Modular Categories with Transitive Galois Actions

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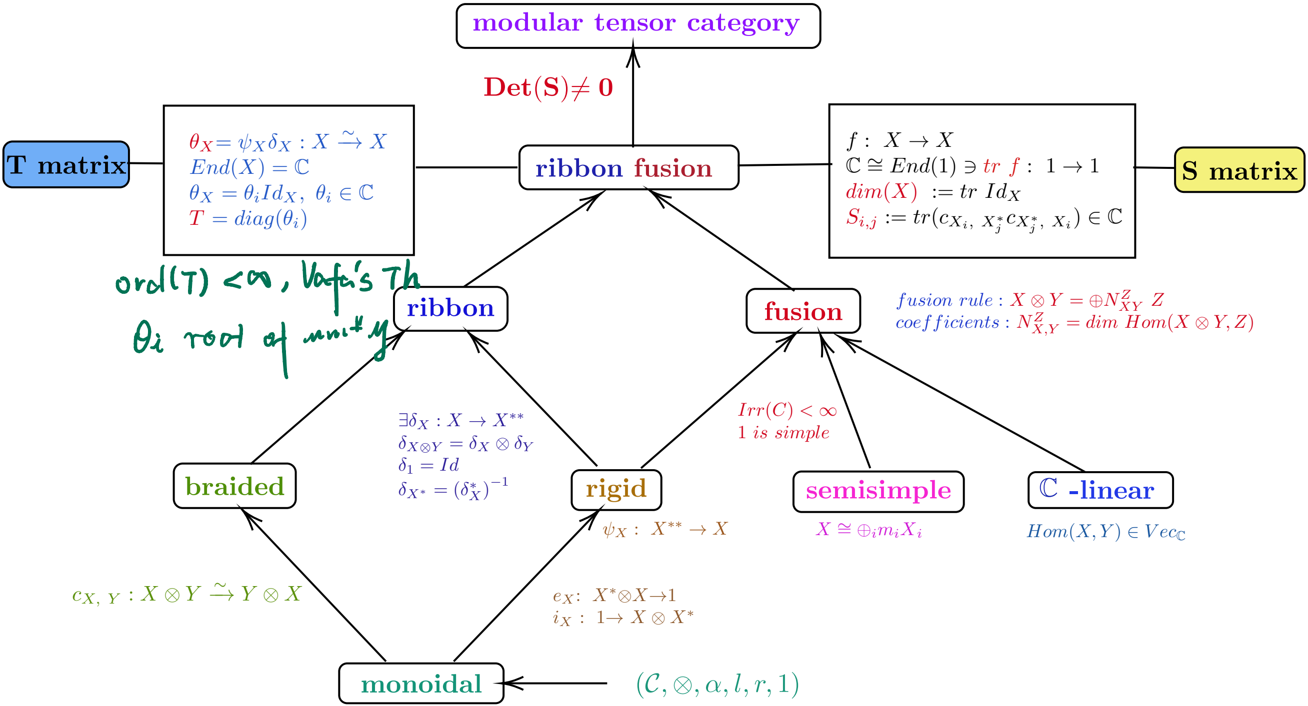
Motivation

Let $\mathcal{C}$ be a modular category with (S, T)

- $\mathrm{Gal}(\mathbb{Q}(S))$ is abelian [de Boer - Goeree 1991]
- $\mathbb{Q}(S) \subset \mathbb{Q}(\zeta_N)$, $N = \mathrm{ord}(T)$. [Ng - Schauenburg 2010].

Classification

- By rank r
  - $r = 4$ [Rowell - Stong - Wang 2009]
  - $r = 5$ [Bruillard - Ng - Rowell - Wang 2016]
  - $r = 6$ [Creamer 2018, Green 2019].
- By # of orbits?
  - 1 orbit?


modular tensor category
Det(S)≠ 0
T matrix
θ_X = ψ_X δ_X : X ⥲ X
End(X) = ℂ
θ_X = θ_i Id_X, θ_i ∈ ℂ
T = diag(θ_i)
ribbon fusion
f : X → X
ℂ ≅ End(1) ∋ tr f : 1 → 1
dim(X) := tr Id_X
S_{i,j} := tr(c_{X_i, X_j^*} c_{X_j^*, X_i}) ∈ ℂ
S matrix
ord(T) < ∞, Vafa's Th
θi root of unity
ribbon
fusion
fusion rule : X ⊗ Y = ⊕N^Z_{XY} Z
coefficients : N^Z_{X,Y} = dim Hom(X ⊗ Y, Z)
∃δ_X : X → X**
δ_{X⊗Y} = δ_X ⊗ δ_Y
δ_1 = Id
δ_{X*} = (δ*_X)^{-1}
Irr(C) < ∞
1 is simple
braided
rigid
semisimple
ℂ -linear
ψ_X : X** → X
X ≅ ⊕_i m_i X_i
Hom(X, Y) ∈ Vec_ℂ
c_{X, Y} : X ⊗ Y ⥲ Y ⊗ X
e_X: X*⊗X→1
i_X : 1→ X ⊗ X*
monoidal
(𝒞, ⊗, α, l, r, 1)

$SL_2(\mathbb{Z})$ Representation

- Let $\mathcal{C}$ be a modular category with (S, T),
- $S^4 = \dim(\mathcal{C})^2 \, Id$, $(ST)^3 = p^+ S^2$, $p^{\pm} = \sum_i \theta_i^{\pm} d_i^2$
- $SL_2(\mathbb{Z}) = \langle s, t \rangle$, $s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- $s^4 = Id$, $(st)^3 = s^2$
- $s \longmapsto S$, $t \longmapsto T$ gives a proj. rep. $\bar{\rho}_{\mathcal{C}}$ of $SL_2(\mathbb{Z}_2)$.

- $\bar{S} := \dfrac{1}{\sqrt{\dim(\mathcal{C})}} S$, $\bar{T} := \dfrac{1}{\gamma} T$, γ is any 3rd root of multiplicative central charge $\xi = \frac{p^+}{\sqrt{\dim(\mathcal{C})}}$

- $s \longmapsto \bar{S}$, $t \longmapsto \bar{T}$ gives a linear rep. ρ of $SL_2(\mathbb{Z})$.

[Dong-Lin-Ng, 2015].

- If $n = \mathrm{ord}(\bar{T})$, then
  - ρ factors through $SL_2(\mathbb{Z}_n)$
  - $\mathrm{im}(\rho) \subset GL_r(\mathbb{Q}(\zeta_n))$

Galois Actions

- $\mathbb{Q}(S) \subset \mathbb{Q}(\zeta_n)$ is an abelian extension. $n = \mathrm{ord}(\bar{T})$.

  Let $G_{\mathcal{C}} = \mathrm{Gal}(\mathbb{Q}(S))$

Character of fusion ring satisfies:

$$\chi(V_i)\chi(V_j) = \sum N_{ij}^{k} \chi(V_k)$$

thus eigenvectors of fusion matrix $(N_i)_{k,j} = N_{i,j}^{k}$

$$SN_iS^{-1} = D_i \Rightarrow SN_i = D_iS$$

$(D_i)_{XY} = \delta_{XY} \frac{S_{iY}}{S_{0Y}}$. Since $\det(S) \neq 0$, these are all characters.

- Consider the fusion ring $K_0(\mathcal{C})$

$$\chi_Y : \mathrm{Irr}(\mathcal{C}) \longrightarrow \mathbb{C}$$

$$X \longmapsto \frac{S_{X,Y}}{S_{1,Y}} = \frac{S_{X,Y}}{d_Y}$$

$\{\chi_Y \mid Y \in \mathrm{Irr}(\mathcal{C})\}$ is the set of all linear charaters of $K_0(\mathcal{C})$

- Notice $\mathbb{Q}(S) = \mathbb{Q}\left(\frac{S_{X,Y}}{S_{1,Y}} \,\middle|\, X, Y \in \mathrm{Irr}(\mathcal{C})\right)$

- $\forall \sigma \in G_{\mathcal{C}}$, $\sigma\left(\frac{S_{X,Y}}{S_{1,Y}}\right) = \frac{S_{X,\hat{\sigma}(Y)}}{S_{1,\hat{\sigma}(Y)}}$

  for some unique $\hat{\sigma} \in \mathrm{Sym}(\mathrm{Irr}(\mathcal{C}))$

Example (Ising)

$$S = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$

- Consider S in the form

$$\begin{bmatrix} 1 & 1 & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -1 & 1 \end{bmatrix},$$

- $\mathbb{Q}(S) = \mathbb{Q}(\sqrt{2})$. $G_{Ising} = \mathbb{Z}_2$.

- G_{Ising} fixes the 2nd simple, interchanges 1st, 3rd.

Transitive Modular Categories

- Def. Modular category $\mathcal{C}$ is transitive if G acts transitively on $\mathrm{Irr}(\mathcal{C})$.

- Eg (Fib category).
  - 2 simple 1. τ. · fusion rule: $\tau \otimes \tau = 1 \oplus \tau$.
  - $$S = \begin{bmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{bmatrix}$$
  - $$\frac{S_{X,Y}}{S_{1,Y}} \to \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$
  - $G_{Fib} = \mathbb{Z}_2 = \langle \sigma \rangle$, $\sigma(\sqrt{5}) = -\sqrt{5}$

Transitivity $\Longrightarrow$ Irreducibility

- Thm (Ng-Wang-Z) If $\mathcal{C}$ is transitive, then the modular rep ρ is irreducible.
  - ρ factors through $SL_2(\mathbb{Z}_n)$
  - Chinese Reminder Thm $\rightsquigarrow$ $SL_2(\mathbb{Z}_{p^k})$
  - Irreducible rep. of $SL_2(\mathbb{Z}_{p^k})$ is known.

    [Nobs. 1976, Nobs Wolfart 1976].

Transitivity $\Longrightarrow$ Fixed-pt-free

Prop: If $\mathcal{C}$ is a transitive modular category, then $|G_{\mathcal{C}}| = |\mathrm{Irr}(\mathcal{C})|$.

- Recall $G_{\mathcal{C}}$ is abelian
- Transitive abelian group action is fixed-pt-free.

- Eg ($\mathrm{Fib} \boxtimes \mathrm{Fib}$)
  - S matrix. $S_{\mathrm{Fib}} \otimes S_{\mathrm{Fib}}$.
  - $\mathbb{Q}(S) = \mathbb{Q}(\sqrt{5})$. $G_{\mathrm{Fib} \boxtimes \mathrm{Fib}} = \mathbb{Z}_2$
  - $|\mathrm{Irr}(\mathrm{Fib} \boxtimes \mathrm{Fib})| = 4$
  - Not transitive.

Deligne Tensor Product

- When $\mathcal{C} \boxtimes \mathcal{D}$ is transitive?

- Corallary. Let $\mathcal{C}$ and $\mathcal{D}$ be modular categories, Then $\mathcal{C} \boxtimes \mathcal{D}$ is transitive iff

  ① Both $\mathcal{C}$ and $\mathcal{D}$ are transitive.

  ② $\mathbb{Q}(\dim(\mathcal{C})) \cap \mathbb{Q}(\dim(\mathcal{D})) = \mathbb{Q}$

Eg: $\mathrm{Fib} \boxtimes PSU(2)_5$

- $PSU(2)_5$ $\quad S = \begin{bmatrix} 1 & d_\alpha & d_\alpha^2 - 1 \\ d_\alpha & -d_\alpha^2 + 1 & 1 \\ d_\alpha^2 - 1 & 1 & d_\alpha \end{bmatrix}$, $\quad d_\alpha = 2\cos(\frac{\pi}{7})$.

- $PSU(2)_5$ is transitive

- $\mathbb{Q}(\sqrt{5}) \cap \mathbb{Q}(d_\alpha) = \mathbb{Q}$. $\qquad$ - $\mathrm{Fib} = PSU(2)_3$.

- $\mathrm{Fib} \boxtimes PSU(2)_5$ is transitive.

Modular Categories $PSU(2)_{p-2}$

- $SU(2)_{p-2}$ from $\mathrm{Rep}(U_q sl_2)$, $q = \exp(\frac{\pi i}{p})$, $p > 3$ prime.
  - Simple obj $\{X_0 = 1, \cdots, X_{p-2}\}$
  - $X_{p-2}^{\otimes 2} = 1$.
  - Not transitive since transitivity $\Rightarrow$ no nontrivial invertible object.
- $PSU(2)_{p-2}$ adjoint subcategory of $SU(2)_{p-2}$.
  - Simple obj $\{X_0, X_2, \cdots, X_{p-3}\}$.
  - $S_{ij} = [(2i+1)(2j+1)]_q$, $[n]_q = \dfrac{q^n - q^{-n}}{q - q^{-1}}$, $q \neq 1$.
  - $PSU(2)_{p-2}$ is transitive.
    - $\dfrac{S_{ij}}{S_{0j}} = [2i+1]_{q^{2j+1}}$, $\forall j$

Since $\gcd(2j+1, 2p) = 1$,

- For $\forall j$, $\exists \sigma$, $\sigma\left(\frac{S_{i,0}}{S_{0,0}}\right) = \frac{S_{i,j}}{S_{0,j}}$

- $PSU(2)_{p-2}$ is prime.

  - A modular category $\mathcal{C}$ is prime if every modular subcategory is $\mathcal{C}$ or Vec.

  - $\langle X_{2k} \rangle \simeq PSU(2)_{p-2}$ for $k \neq 0$.

Thm (Prime Transitive Modular Categories)

Let $\mathcal{C}$ be a nontrivial modular category.

Then $\mathcal{C}$ is prime and transitive iff

$\mathrm{ord}(T) = p$ is a prime > 3 and $\mathcal{C}$ is

equivalent to a Galois conjugate of $PSU(2)_{p-2}$

as modular categories. $\quad q \to q^{\ell}$, $\gcd(\ell, 2p) = 1$.

Factorization of Modular Categories

Thm (Müger) Every modular category is equivalent to a finite Deligne product of prime ones.

Prop (DMNO) If $\mathcal{C}_{pt} \simeq \mathrm{Vec}$, then the prime factorization is unique up to a permutation of factors.

Factorization of Transitive Modular Categories

Thm (Ng - Wang - Z)

Let $\mathcal{C}$ be a transitive modular category. Then

① every fusion subcategory of $\mathcal{C}$ is a transitive modular subcategory, and

② the prime factorization of $\mathcal{C}$ is unique up to a permutation of factors.

Summary

Transitivity

- Irreducible modular rep
- fixed-pt-free
- $\mathcal{C}_{pt} \simeq \mathrm{Vec}$
- Unique factorization.

Classification of Transitive Modular Categories

Thm (Ng - Wang - Z)

Let $\mathcal{C}$ be a nontrivial modular category.

Then $\mathcal{C}$ is transitive iff $\mathcal{C}$ is equivalent to a Deligne product of prime transitive modular categories whose T-matrices have distinct orders.

In particular, $\operatorname{ord}(T_{\mathcal{C}})$ is a square-free odd integer > 3.

Thank you !