Virtual braid groups, virtual twin groups and crystallographic groups

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This is a virtual talk from a virtual research project about virtual objects

Joint work with *Paulo Cesar Cerqueira dos Santos Júnior* (UFBA)

`Online seminar: Classical Knots, Virtual Knots, and Algebraic Structures Related to Knots`

April, 2022

First connection...

Gonçalves, D. L., Guaschi, J., and Ocampo, O. *A quotient of the artin braid groups related to crystallographic groups*. Journal of Algebra **474** (2017), 393–423.

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Some related works

- Marin, I. *Crystallographic groups and flat manifolds from complex reflection groups*. Geom. Dedicata **182** (2016), 233–247.
- Gonçalves, D. L., Guaschi, J., Ocampo, O., and e Pereiro, C. M. *Crystallographic groups and flat manifolds from surface braid groups*. Topology and its Applications **293** (2021), 107560.
- Diniz, R. S. *Grupos de tranças de superfícies finitamente perfuradas e grupos cristalográficos*. PhD thesis, Universidade Federal de São Carlos, 2020.
- Bardakov, V., Emel'yanenkov, I., Ivanov, M., Kozlovskaya, T., Nasybullov, T., and Vesnin, A. *Virtual and universal braid groups, their quotients and representations*. To appear in Journal of group theory.

Natural question

It is natural to ask which *braid-like groups* have quotients isomorphic to crystallographic groups.

In this talk we deal with this problem for the case of virtual braid and virtual twin groups and some related groups. It is based on the paper [OS].

1

[1] [OS] Oscar Ocampo and Paulo Cesar Cerqueira dos Santos Júnior, Virtual braid groups, virtual twin groups and crystallographic groups (2021), arXiv:2110.02392 .

We shall use the following presentation of VB_n (see [BB, Theorem 4][2]):

The group VB_n admits the following group presentation:

- Generators: σ_i, ρ_i where $i = 1, 2, \dots, n-1$.
- Relations:

$$
\begin{array}{rl}
\text{(AR1)} & \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1, 2, \dots, n-2; \\
\text{(AR2)} & \sigma_i\sigma_j = \sigma_j\sigma_i,\ \mid i-j \mid \geq 2; \\
\text{(SR1)} & \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1},\ i = 1, 2, \dots, n-2; \\
\text{(SR2)} & \rho_i\rho_j = \rho_j\rho_i,\ \mid i-j \mid \geq 2; \\
\text{(SR3)} & \rho_i^2 = 1,\ i = 1, 2, \dots, n-1; \\
\text{(MR1)} & \sigma_i\rho_j = \rho_j\sigma_i,\ \mid i-j \mid \geq 2; \\
\text{(MR2)} & \rho_i\rho_{i+1}\sigma_i = \sigma_{i+1}\rho_i\rho_{i+1},\ i = 1, 2, \dots, n-2.
\end{array}
$$

It is clear that there are embeddings of the Artin braid group B_n and of the symmetric group S_n into VB_n.

[2] [BB] Bardakov, V. G., and Bellingeri, P. Combinatorial properties of virtual braids, *Topology and its Applications* **156**, 6 (2009), 1071–1082.

Let $n \geq 2$ and let $\pi_p \colon VB_n \longrightarrow S_n$ be the homomorphism defined by $\pi_p(\sigma_i) = \pi_p(\rho_i) = \tau_i$ for $i = 1, \ldots, n-1$ and $\tau_i = (i, i+1)$.

The *pure virtual braid group* VP_n is defined to be kernel of π_p, from which we obtain the following short exact sequence

$$1 \longrightarrow VP_n \longrightarrow VB_n \xrightarrow{\pi_p} S_n \longrightarrow 1 \,. \quad (1)$$

As mentioned in [Ba][3] the virtual braid group admits a decomposition as semi-direct product $VB_n = VP_n \rtimes S_n$, the map $\iota \colon S_n \longrightarrow VB_n$ given by $\iota(\tau_i) = \rho_i$, for $i = 1, \ldots, n-1$, is naturally a section for π_p.

The group VP_n admits the following group presentation (see [Ba, Theorem 1]):

- Generators: $\lambda_{i,j}$ for $1 \leq i \neq j \leq n$.
- Relations:
  - $\lambda_{i,j}\lambda_{k,l} = \lambda_{k,l}\lambda_{i,j}$ for i, j, k, l distinct.
  - $\lambda_{k,i}(\lambda_{k,j}\lambda_{i,j}) = (\lambda_{i,j}\lambda_{k,j})\lambda_{k,i}$ for i, j, k distinct.

[3] [Ba] Bardakov, V. G. The virtual and universal braids, *Fundamenta Mathematicae* **184** (2004), 1–18.

The split short exact sequence (1) provides, for every $n \geq 2$, the following split short exact sequence

$$1 \longrightarrow VP_n/\Gamma_2(VP_n) \longrightarrow VB_n/\Gamma_2(VP_n) \xrightarrow{\overline{\pi}_p} S_n \longrightarrow 1. \qquad (2)$$

From the presentation of VP_n it is straightforward to show that $VP_n/\Gamma_2(VP_n)$ is isomorphic to the free Abelian group $\mathbb{Z}^{n(n-1)}$ and generated by the set

$$\{\lambda_{i,j} \mid 1 \leq i \neq j \leq n\} \qquad (3)$$

From Lemma 1 of [Ba], using the presentation of VP_n given before, we conclude that the action by conjugation of $S_n = \langle \rho_1, \ldots, \rho_{n-1} \rangle$ on VP_n is described as follows: let $w \in S_n$ and let $\lambda_{i,j}$ be a generator of VP_n, with $1 \leq i \neq j \leq n$, then

$$w \cdot \lambda_{i,j} = w\lambda_{i,j}w^{-1} = \lambda_{w^{-1}(i),w^{-1}(j)}. \qquad (4)$$

From this, we conclude that the action of S_n on $VP_n/\Gamma_2(VP_n)$ is injective.

Algebraic Characterization of crystallographic groups

Let Φ be a finite group, let Π be any group and let $m \geq 1$ an integer.
Π is a crystallographic group of dimension m if and only if there exists a short exact sequence

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1 \qquad (5)$$

such that the rank m integral representation $\Theta\colon \Phi \to Aut(\mathbb{Z}^m)$, induced by the action by conjugation, is faithful.

- The group Φ is called *the holonomy group of* Π.
- Θ is called *the holonomy representation of* Π.
- A torsion-free crystallographic group is called a *Bieberbach group*.

Algebraic Characterization of crystallographic groups

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$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1 \tag{6}$$

such that the rank m integral representation $\Theta\colon \Phi \to Aut(\mathbb{Z}^m)$, induced by the action by conjugation, is faithful.

Theorem 1

Let $n \geq 2$. There is a split short exact sequence:

$$1 \longrightarrow \mathbb{Z}^{n(n-1)} \longrightarrow VB_n/\Gamma_2(VP_n) \xrightarrow{\overline{\pi}_p} S_n \longrightarrow 1 \tag{7}$$

and the middle group is a crystallographic group.

Theorem 1

Let $n \geq 2$. There is a split short exact sequence:

$$1 \longrightarrow \mathbb{Z}^{n(n-1)} \longrightarrow VB_n/\Gamma_2(VP_n) \xrightarrow{\overline{\pi}_p} S_n \longrightarrow 1 \qquad (8)$$

and the middle group is a crystallographic group.

We note that in [BEIKNV, Section 3][4] the authors proved that

- there is a decomposition $VB_n/\Gamma_2(VP_n) = VP_n/\Gamma_2(VP_n) \rtimes S_n$
- $VB_n/\Gamma_2(VP_n)$ it is a crystallographic group,
- and also they studied linear representations of this group.

In our work we reprove the connection between virtual braid groups and crystallographic groups and then we study structural aspects of $VB_n/\Gamma_2(VP_n)$.

[4] [BEIKNV] Bardakov, V., Emel'yanenkov, I., Ivanov, M., Kozlovskaya, T., Nasybullov, T., and Vesnin, A. *Virtual and universal braid groups, their quotients and representations*. To appear in Journal of group theory.

In [BEER, Prop. 4.8][5] was shown that the restriction of the homomorphism $\varphi_n \colon B_n \hookrightarrow VB_n$, $\sigma_i \to \sigma_i$ for $1 \leq i \leq n-1$, to the respective pure groups is injective: $P_n \hookrightarrow VP_n$, $A_{i,j} \mapsto (\lambda_{j-1,j} \dots \lambda_{i+1,j})\lambda_{i,j}\lambda_{j,i}(\lambda_{j-1,j} \dots \lambda_{i+1,j})^{-1}$..

From the following commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & P_n/\Gamma_2(P_n) & \longrightarrow & B_n/\Gamma_2(P_n) & \longrightarrow & S_n & \longrightarrow & 1 \\
 & & \downarrow \widetilde{\varphi_{n|}} & & \downarrow \widetilde{\varphi_n} & & \| & & \\
1 & \longrightarrow & VP_n/\Gamma_2(VP_n) & \longrightarrow & VB_n/\Gamma_2(VP_n) & \longrightarrow & S_n & \longrightarrow & 1.
\end{array}$$

and using the five lemma we conclude the following result.

Proposition 2

The embedding of B_n into VB_n induces an embedding of $B_n/\Gamma_2(P_n)$ into $VB_n/\Gamma_2(VP_n)$.

[5] [BEER] Bartholdi, L., Enriquez, B., Etingof, P., and Rains, E. *Groups and Lie algebras corresponding to the Yang-Baxter equations*, J. Algebra **305**, 2 (2006), 742–764.

Proposition 2

The embedding of B_n into VB_n induces an embedding of $B_n/\Gamma_2(P_n)$ into $VB_n/\Gamma_2(VP_n)$.

- It is natural to ask which elements in the quotient of the virtual braid group $VB_n/\Gamma_2(VP_n)$ have finite order, as mentioned in [BEIKNV, Question 3.1].
- Also, in [BEIKNV, Question 3.2] the authors asked about the realization of Bieberbach groups (i.e. torsion-free crystallographic groups) inside $VB_n/\Gamma_2(VP_n)$.
- Follows, from Proposition 2, that there are infinite elements of finite order in $VB_n/\Gamma_2(VP_n)$ coming from $B_n/\Gamma_2(P_n)$ (see [GGO, Theorem 3][6]) and also the Bieberbach groups realized in $B_n/\Gamma_2(P_n)$ are naturally realized in $VB_n/\Gamma_2(VP_n)$.

We shall prove that there are other elements of finite order and Bieberbach subgroups in $VB_n/\Gamma_2(VP_n)$.

[6] [GGO] D. L. Gonçalves, J. Guaschi and O. Ocampo, A quotient of the Artin braid groups related to crystallographic groups, *J. Algebra* **474** (2017), 393–423.

The general technique to study finite order elements and conjugacy classes of elements in $VB_n/\Gamma_2(VP_n)$ is given by solving integral equations on the exponents of the expressions of the words by the generators of $VP_n/\Gamma_2(VP_n)$.

Theorem 3

Let $2 \leq t \leq n$ and consider a collection $1 \leq r_1 < r_2 < \cdots < r_{t-1} \leq n$ of consecutive integers. Let $\overline{\pi}_p(\rho_{r_1}\rho_{r_2}\ldots\rho_{r_{t-1}})^{-1} = \theta$ and let T_θ be a transversal of the action of $\rho_{r_1}\rho_{r_2}\ldots\rho_{r_{t-1}}$ on $\{\lambda_{i,j} \mid 1 \leq i \neq j \leq n\}$. The element

$$\prod_{1\leq i\neq j\leq n} \lambda_{i,j}^{a_{i,j}}(\rho_{r_1}\rho_{r_2}\ldots\rho_{r_{t-1}}),$$

where $a_{i,j} \in \mathbb{Z}$, has order t in $VB_n/\Gamma_2(VP_n)$ if and only if $\sum_{\lambda_{i_1,j_1}\in\mathscr{O}_\theta(\lambda_{i,j})} a_{i_1,j_1} = 0$ for all $\lambda_{i,j} \in T_\theta$.

Proposition 4

Let β_1, β_2 elements of order τ in $VB_n/\Gamma_2(VP_n)$. Then β_1 and β_2 are conjugate in $VB_n/\Gamma_2(VP_n)$ if and only if $\overline{\pi}_p(\beta_1)$ and $\overline{\pi}_p(\beta_2)$ are conjugate in S_n.

Sketch of the proof of Theorem 3

To solve the following equation in $VP_n/\Gamma_2(VP_n)$

$$\left(\prod_{1\leq i\neq j\leq n} \lambda_{i,j}^{a_{i,j}}(\rho_1\rho_2\dots\rho_{t-1})\right)^t = \prod_{1\leq i\neq j\leq n} \lambda_{i,j}^{a_{i,j}} \prod_{1\leq i\neq j\leq n} \lambda_{\theta(i),\theta(j)}^{a_{i,j}} \prod_{1\leq i\neq j\leq n} \lambda_{\theta^2(i),\theta^2(j)}^{a_{i,j}} \cdots \prod_{1\leq i\neq j\leq n} \lambda_{\theta^{t-1}(i),\theta^{t-1}(j)}^{a_{i,j}} = 1$$

is equivalent to solve the system of integer equations

$$\begin{cases} a_{i_1,j_1} + a_{\theta^{-1}(i_1),\theta^{-1}(j_1)} + \cdots + a_{\theta^{-(t-1)}(i_1),\theta^{-(t-1)}(j_1)} = 0 \\ a_{i_2,j_2} + a_{\theta^{-1}(i_2),\theta^{-1}(j_2)} + \cdots + a_{\theta^{-(t-1)}(i_2),\theta^{-(t-1)}(j_2)} = 0 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{i_m,j_m} + a_{\theta^{-1}(i_r),\theta^{-1}(j_r)} + \cdots + a_{\theta^{-(t-1)}(i_r),\theta^{-(t-1)}(j_r)} = 0 \end{cases}$$

i.e. $\left(\prod_{1\leq i\neq j\leq n} \lambda_{i,j}^{a_{i,j}}(\rho_1\rho_2\dots\rho_{t-1})\right)^t = 1$ if and only if $\sum_{\lambda_{r,s}\in\mathcal{O}_\theta(\lambda_{i_m,j_m})} a_{r,s} = 0$ for all $\lambda_{i_m,j_m} \in T_\theta$. □

Corollary 5

Let $m, t \in \mathbb{N}$ and let $n_1, n_2, \ldots, n_t$ be positive integers, not necessarily distinct, greater than or equal to 2 for which $\sum_{i=1}^{t} n_i \leq m$. Then $VB_m/\Gamma_2(VP_m)$ possesses infinite elements of order $\operatorname{lcm}(n_1, \ldots, n_t)$. Further, there exists such an element whose related permutation has cycle type $(n_1, n_2, \ldots, n_t)$.

Let $n \geq 2$. Let $\alpha\colon \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_n)$ be a homomorphism, then α is determined by the image of a generator x of $\mathbb{Z}$. Further, there is $k \in \{1, \ldots, n-1\}$ such that $\gcd(n,k) = 1$ and $\alpha(x) = \alpha_k$ where $\alpha_k\colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ is given by $\delta \longmapsto \delta^k$.

A group G is *virtually cyclic* if it has a cyclic subgroup H of finite index.

Again, we transform a realization problem in $VB_n/\Gamma_2(VP_n)$ into a problem of solving systems of equations in the free abelian group $VP_n/\Gamma_2(VP_n) = \mathbb{Z}^{n(n-1)}$ obtaining the following result.

Theorem 6

Let $n \geq 2$. Then, for every $k \in \{1, \ldots, n-1\}$ such that $\gcd(n,k) = 1$, the virtually cyclic group $\mathbb{Z}_n \rtimes_{\alpha_k} \mathbb{Z}$ can be realized in $VB_n/\Gamma_2(VP_n)$.

- Torsion-free crystallographic groups, also called *Bieberbach groups*, are notably interesting from its relation with Riemannian geometry and dynamical systems.

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- It is well known ([W, Section 3.3][7]) from the Bieberbach theorems that, via the fundamental group, there is a correspondence between the collection of Bieberbach groups of dimension m and the collection of flat compact connected Riemannian manifolds of dimension m.

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- A lot of information of a flat Riemannian manifold is recorded algebraically in its fundamental group (that is a Bieberbach group) or in its holonomy representation.
- So, it is very interesting to explore algebraic properties of Bieberbach groups in order to get information of their respective flat Riemannian manifolds.

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Theorem 7

Let G_n be the cyclic subgroup of S_n generated by $(1, n, n-1, \ldots, 2)$. There is a Bieberbach subgroup $\widetilde{G_n}$ in $VB_n/\Gamma_2(VP_n)$ of dimension $n(n-1)$ and holonomy group G_n. Further, the center $Z(\widetilde{G_n})$ is a free Abelian group of rank $n-1$.

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Sketch of the proof

We need to find a "nice" subgroup L of $VP_n/\Gamma_2(VP_n)$ such that the following short exact sequence

$$1 \longrightarrow L \longrightarrow \widetilde{G_n} \longrightarrow G_n \longrightarrow 1$$

satisfies the definition of a Bieberbach group. It works well taking

$$L = \langle \prod_{\lambda_{r,s}\in\mathscr{O}_\tau(\lambda_{1,2})} \lambda_{r,s},\ \lambda_{i,j}^n \mid 1 \leq i \neq j \leq n \text{ and } (i,j) \neq (1,2)\rangle.$$

To compute $Z(\widetilde{G_n})$ we use Lemma 5.2 of [Sz][a], which ensures that the center of $\widetilde{G_n}$ is formed by the elements of L fixed by the action of G_n.

[a] [Sz] Szczepański, A. Geometry of crystallographic groups, volume 4 of algebra and discrete mathematics, 2012.

- The *twin groups* T_n (also known as *planar braid groups*), for $n \geq 2$, form a special class of right-angled Coxeter groups and first appeared in the work of Shabat and Voevodsky [SV][8], wherein they were referred as *Grothendieck cartographical groups*.

[8] [SV] G. B. Shabat and V. A. Voevodsky, Drawing curves over number fields, The Grothendieck Festschrift, Vol. III, 199–227, Progr. Math., 88, Birkhauser Boston, Boston, MA, 1990.

[9] [K] M. Khovanov, *Real* $K(\pi,1)$ *arrangements from finite root systems*, Math. Res. Lett. **3** (1996), no. 2, 261–274

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- Later, twin groups were brought to light by Khovanov [K][9] under the present name, who also gave a geometric interpretation of these groups similar to the one for classical braid groups, as an $\mathbb{R}$-analogue of Artin's pure braid group on n strands.

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- The *virtual twin group* VT_n was introduced in [BSV][10] as an abstract generalization of twin groups, together with the *pure virtual twin group* PVT_n which is defined as the kernel of the natural surjection from VT_n onto S_n.

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- The *virtual twin group* VT_n was introduced in [BSV][10] as an abstract generalization of twin groups, together with the *pure virtual twin group* PVT_n which is defined as the kernel of the natural surjection from VT_n onto S_n.
- Then, in [NNS][11], VT_n and PVT_n were studied with more details.

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[8] [SV] G. B. Shabat and V. A. Voevodsky, Drawing curves over number fields, The Grothendieck Festschrift, Vol. III, 199–227, Progr. Math., 88, Birkhauser Boston, Boston, MA, 1990.

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Using the presentations of VT_n, PVT_n and the description of the action by conjugation of S_n on PVT_n given by Naik, Nanda and Singh [NNS] we prove the following result.

Theorem 8

Let $n \geq 2$. There is a split short exact sequence:

$$1 \longrightarrow \mathbb{Z}^{n(n-1)/2} \longrightarrow VT_n/\Gamma_2(PVT_n) \xrightarrow{\overline{\pi}} S_n \longrightarrow 1 \tag{9}$$

and the middle group is a crystallographic group.

Remark

Similar structural results, as the ones given for $VB_n/\Gamma_2(VP_n)$, were proved for the quotient $VT_n/\Gamma_2(PVT_n)$.

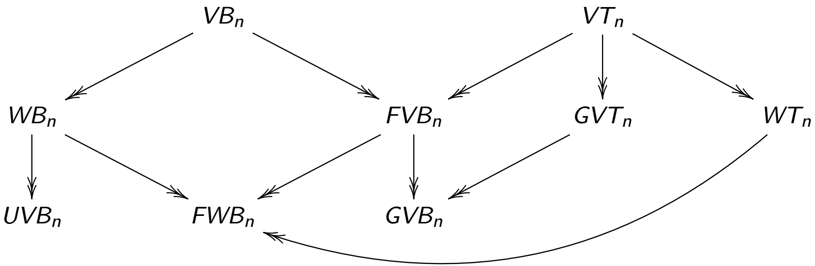
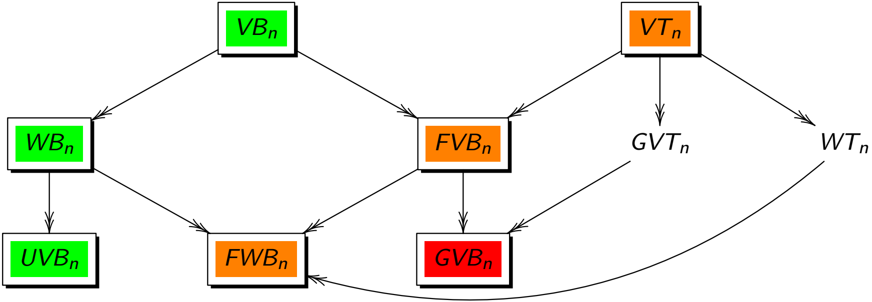


Figure: Quotients of virtual braid and virtual twin groups



Theorem 9

Let $n \geq 2$.

1. The groups $WB_n/\Gamma_2(WP_n)$ and $UVB_n/\Gamma_2(UVP_n)$ are isomorphic to $VB_n/\Gamma_2(VP_n)$.
2. The groups FWB_n and $FVB_n/\Gamma_2(FVP_n)$ are isomorphic to $VT_n/\Gamma_2(PVT_n)$.

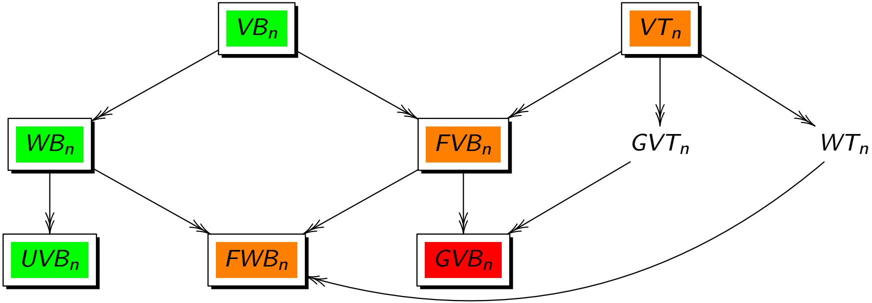
Theorem 9

Let $n \geq 2$.

1. The groups $WB_n/\Gamma_2(WP_n)$ and $UVB_n/\Gamma_2(UVP_n)$ are isomorphic to $VB_n/\Gamma_2(VP_n)$.
2. The groups FWB_n and $FVB_n/\Gamma_2(FVP_n)$ are isomorphic to $VT_n/\Gamma_2(PVT_n)$.

Remark

- As a consequence, the results obtained for the quotients $VB_n/\Gamma_2(VP_n)$ and $VT_n/\Gamma_2(PVT_n)$ also holds for the other quotient groups, respectively.
- The isomorphisms of the first item were pointed out to us by Paolo Bellingeri and John Guaschi in personal communications.



Remark

The respective quotient of the *virtual Gauss braid group* $GVB_n/\Gamma_2(GVP_n)$ is not a crystallographic group since the abelianization of the Gauss virtual pure braid group has finite order elements.

However, concerning structural aspects, using the same techniques we can state and prove for the quotient group $GVB_n/\Gamma_2(GVP_n)$ equivalent versions of the results obtained for $VB_n/\Gamma_2(VP_n)$.

Thanks for your attention!