The reductivity of Knot projections

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CKVK*Seminar

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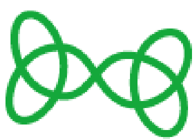
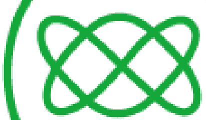
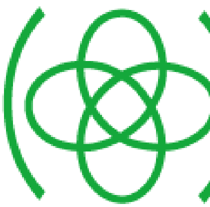
§0. Outline

P: a knot projection

r(P): the *reductivity* of P

Example

$r\Big($  $\Big) = 0$ $r\Big($  $\Big) = 1$ $r\Big($  $\Big) = 2$ $r\Big($  $\Big) = 3$

Theorem (S.)

$$r(P) \leq 4 \quad (^{\forall} P)$$

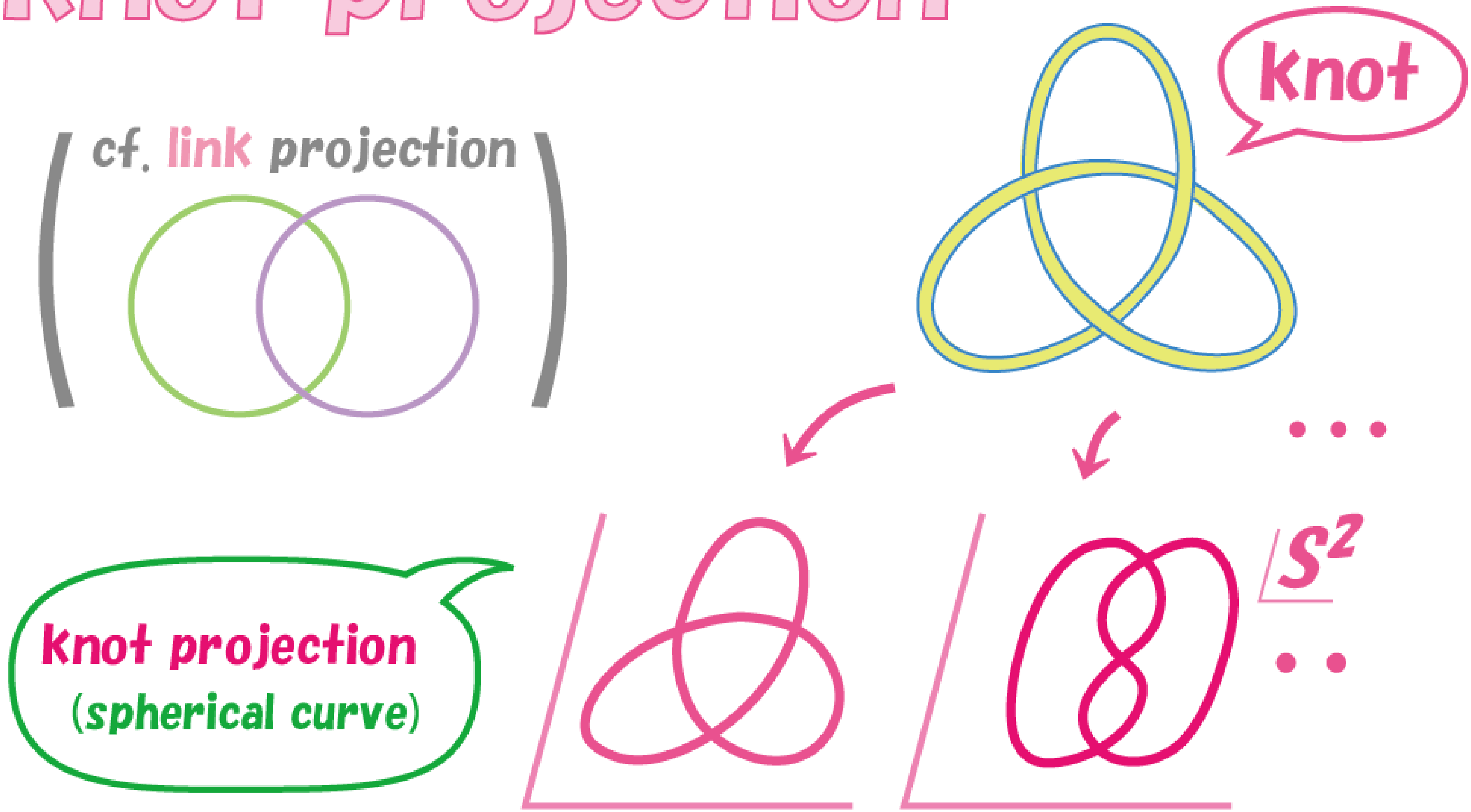
Reductivity problem

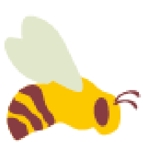
$$\exists ? P \text{ s.t. } r(P) = 4$$

Contents

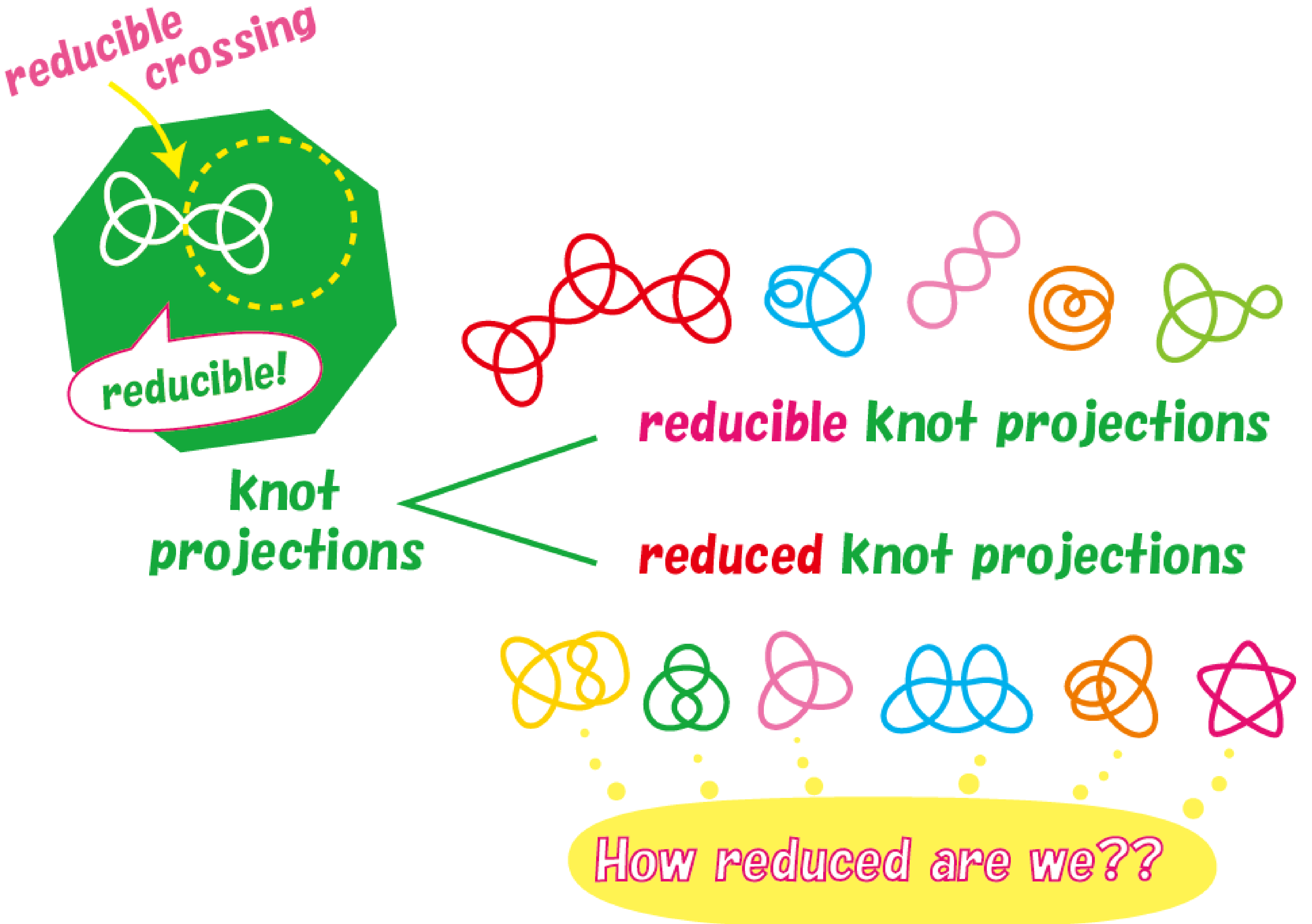
§1. Knot projections

Knot projection





We consider knot projections which have at least one crossing.

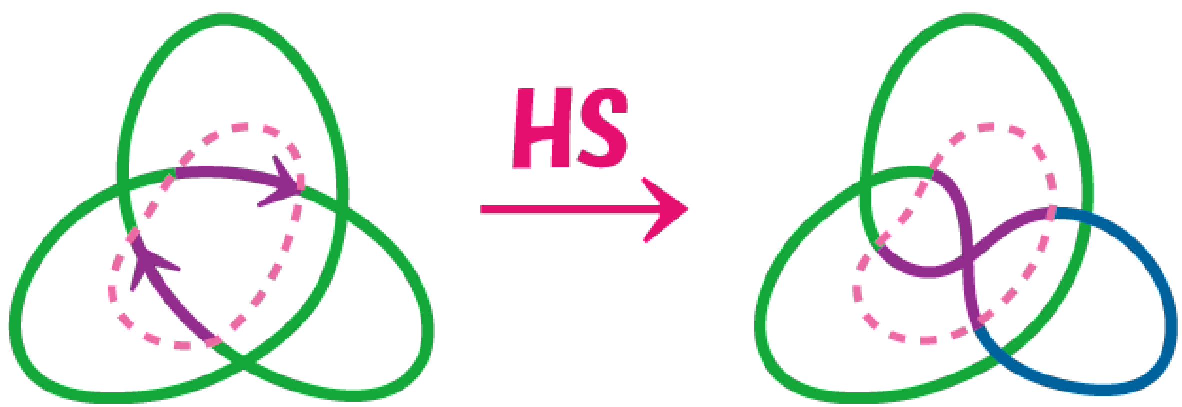


reducible crossing
reducible!
knot projections
reducible knot projections
reduced knot projections
How reduced are we??

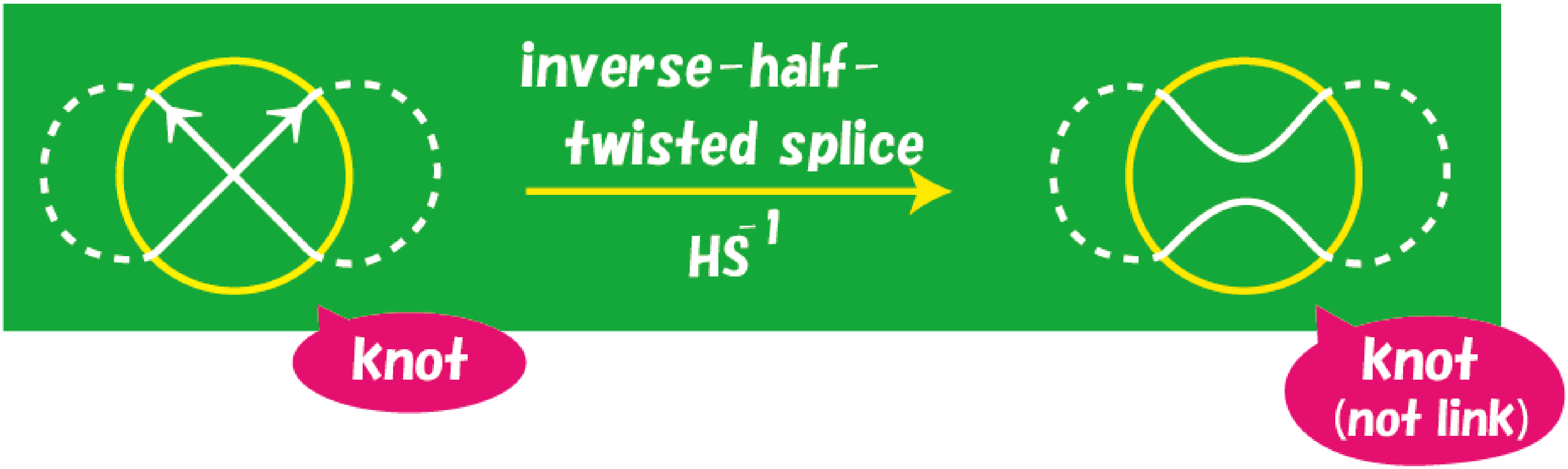
§2. Half-twisted splice

Half-twisted splice (HS)

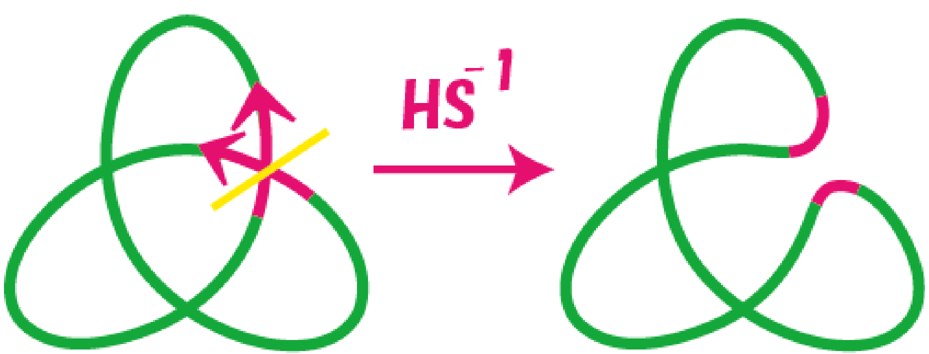
Example

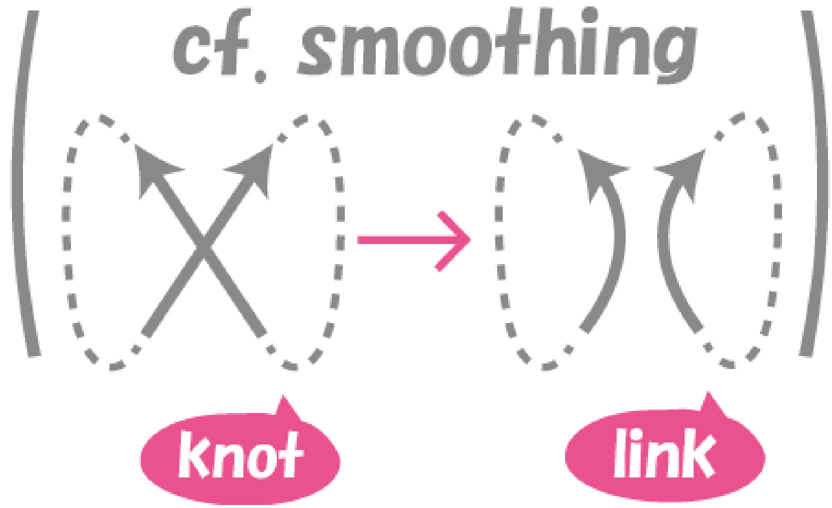


Inverse-half-twisted splice (HS^{-1})



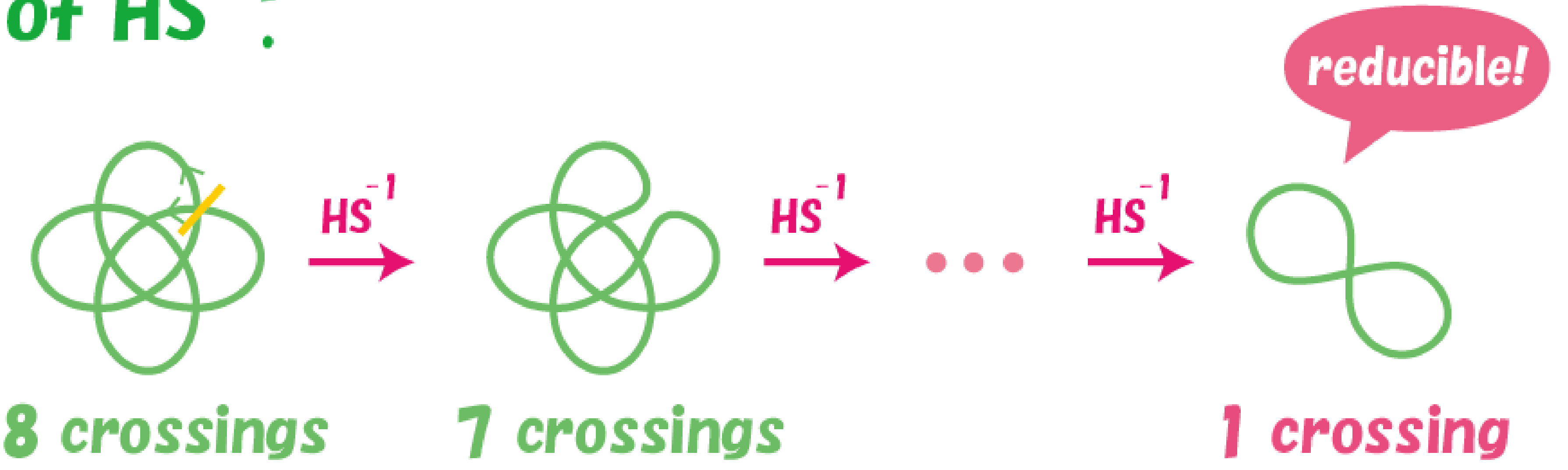
Example



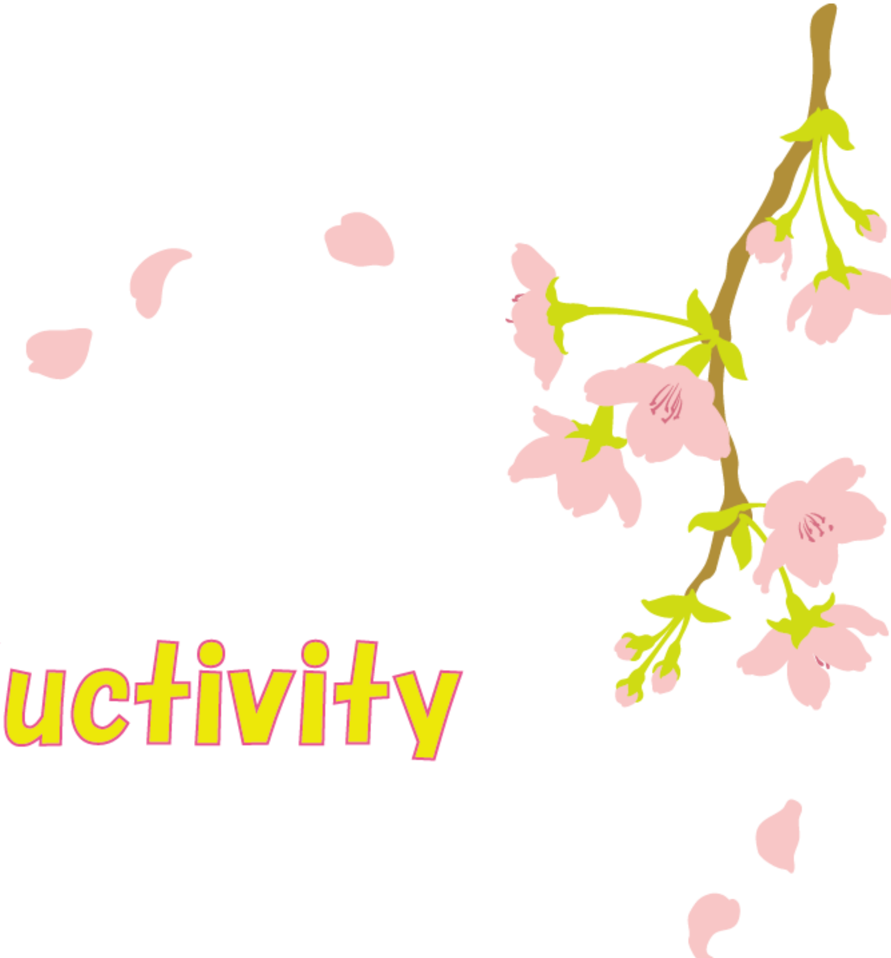


Remark

We can obtain a reducible knot projection from any knot projection by a finite number of HS^{-1}!



§3. Reductivity

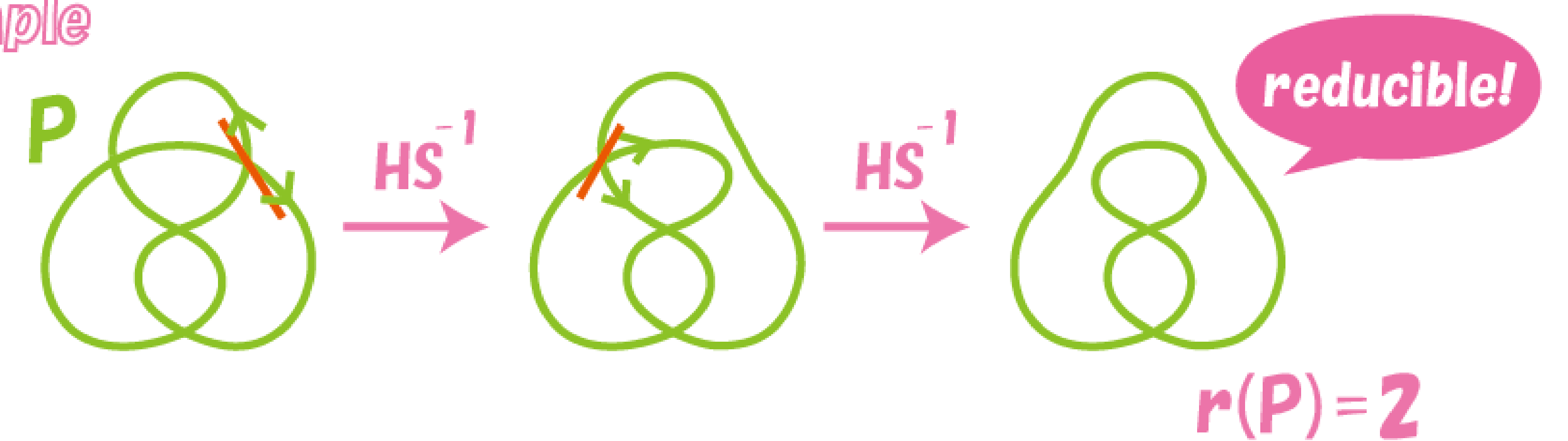


Reductivity -how much reduced??

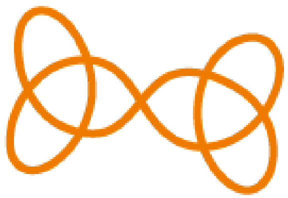
Definition P: a knot projection

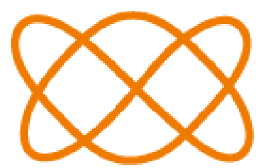
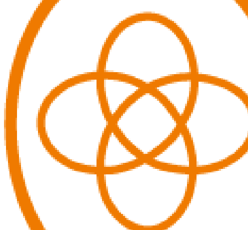
The **reductivity** r(P) of P is the minimal number of HS^{-1} which are needed to obtain a reducible knot projection from P.

Example



Example

r() = 0 r() = 1

r() = 2 r() = 3

There exist infinitely many knot projections P with $r(P) = 0$, 1, 2, and 3.

Reductivity is four or less

Theorem 1 (S)

$$r(P) \leq 4 \quad ({}^{\forall}P)$$

Reductivity problem

$$^{\exists ?}P \text{ s.t. } r(P) = 4$$

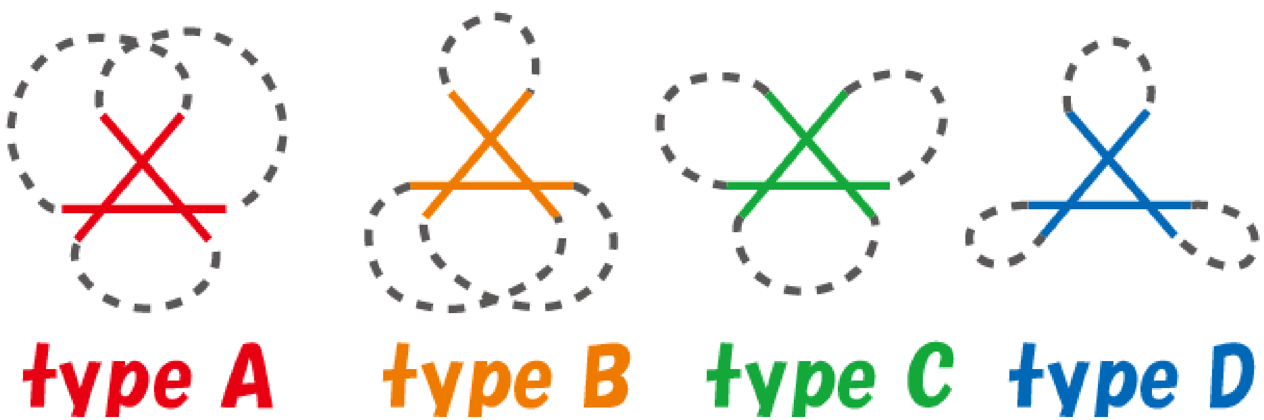
Reference: A. Shimizu, The reductivity of spherical curves, Topology and its Applications 196 (2015).

§4. 2-gons & 3-gons

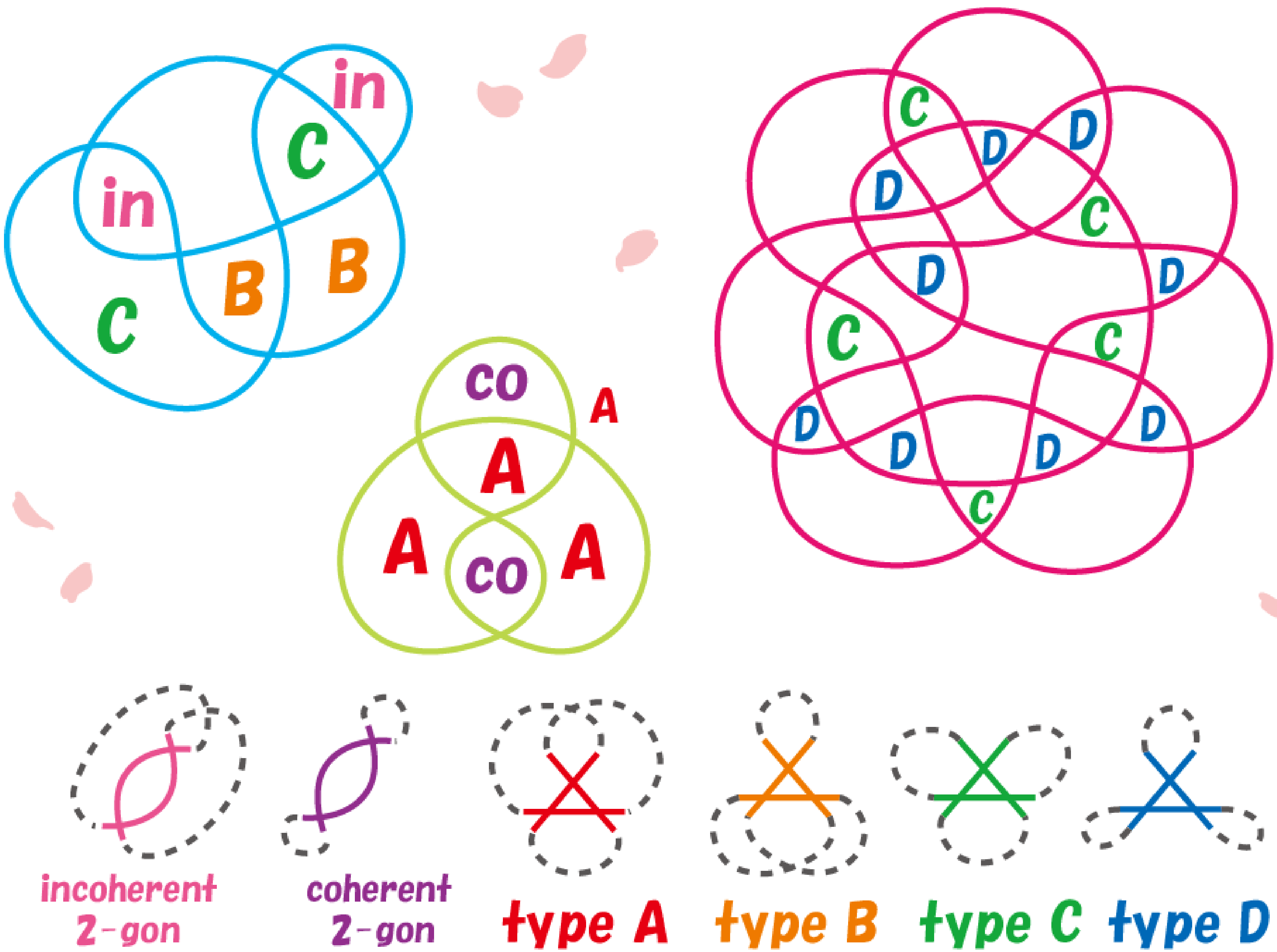
2-gons & 3-gons

There are two types of 2-gons:

There are four types of 3-gons:



Example


in
C
in
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B
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co
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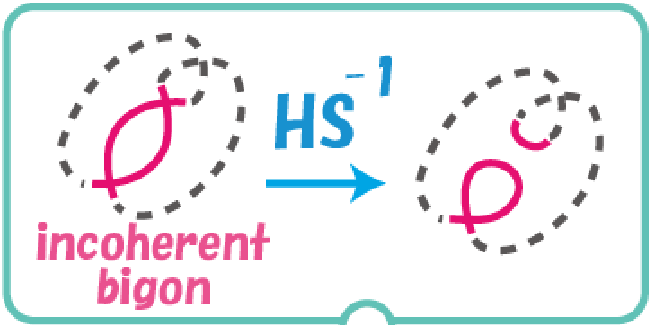
A
A
co
C
D
D
D
D
C
D
D
C
C
D
D
D
D
C
incoherent 2-gon
coherent 2-gon
type A
type B
type C
type D

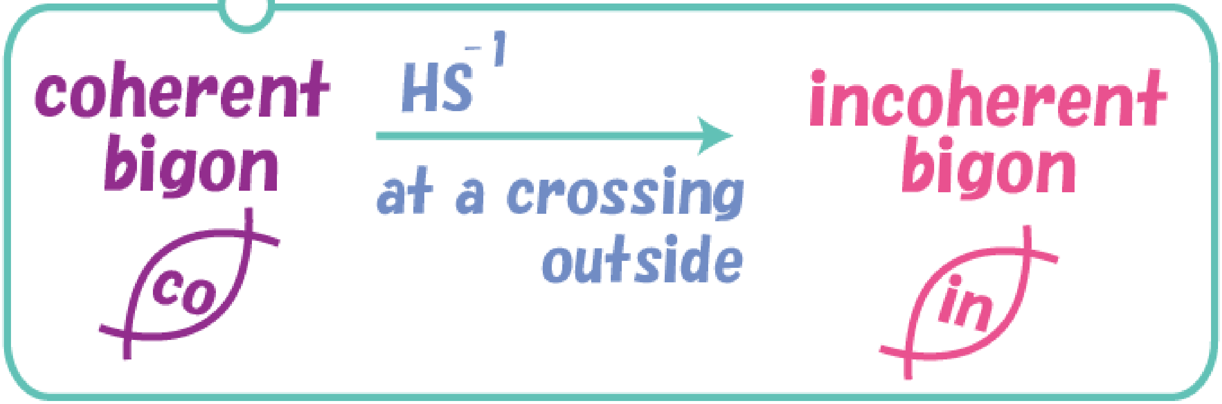
2-gons

Lemma 2



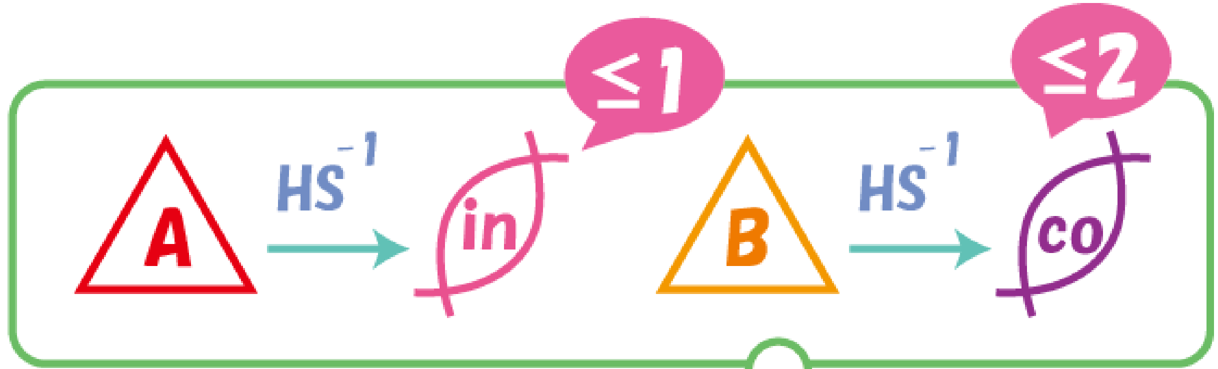
If P has an incoherent 2-gon, then $r(P) \leq 1$.

If P has a coherent 2-gon, then $r(P) \leq 2$.



3-gons

Lemma 3

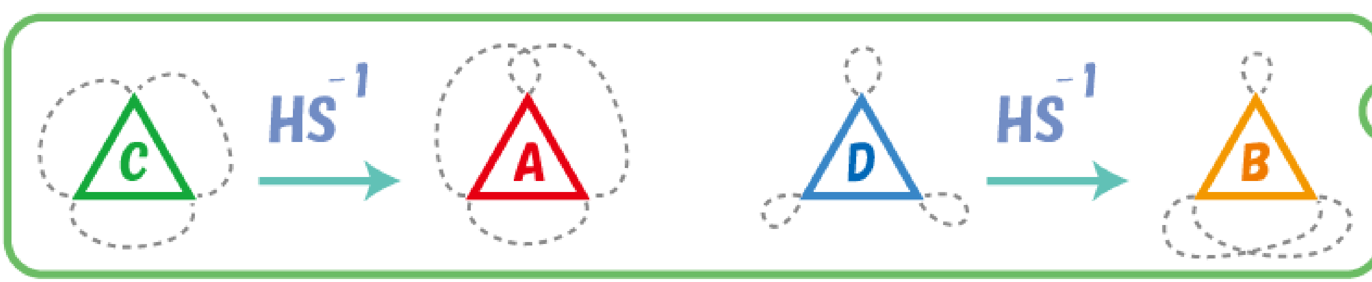


If P has a 3-gon of type A, then $r(P) \leq 2$.

If P has a 3-gon of type B, then $r(P) \leq 3$.

If P has a 3-gon of type C, then $r(P) \leq 3$.

If P has a 3-gon of type D, then $r(P) \leq 4$.



Corollary 4

If P has at least one of

incoherent 2-gon, coherent 2-gon, 3-gon of type A, 3-gon of type B, 3-gon of type C,

then $r(P) \leq 3$.

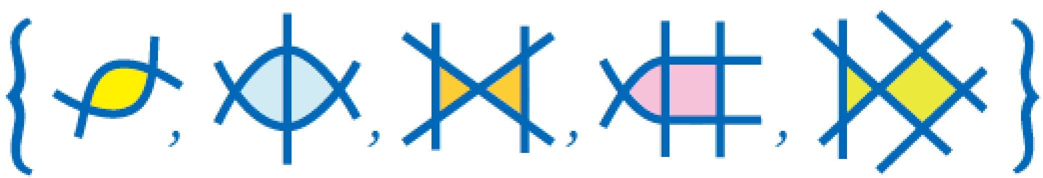
§5. Unavoidable sets

Definition

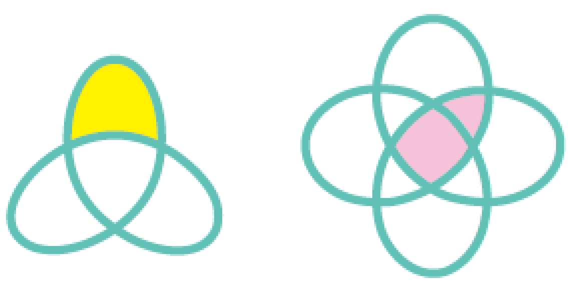
S: a set consisting of parts of knot projections

S is an **unavoidable set for a knot proj.** if every knot projection has at least one of the parts in S.

Example:



is an unavoidable set for a **reduced** knot projection.

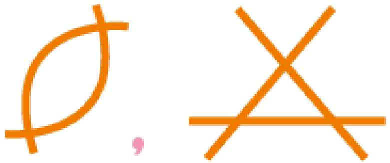


prove later

AST's theorem

Theorem (Adams-Shinjo-Tanaka)

Every reduced knot projection has a 2-gon or 3-gon.

i.e., {  } is an unavoidable set for a reduced knot projection.

Reference: C. C. Adams, R. Shinjo and K. Tanaka, Complementary regions of knot and link diagrams, Ann. Comb. 15 (2011), 549-563.

Proof of AST's theorem

P: a reduced knot projection

C_n: the number of n-gons of P

Euler's characteristic

$$v - e + f = 2$$

# of crossings $\sum_k \frac{kC_k}{4}$ → v

# of edges $\sum_k \frac{kC_k}{2}$ → e

# of regions $\sum_k C_k$ → f

$$\rightarrow 2C_2 + C_3 = 8 + C_5 + 2C_6 + 3C_7 + \cdots$$

AST's formula

$$\rightarrow C_2 > 0 \text{ or } C_3 > 0$$

Proof of Theorem 1

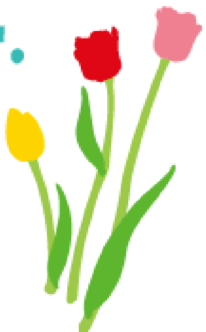
"reductivity is four or less"

If P is reducible, then $r(P)=0$.
(by definition)

If P is reduced, P has a 2-gon or 3-gon.
(by AST's theorem)

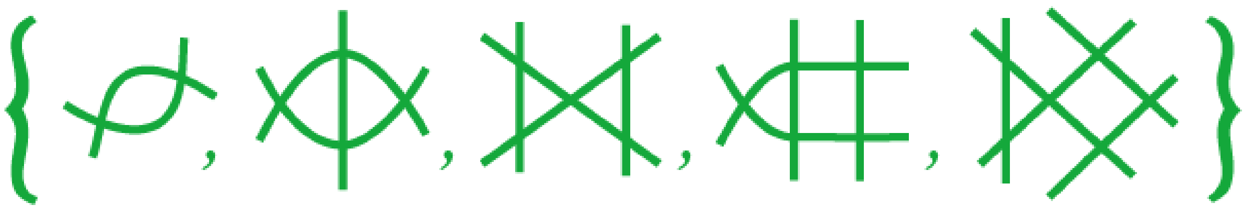
If P has a 2-gon, then $r(P)\leq 2$.
(by Lemma 2)

If P has a 3-gon, then $r(P)\leq 4$.
(by Lemma 3)



Further unavoidable set

Lemma 5



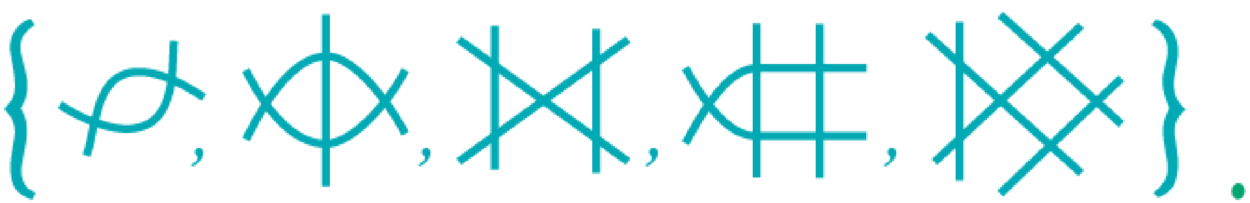
is an unavoidable set
for a reduced knot projection.

Proof of Lemma 5

Use the "discharging method"
from graph theory
(four-color theorem)!

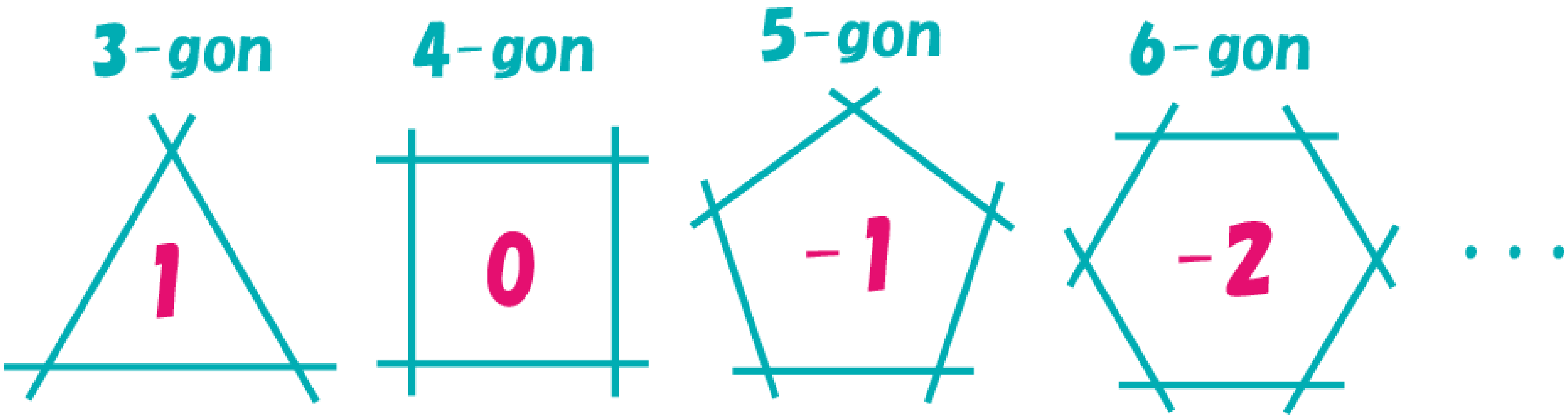
P: a reduced knot projection

Assume P does not have any part in



Then, ...

Give "charge" (4−n) to each n-gon.



Then the total charge is...

$$c_3 - c_5 - 2c_6 - 3c_7 - \cdots$$

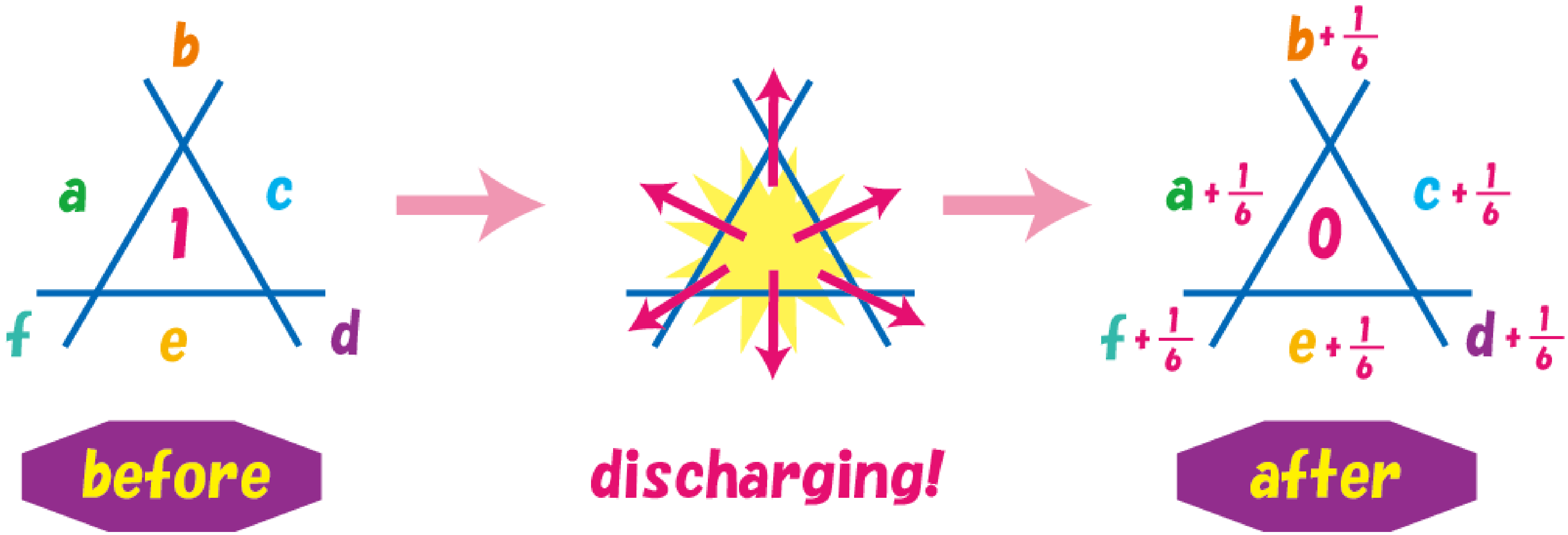
$$= 8$$

c_n: the number of n-gons

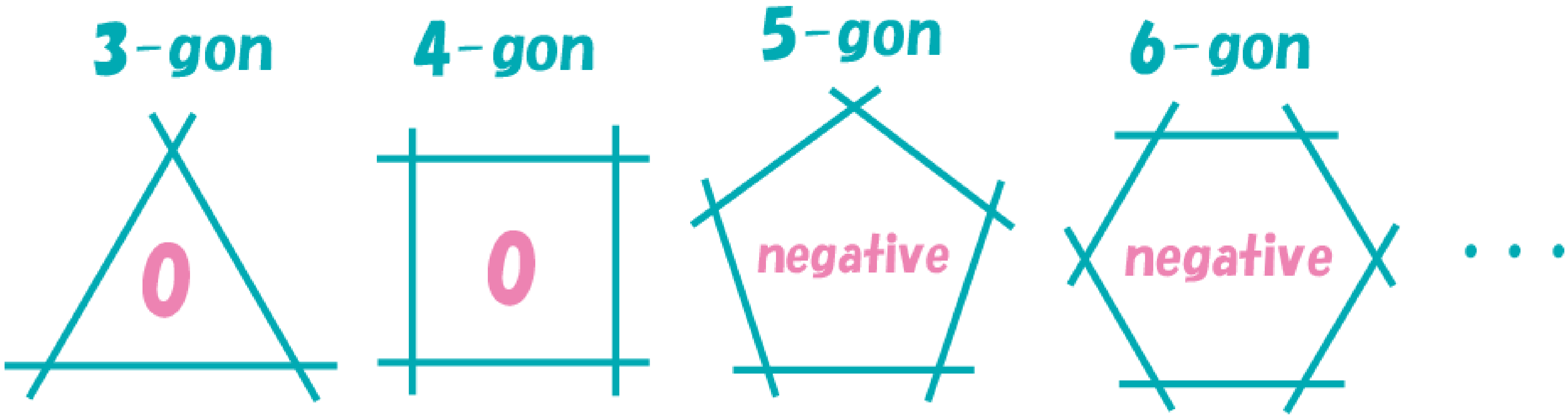
AST's formula

$$2c_2 + c_3 = 8 + c_5 + 2c_6 + 3c_7 + \cdots$$

"Discharging" at every 3-gon to the neighbor six regions by $\frac{1}{6}$.



After discharging...



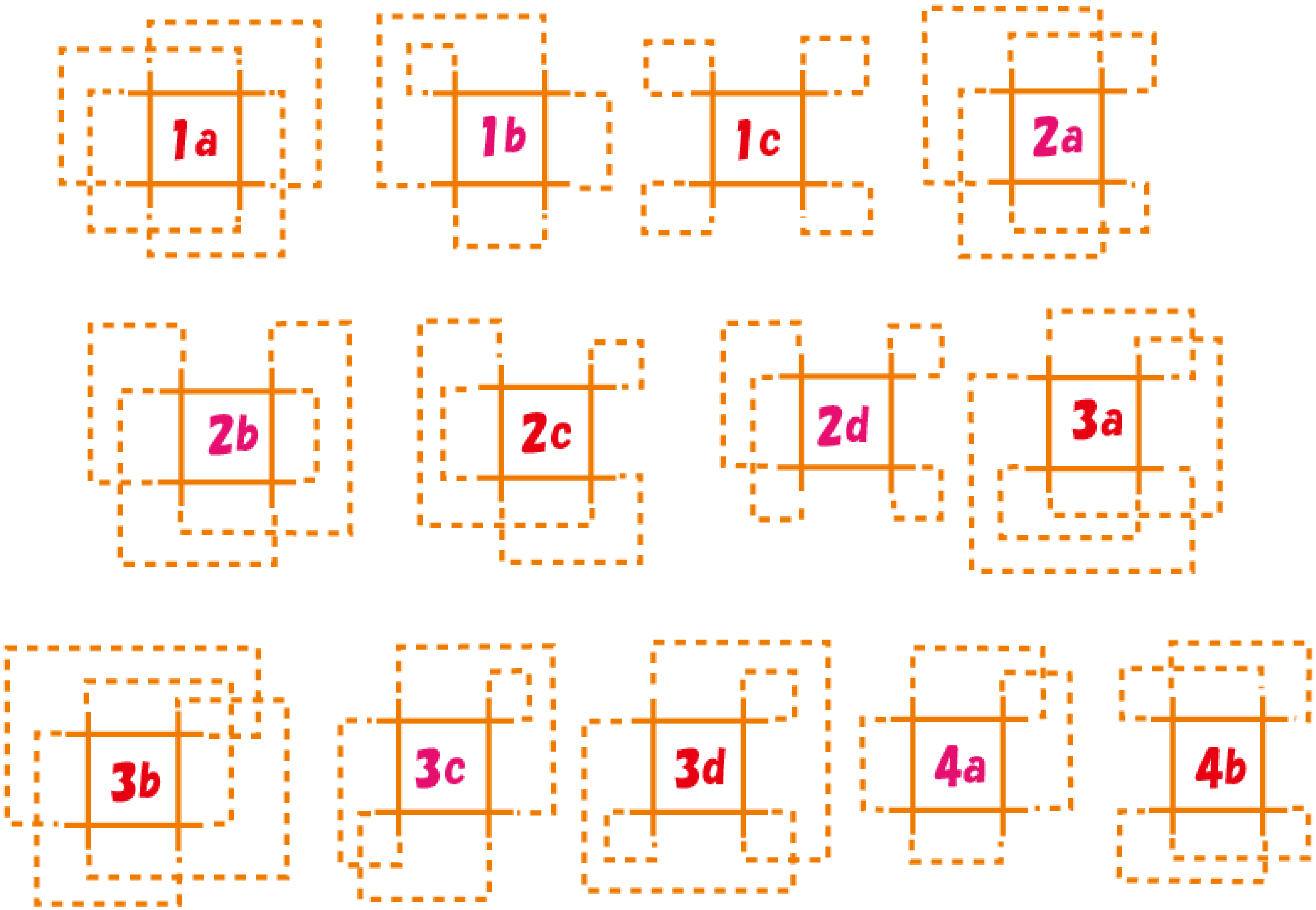
Contradicts that the total charge is 8.

Hence  is an un-avoidable set for a reduced knot proj.

§6. 4-gons & 5-gons

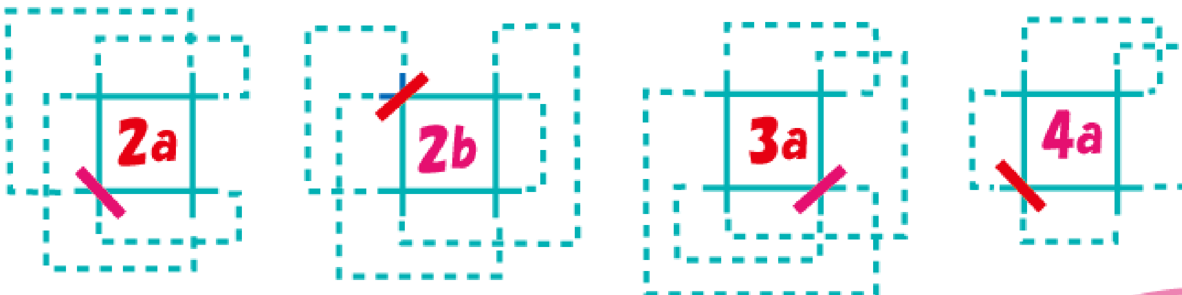
4-gons

There are 13 types of 4-gons:



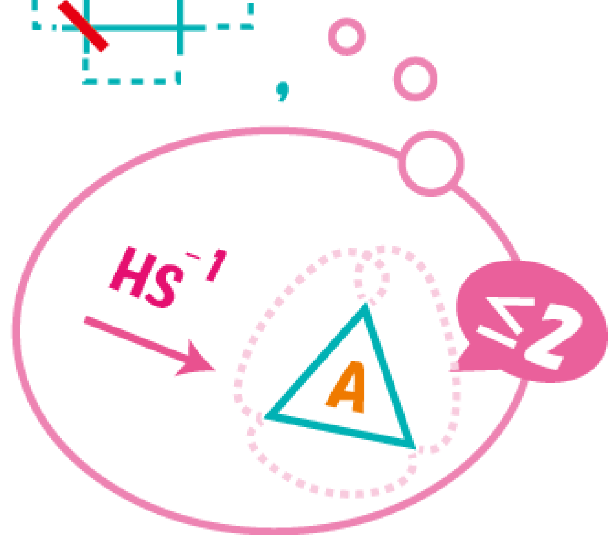
Lemma 6

If a knot projection P has one of



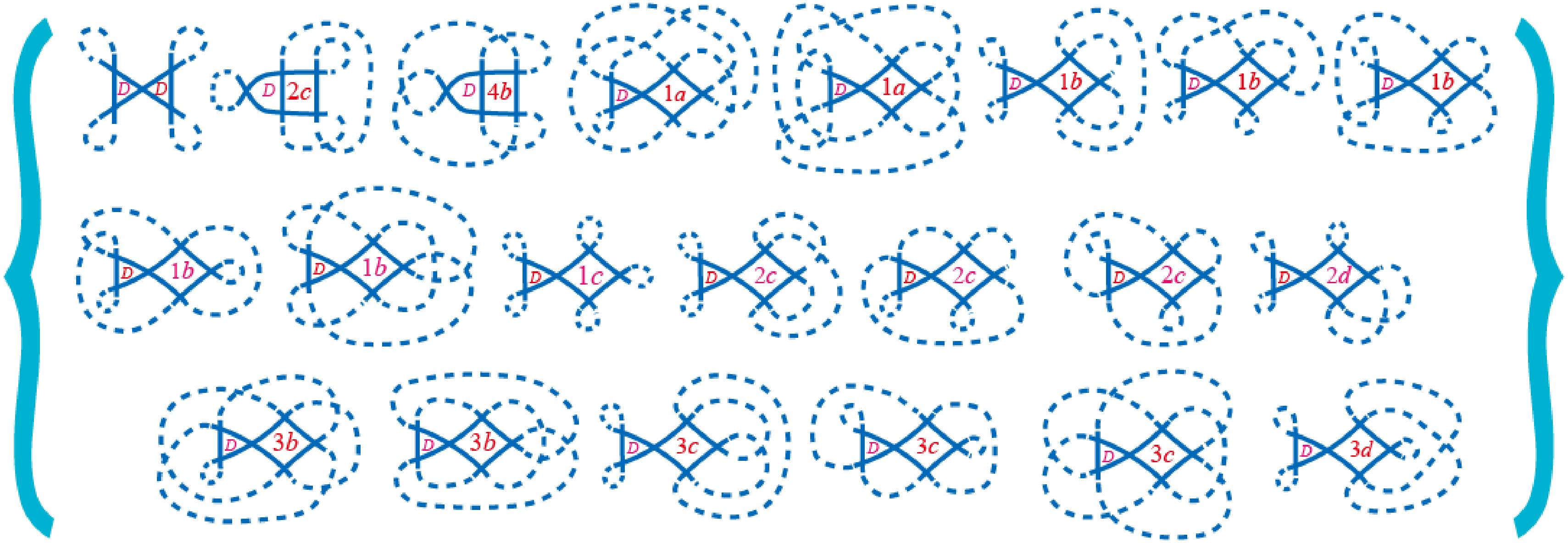
,

then $r(P) \leq 3$.



Unavoidable set for P with r(P)=4

Theorem 7 (Onoda-S)

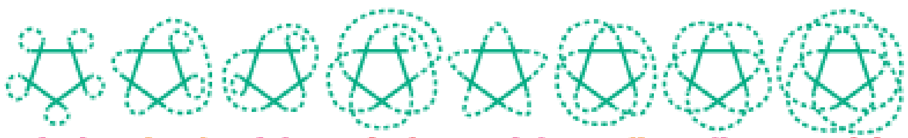


is an unavoidable set for a knot projection with reductivity four.

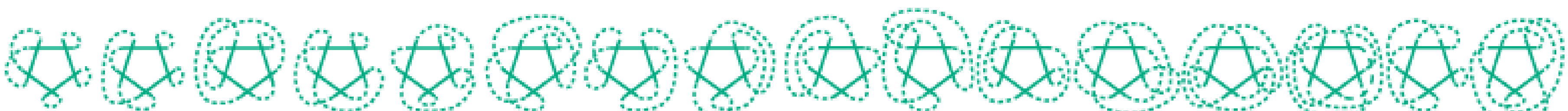
Reference: Y. Onoda and A. Shimizu, The reductivity of spherical curves Part II: 4-gons, Tokyo J. Math. 41 (2018), 51-63.

5-gons

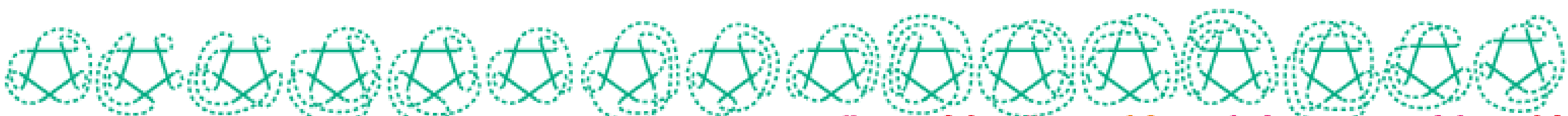
There are 56 types of 5-gons:



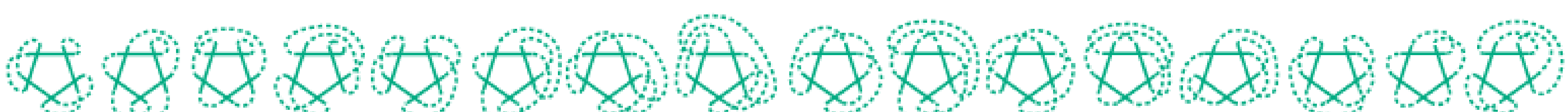
1abcde 1abced 1abdec 1abedc 1acebd 1acedb 1adbec 1aedcb



2abcde 2abced 2abdce 2abdec 2abecd 2abedc 2acbde 2acbed 2acebd 2acedb 2adbce 2adbec 2adcbe 2adceb 2aedbc 2aedcb



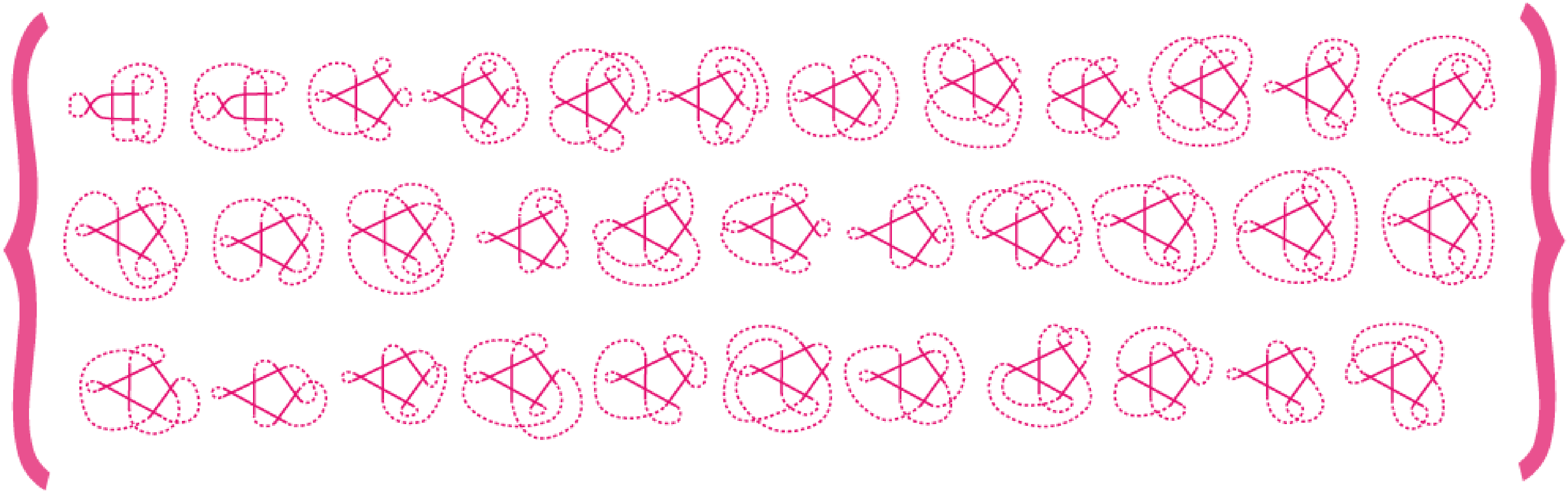
3abcde 3abced 3abdce 3abdec 3abecd 3abedc 3acbde 3acbed 3acdbe 3acebd 3adbec 3adcbe 3adceb 3adecb 3aebdc 3aedcb



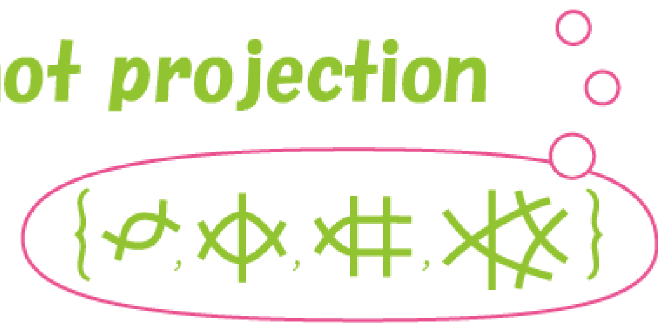
4abcde 4abced 4abdec 4abedc 4acbde 4acbed 4acdbe 4acdeb 4acebd 4acedb 4adbce 4adbec 4adcbe 4adecb 4aecbd 4aedcb

Unavoidable set for P with r(P)=4

Theorem 8 (Kashiwabara-S)



is an unavoidable set for a knot projection with reductivity four.

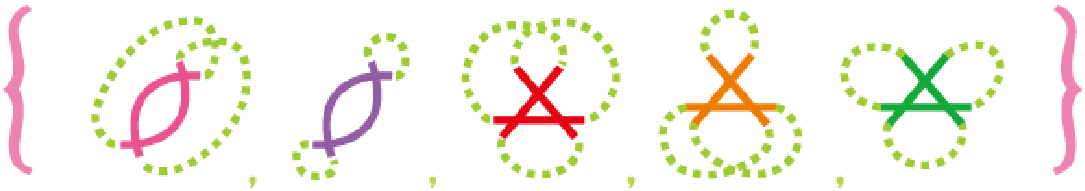


Reference: K. Kashiwabara and A. Shimizu, A note on unavoidable sets for a spherical curve of reductivity four, Kyungpook Math. J. 59 (2019), 821-834.

§7. 2-gons & 3-gons again

Question

Is



incoherent 2-gon, coherent 2-gon, 3-gon of type A, 3-gon of type B, 3-gon of type C

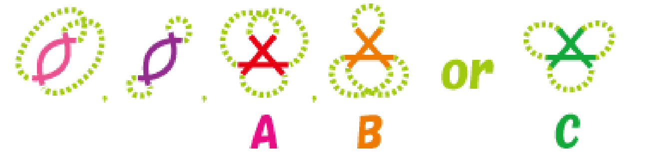
except

3-gon of type D

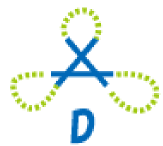
an unavoidable set for a reduced knot projection?

(If so, the reductivity problem is to be solved negatively, i.e., $r(P) \leq 3$ for any P.)

NO!

This does not have 

,

and has only 

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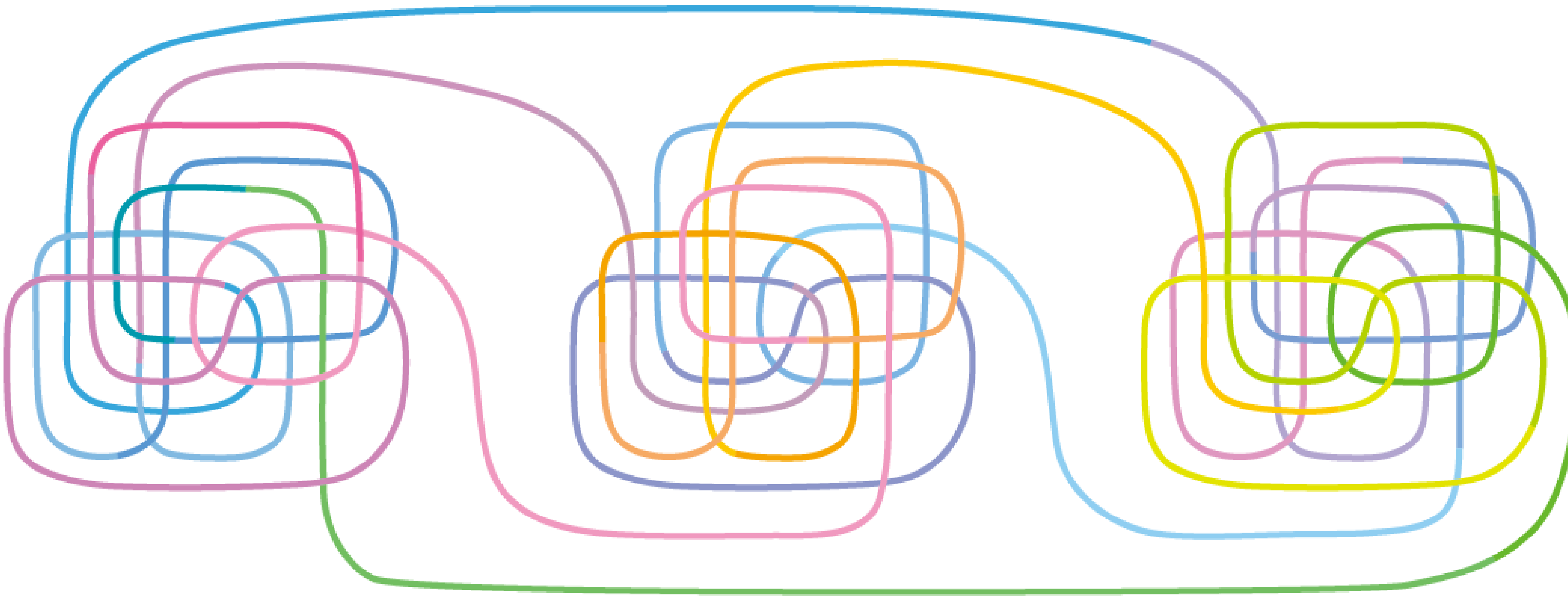
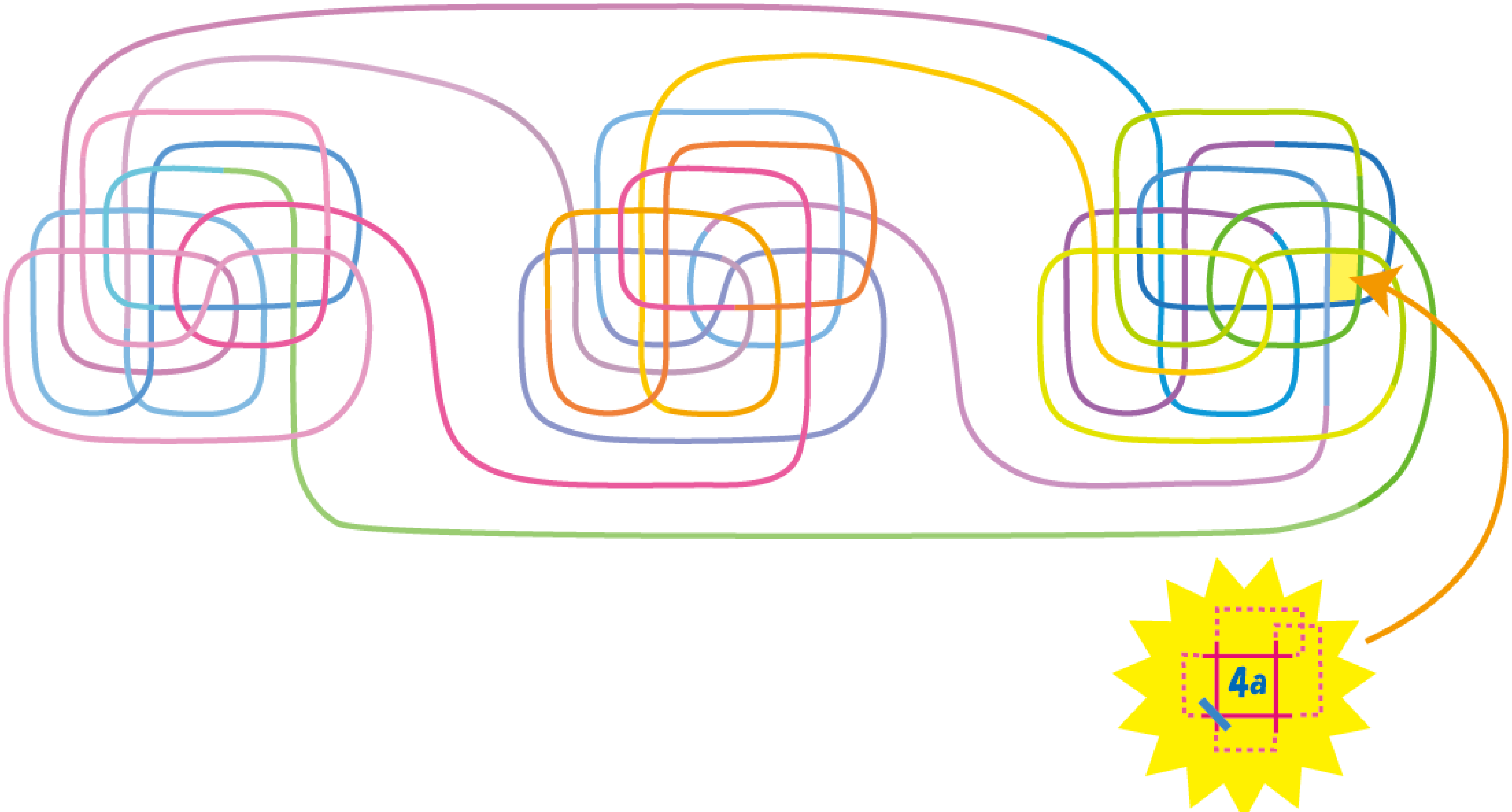


Reference: K. Kashiwabara and A. Shimizu, A note on unavoidable sets for a knot projection of reductvity four, Kyungpook Math. J. 59 (2019), 821-834.

However, the reductivity is not four!



Reductivity problem

$\exists ?\ P$ s.t. $r(P)=4$

Thank you

for watching!