Entanglement bootstrap: from gapped many-body ground states to emergent laws

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UC San Diego

10/09/2020

Quantum Symmetry Student Seminar
at the Ohio State University

based on arXiv: 2008.11793

UC San Diego

Outline:

based on:
Shi, Kim 2020 (arxiv: 2008.11793)
Shi, Kato, Kim 2019 (arxiv: 1906.09376)

Motivation and introduction

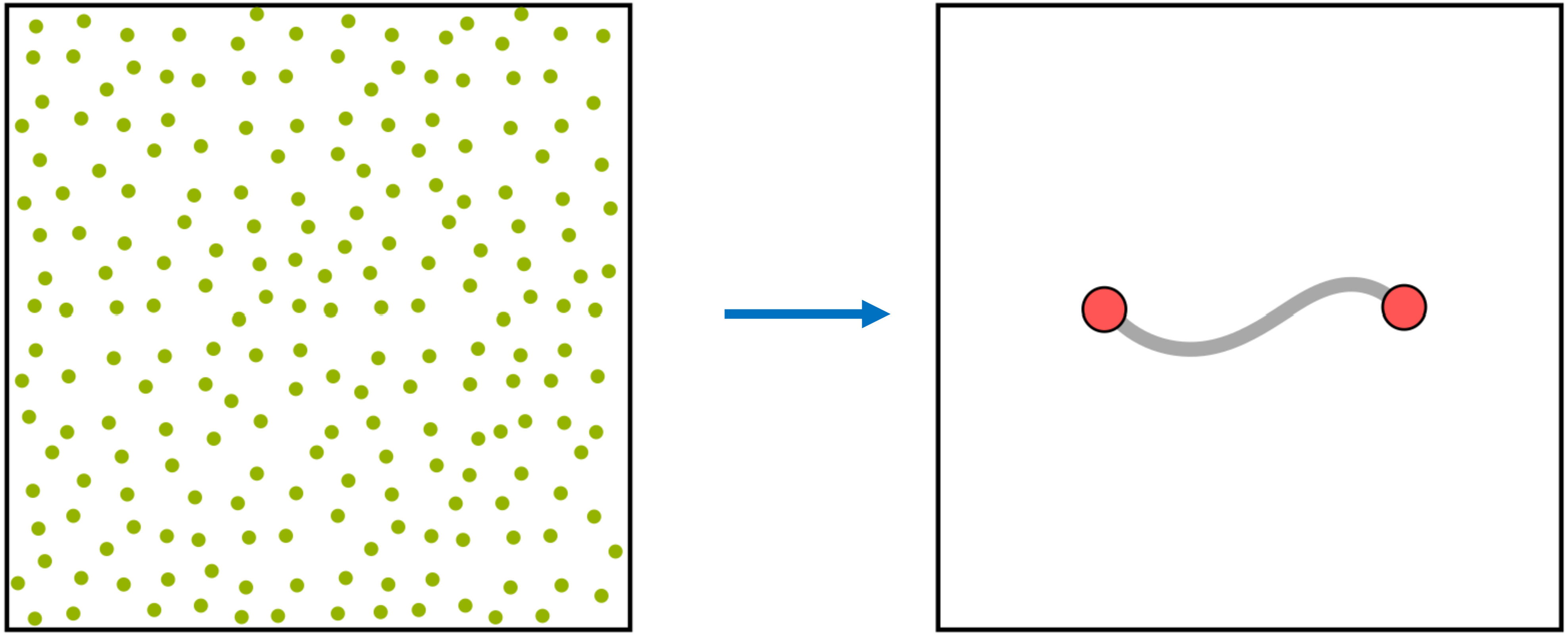
quantum many-body systems and emergent laws (emergent beauty)

Entanglement bootstrap:

1. axioms – ground state entanglement entropy
2. information convex set
3. emergent anyon theory (derive)
   - anyon types (simple objects)
   - fusion multiplicities (finite dimensional Hilbert spaces)
   - braiding, topological entanglement etc (omitted)
4. gapped domain wall – a new superselection sector

Summary and outlook

Interacting quantum many-body systems



microscopic degrees of freedom
(usually complicated)

emergent (effective) theory at large
length / low energies
(beautiful mathematical structure)

Motivation: emergent physical laws

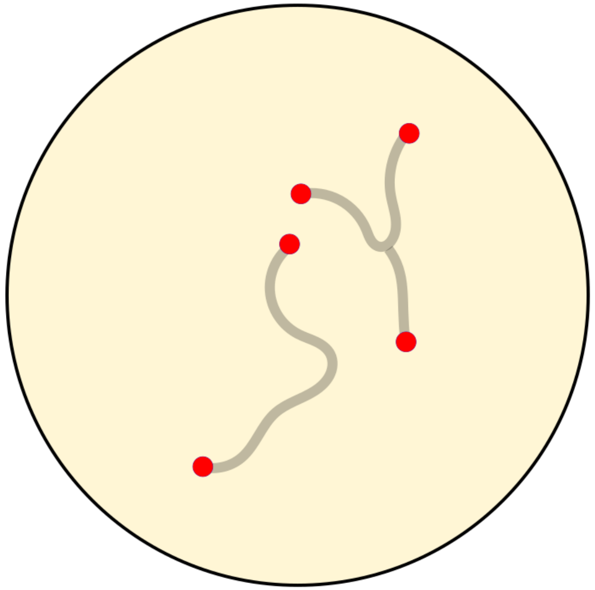
Interacting quantum many-body systems may possess exotic phenomena:

---------- our focus: 2D gapped systems without symmetry

2D topologically ordered system → anyons

symmetry is not essential at the microscopic level

Example: the toric code model
a large Hilbert space:
$\dim H = 2^N$
N is the number of links



emergent law:
4 anyon types: $\{1, e, m, f\}$
with nontrivial fusion/braiding
or equivalently
UMTC: $Z(\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2})$
with 4 simple objects

Motivation: for 2D gapped systems

A well-known conjecture for 2D gapped systems: *Kitaev 2005 Appendix E*

$(UMTC, c_-)$ classifies 2D bosonic gapped phases without symmetries.

UMTC = unitary modular tensor category.

It is a category with the following properties:
monoidal, semi-simple, linear, rigid, spherical, braided, and non-degenerate.
(Did I get these right?)

*Here, UMTC is the unitary modular tensor category and c_- is the chiral central charge

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Brief introduction of UMTC:

Superselection sectors: $\mathcal{C} = \{1, a, b, c, \cdots\}$

vacuum 1, antiparticles are defined.

Fusion rules: $a \times b = \sum_c N_{ab}^c \, c \,,$

a set of conditions are satisfied by N_{ab}^c

The quantum dimension:

$$d_a d_b = \sum_c N_{ab}^c d_c, \text{ and } \mathcal{D} = \sqrt{\sum_a d_a^2}.$$

Modular matrices: S-matrix (mutual braiding) and T-matrix (the topological spins.)

F-symbols and R-symbols

Modularity: The additional requirement that the S-matrix is unitary.

*Here, UMTC is the unitary modular tensor category and c_- is the chiral central charge

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Example: Fibonacci anyon model:

from *Rowell, Stong, Wang 2007*

(here $\varphi = \frac{1+\sqrt{5}}{2}$)

Anyon types: $\{1, \tau\}$
Fusion rules: $\tau^2 = 1 + \tau$
Quantum dimensions: $\{1, \varphi\}$

Twists: $\theta_1 = 1, \theta_\tau = e^{\frac{4\pi i}{5}}$
Total quantum order: $D = 2\cos(\frac{\pi}{10}) = \frac{\sqrt{5}}{2\sin(\frac{\pi}{5})}$
Topological central charge: $c = \frac{14}{5}$
Braidings: $R_1^{\tau\tau} = e^{-\frac{4\pi i}{5}}, R_\tau^{\tau\tau} = e^{\frac{3\pi i}{5}}$
S-matrix: $S = \frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix}$,
F-matrices: $F_\tau^{\tau,\tau,\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}$

Motivation: for 2D gapped systems

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When and why do we expect the anyon theory to emerge?

Can we derive these laws from somewhere?

(even nicer if we can discover a new physics)

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Entanglement bootstrap **is an attempt to answer these questions:**

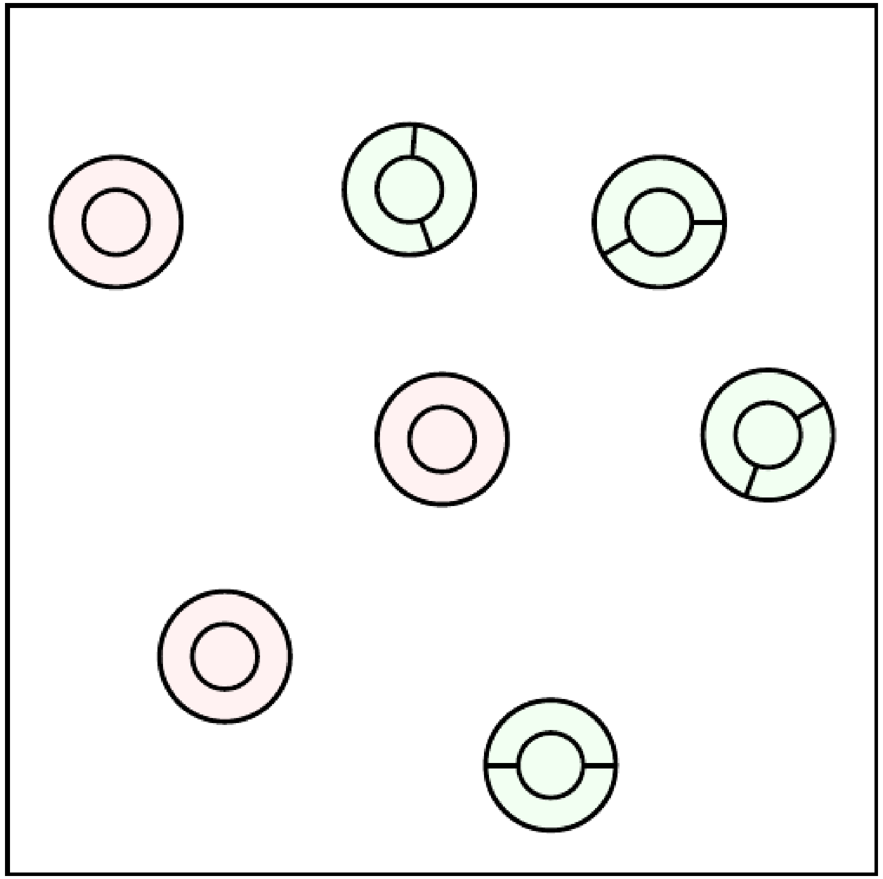
Want: (1) physically motivated assumptions

(2) cover a large class of phases

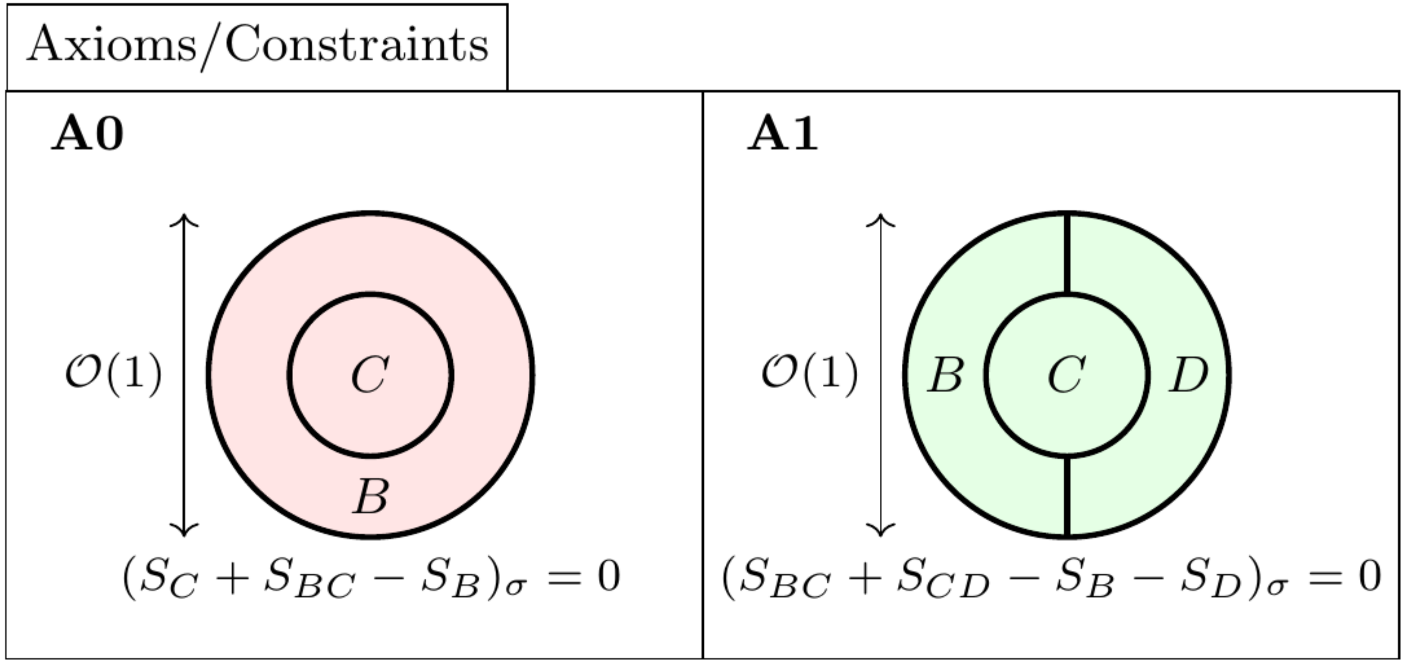
(3) nontrivial predictions

(4) would be nicer if it is good math

Hint (hindsight): start from an entanglement area law



bound-sized disks
reference state $\sigma = |\psi\rangle\langle\psi|$



axioms contemplated by Isaac Kim 2014, and 2015 (slides)

*Here, S_A is the entanglement entropy.
*We assume that the total Hilbert space is a tensor product of finite dimensional ones – suitable for bosonic.

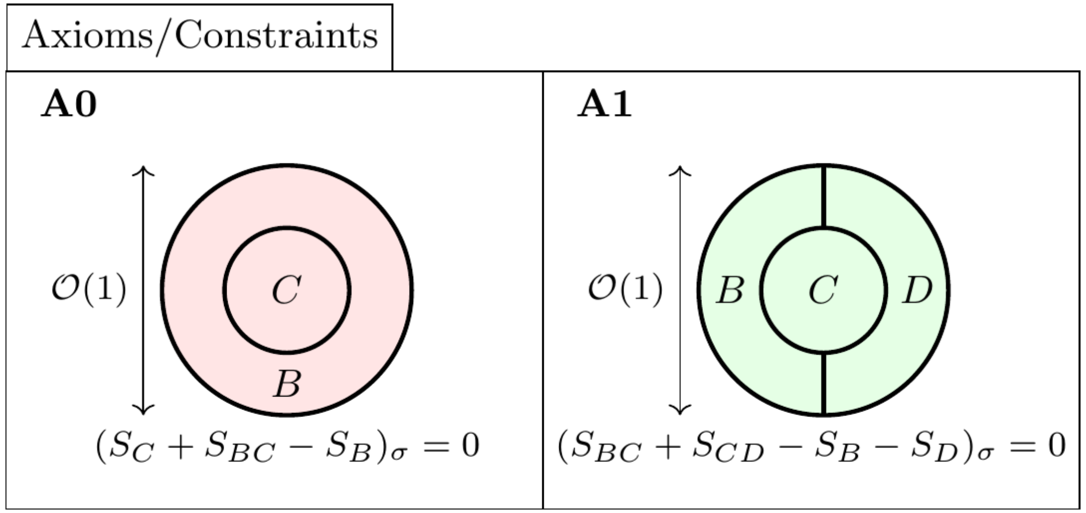
Remark: area law for gapped phases

$$S(A) = \alpha\ell - \gamma$$

Kitaev, Preskill 2005
Levin, Wen 2005

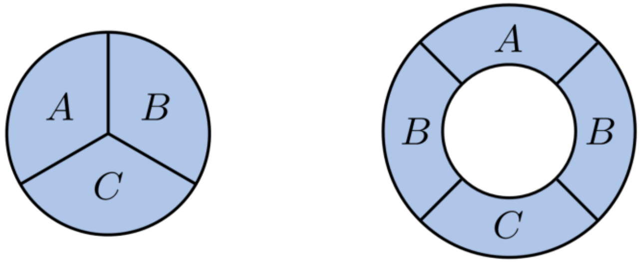
a famous (conjectured) form of area law

imply



the input ground state $|\psi\rangle$ determines the value of γ

imply



linear combination of entropy
(total quantum dimension)

$$\gamma = \ln \mathcal{D}$$

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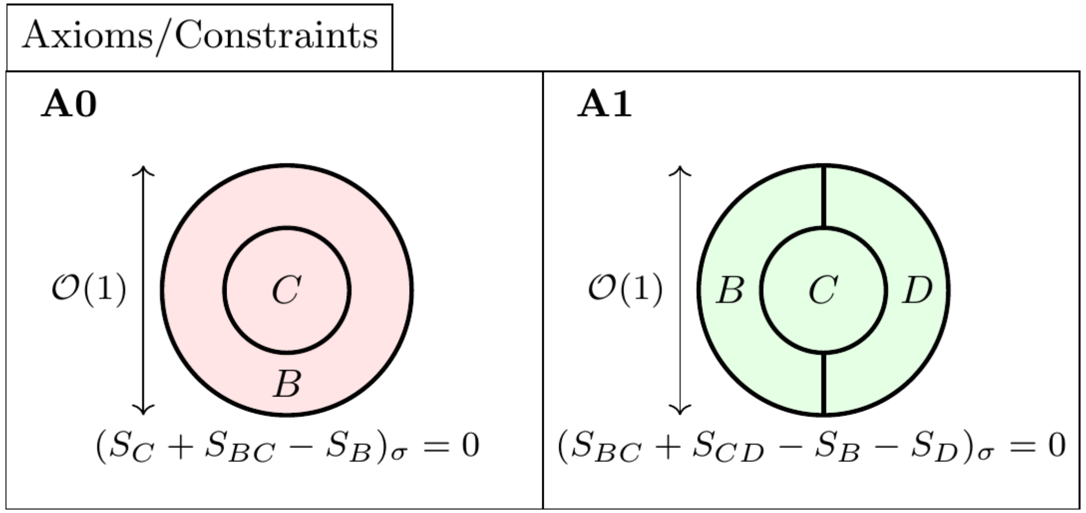


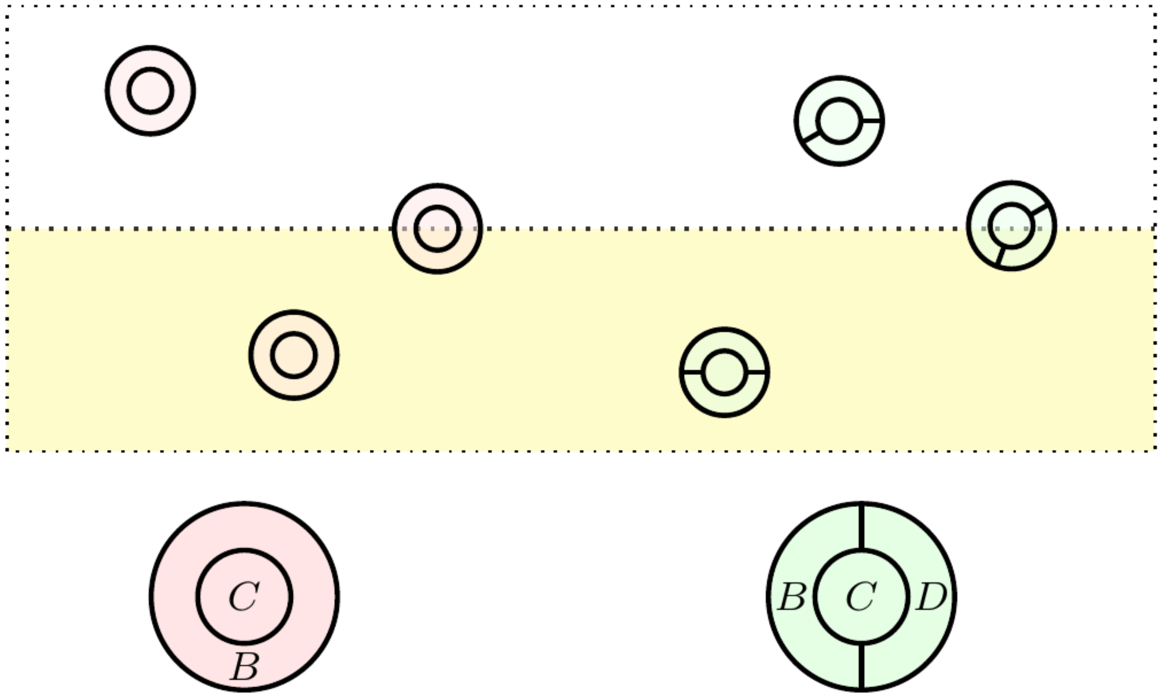
the input ground state $|\psi\rangle$ determines the value of γ

imply?

- some of the axioms of UMTC?
- the full set of axioms of UMTC?
- uncover unknowns?

Can easily generalize!

gapped domain wall version of the axioms



$(S_C + S_{BC} - S_B)_\sigma = 0$ $\quad$ $(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$

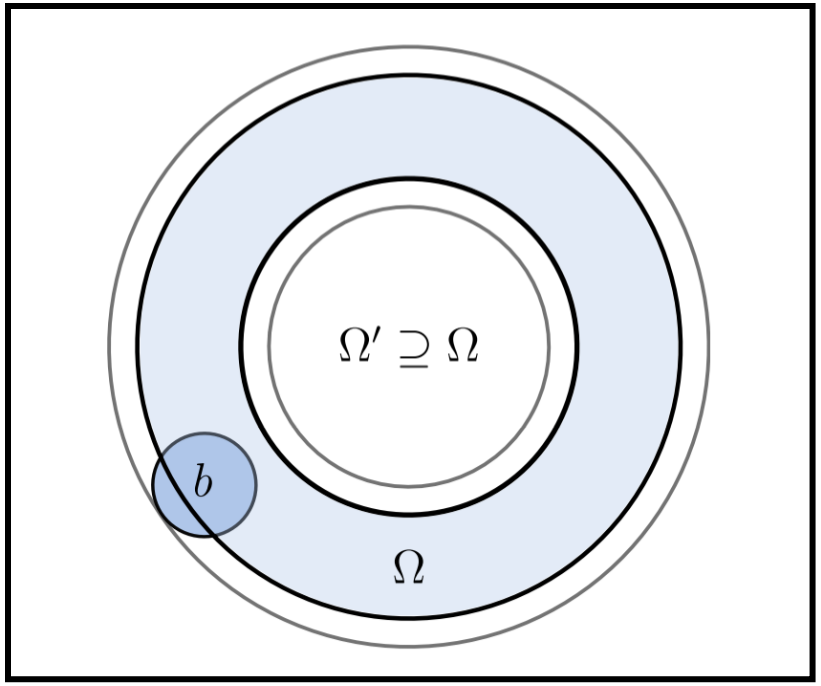
we shall argue: the framework can discover unknowns

Information convex set

Consider *a set of density matrices,* which we call as the **information convex set**:

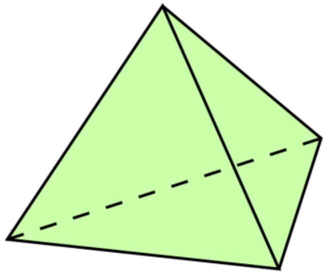
$$\Sigma(\Omega) \equiv \{\rho_\Omega | \rho_\Omega = \mathrm{Tr}_{\Omega' \backslash \Omega}\, \rho_{\Omega'}, \rho_{\Omega'} \in \tilde{\Sigma}(\Omega')\},$$

here, Ω' is Ω plus an extra layer. Elements in $\tilde{\Sigma}(\Omega')$ are locally indistinguishable from the ground state.



Ω can have any topology, annulus is an example

Fact: The information convex set is a convex set.

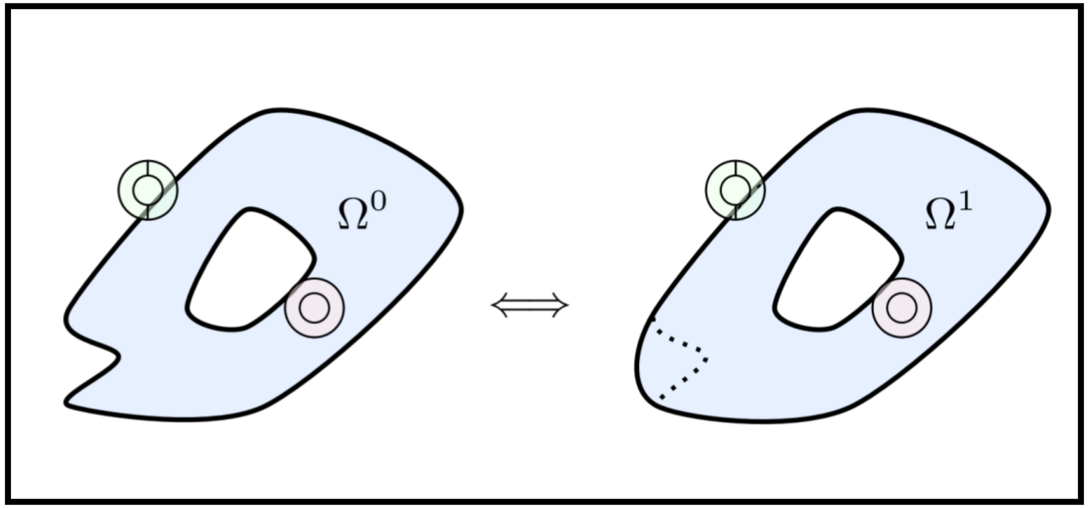
$\Sigma(\Omega) =$ 

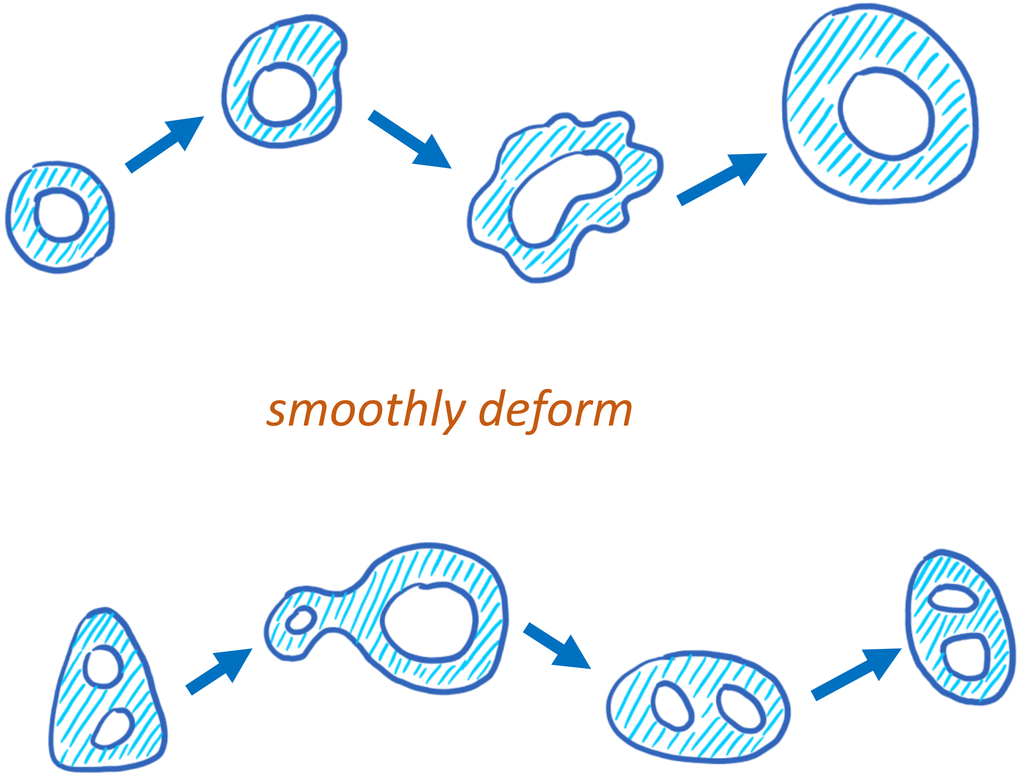
a convex set, the structure will be studied

How does entanglement bootstrap work?

The isomorphism theorem:

$$\Sigma(\Omega^0) \cong \Sigma(\Omega^1)$$





*preserves the entropy difference and fidelity.

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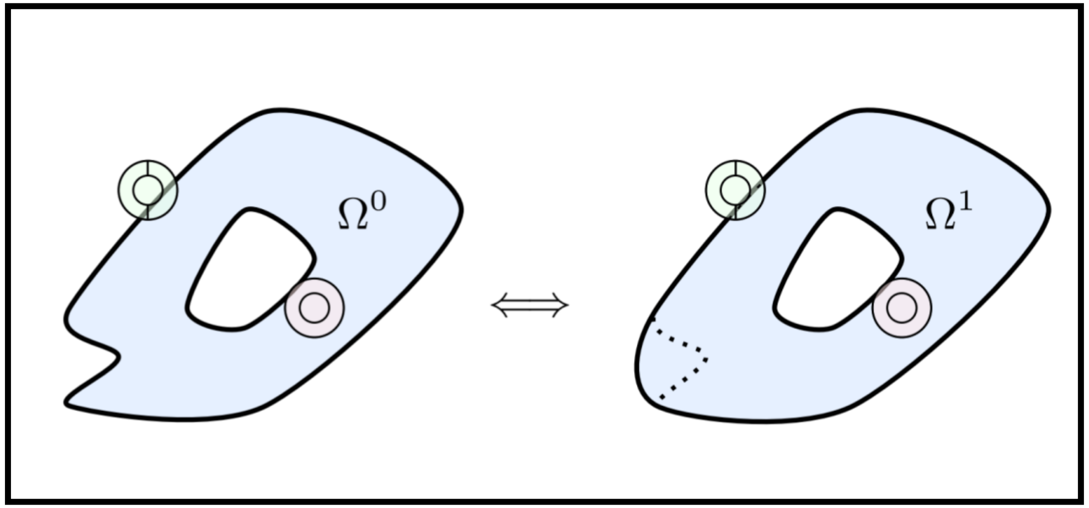

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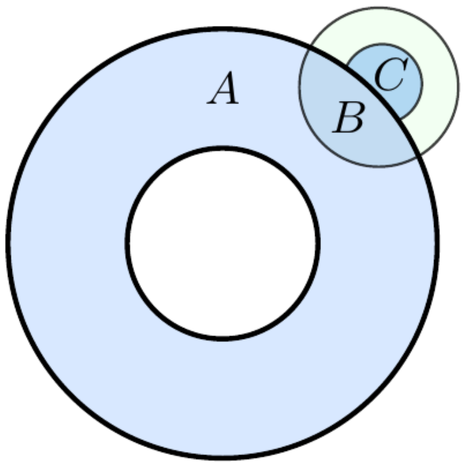
Proof sketch:



Apply axiom **A1** to the green disk containing BC.

SSA

$$I(A:C|B) = 0$$

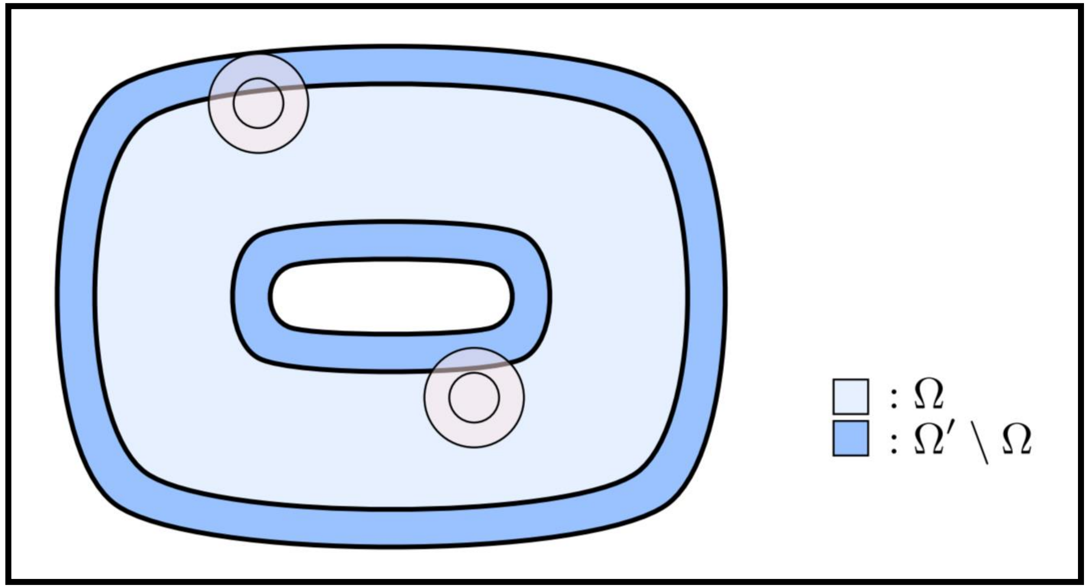
smooth deformation: $AB \to ABC$

$$I(A:C|B) \equiv S_{AB} + S_{BC} - S_B - S_{ABC}$$

SSA=the strong subadditivity: $I(A:C|B) \geq 0$

How does entanglement bootstrap work?

Factorization property:



Proof sketch:
We first apply **A1** to extend **A0** to larger regions.
Then we use the extended version of **A0**.

For any extreme point $\rho^{\langle e \rangle}_{\Omega'} \in \Sigma(\Omega')$,

$$\left(S_{\Omega} + S_{\Omega'} - S_{\Omega' \setminus \Omega}\right)_{\rho^{\langle e \rangle}} = 0.$$

put strong constraint on the correlation of elements in the information convex sets

*preserves the entropy difference and fidelity.

How does entanglement bootstrap work? – main results

Bulk:

1. **Superselection sectors (bulk)** – defined
2. **Fusion multiplicities** – well-defined

Domain wall:

1. **A new superselection sector**

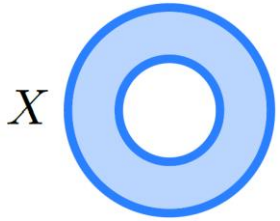
Not covered in the talk:

1. **Axioms about the fusion rules** – antiparticles and identities
2. **Topological entanglement entropy**
3. **unitary string operators**
4. **nontrivial mutual braiding statistics**

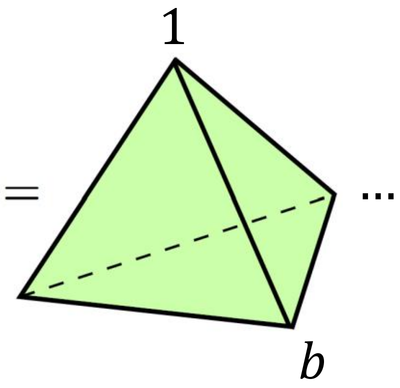
The main results – with proof sketch

Superselection sectors (anyon types) are well-defined:

The simplex theorem: For an annulus X,



$\Sigma(X) =$



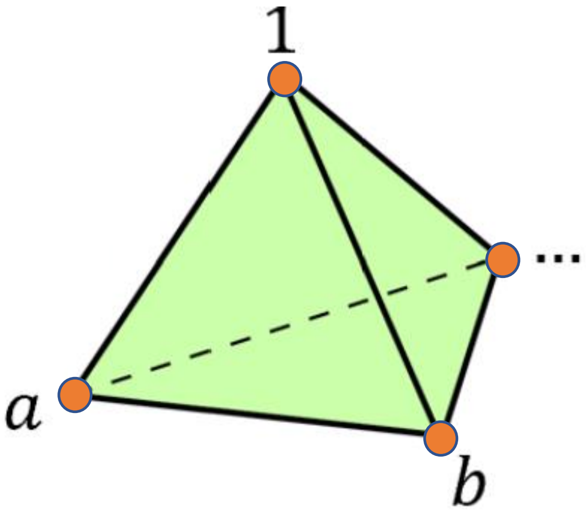
$$\Sigma(X) = \{\rho_X | \rho_X = \sum_a p_a \sigma_X^a\}$$

$$\sigma_X^a \perp \sigma_X^b \text{ for } a \neq b.$$

There is a unique vacuum 1.

Anyon types: $\mathcal{C} = \{1, a, b, \cdots\}.$

Extreme points:



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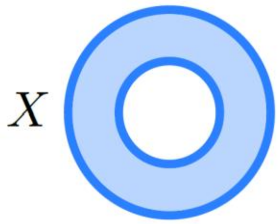

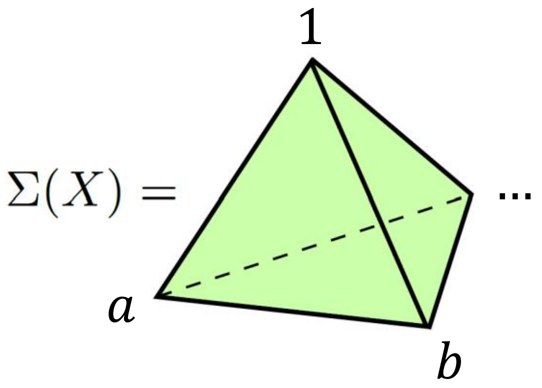

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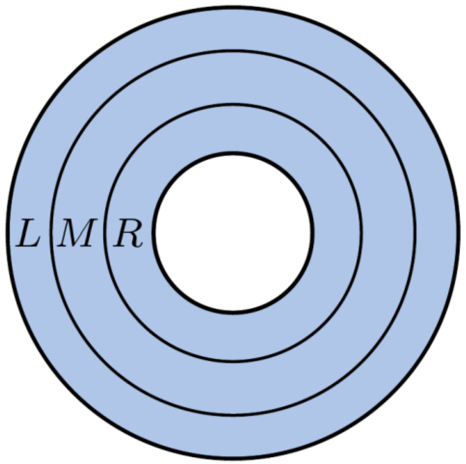
$X = LMR$



$$\Sigma(L) \cong \Sigma(X) \cong \Sigma(R)$$

the fidelity $F(\rho, \sigma)$ is preserved

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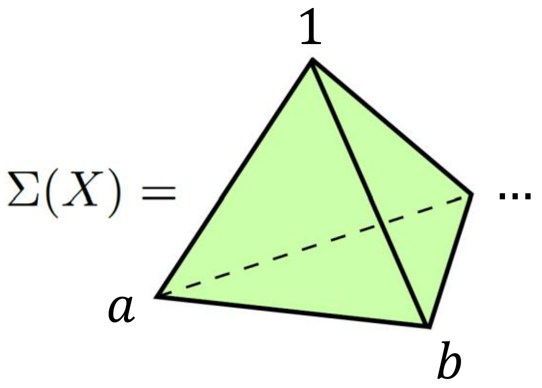

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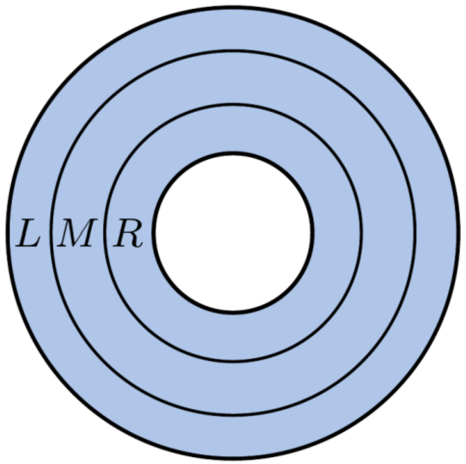

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$F = F_L = F_R = F_{LMR}$ (isomorphism)

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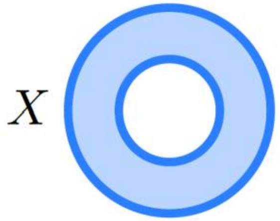

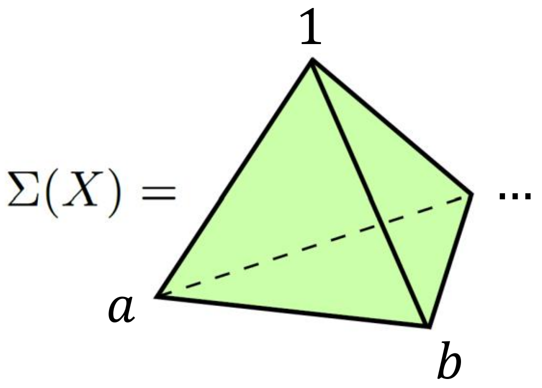

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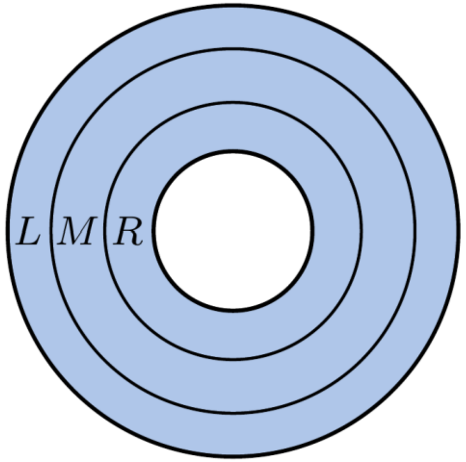

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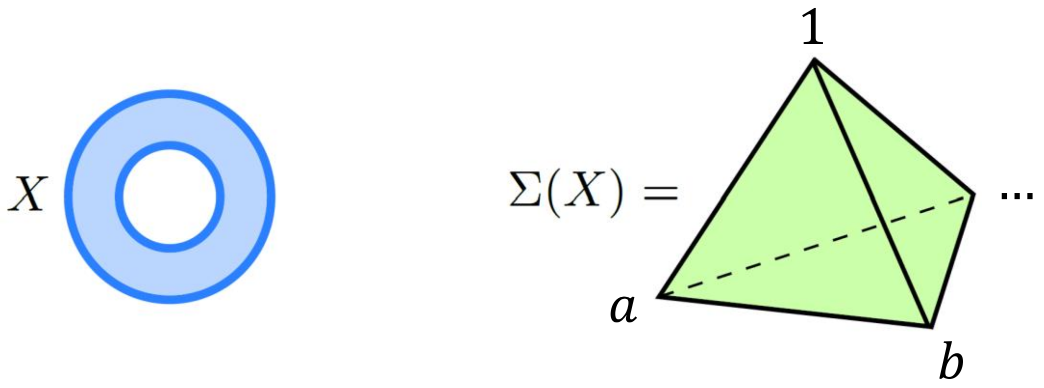

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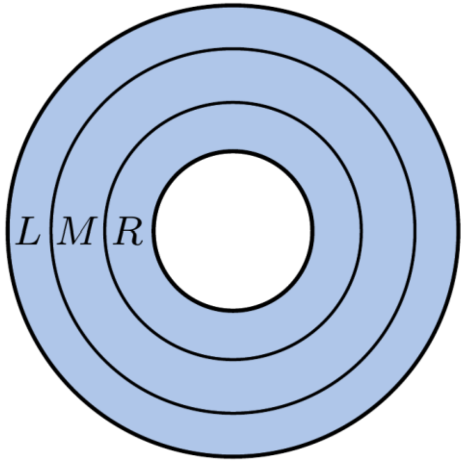

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$F_{LMR} \leq F_{LR}.$ (monotonicity of fidelity)

$F_{LR} = F_L F_R.$ (for the extreme points)

The main results – with proof sketch

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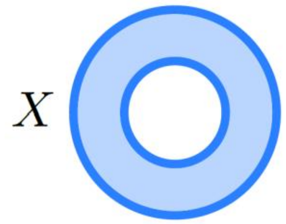

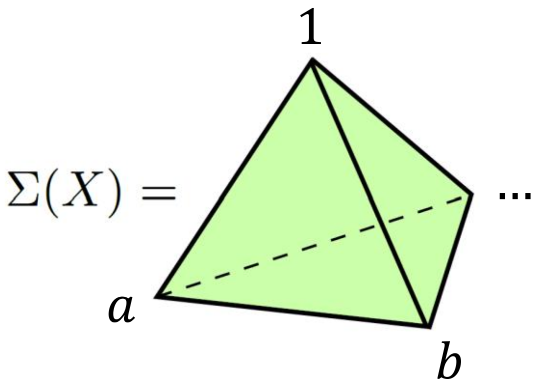

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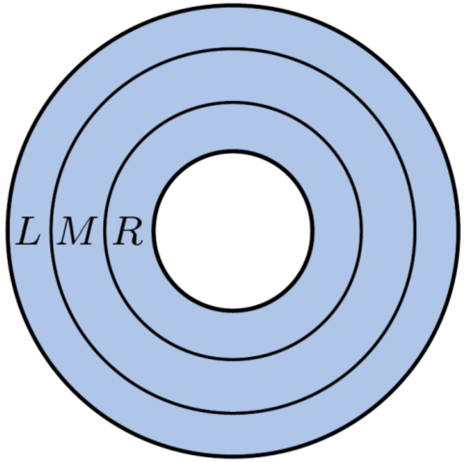

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but $F \in [0, 1]$

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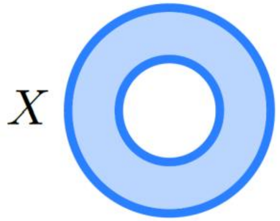

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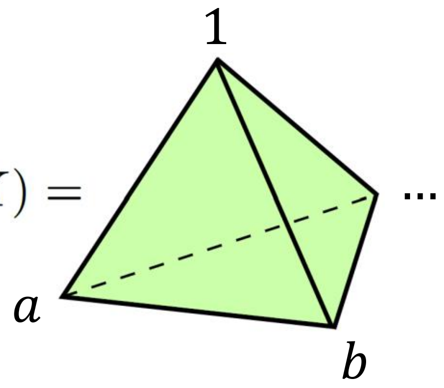

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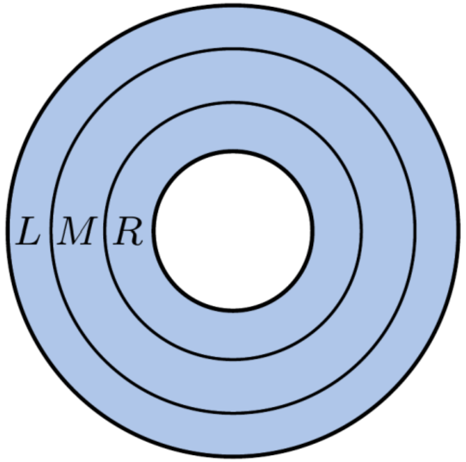

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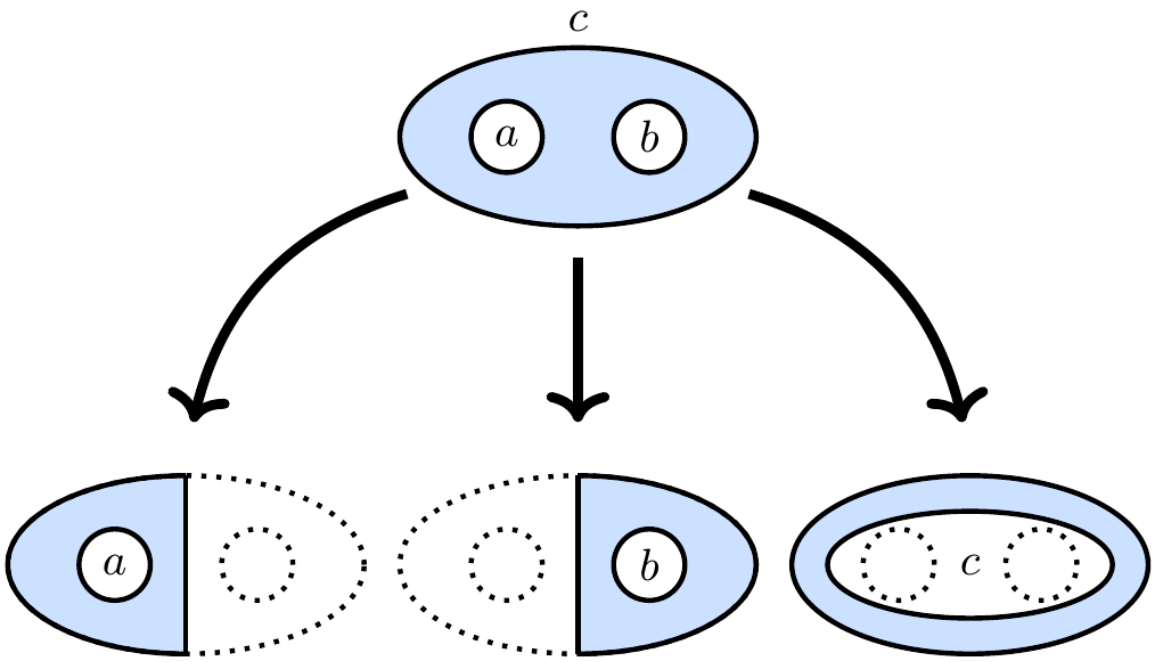
The main results – with proof sketch

Fusion multiplicities are will defined:

Theorem: (Hilbert space theorem)
2-hole disk Y, specifying sectors (a, b, c)



$$\Sigma_{ab}^{c}(Y) \subseteq \Sigma(Y)$$

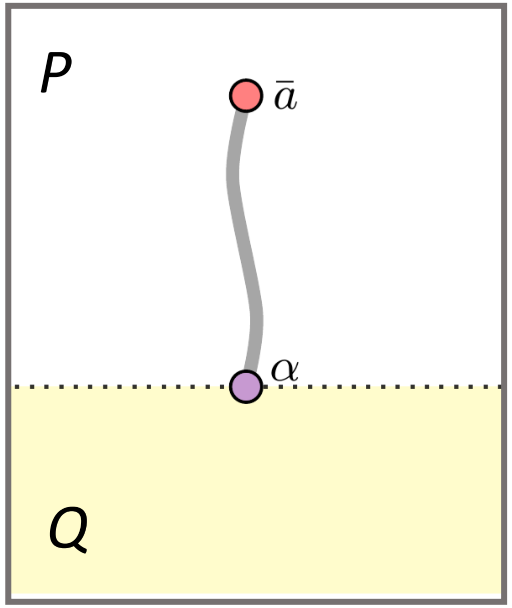
$\Sigma_{ab}^{c}(Y) \cong$  $a \times b \to c$

N_{ab}^{c} dimensional

$$a \times b = \sum_{c} N_{ab}^{c}\, c.$$

axiom **A0** is responsible for the quantum coherence

Gapped domain walls between 2D topologically ordered systems



anyons (bulk)
wall excitations (wall)

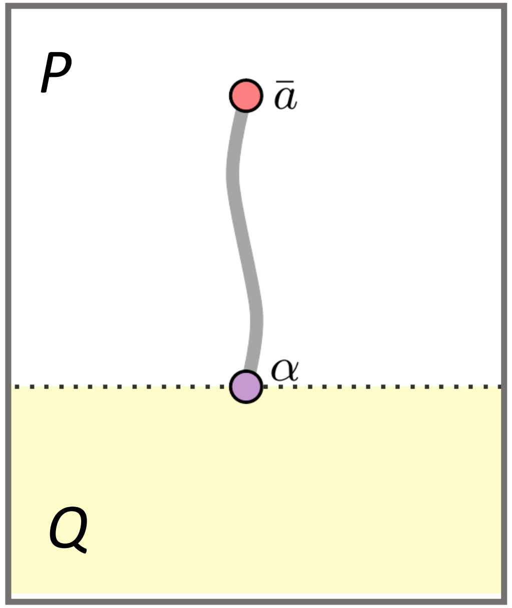
these excitations are distinct
Kitaev, Kong 2011

At the physical level:

- anyon can fuse
- anyons can braid
- anyons can fuse onto the wall
- wall excitations can fuse
- etc

What is the general mathematical theory behind these physical phenomena?

Gapped domain walls between 2D topologically ordered systems



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Proposals of the underlying math theory:

Kong 2013 arxiv: 1307.8244
also: Fuchs, Schweigert, Valentino 2012

$$\mathcal{C} \xrightarrow{L} \mathcal{E} \xleftarrow{R} \mathcal{D}$$

$$\mathcal{C} \boxtimes \overline{\mathcal{D}} \xrightarrow[\simeq]{G} Z(\mathcal{E}), \quad L \boxtimes R : \mathcal{C} \boxtimes \overline{\mathcal{D}} \to \mathcal{E}, \quad \text{forget} : Z(\mathcal{E}) \to \mathcal{E}$$

$\mathcal{C}$ is the UMTC for P

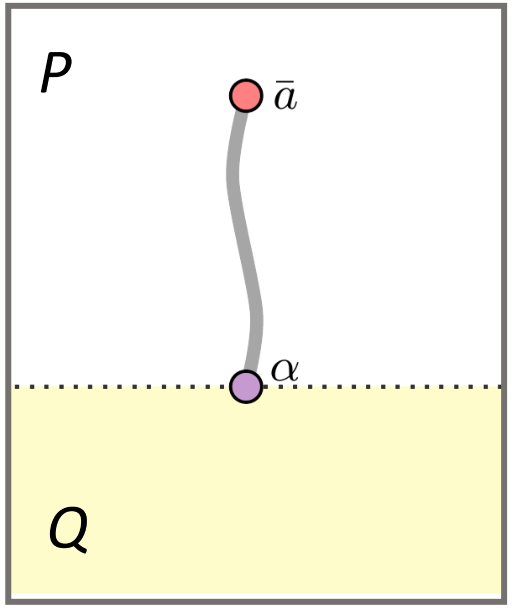
$\mathcal{D}$ is the UMTC for Q

$\mathcal{E}$ is a module category for the domain wall

for one-step condensation: (*Kong 2013*)

$$\mathcal{E} \simeq \mathcal{C}_A \ \ as\ (unitary)\ spherical\ fusion\ categories$$

Gapped domain walls between 2D topologically ordered systems



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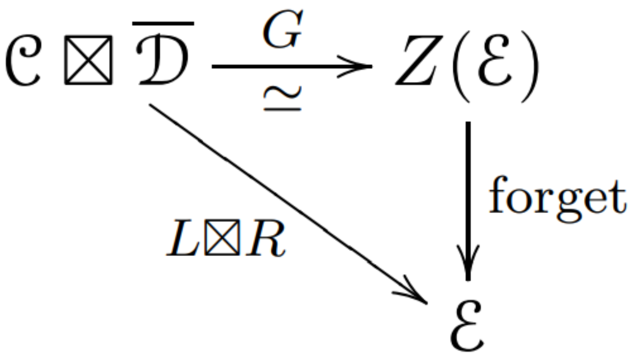
also: Fuchs, Schweigert, Valentino 2012

$$\mathcal{C} \xrightarrow{L} \mathcal{E} \xleftarrow{R} \mathcal{D}$$



$\mathcal{C}$ is the UMTC for P

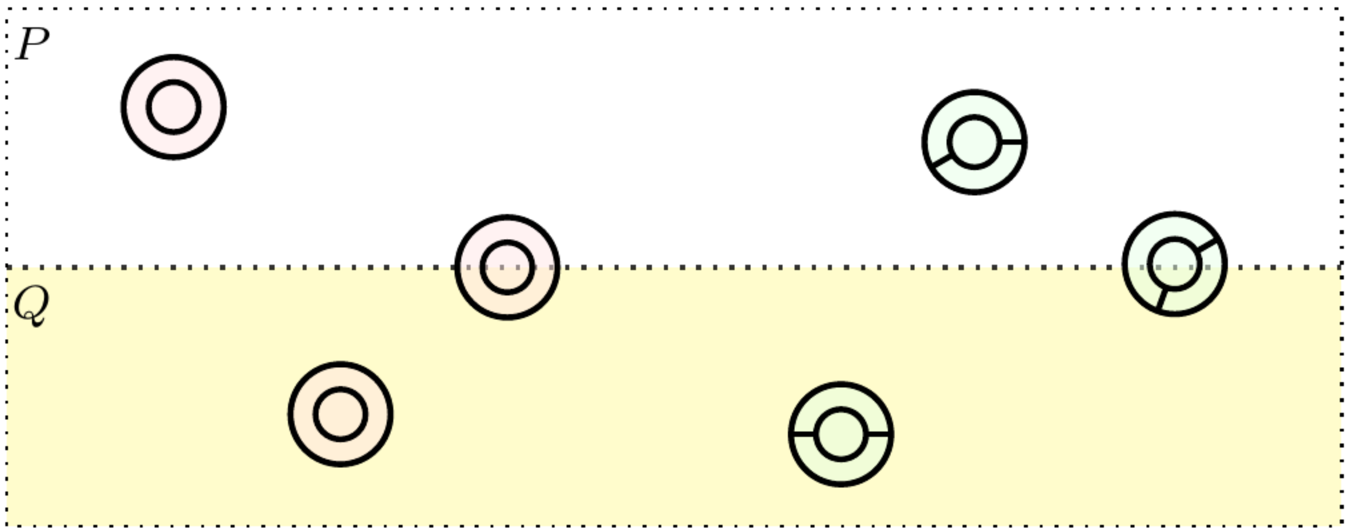
$\mathcal{D}$ is the UMTC for Q

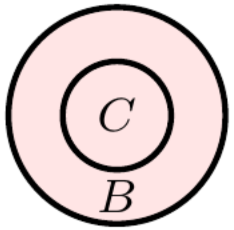
$\mathcal{E}$ is a module category for the domain wall

With entanglement bootstrap:
We identify things that appear to no be characterized by these.

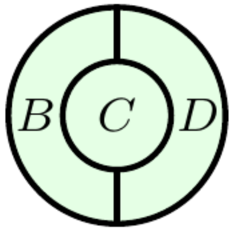
Entanglement bootstrap – domain wall

Axioms recap:



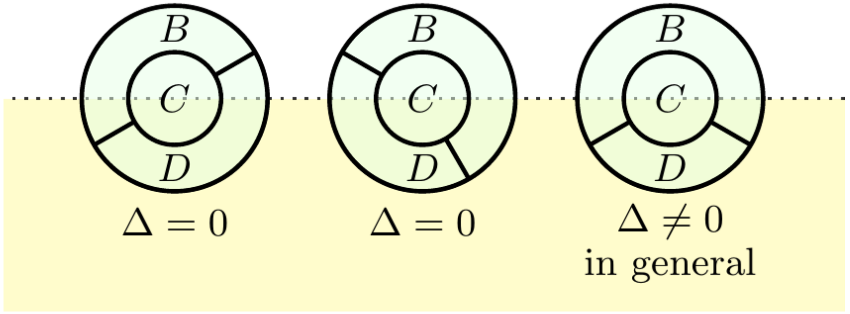


$$(S_C + S_{BC} - S_B)_\sigma = 0$$



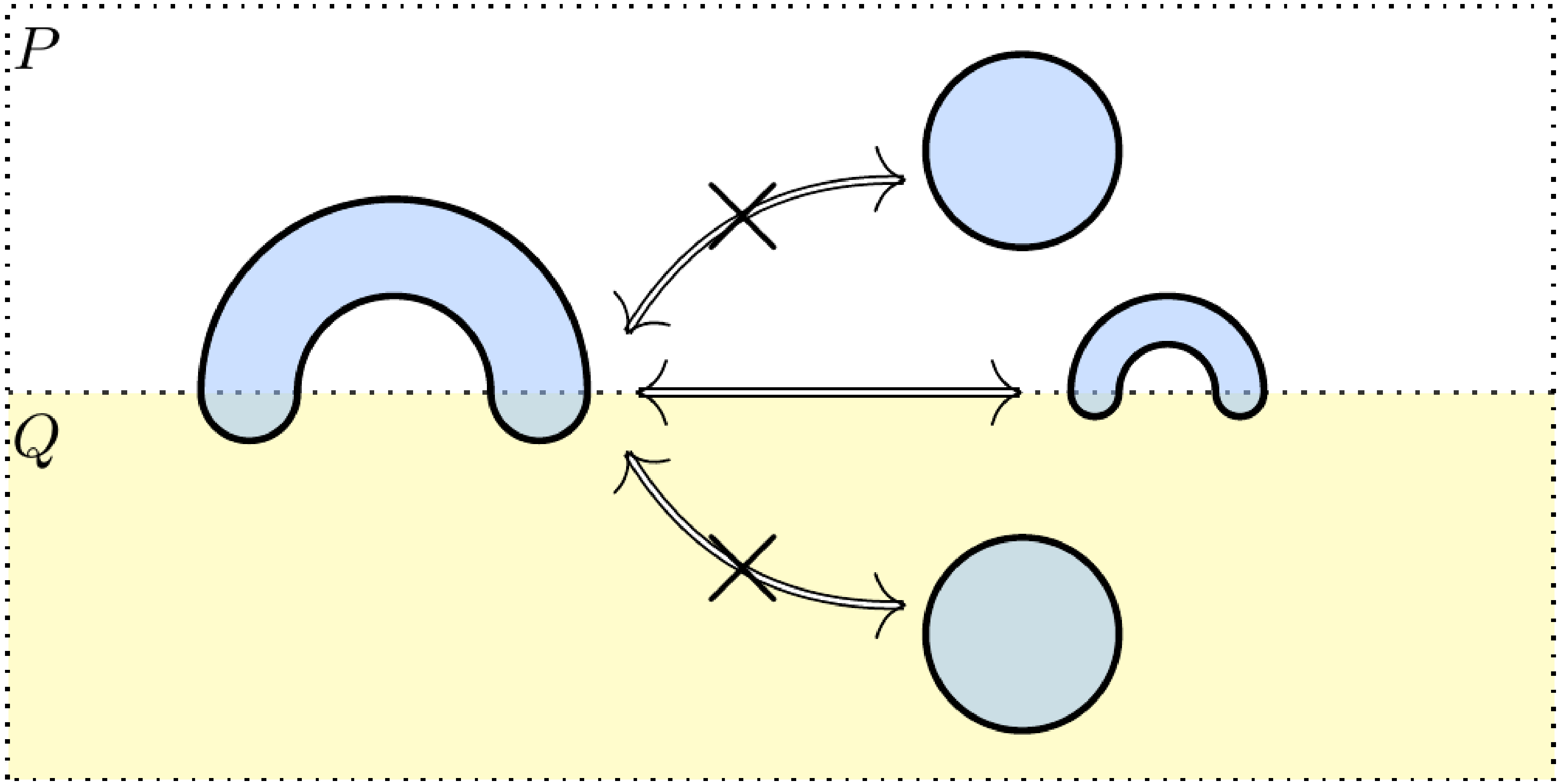
$$(S_{BC} + S_{CD} - S_B - S_D)_\sigma = 0$$

$$\Delta := (S_{BC} + S_{CD} - S_B - S_D)_\sigma$$

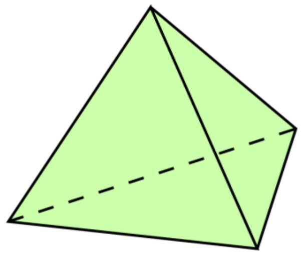


The main results – domain wall

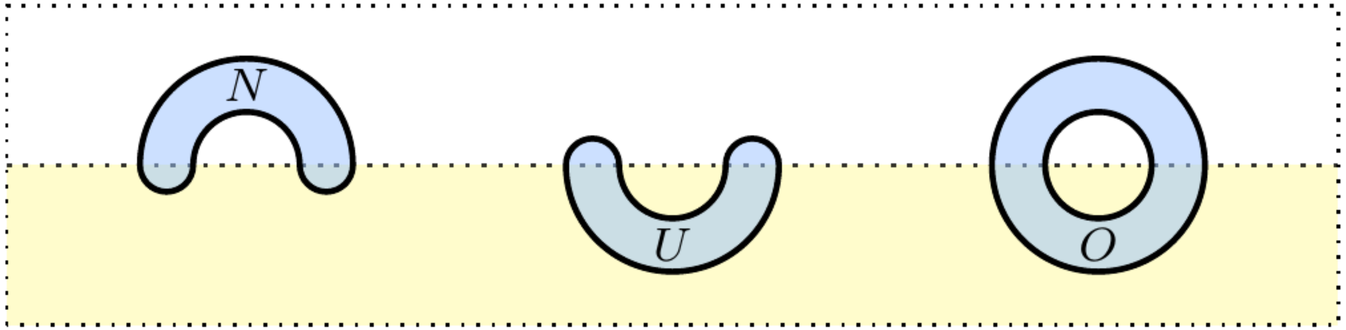
Isomorphism theorem for the gapped domain wall:



The main results – domain wall



Superselection sectors on the domain wall:

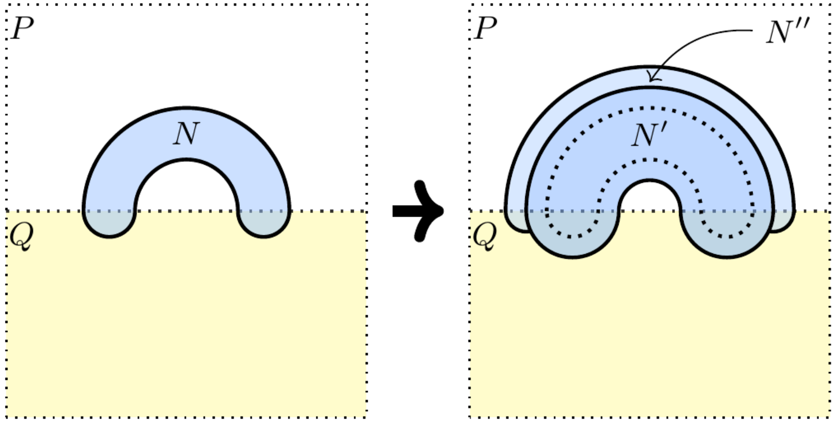


$$\mathcal{C}_N = \{1, n, \cdots\} \quad \mathcal{C}_U = \{1, u, \cdots\} \quad \mathcal{C}_O = \{1, \alpha, \beta, \cdots\}$$

- the 1st and 2nd sectors are new to our knowledge
- these can in principle be measured

- the 3rd set is the known superselection sectors on gapped domain walls *Kitaev, Kong 2011*

Idea of the proof:
(simplex theorem generalized)

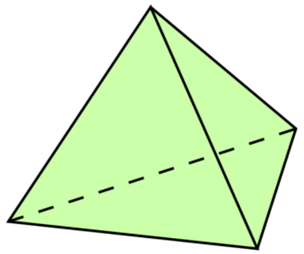


for a pair of extreme points of $\Sigma(N)$:

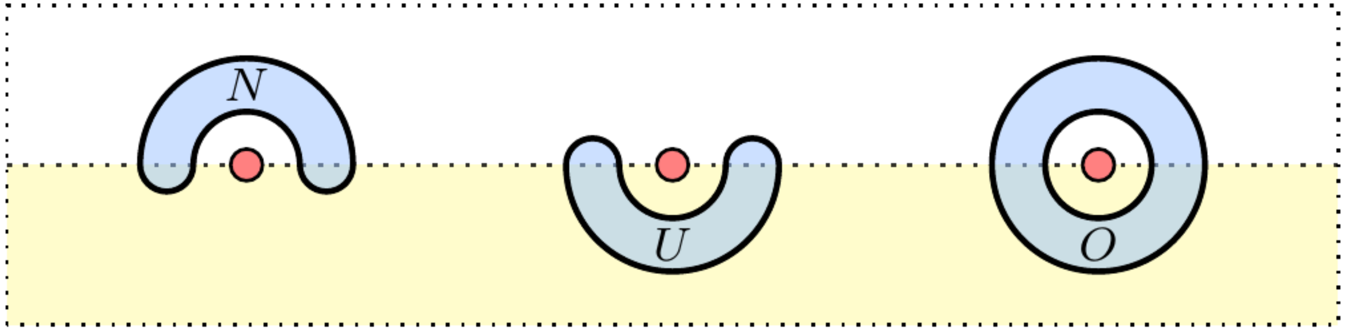
$$F \leq F^2 \text{ but } F \in [0,1]$$

$$\longrightarrow F \in \{0, 1\}$$

The main results – domain wall



Interpretation: parton sectors and composite sectors:



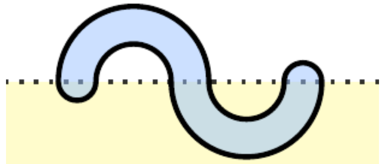
$$\mathcal{C}_N = \{1, n, \cdots\} \quad \mathcal{C}_U = \{1, u, \cdots\} \quad \mathcal{C}_O = \{1, \alpha, \beta, \cdots\}$$

- the 1st and 2nd sets are not local excitations
- they are "parton sectors" that composite other domain wall sectors

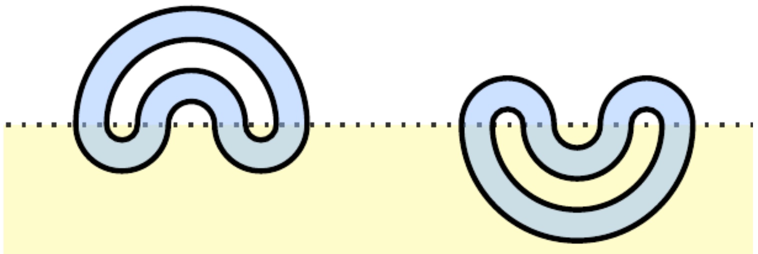
Theorem:

$$\mathcal{C}_O = \bigcup_{\substack{n \in \mathcal{C}_N \\ u \in \mathcal{C}_U}} \mathcal{C}_O^{[n,u]}$$

Other examples of composite sectors:



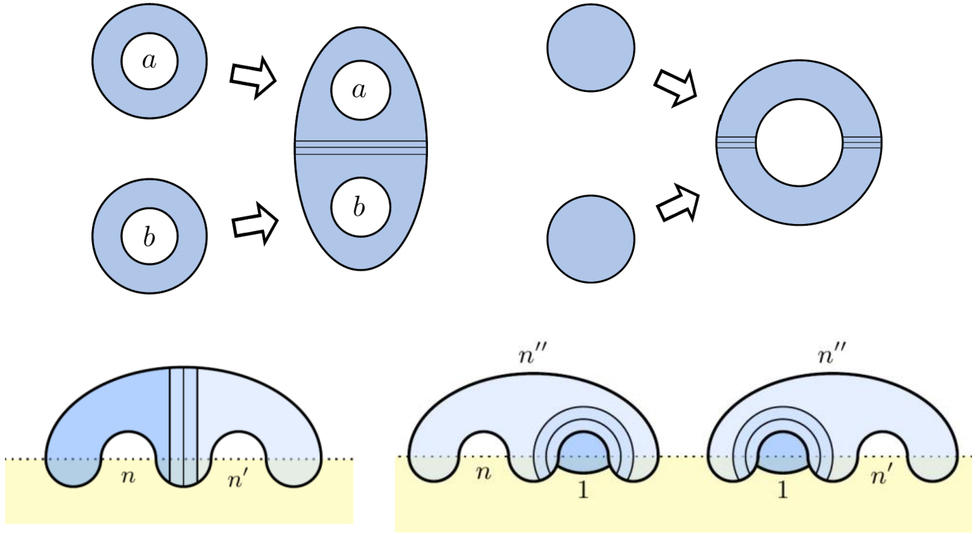
$$\mathcal{C}_S = \{1, s, \cdots\}$$



$$\mathcal{C}_{\mathbb{N}} = \{1, \mathcal{N}, \cdots\} \quad \mathcal{C}_{\mathbb{U}} = \{1, \mathcal{U}, \cdots\}$$

Things did not cover

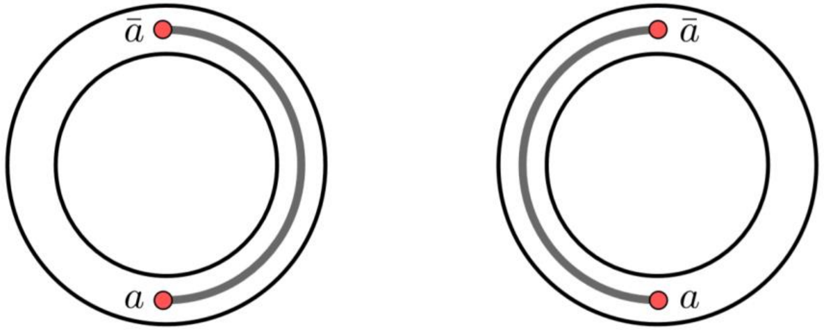
Merging density matrices:



can change the topology, derive:

- define anti-sectors
- quantum dimension
- fusion rules and consistency
- topological entanglement entropy

String operators and nontrivial mutual braiding:



$$S_{ab} \equiv \frac{d_a d_b}{\mathcal{D}} f_{ab},$$

$$f_{ab} \equiv \mathrm{Tr}(U_L^{a\dagger} U_R^a \sigma_X^b).$$

Shi 2019
arxiv: 1911.01470

The Verlinde formula can be derived, which implies that the S-matrix is unitary.

(expected to go further, e.g., F, R-symbols)

Summary:

Entanglement provides a mechanism for the emergence of physical laws, in gapped systems!

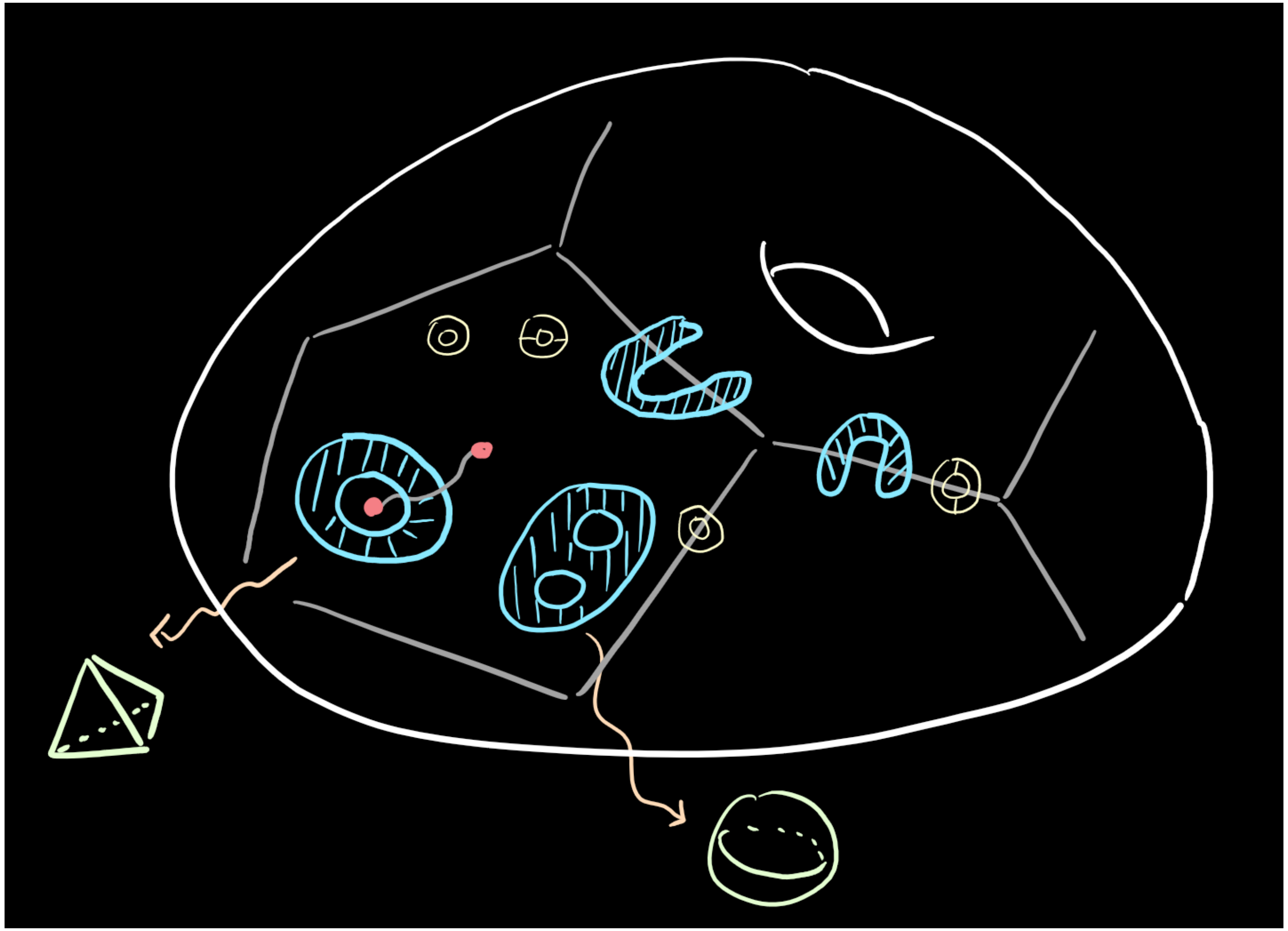
Input: A quantum state that satisfies axioms **A0** and **A1**.

Output-1: The anyon theory emerges above the length scale involved in the axioms.

Output-2: Anyon data can be extracted "locally" from the many-body ground state.

Domain wall: Appears to uncover a new physical object.

How to describe? Is category theory continue to be the right framework?

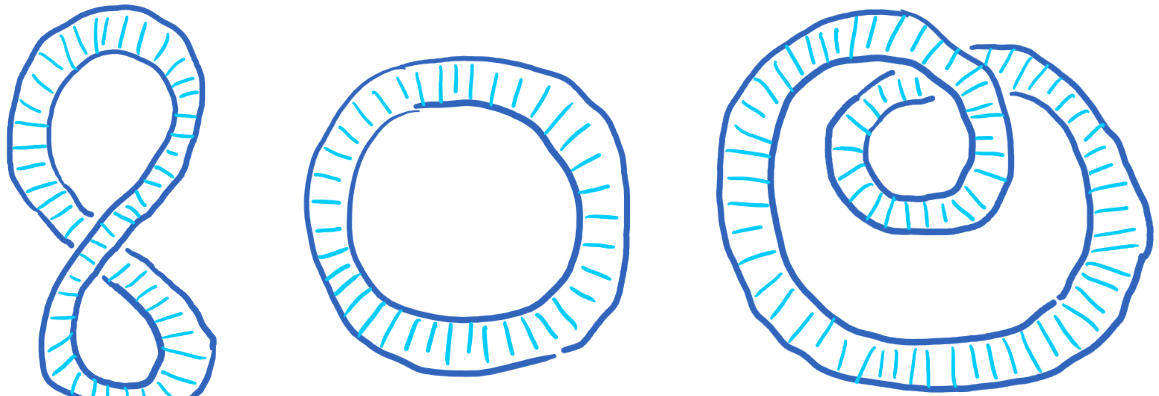


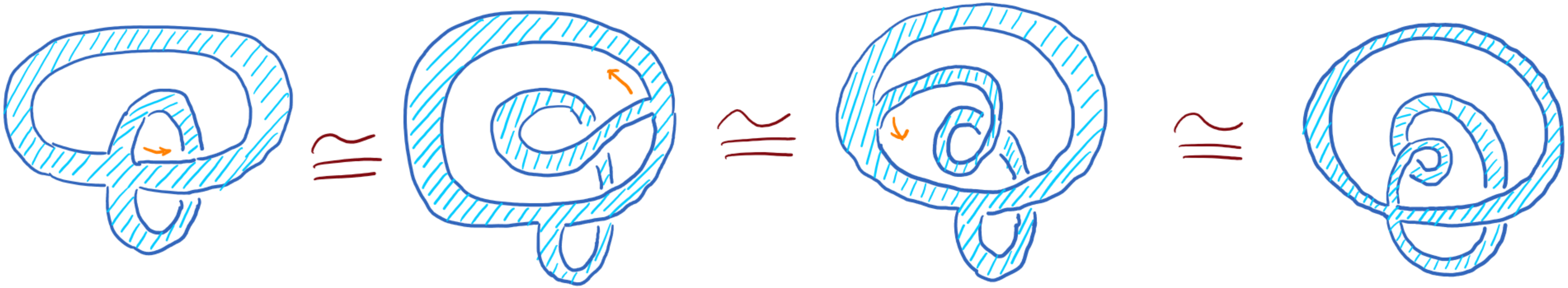
Thank you!
Questions please.

below are backup slides

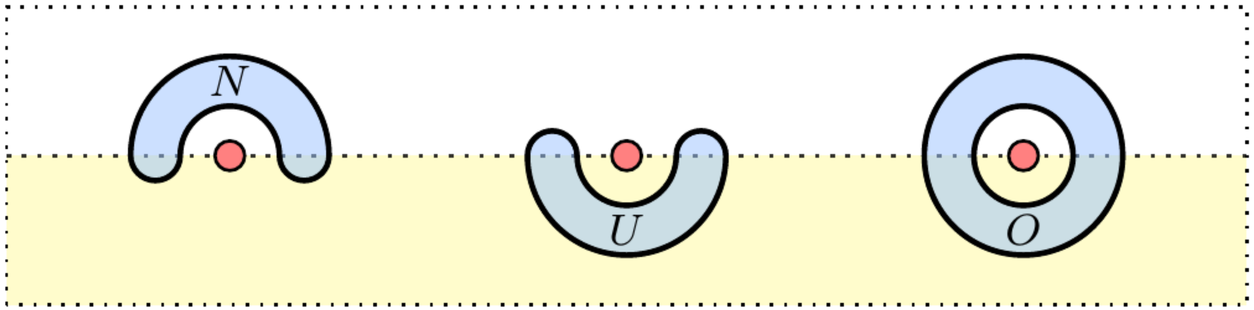
A question: what branch of math is related to these?

Regions that are not subsystems:





Explicit data of domain wall:



A gapped domain wall between two non-Abelian (S_3) quantum double models:

$a \in \mathcal{C}_P$ (or $x \in \mathcal{C}_Q$)	1	A	J^w	J^x	J^y	J^z	K^a	K^b
N_a^1 (or N_x^1)	1	0	1	0	0	0	0	0
d_a (or d_x)	1	1	2	2	2	2	3	3

bulk data

Set	Labels	Quantum dimensions
$\mathcal{C}_N$	$\{1, n\}$	$\{1, \sqrt{2}\}$
$\mathcal{C}_U$	$\{1, u\}$	$\{1, \sqrt{2}\}$
$\mathcal{C}_O^{[1,1]}$	$\{\cdots\}$	$\{1, 1, 1, 1\}$
$\mathcal{C}_O^{[n,1]}$	$\{\cdots\}$	$\{2, 2\}$
$\mathcal{C}_O^{[1,u]}$	$\{\cdots\}$	$\{2, 2\}$
$\mathcal{C}_O^{[n,u]}$	$\{\cdots\}$	$\{2, 2, 2, 2\}$

domain wall data

Remark: area law for gapped phases

Reduced density matrix:

$$\rho_A = \mathrm{Tr}_B |\psi_{AB}\rangle\langle\psi_{AB}|.$$

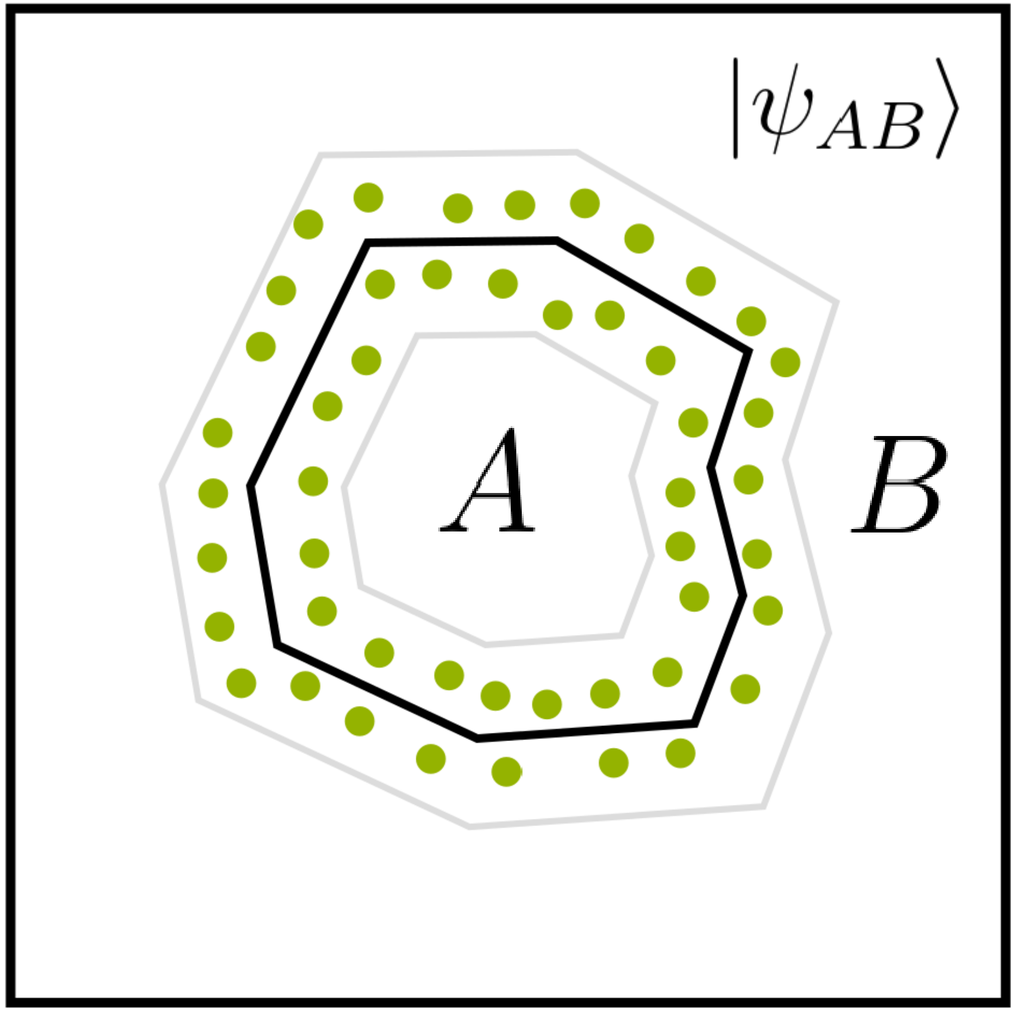
von Neumann entropy:

$$S(\rho_A) = -\mathrm{Tr}_A(\rho_A \ln \rho_A)$$

entanglement area law:

$$S(\rho_A) = \alpha \ell + \cdots$$

*area means the boundary length of A.

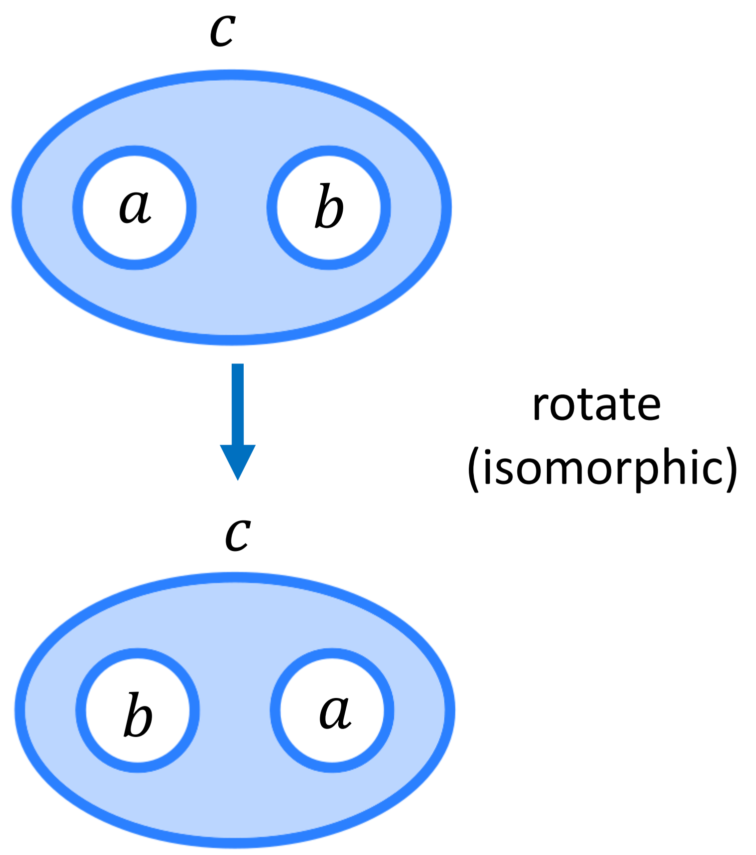


The main results – with proof sketch

3. Extracting fusion multiplicities
4. Axioms about the fusion rules

We derive the following rules for fusion multiplicities:

1. $N_{ab}^{c} = N_{ba}^{c}$.
2. $N_{1a}^{c} = \delta_{a,c}$.
3. $N_{ab}^{1} = \delta_{b,\bar{a}}$.
4. $N_{ab}^{c} = N_{\bar{b}\bar{a}}^{\bar{c}}$.
5. $\sum_i N_{ai}^{d} N_{bc}^{i} = \sum_j N_{ab}^{j} N_{jc}^{d}$.



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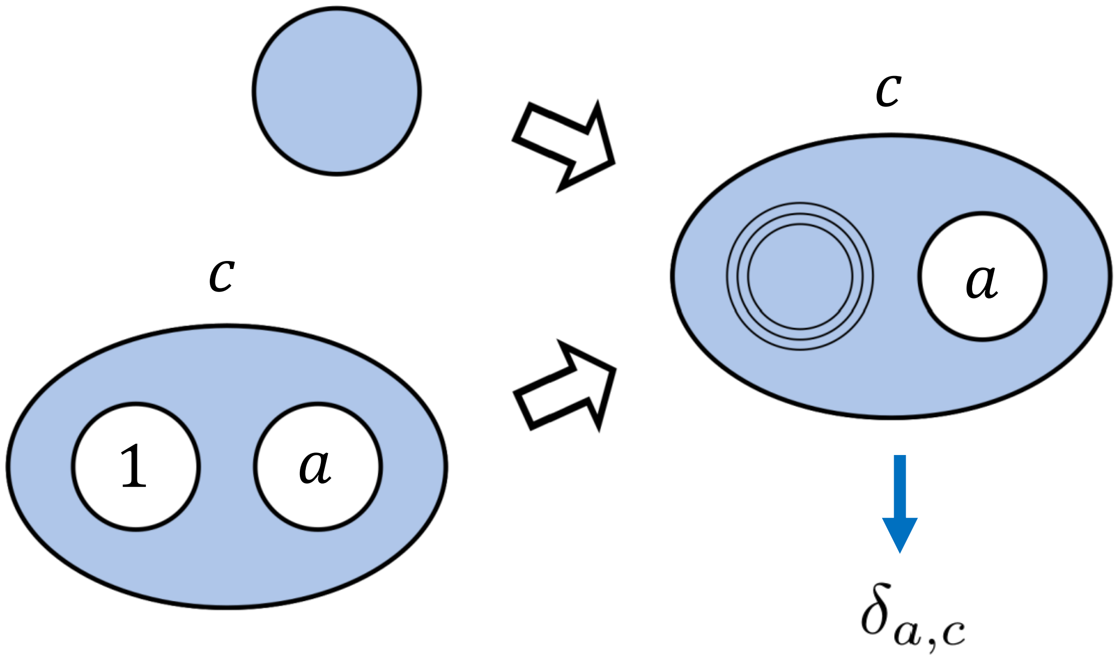
Patching a hole:



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A new technique is needed!

↓

Calculating the entropy in different ways.

↓

consistency conditions

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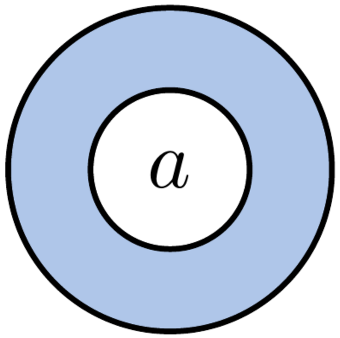
Prepare for the proof: a topological piece of entropy



Define $f(a) \equiv \dfrac{S(\sigma_X^a) - S(\sigma_X^1)}{2}$ for annulus X.

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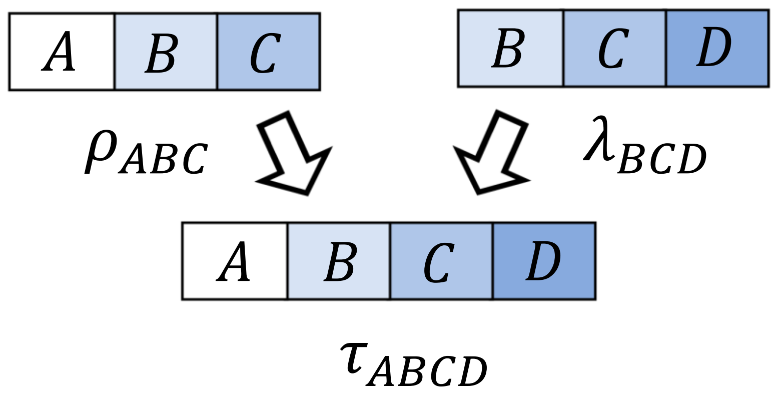
5. $\sum_i N_{ai}^{d} N_{bc}^{i} = \sum_j N_{ab}^{j} N_{jc}^{d}$.

The merging lemma: *Kato, Furrer, Murao 2015*

merge a pair of quantum Markov states:



*Details:
1. τ is a quantum Markov state
2. preserves entropy difference

The main results – with proof sketch

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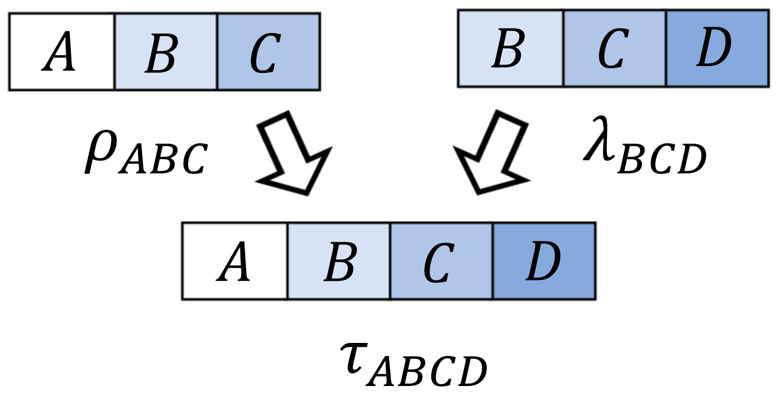

Merging theorem: *Shi, Kato, Kim 2019*

$\Sigma(ABC)$, $\Sigma(BCD)$ $\Rightarrow$ $\Sigma(ABCD)$

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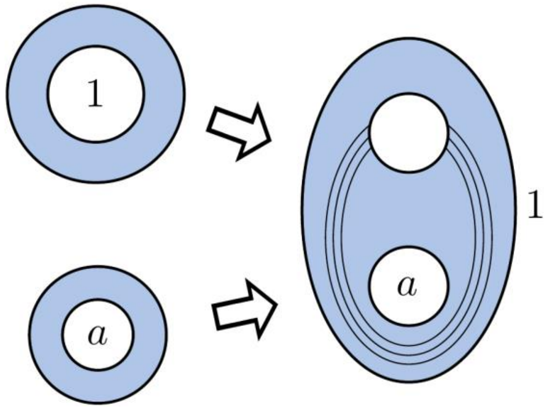
Computing the entropy difference in two ways:

the entropy difference between $(1, a)$ and (1,1)



$$2f(a) = f(a) + \ln\Big(\sum_b N_{ab}^{1} e^{f(b)}\Big)$$

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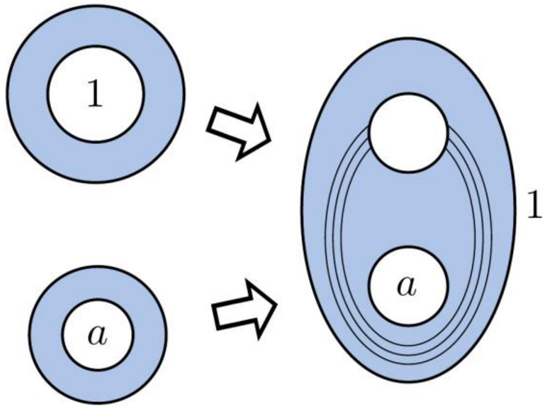

$$e^{f(a)} = \sum_b N_{ab}^{1} e^{f(b)}$$

The main results – with proof sketch

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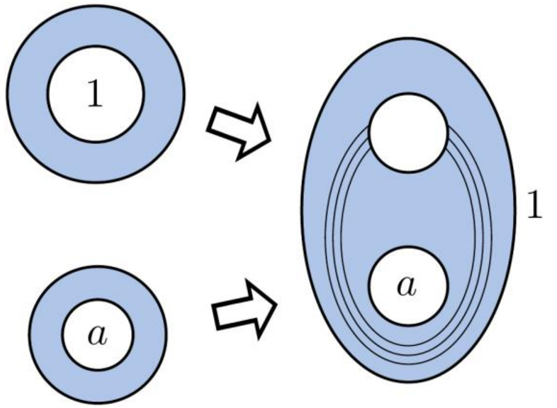

$$e^{f(a)} = \sum_b N_{ab}^1 e^{f(b)}$$

For each $a \in \mathcal{C}$, there is a unique b, s.t.:

$N_{ab}^1 \geq 1, \quad e^{f(a)} \geq e^{f(b)} \qquad e^{f(a)} \leq e^{f(b)}$

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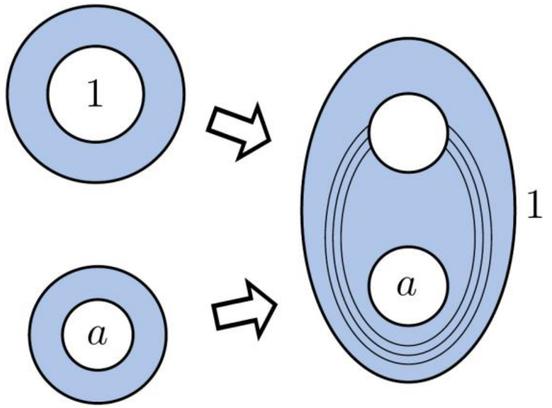


$$e^{f(a)} = \sum_b N_{ab}^{1} e^{f(b)}$$

For each $a \in \mathcal{C}$, there is a unique $\bar{a}$, s.t.:

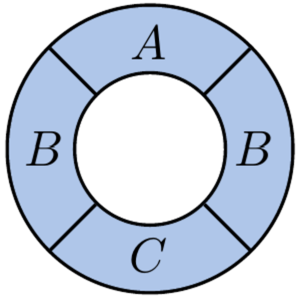
$$f(a) = f(\bar{a}), \text{ and } N_{ab}^{1} = \delta_{b,\bar{a}}.$$

The main results – with proof sketch

4. Axioms about the fusion rules
5. The topological entanglement entropy

The value of TEE: $\gamma = \ln \mathcal{D}$

To be concrete, we consider the Levin-Wen partition:



$$I(A:C|B) = 2\ln \mathcal{D}$$

The main results – with proof sketch

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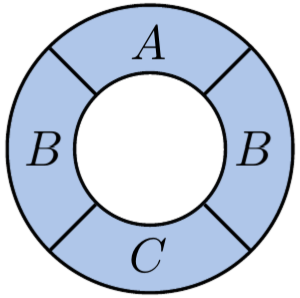
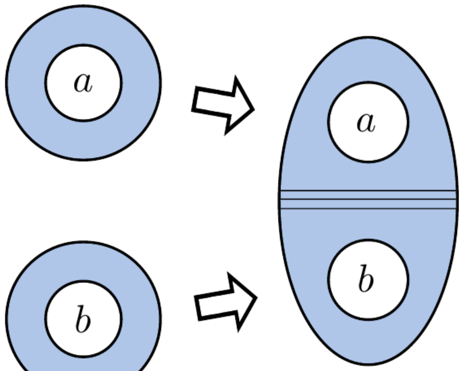


$$I(A:C|B) = 2\ln \mathcal{D}$$

Observation 1: $f(a) = \ln d_a.$



$$e^{f(a)}e^{f(b)} = \sum_c N_{ab}^c e^{f(c)}.$$

uniquely determine

The main results – with proof sketch

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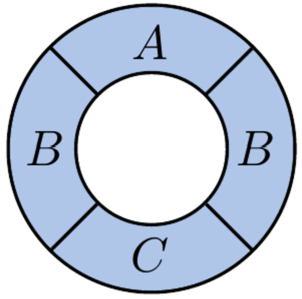
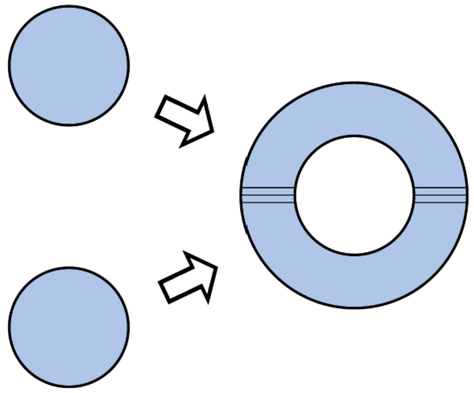


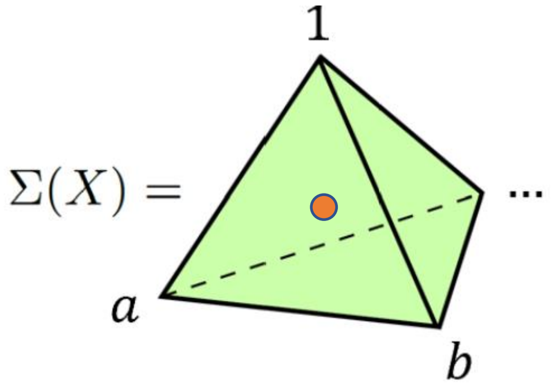
$$I(A:C|B) = 2\ln \mathcal{D}$$

Observation 1: $f(a) = \ln d_a.$

Observation 2: The "center" of $\Sigma(X)$, $X = ABC$, has

$$I(A:C|B)_{\sigma^*} = 0.$$





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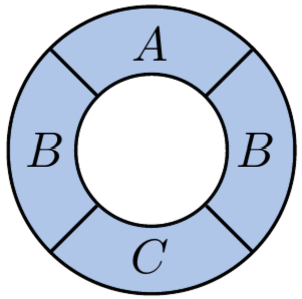

$$I(A:C|B) = 2\ln \mathcal{D}$$

Observation 1: $f(a) = \ln d_a.$

Observation 2: The "center" of $\Sigma(X)$, $X = ABC$, has

$$I(A:C|B)_{\sigma^*} = 0.$$

Observation 3: TEE is the entropy difference of the "center" and the "corner" of $\Sigma(X)$

$$2\ln \mathcal{D} = S(\sigma_X^*) - S(\sigma_X^1).$$

$\Sigma(X) =$