Crossing ball structures
  - Re-plumbing definite surfaces
- Geometric proof of the flyping theorem
- Related problems

Spanning surfaces

Conventions: Let $D, D' \subset S^2$ be reduced alternating diagrams of a prime alternating knot $L \subset S^3$; νL, νF, and νS^2 denote closed regular neighborhoods.

Definition: A **spanning surface** is a properly embedded surface $F \subset S^3 \setminus \mathring{\nu} L$ such that ∂F intersects each meridian on $\partial \nu L$ transversally in one point, and F is compact and connected, but not necessarily orientable.

Definition: $\beta_1(F) = \text{rank } H_1(F)$.

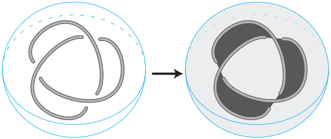


Observation

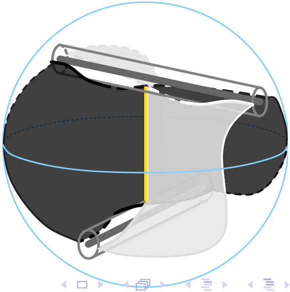
If α consists of properly embedded disjoint arcs in a spanning surface F and $F' = F \setminus \mathring{\nu}\alpha$ is a disk, then $\beta_1(F) = |\alpha|$.

Chessboard surfaces

Color the regions of $S^2 \setminus D$ black and white in chessboard fashion and construct spanning surfaces B and W for L like this:



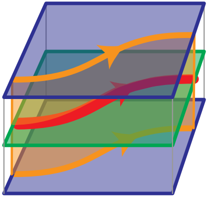
B and W are called the **chessboard** surfaces from D. They intersect in *vertical arcs* which project to the the crossings of D:



The Gordon-Litherland pairing on a spanning surface F

Denote projection $p : \nu F \to \boldsymbol{F}$. Given any oriented simple closed curve (s.c.c.) $\boldsymbol{\gamma} \subset F$, denote $\widetilde{\gamma} = \partial(p^{-1}(\gamma))$, and orient $\widetilde{\gamma}$ following $\boldsymbol{\gamma}$.

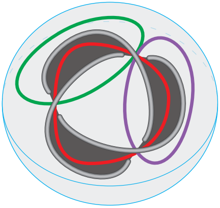
Gordon-Litherland define a symmetric bilinear pairing



$$\langle \cdot , \cdot \rangle : H_1(F) \times H_1(F) \to \mathbb{Z}$$

$$\langle [\alpha], [\beta] \rangle = \mathrm{lk}(\alpha, \widetilde{\beta}).$$

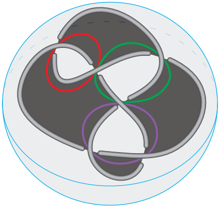
The **framing** of a s.c.c. $\gamma \subset F$ is $\frac{1}{2}\langle [\gamma], [\gamma] \rangle$.



Examples: The pairings for B and W shown left are represented by

$\begin{bmatrix} 3 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$,

that for B right by $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

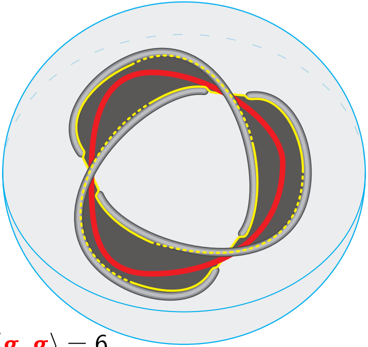


Boundary slopes

The **euler number** $e(F)$ is the algebraic self-intersection number of the properly embedded surface obtained by perturbing F in B^4. Alternatively,

$$-e(F) = \frac{1}{2}\sum_{i=1}^{m}\langle[\ell_i],[\ell_i]\rangle,$$

where $\partial F = \ell_1 \sqcup \ldots \sqcup \ell_m$. Call $s(F) = -e(F)$ the **slope** of F.



<u>**Example:**</u> B and W shown right have $s(W) = 0$ and $s(B) = 6$. This is because W is orientable and, denoting a generator of $H_1(B)$ by $\boldsymbol{g}$:

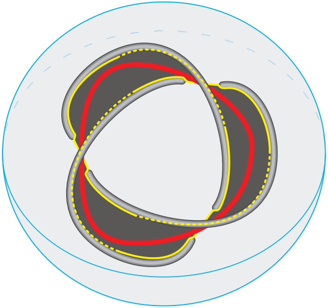
$$s(B) = \tfrac{1}{2}\langle[\partial B],[\partial B]\rangle = \tfrac{1}{2}\langle 2\boldsymbol{g}, 2\boldsymbol{g}\rangle = 2\langle \boldsymbol{g},\boldsymbol{g}\rangle = 6.$$

Boundary slopes and signatures

If F spans a knot L and $\widehat{L}$ is a pushoff of L in F, then the slope of F is

$$s(F) = -\frac{1}{2}\langle [L], [L] \rangle = \mathrm{lk}\left(L, \widehat{L}\right),$$

which equals the framing of L in F. The **signature of *F***, denoted $\sigma(F)$, is the number of positive eigenvalues of $\langle \cdot, \cdot \rangle$ minus the number of negative eigenvalues.



Gordon-Litherland show that the quantity $\sigma(F) - \frac{1}{2}s(F)$ depends only on L. This is called the **knot signature**, denoted $\sigma(L)$.

Example: The surfaces B and W shown have slopes $s(B) = 6$ and $s(W) = 0$ and signatures $\sigma(B) = 1$ and $s(W) = -2$. Thus

$\sigma\Bigg($  $\Bigg) = \begin{cases} \sigma(B) - \frac{1}{2}s(B) = 1 - 3 = -2 \\ \sigma(W) - \frac{1}{2}s(W) = -2 - 0 = -2. \end{cases}$

Definite surfaces and Greene's characterization

Definition: F is **positive-definite** if $\langle \alpha, \alpha \rangle > 0$ for nonzero $\alpha \in H_1(F)$. This holds IFF $\sigma(F) = \beta_1(F)$, also IFF for each s.c.c. $\gamma \subset F$:

- The **framing** of γ in F is positive, or
- γ bounds an orientable subsurface of F.

Greene's characterization of alternating diagrams

A knot **diagram is alternating** IFF *its chessboard surfaces are definite surfaces of opposite signs.*

Greene's characterization of alternating links

If *B and W are positive- and negative-***definite spanning surfaces** *for a knot $L \subset S^3$,* **then** *L has an* **alternating diagram** *D whose chessboard surfaces are isotopic to B and W.*

Moreover, D is **reduced** IFF $\langle \alpha, \alpha \rangle \neq \pm 1$ *for all α in $H_1(B)$, $H_1(W)$.*

Convention: The chessboard surfaces from D and D' are B, W and B', W', with B, B' positive-definite and W, W' negative-definite.

Recall that the knot signature $\sigma(L) = \sigma(F) - \frac{1}{2}s(F)$ depends only on L, and that $\pm$-definite surfaces $F_\pm$ satisfy $\sigma(F_\pm) = \pm\beta_1(F_\pm)$. This implies:

Slope difference lemma

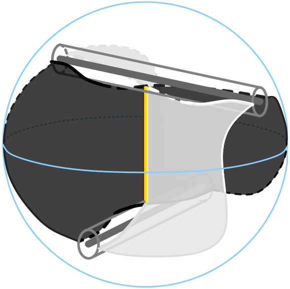
If $F_\pm$, respectively, are $\pm$-definite spanning surfaces for L, then

$$s(F_+) - s(F_-) = 2(\beta_1(F_+) + \beta_1(F_-)).$$

I use the slope difference lemma and cut-and-paste arguments to prove:

Definite intersection lemma (K)

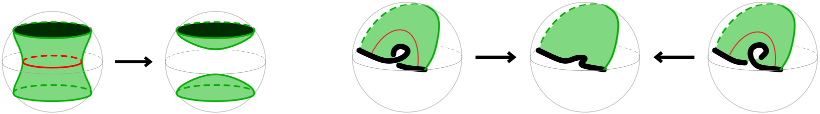
If α is a non-∂-parallel arc of $B \cap W$, then $i(\partial B, \partial W)_{\nu\partial\alpha} = 2$.



Two notions of essential surfaces

Definitions:

- F is **geometrically essential** if $\nexists$:



- F is $\boldsymbol{\pi_1}$**-essential** if $F \hookrightarrow S^3 \setminus L$ induces an injection of fundamental groups, and F is not a mobius band spanning the unknot.

Remarks: The following facts are classical applications of Dehn's Lemma:

(1) If F is π_1-essential, then F is geometrically essential.

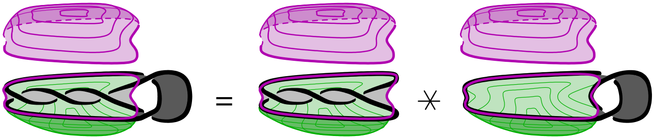
(2) If F is **2-sided** and geometrically essential, then F is π_1-essential.

Plumbing and re-plumbing

Let $V \subset S^3 \setminus \setminus F$ be a properly embedded disk s.t.

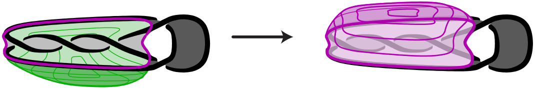
- ∂V bounds a disk $U \subset F$.
- Denoting $S^3 \setminus \setminus (U \cup V) = Y_1 \sqcup Y_2$, neither $F_i = F \cap Y_i$ is a disk.

Then V is a **plumbing cap** for F, and U is its **shadow**.



Say that F is obtained by (generalized) **plumbing** F_1 and F_2 along U, denoted $F_1 * F_2 = F$. This operation is also called **Murasugi sum**.

The operation $F \to F' = (F \setminus U) \cup V$ is called **re-plumbing**, and can also be realized via proper isotopy through the 4-ball:



Murasugi sum is a natural geometric operation

A **Seifert surface** is an oriented spanning surface.

Theorem (Gabai 1985 [3, 4])

Let $F_1 * F_2 = F$ be a Murasugi sum—i.e. (generalized) plumbing—of Seifert surfaces, $\partial F_i = L_i$, $\partial F = L$. Then:

(1) F is **essential** *if F_1 and F_2 are essential.*

(2) F has **minimal** *genus* IFF *F_1 and F_2 both have minimal genus.*

(3) L is a **fibered knot with fiber** *F* IFF *each L_i is fibered with fiber F_i.*

(4) $S^3 \setminus \mathring{\nu} L$ has a **nice codimension 1 foliation** IFF *both $S^3 \setminus \mathring{\nu} L_i$ do.*

Property (1) also holds for arbitrary (1- and 2-sided) spanning surfaces:

Theorem (Ozawa 2011 [15])

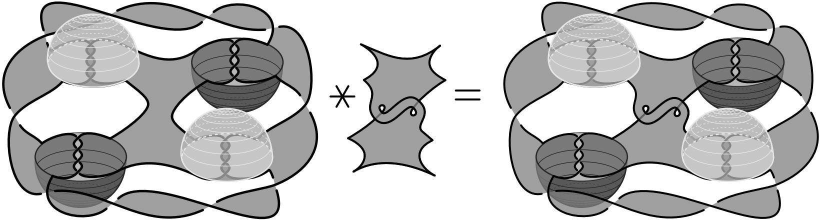
Let $F_1 * F_2 = F$ be a Murasugi sum of spanning surfaces. If F_1 and F_2 are **π_1-essential**, *then F is π_1-essential.*

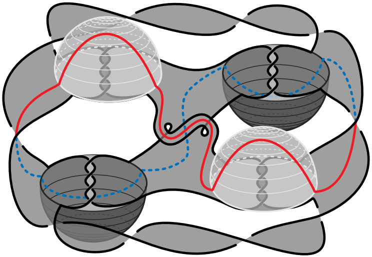
Changing "π_1-essential" to "geometrically essential" makes this false...

Plumbing needn't respect geometric essentiality

Theorem (K)

A Murasugi sum of geometrically essential surfaces need not be geometrically essential.

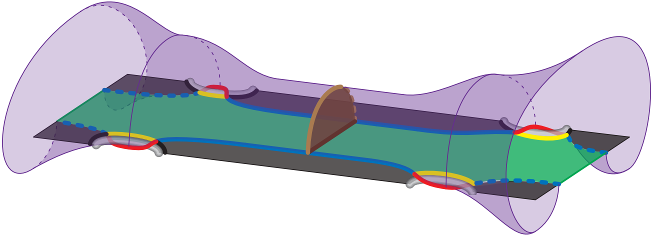




Irreducible plumbing caps V for F

A plumbing cap V is **acceptable** if no arc of $\partial V \cap F$ is ∂-parallel and no arc of $\partial V \cap \partial \nu L$ is parallel in $\partial \nu L$ to ∂F.

If there is a properly embedded disk $\boldsymbol{X} \subset S^3 \setminus \setminus (\nu L \cup F \cup V)$ like the one shown below, then then V is **reducible**; if not, then V is **irreducible**.

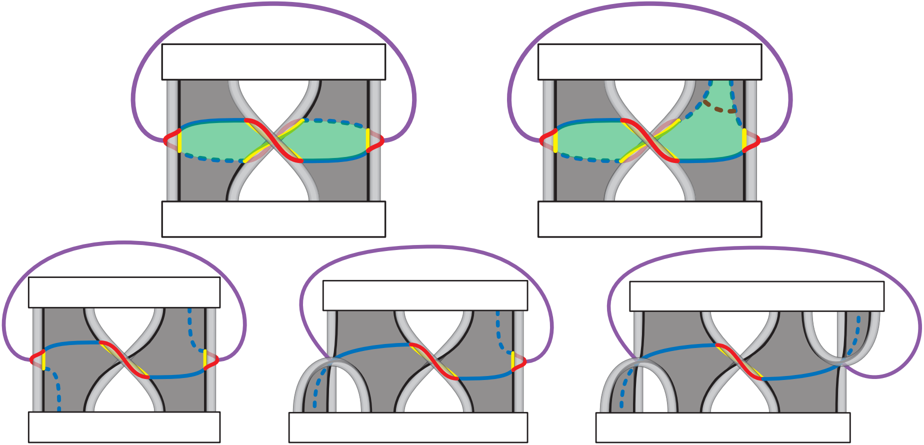


Lemma

If F and F' are related by a sequence of re-plumbing moves, then each move in some such a sequence follows an **acceptable**, **irreducible** *cap.*

Apparent plumbing cap theorem

If V is an irreducible plumbing cap for B in "standard position," then V is apparent in D, as shown top-left:



Sketch of proof.

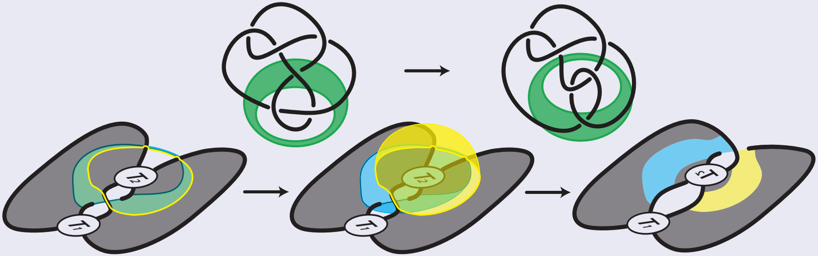
Let V_0 be an outermost disk of $V \setminus W$ (bottom row). If $|V \cap W| = 1$, done. Else, (top-right), and V is reducible. $\square$

Apparent plumbing caps correspond to flypes.

Proposition

If a flype $D_0 \to D_1$ follows a plumbing cap V for B_0, then re-plumbing B_0 along V gives a surface isotopic to B_1; also, W_0 is isotopic to W_1.

Proof.



$\square$

We have shown that apparent plumbing caps correspond to flypes and:

Lemma: Any re-plumbing sequence can be refined to one in which each move follows an acceptable, irreducible cap.

Apparent plumbing cap theorem: If V is an irreducible plumbing cap for B in standard position, then V is apparent in D.

Proposition: If $D_0 \to D_1$ is a flype (along an apparent plumbing cap), then W_0 and W_1 are related by re-plumbing or isotopy, as are B_0 and B_1.

Flyping re-plumbing theorem

D and D' are related by flypes IFF *B and B' are related by re-plumbing and isotopy moves, as are W and W'.*

Proof.

The proposition gives one direction. For the converse, the lemma and theorem give re-plumbing sequences $B = B_0 \to \cdots \to B_m = B'$ and $W = W_m \to \cdots \to W_n = W'$ along apparent plumbing caps. The proposition then gives a flyping sequence

$$\underset{B,W}{D} = D_0 \to \cdots \to \underset{B',W}{D_m} \to \cdots \to \underset{B',W'}{D_{m+n}}.$$

□

Logical interlude: how to prove the flyping theorem

Flyping re-plumbing theorem (shown)

D and D' are related by flypes if and only if B and B' are related by isotopy and re-plumbing moves, as are W and W'.

Definite re-plumbing theorem (still need to show)

Any essential positive- (resp. negative-) definite surface spanning L is related to B (resp. W) by isotopy and re-plumbing moves.

Flyping theorem (will then follow)

All reduced alternating diagrams of L are related by flypes.

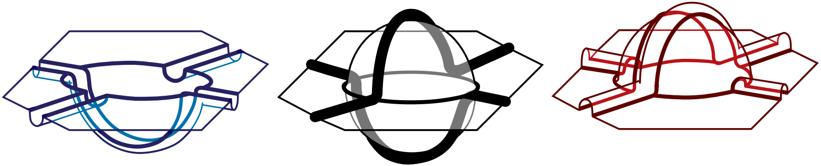
Proof of the flyping theorem (assuming definite re-plumbing theorem).

Let D, D' be reduced alternating diagrams of a prime knot L with respective chessboard surfaces B, W and B', W', where B, B' are positive-definite. The definite re-plumbing theorem implies that B and B' are related by re-plumbing and isotopy moves, as are W and W'. Thus, by the flyping re-plumbing theorem, D and D' are related by flypes. $\square$

Crossing ball setup

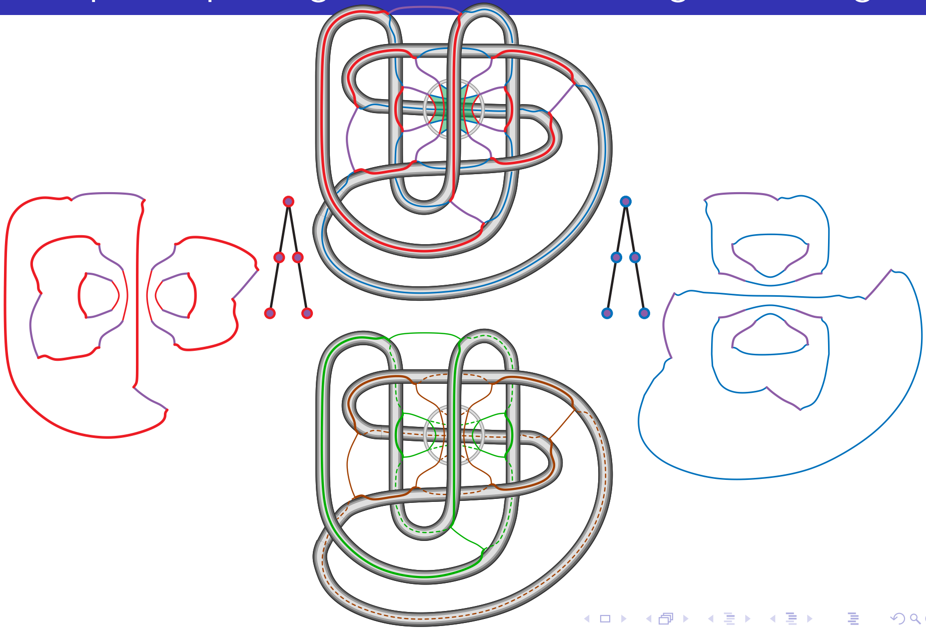
Construct a tiny closed **crossing ball** C_t at each crossing point c_t of D, and denote $C = \bigsqcup_{t=1}^{n} C_t$. Adjust D to embed L in $(S^2 \setminus \mathrm{int}(C)) \cup \partial C$.

Denote the two balls of $S^3 \setminus \setminus (S^2 \cup C \cup \nu L)$ by $H_\pm$, with $\boldsymbol{\partial H_+ = S_+}$ and $\boldsymbol{\partial H_- = S_-}$.



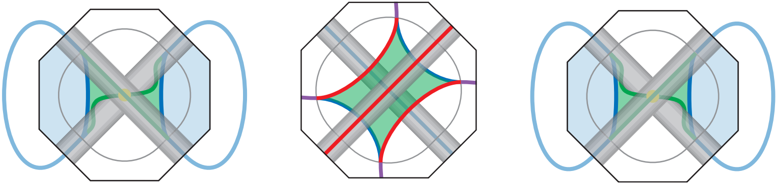
To prove the definite re-plumbing theorem, we will put an arbitrary essential positive-definite surface F in a "standard position" and consider innermost disks, etc.

Example: a spanning surface in the crossing ball setting

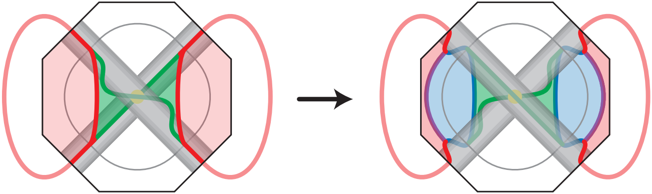


If F is in standard position, then:

- Each component of $F \cap C$ is a **crossing band** or a **saddle disk**:



- Each crossing band in F is disjoint from S_+:



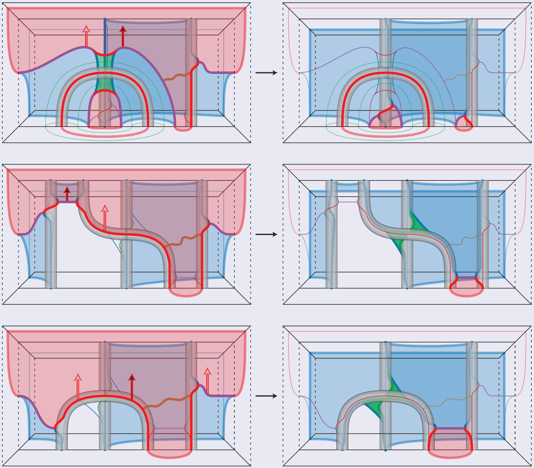
Define the **complexity** of F to be

$$||F|| = \#(\text{crossings without crossing bands}) + \#(\text{saddle disks}).$$

Definite re-plumbing theorem

Any essential positive-definite spanning surface F for L is related to B by isotopy and re-plumbing moves.

Sketch of proof.



Isotop F into standard position with $||F||$ minimized. Modify an innermost circle of $F \cap S_+$ to get an annulus $A \subset S^2$. Cut A it into rectangles A_i. Each prism $\pi^{-1}(A_i)$ intersects F as shown left. As shown, there is a re-plumbing move which decreases $||F||$. Repeat this process until $||F|| = 0$, whence F is isotopic to B. $\square$

Flyping re-plumbing theorem (shown)

D and D' are related by flypes if and only if B and B' are related by isotopy and re-plumbing moves, as are W and W'.

Definite re-plumbing theorem (shown)

Any essential positive- (resp. negative-) definite surface spanning L is related to B (resp. W) by isotopy and re-plumbing moves.

Flyping theorem (shown)

All reduced alternating diagrams of L are related by flypes.

The flyping theorem immediately gives a new proof of the same part of Tait's conjectures that Greene proved:

Theorem

Any two reduced alternating diagrams of the same knot have the **same crossing number and writhe**.

Yet, it **does not follow** that any reduced alternating diagram **minimizes crossings**. All existing proofs of this fact use the Jones polynomial.

Geometric proofs: open problems

Theorem

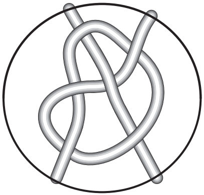
Any two reduced alternating diagrams of the same knot have the **same crossing number and writhe**.

Open problem

Give an entirely geometric proof that any reduced alternating knot diagram realizes the underlying knot's **crossing number**.

Open problem

Give an entirely geometric proof that any reduced alternating **tangle diagram** *realizes the underlying tangle's crossing number.*



Open problem

Give an entirely geometric proof that any **adequate** *knot diagram realizes the underlying knot's crossing number.*

Open problem

Give an entirely geometric proof that any (reduced alternating knot / reduced alternating tangle / adequate) diagram minimizes crossings.

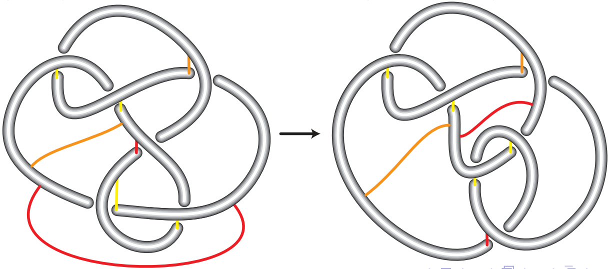
One approach to these problems is to translate statements about diagrams to statements about chessboard surfaces, a la:

Howie's characterization of alternating knots

A knot in S^3 is alternating iff it has spanning surfaces $F_\pm$ which satisfy

$$2(\beta_1(F_+) + \beta_1(F_-)) = s(F_+) - s(F_-).$$

Alternatively, using the flyping theorem, one can extend any alternating knot to a spatial graph in a way that captures all symmetries, and all alternating diagrams, of the knot. Crossings become geometric objects:



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Thank you!