Links in homology spheres are homotopic to slice links - an application of the relative Whitney trick

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classical knots, virtual knots, and algebraic structures related to knots,
The Ohio State University.

8 February 2021

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1. Definitions and statement of the main theorem
2. The story for knots (Austin-Rolfsen '99)
3. Key tool: A sufficient condition for sliceness (J.C. Cha-M.H. Kim-M. Powell '20)
4. The (relative) Whitney trick
5. Proof of the main theorem.

Slice links in S^3.

This project is part of a program of asking about the difference between knot (link) concordance in S^3 and concordance for knots (and links) in homology spheres.

The central question of knot (and link) concordance is as follows:

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- I'll be (mostly) interested in the topological setting today.
- What does it mean for a link in a homology sphere to be slice?

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- Example: Any knot K in S^3 bounds a P.L (but non-locally flat) disk, cone(K), in $B^4 = \text{cone}(S^3)$. [**Ak91**, **Le16**] Not so for knots in homology spheres.

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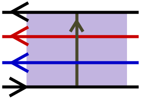

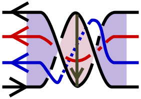

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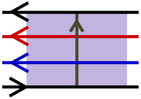

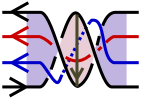

- Now $\Delta_K(t) = \pm t^p$, so K is slice.

Goal: Do the same for links.

Key tool: A sufficient condition for sliceness.

Theorem (Cha-Kim-Powell [CKP2020])

Let L be a boundary link in S^3. If L bounds a Seifert surface admitting a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^{g}$ so that for all j, $\alpha_j \cup \bigcup_i \beta_i^+$ and $\beta_j \cup \bigcup_i \beta_i^+$ are link-homotopically trivial then L is (freely) slice.

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- $J \subseteq S^3$ is link-homotopically trivial if there is a sequence of self-crossing changes reducing J to the unlink.
- $J \subseteq S^3$ is link-homotopically trivial iff J bounds a disjoint union of immersed disks in B^4. *(4D-homotopically trivial)*

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Let L be a boundary link in Y. If L bounds a Seifert surface admitting a symplectic basis $\{\alpha_i, \beta_i\}$ so that for all j, $\alpha_j \cup \bigcup_i \beta_i^+$ and $\beta_j \cup \bigcup_i \beta_i^+$ are 4D-link-homotopically trivial then L is (freely) slice.

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- A link $J \subseteq Y$ is 4D-link-homotopically trivial if its components bound disjoint immersed disks in $\mathcal{B}_Y$.

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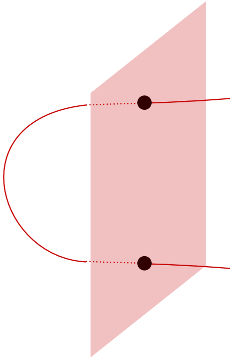
- The key to this lemma is the **relative Whitney trick**.

The Whitney trick (in dimension 4)

Goal: Find "cancelling" points in the intersection of surfaces in a 4-manifold and removed them by a homotopy.

Let S_1 and S_2 be immersed surfaces in a 4-manifold, intersecting transversely in a points p, q.



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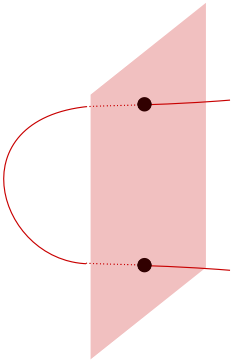

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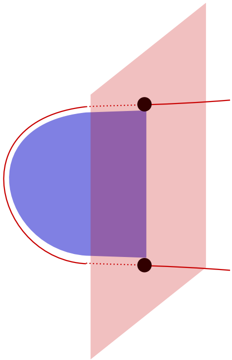

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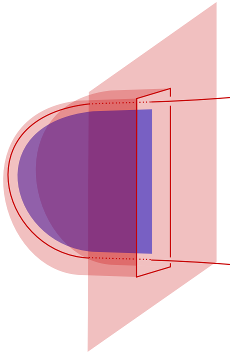

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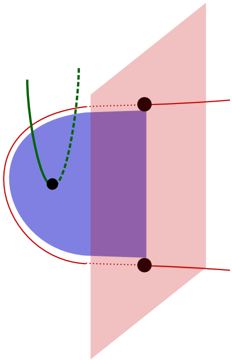

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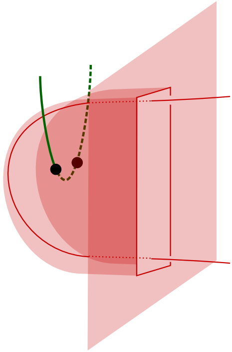

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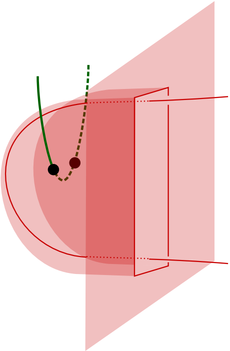

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This trick works very well in high dimensions and is used in eg. the Whitney embedding theorem and the h-cobordism theorem.

The relative Whitney trick.

Goal: At the cost of changing the boundary of a surface by a homotopy, remove a point of intersection.

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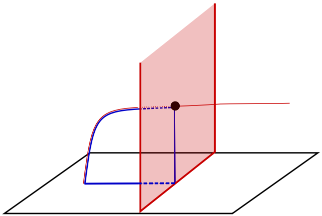

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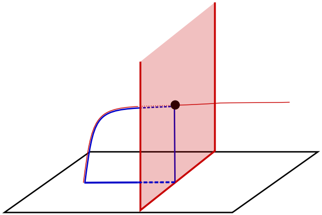

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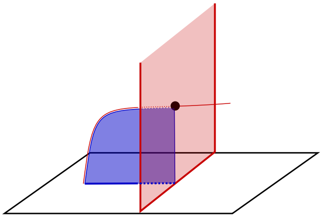

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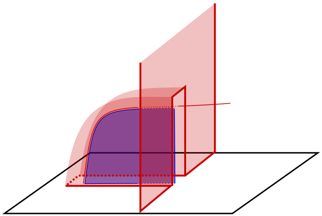

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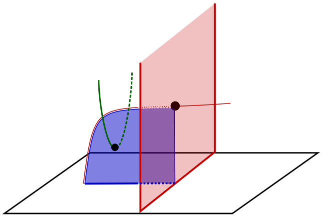

If $a_1 * a_2 * a_3$ bounds an immersed disk Δ then Δ is a relative Whitney disk. (Any framing on $a_1 * a_2$ extends over Δ.)

Slide S_1 over Δ to remove this point of intersection.

Be careful! You might add new points of intersection.

The relative Whitney trick.

Goal: At the cost of changing the boundary of a surface by a homotopy, remove a point of intersection.

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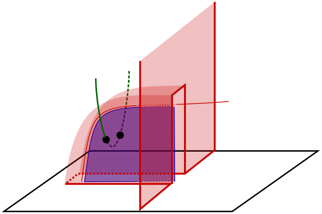

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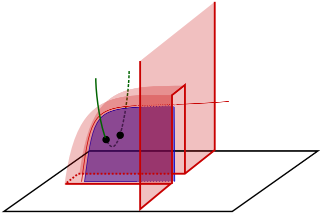

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Proof of the disjoint immersion lemma

Lemma (Homotopy trivializing lemma)

Any link in a homology sphere is homotopic to a 4D-homotopically trivial link.

Proof for a 2-component link:

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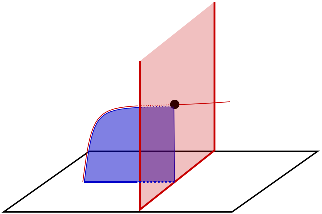

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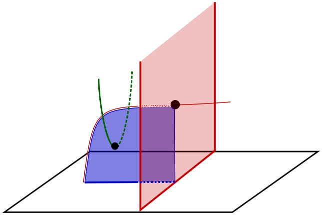

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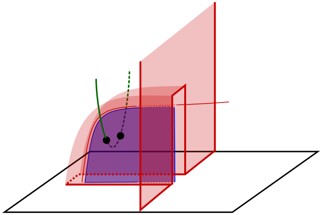

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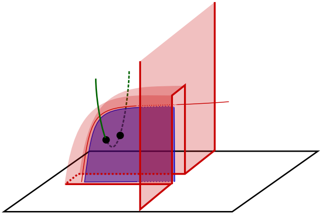

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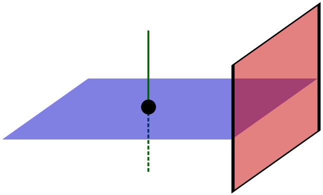

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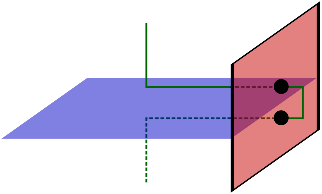

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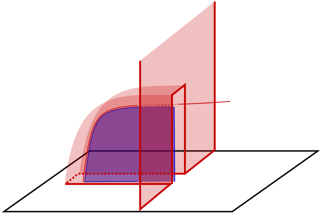

- Problem: Δ might intersect D_2. If so, $|D_1 \cap D_2|$ increases.
- Solution: Use a finger move to kill this intersection and increase number of double points in D_2
- After removing all points from $D_2 \cap \Delta$ perform the relative Whitney Move. $|D_1 \cap D_2|$ reduces. The number of double points of D_1 increases.

Proof of the disjoint immersion lemma

Lemma (Homotopy trivializing lemma)

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Proof for a 3-component link.

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- $\mathcal{B}_Y$ is simply connected: L bounds immersed disks D_1, D_2, D_3.
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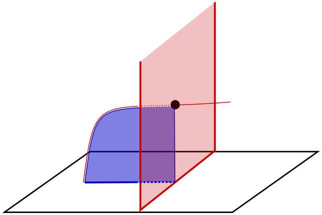

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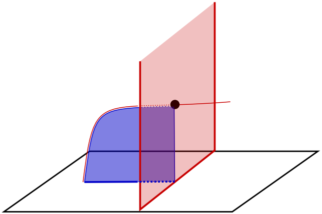

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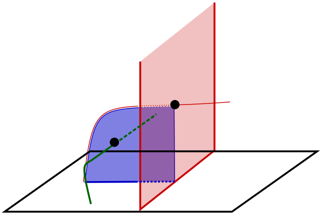

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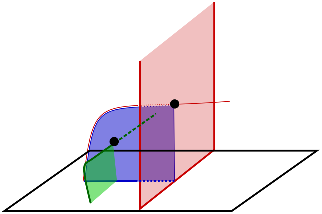

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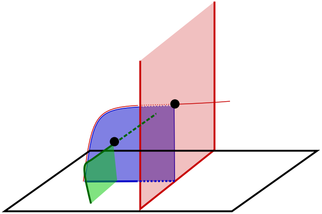

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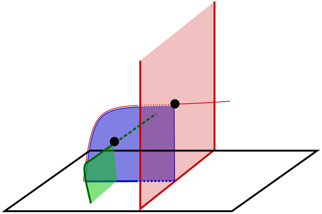
- Solution: Find a "higher order" relative Whitney disk Δ_q
- Use Finger moves to remove points in $\Delta_q \cap D_3$ - more self intersections of D_3.

Proof of the disjoint immersion lemma

Proof for a 3-component link. (Completed)

So far:

- $p \in D_1 \cap D_2$. Δ_p is a rel. Whitney disk.
- $q \in \Delta_p \cap D_3$. Δ_q is a rel. Whitney disk.
- $\Delta_q \cap D_3 = \emptyset$ is disjoint.



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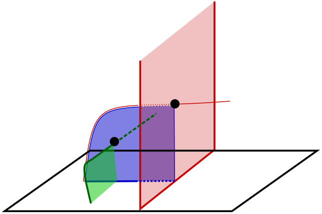

- Use the relative Whitney trick to reduce $\Delta_p \cap D_3$: $\Delta_p \cap D_1$ and $\Delta_p \cap D_2$ grow. More double points in Δ_p. Iterate until $\Delta_p \cap D_3 = \emptyset$
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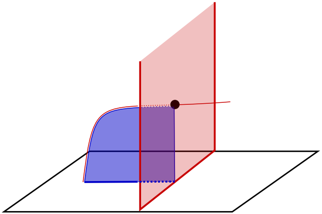

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- Use the finger move to get $\Delta_p \cap D_2 = \emptyset$. Use the Relative Whitney trick to reduce $D_1 \cap D_2$ by 1 while preserving $D_1 \cap D_3$ and $D_2 \cap D_3$.
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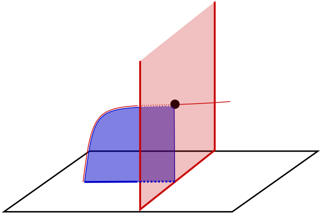

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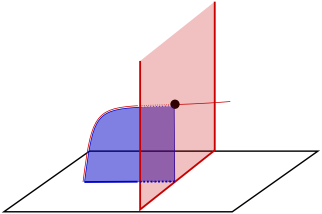

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- Iterate to remove every point from $D_1 \cap D_2$, $D_1 \cap D_3$ and $D_2 \cap D_3$.
- The proof for links of more components is only more complicated by book-keeping. The philosophy is: Whenever you see an intersection that a finger-move can't fix, find a relative Whitney disk.

Proof of the main theorem

Theorem (Goal)

Any link in a homology sphere is homotopic to a slice link.

-
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Consequence (End result of this slide.)

Any link $L \subseteq Y$ is homotopic to a link J with $J \cup J^+$ 4D-homotopically trivial.

Proof of the main theorem

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Any link in a homology sphere is homotopic to a slice link.

- Y is a homology sphere $\implies L_i \sim \prod_{j=1}^{g_i}[\alpha_j, \beta_j]$ in π_1. Use this to build a Seifert surface Σ. (All links are homotopic to boundary links)
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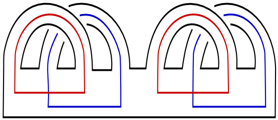

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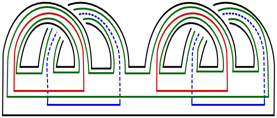
- By homotopy trivializing Lemma, up to homotopy $\bigcup \alpha_i \cup \beta_i^+$ bounds disjoint immersed disks. This changes L by a homotopy. Surger Σ and Σ^+.



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Theorem

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Apply the Cha-Kim-Powell Theorem:

Theorem

Let L be a boundary link in Y. If L bounds a Seifert surface admitting a symplectic basis $\{\alpha_i, \beta_i\}$ so that for all j, $\bigcup_i \beta_i^+ \cup \alpha_j$ and $\bigcup_i \beta_i^+ \cup \beta_j$ are 4D-link-homotopically trivial then L is freely slice.

This completes the proof.

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By the "consequence" a homotopy arranges that $\bigcup \alpha_i \cup \alpha_i^+ \cup \beta_i \cup \beta_i^+$ is 4D-homotopically trivial. This changes L by a homotopy.

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Some observations / questions

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  - Consequence:
  - Question: Partial answer:
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- Observation: For knots this is a 4-D analogue of Dehn's lemma.
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Some observations / questions

- Observation: In the proof of the homotopy trivalizing lemma each point in $D_i \cap D_j$ was undone with one crossing change between L_i and L_j.
  - Consequence: The minimal number of intersections between immersed disks bounded by L is the same as the number of crossing changes needed to transform L to a 4D-homotopically trivial link. Call it $n(L)$.
  - Question: What is $n(L)$? Partial answer: For all m-component links $\sum |\operatorname{lk}(L_i, L_j)| \leq n(L) \leq \sum |\operatorname{lk}(L_i, L_j)| + C_m$. $C_2 = 0$, $C_3 = 2$, $C_m =$???
- Observation: For knots this is a 4-D analogue of Dehn's lemma.
  - Question: If $J \subseteq M^3$ is nullhomotopic in W^4 then is J homotopic to a knot (or link) which bounds (disjoint) smooth/locally flat embedded disk(s) in W

Thanks for listening!

See on on Feb 8!

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