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Bundles & Characteristic Classes

Summer 2021

Contents

Lectures

Lecture 1

1.1 Conventions & Tentative Plan

Convention.

(a) Denote CAT your favorite choice between the category of topological spaces Top and smooth manifolds with corners DIFF. This means that a CAT group is a Lie group when CAT $=$ DIFF the category of smooth manifolds or a topological group when CAT is paracompact Hausdorff spaces.
(b) All maps, actions and objects considered belong to your favorite choice of CAT. An isomorphism of topological spaces is a homeomorphism and an isomorphism of smooth manifolds is a diffeomorphism.
(c) We will consider group actions which are faithful. However, following Steenrod, we will call such group actions ***effective***.

Let us now establish a definition that often comes in when discussing principal G-bundles. We will not use it, but it is nice to know in this context.

Definition (G-Torsor). Let G be a group and suppose G acts on a space F. Then we say F is a ***G-torsor*** if G acts effectively on F *and* the map $G \times F \to F \times F$ defined by sending $(g, x) \mapsto (x, gx)$ is an isomorphism. An analogous definition works for a right action.

Exercise 1. *Show that if F is a G-torsor, then $F \cong G$ and that this isomorphism may be taken to be G-equivariant.*

Example 1. Let G be a topological group that does not have the trivial topology—for example, consider $G = \mathrm{GL}_n(\mathbf{R})$ topologized as a subset of $\mathbf{R}^{n^2}$. Let G_{t} be the space with the same underlying set as G but equipped with the trivial topology (only G and $\emptyset$ are open). Then $G \curvearrowright G_{\mathrm{t}}$ effectively by left translation but $G \ncong G_{\mathrm{t}}$. Hence, G_{t} is not a G-torsor.

Plan.

(1) Bundles.
(2) Characteristic Classes.
(3) Up to participants. Possibilities: Chern-Weil theory, characteristic classes of surface bundles or bordism. Based on participant backgrounds, it looks like we'll be talking about bordism, time permitting.
(4) I will provide an appendix with the some of the technical background for participants who have not been exposed it. I will suppose at least some prerequisites while writing it.

1.2 Fiber Bundles and G-Bundles

Reminder. To simplify our lives, all group actions considered henceforth will be faithful. Following Steenrod will call such group actions ***effective***.

The basic building block for the things we are interested in is the notion of a fiber bundle.

Definition (Fiber Bundle). A ***fiber bundle*** over a ***base space*** B with ***fiber*** F and ***total space*** E is a map $p\colon E \to B$ satisfying the following local triviality condition: for each $x \in B$, there is an open nbhd U of x an isomorphism $\varphi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U & = & U \end{array}$$

We call such a nbhd ***trivializing*** and the map φ a ***trivialization*** or a ***bundle chart***. A particular fixed choice of such trivializations covering B, say $\mathscr{A} = \{(\varphi_i, U_i)\}_{i \in I}$ is called an ***atlas*** or a ***bundle atlas***. We can denote this information as $(E, p, B, F, \mathscr{A})$.

Remark. The data of a fiber bundle makes precise the idea of one space (the fiber) "continuously/smoothly" parameterizing another space (the base space). But often times (and in particular in geometric situations) the fibers we consider have *more* structure than simply being a space.

The basic object we are actually interested in are fiber bundles with a specified structure group.

Definition. Let G be group. A ***fiber bundle with structure group*** G or a G***-bundle*** is a fiber bundle $p\colon E \to B$ with fiber F and a left G-action on F along with the additional data of a G***-atlas***.

A G-atlas is a collection of trivializations $\mathscr{A} = \left\{(U_i, \varphi_i\colon p^{-1}(U_i) \cong U_i \times F)\right\}$ with each U_i open and covering B which we require to satisfy the following compatibility criterion: for each transition map $\varphi_{ij} \overset{\text{def}}{=} \varphi_i \varphi_j^{-1}$, there exists a map $g_{ij}\colon U_i \cap U_j \to G$ such that

$$\varphi_{ij} = \varphi_i \varphi_j^{-1}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F \qquad \varphi_{ij}(x, f) = (x, f \cdot g_{ij}(x)).$$

We call each g_{ij} a ***transition function***. We will call each φ_{ij} a ***transition*** or ***transition map*** to differentiate between the two.

Warning. The transition functions g_{ij} are required to be CAT.

Notation. We will often denote this information as $(E, p, B, G, F, \mathscr{A})$. It's useful to think of this schematically as

$$E \xrightarrow{p} B \qquad G \curvearrowright F \qquad \{(\varphi_i, U_i)\}_{i \in I}\,.$$

Example 2. Consider the case where $G = \mathrm{GL}_n(\mathbf{R})$ and $F = \mathbf{R}^n$—here, we let $G \curvearrowright F$ in the usual way by linear isomorphisms. Then a ***vector bundle of rank*** n is precisely a fiber bundle with structure group $\mathrm{GL}_n(\mathbf{R})$ and fiber $\mathbf{R}^n$. Thus, G-bundles are a vast generalization of vector bundles.

Exercise 2. *In the example above, we assumed that the action of* $\mathrm{GL}_n(\mathbf{R})$ *on* $\mathbf{R}^n$ *was effective. Verify this.*

Proposition 1.2.1. *Let* $\xi = (E, p, B, G, F, \mathscr{A})$ *be a* G*-bundle and denote the identity element of* G *by* e*. The transition functions for* ξ *satisfy the following compatibility condition. For any choice of indices* i, j, k *for which the relevant intersections of the sets* U_i, U_j *and* U_k *are non-empty,*

(1) $g_{ij} = g_{ik} g_{kj}$ *on* $U_i \cap U_k \cap U_j$*;*
(2) $g_{ii} = e$*;*
(3) $g_{ij} = g_{ji}^{-1}$*, where the inverse means the inverse group element.*

The conditions ***(a)****,* ***(b)*** *and* ***(c)*** *together constitute what are called the* ***cocycle conditions****. The inversion and the multiplication all happen pointwise in* G*.*

Proof. We have $\varphi_i \varphi_j^{-1} = \varphi_i \varphi_k^{-1} \varphi_k \varphi_j^{-1}$ and the left-hand side has the form $(b, v) \mapsto (b, g_{ij}(b)v)$ whereas the right-hand side has the form $(b, v) \mapsto (b, g_{ik}(b) g_{kj}(b)v)$ so we conclude that $g_{ij}(b)v = g_{ik}(b) g_{kj}(b)v$ for all acceptable choices of $(b, v) \in B \times F$. Since G acts faithfully on F, this means that $g_{ij} = g_{ik} g_{kj}$.

$\varphi_i \varphi_i^{-1} = \mathrm{id}$, so since G acts faithfully on F, one also concludes that $g_{ii} = e$. $g_{ij} = g_{ji}^{-1}$ follows similarly by faithfulness since $\varphi_{ij} = \varphi_{ji}^{-1}$. ∎

Exercise 3 ((∗), ⋆). *Fix a* G*-bundle* ξ*, show that the maps* g_{ij} *are in fact unique.* [Hint: Use that the left G action is effective.]

Remark. When G *does not* act faithfully, we must consider the maps g_{ij} as part of the data of the G-bundle since they are only unique up to elements of the kernel of the corresponding group-homomorphism $G \to \mathrm{Aut}_{\mathsf{CAT}}(F)$.

Definition. Say a G-atlas $\mathscr{A}$ for a G-bundle ξ is ***maximal*** if there does not exist a G-atlas $\mathscr{B}$ with $\mathscr{A} \subset \mathscr{B}$. Equivalently, whenever a trivialization (φ, U) satisfies the above compatibility criterion for every trivialization of $\mathscr{A}$, then (φ, U) is already in $\mathscr{A}$ (since we could otherwise append it to $\mathscr{A}$ to produce a larger G-atlas).

Exercise 4 ((∗), ⋆⋆). *Let* ξ *be a fiber bundle.*

(a) *Show that any* G*-atlas for* ξ *is contained in a unique maximal* G*-atlas.* [Hint: Look up the proof that any atlas for a manifold is contained in a unique maximal atlas and try to repeat the argument.]

(b) *Define a relation on the set of G-atlases for ξ by saying that two G-atlases $\mathscr{A}$ and $\mathscr{B}$ are equivalent and write $\mathscr{A} \sim \mathscr{B}$ if $\mathscr{A}$ and $\mathscr{B}$ are contained in the same maximal atlas. Show that $\sim$ is an equivalence relation on G-atlases for ξ.*

(c) *Show that if $\mathscr{A} \sim \mathscr{B}$, then $\mathscr{A} \cup \mathscr{B}$ is another G-atlas for ξ equivalent to both $\mathscr{A}$ and $\mathscr{B}$.*

(d) *Suppose $\mathscr{A}$ and $\mathscr{B}$ are inequivalent G-atlases for the fiber bundle ξ. Show that there exists G-atlases $\mathscr{A}'$ and $\mathscr{B}'$ with $\mathscr{A} \sim \mathscr{A}'$ and $\mathscr{B} \sim \mathscr{B}'$ such that the trivializing open nbhds of $\mathscr{A}'$ and $\mathscr{B}'$ are the same. In other words, there is an index set I such that $\mathscr{A}' = \{(\varphi_i^A, U_i)\}_{i\in I}$ and $\mathscr{B}' = \{(\varphi_i^B, U_i)\}_{i\in I}$.*

Remark. As a consequence of this exercise, we will often suppress the G-atlas of a G-bundle and we will assume that a G-bundle comes equipped with a specified choice of an *equivalence class* of G-bundle atlases, rather than a G-atlas itself. Later, in **Exercise 7**, you will show that the definition of a morphism of G-bundles is independent of the equivalence classes of G-bundle atlases used.

1.3 Morphisms of Bundles

Let us begin with the simplest case.

Definition. Let $\xi = (E, p, B, F)$ and $\xi' = (E', p', B', F')$ be fiber bundles. A ***morphism*** $\xi \to \xi'$ is a pair of map $(\widetilde{f}, f)\colon (E, B) \to (E', B')$ making TFDC:

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

Exercise 5. *Show that f is completely determined by $\widetilde{f}$.*

Exercise 6. *Use commutativity of the above square to show that the map $E \to E'$ is a fiber-preserving map in the sense that the fiber in E over $b \in B$ is mapped to the fiber in E' over $f(b) \in B'$.*

Let us recall the motivating example given in the lecture for how we will define bundle morphisms, which we flesh out in a little more detail. The idea is that we should want the bundle morphism to “come from” the group structure somehow.

Reminder. Consider the case of $G = \mathrm{GL}_n(\mathbf{R})$—the group $(n \times n)$ invertible matrices topologized as a subset of $\mathbf{R}^{n^2}$—and $F = \mathbf{R}^n$. Then $\mathrm{GL}_n(\mathbf{R}) \curvearrowright \mathbf{R}^n$ in the evident way by linear isomorphisms. Let $A \in \mathrm{GL}_n(\mathbf{R})$.

If there is any justice in the world, then given a map $f\colon X \to Y$, we should like to say that the following is a bundle morphism

$$\begin{array}{ccc} X \times \mathbf{R}^n & \xrightarrow{f\times(v\mapsto Av)} & Y \times \mathbf{R}^n \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the vertical maps are the projections onto the first factor. For instance, A could be the matrix rI where I is the identity matrix, in which case Av is simply multiplication by r.

If we are given a vector bundle $E \to B$, then for any trivialization $\varphi_{ij}\colon (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$, the commutative diagram

$$\begin{array}{ccc} (U_i \cap U_j) \times F & \xrightarrow{\varphi_{ij}} & (U_i \cap U_j) \times F \\ \downarrow & & \downarrow \\ U_i \cap U_j & = & U_i \cap U_j \end{array}$$

ought also to constitute a bundle map.

With this example in mind, we provide our definition of a morphism of G-bundles.

Warning. For simplicity, we first restrict to the case of the same fiber in the definition below.

Definition (Morphisms of G-Bundles). Let $\xi' = (E', p', B', G, F, \mathscr{A}')$ and $\xi = (E, p, B, G, F, \mathscr{A})$ be two G-bundles, say with G acting on the left of the fibers. A ***morphism*** $(\widetilde{f}, f)\colon \xi \to \xi'$ is a tuple of maps,

$$\widetilde{f}\colon E \to E' \qquad f\colon B \to B'$$

which we require to satisfy the following local property.

For each $(\varphi_i, U_i) \in \mathscr{A}$ and each $(\varphi'_k, U'_k) \in \mathscr{A}'$, there exists a map $\overline{g}_{ki}\colon U_i \cap f^{-1}(U'_k) \to G$ such that

$$\varphi'_k \circ \widetilde{f} \circ \varphi_i^{-1}\colon U_i \cap f^{-1}(U'_k) \times F \to f(U_i \cap f^{-1}(U'_k)) \times F$$

has the form

$$(x, v) \mapsto (f(x), \overline{g}_{ki}(x) \cdot v).$$

Remark. This morphism arises as the dashed arrow in the commutative diagram

$$\begin{array}{ccccc}
p^{-1}(U_i \cap f^{-1}(U'_k)) & \xrightarrow{\widetilde{f}} & (p')^{-1}(f(U_i \cap f^{-1}(U'_k))) & \xrightarrow{\subset} & (p')^{-1}(U'_k) \\
\varphi_i^{-1}\uparrow & & \downarrow \varphi'_k & & \downarrow \varphi'_k \\
(U_i \cap f^{-1}(U'_k)) \times F & \dashrightarrow & f(U_i \cap f^{-1}(U'_k)) \times F' & \xrightarrow[\subset]{} & U'_k \times F'
\end{array}$$

Definition. Define Bun_G^F the ***category of G-bundles with fiber F*** to have as its objects G-bundles with fiber F and morphisms as above.

We similarly define $\mathsf{Bun}_G^F(B)$ to be the ***category of G-bundles with fiber F over B*** to have as its objects G-bundles with base space B and fiber F. The morphisms are as above except, in addition, we require $f = \mathrm{id}_B$. Thus, all of the action happens on the total space.

Warning. As stated in the **Conventions & Tentative Plan** section of the first lecture's notes, when CAT is paracompact Hausdorff spaces, we must allow the total space of the G-bundle to be non-paracompact. In other words, for CAT paracompact Hausdorff spaces, Bun_G^F is G-bundles where $G, F, B \in \mathsf{CAT}$, but we place no such restriction on the total spaces of the G-bundles. We could also assume all bundles in sight are numerable but we will only address this in **Lecture 6**.

Of course, there's a little more to do here—clearly, we must verify that composites of bundle morphisms are themselves bundle morphisms!

Claim 1. This definition does in fact form a category.

Proof. Say we consider $\xi_1 \xrightarrow{(\widetilde{g}, g)} \xi_2 \xrightarrow{(\widetilde{f}, f)} \xi_3$ where the bundle atlas for ξ_j is $\mathscr{A}_j = \{(\varphi_{j_i}, U_{j_i})\}_{i \in I_j}$. We will begin the proof we an investigatory first step.

Since $(\widetilde{g}, g)$ is a G-bundle morphism, $\varphi_{2_j} \circ \widetilde{g} \circ \varphi_{1_i}^{-1}$ has the form $(b_1, v) \mapsto (g(b_1), \overline{g}_{2_j 1_i}(b) \cdot v)$ on $U_{1_i} \cap g^{-1}(U_{2_j})$. Similarly, since $(\widetilde{f}, f)$ is a G-bundle morphism, $\varphi_{3_k} \circ \widetilde{f} \circ \varphi_{2_j}^{-1}$ has the form $(b_2, v) \mapsto (f(b_2), \overline{g}_{3_k 2_j}(b_2) \cdot v)$ on $U_{2_j} \cap f^{-1}(U_{3_k})$ and so, composing these,

$$\varphi_{3_k} \widetilde{f} \varphi_{2_j}^{-1} \varphi_{2_j} \widetilde{g} \varphi_{1_i} \qquad (b_1, v) \mapsto (f(g(b_1)), \overline{g}_{3_k 2_j}(g(b_1)) \overline{g}_{2_j 1_i}(b) \cdot v)$$

on $U_{1_i} \cap g^{-1}(U_{2_j} \cap f^{-1}(U_{3_k})) = U_{1_i} \cap g^{-1}(U_{2_j}) \cap (f \circ g)^{-1}(U_{3_k})$.

This suggests we define

$$\overline{g}_{3_k 1_i} \overset{\text{def}}{=} (\overline{g}_{3_k 2_j} \circ g) \overline{g}_{2_j 1_i}$$

on $U_{1_i} \cap g^{-1}(U_{2_j}) \cap (f \circ g)^{-1}(U_{3_k})$ —this is CAT because everything in sight is CAT. However, at this point, we need to extend $\overline{g}_{3_k 2_j}$ to a CAT map defined on all of $U_{1_i} \cap (f \circ g)^{-1}(U_{3_k})$. The idea will be to glue things together.

For each $x \in (f \circ g)^{-1}(U_{3_k})$, pick $U_{2_{j,x}}$ be any trivializing open nbhd of ξ_2 such that $x \in g^{-1}(U_{2_{j,x}})$—a little thought shows that we can always find such a set. For such an open set, the procedure above produces

$$\overline{g}_{3_k 1_{i,x}} \overset{\text{def}}{=} (\overline{g}_{3_k 2_{j,x}} \circ g) \overline{g}_{2_{j,x} 1_i}$$

on $U_{1_i} \cap g^{-1}(U_{2_{j,x}}) \cap (f \circ g)^{-1}(U_{3_k})$. Suppose we also have $y \in (f \circ g)^{-1}(U_{3_k})$ such that $g^{-1}(U_{2_{j,y}}) \cap g^{-1}(U_{2_{j,x}}) \neq \varnothing$. Note that the composites that furnish $\overline{g}_{3_k 1_{i,x}}$ and $\overline{g}_{3_k 1_{i,y}}$ are in fact *equal* on their common domain. Indeed, $\overline{g}_{3_k 1_{i,x}}$ arises from

$$\varphi_{3_k} \widetilde{f} \varphi_{2_{j,x}}^{-1} \varphi_{2_{j,x}} \widetilde{g} \varphi_{1_i} = \varphi_{3_k} \widetilde{f} \widetilde{g} \varphi_{1_i}$$

whereas $\overline{g}_{3_k 1_{i,y}}$ arises from

$$\varphi_{3_k}\widetilde{f}\varphi_{2_{j,y}}^{-1}\varphi_{2_{j,u}}\widetilde{g}\varphi_{1_i} = \varphi_{3_k}\widetilde{f}\widetilde{g}\varphi_{1_i}.$$

Hence,

$$\overline{g}_{3_k1_i,y} = \overline{g}_{3_k1_i,x} \qquad \text{on the open set} \qquad U_{1_i} \cap g^{-1}(U_{2_{j,x}}) \cap g^{-1}(U_{2_{j,y}}) \cap (f\circ g)^{-1}(U_{3_k}).$$

Finally, define

$$\overline{g}_{3_k1_k}(b) = \overline{g}_{3_k1_k,b}(b).$$

It follows from what we have just shown that this is well-defined. To see that it is smooth when $\mathsf{CAT} = \mathsf{DIFF}$, note that smoothness is a local property and that upon restriction to the open nbhd $U_{1_i} \cap g^{-1}(U_{2_{j,b}}) \cap (f\circ g)^{-1}(U_{k_3})$, $\overline{g}_{3_k1_k} = \overline{g}_{3_k1_k,b}$ and $\overline{g}_{3_k1_k,b}$ is smooth by assumption. ■

The following exercise is dependent on the results of **Exercise 4**.

Exercise 7 ((∗), ⋆⋆). *Let $\xi, \xi' \in \mathsf{Bun}_G^F$ with fixed bundle atlases $\mathscr{A}$ and $\mathscr{A}'$, respectively.*

- **(i)** *Show that $(\widetilde{f}, f)\colon \xi \to \xi'$ is a morphism of G-bundles with respect to the atlases $\mathscr{A}$ and $\mathscr{A}'$* ***iff*** *for any other choices of G-bundle atlases $\mathscr{B} \sim \mathscr{A}$ and $\mathscr{B} \sim \mathscr{A}'$ in the same equivalence classes, $(\widetilde{f}, f)\colon \xi \to \xi'$ is a morphism of G-bundles with respect to the atlases $\mathscr{B}$ and $\mathscr{B}'$.*
- **(ii)** *Conclude that the definition of a morphism of G-bundles is independent of the equivalence class of G-bundle atlases used.*

Remark. As a consequence of **Exercise 7**, it is easy to see that the category Bun_G^F is categorically equivalent to its quotient formed by identifying G-bundles over the same base space and having G-atlases in the same equivalence class.

Exercise 8 ((∗), ⋆). *Show that the $\overline{g}_{ki}$ are unique if they exist.* [Hint: Use that the action of G on F is effective.]

Remark. When G *does not* act faithfully, we must consider the $\overline{g}_{ki}$ as part of the data. Since we are assuming G acts effectively (hence, faithfully), we are in a situation where we need only stipulate that the $\overline{g}_{ki}$ exist.

Exercise 9 ((∗), ⋆). *Let $\xi = (E, p, B, G, F)$ and $\xi' = (E', p', B', G, F)$ be two objects in Bun_G^F. Given a morphism $(\widetilde{f}, f)\colon \xi \to \xi'$ in Bun_G^F, show that for each $b \in B$ the restriction $\widetilde{f}\big| p^{-1}(b)$ is an isomorphism from $p^{-1}(b)$ to $(p')^{-1}(f(b))$.*

Remark. In fact, something somewhat unexpected is true. Let $\xi = (E, p, B, G, F)$ and $\xi' = (E', p', B', G, F)$ be two objects in Bun_G^F. If $(\widetilde{f}, f)\colon \xi \to \xi'$ is a morphism of G-bundles, then the square

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

is a pullback in your favorite choice of CAT. This is the source of the “naturality” criterion for characteristic classes. We shall defer the proof of this for later.

Exercise 10. *Given a morphism $(\widetilde{f}, f)\colon \xi \to \xi'$ in Bun_G^F, show that the map f on the base space is completely determined by the map $\widetilde{f}$ on the total space.*

It is possible to vastly generalize this, though we will not have any occasion to use this but we will use the idea when we consider reductions of structure group. There is no standard definition for the following, so feel free to come up with your favorite variation!

Definition. Let Bun be the category whose objects are ***structured fiber bundles***—that is, fiber bundles with a structure group (recall that we assume all group actions on the fiber are effective). Given $\xi, \xi' \in \mathsf{Bun}$, $\xi = (E, p, B, G, F, \mathscr{A})$, $\xi' = (E', p', B', H, F', \mathscr{A}')$, a ***morphism*** $\xi \to \xi'$ is a *quadruple* $(\varphi, \overline{f}, \widetilde{f}, f)\colon \xi \to \xi'$,

$$\varphi\colon G \to H \qquad \overline{f}\colon F \to F' \qquad \widetilde{f}\colon E \to E' \qquad f\colon B \to B'$$

which we require to satisfy the following properties.

- **(a)** φ is a group-homomorphism.
- **(b)** $\overline{f}(g \cdot v) = \varphi(g) \cdot \overline{f}(v)$.
- **(c)** For each $U_i \in \mathscr{A}$ and each $U_k' \in \mathscr{A}'$, $U_i \cap f^{-1}(U_k')$, there exists a map $\overline{h}_{ki}\colon U_i \cap f^{-1}(U_k') \to H$ such that

$$\varphi_k' \circ \widetilde{f} \circ \varphi_i^{-1}\colon U_i \cap f^{-1}(U_k') \times F \to f(U_i \cap f^{-1}(U_k')) \times F' \subset B \times F'$$

has the form

$$(x, v) \mapsto (f(x), \overline{h}_{ki}(x) \cdot \overline{f}(v)).$$

While intuitively straightforward, these definitions or unwieldy. The following theorem saves us by furnishing a more reasonable criterion for providing isomorphisms in the category $\mathsf{Bun}_G^F(B)$.

Theorem 1.3.1. *Fix a choice for* CAT *to work in. Let* $\xi = (E, p, B, G, F, \mathscr{A})$ *and* $\xi' = (E', q, B, G, F, \mathscr{A}')$ *be two* G*-bundles in* Bun_G^F *over* B *with commonly refined atlases* $\mathscr{A} = \{(U_i, \varphi_i)\}$ *and* $\mathscr{A}' = \{(U_i, \psi_i)\}$*. Denote the transition functions* $\{g_{ij}\}$ *for* ξ *and* $\{g'_{ij}\}$ *for* ξ'*.*

(a) ξ *and* ξ' *are isomorphic as* G*-bundles over* B ***iff*** *there are functions* $g_i\colon U_i \to G$ *such that for all* i, j*,*

$$g'_{ij} = g_i^{-1} g_{ij} g_j,$$

the multiplication and inversion pointwise in G*. In particular, the isomorphism in the* $(\Leftarrow)$ *direction is given by* $f\colon E \to E'$ *defined by letting* $f_i\colon U_i \times F \to U_i \times F$ *be* $f_i(b, v) = (b, g_i^{-1}(b)v)$ *and* $f|\,p^{-1}(U_i) = \psi_i^{-1} f_i \varphi_i$*.*

(b) *The conclusion of* ***(a)*** *is independent of the choice of transition functions for* ξ *and* ξ'*.*

(c) *In particular, the isomorphism of* ***(a)*** *is given by defining* $f_i\colon U_i \times F \to U_i \times F$ *as* $f_i(b, v) = (b, g_i^{-1}(b)v)$ *and then setting* f *to be* $\psi_i^{-1} \circ f_i \circ \varphi_i$ *on* $p^{-1}(U_i)$*.*

Remarks.

(i) Note that the atlases are assumed to have the same trivializing open nbhds. This can always be arranged by taking intersections of the trivializing open nbhds in each atlas as you are asked to show in one of the preceding exercises.

(ii) To say that E and E' are isomorphic over B means that the isomorphism of G-bundles $E \to E'$ has the form (f, id_B).

(iii) It is worth reiterating that g_i^{-1} indicates pointwise inversion in G of g_i and $g_i^{-1} g_{ij} g_j$ indicates pointwise multiplication in G of the functions.

Proof. $(\Leftarrow)$ Define $f\colon E \to E'$ as follows. In the bundle coordinates of (U_i, φ_i), we define $f_i\colon U_i \times F \to U_i \times F$ by

$$f_i(b, v) = (b, g_i^{-1}(b)v)$$

and define

$$f\colon E \to E' \qquad \text{by letting} \qquad f|\,p^{-1}(U_i) = \psi_i^{-1} \circ f_i \circ \varphi_i.$$

Once we know this is a well-defined expression for f and morphism of G-bundles, we will have that f is an isomorphism since it is locally an isomorphism.

To see this is well-defined, we must check that $\psi_i^{-1} \circ f_i \circ \varphi_i$ and $\psi_j^{-1} \circ f_j \circ \varphi_j$ agree on $p^{-1}(U_i \cap U_j)$. Hence, it suffices to show

$$\psi_j \psi_i^{-1} f_i \varphi_i \varphi_j^{-1} = f_j,$$

or, written another way,

$$\psi_{ji} f_i \varphi_{ij} = f_j.$$

Taking $(b, v) \in U_i \cap U_j \times F$, the left-hand side sends

$$(b, v) \mapsto (b, g'_{ji}(b) g_i^{-1}(b) g_{ij}(b) v).$$

Since we know that $g'_{ji} = (g_j)^{-1} g_{ji} g_i$, we have the following string of equalities

$$\begin{aligned} \psi_{ji} f_i \varphi_{ij}(b, v) &= (b, g'_{ji}(b) g_i^{-1}(b) g_{ij}(b) v) = (b, (g_j)^{-1}(b) g_{ji}(b) g_i(b) g_i^{-1}(b) g_{ij}(b) v) = (b, g_j^{-1}(b) g_{ji}(b) g_{ij}(b) v) \\ &= (b, g_j^{-1}(b) g_{ji}(b) g_{ij}(b) v) = (b, g_j^{-1}(b) v) = f_j(b, v). \end{aligned}$$

Thus, f is well-defined and an isomorphism in the underlying category of spaces. It remains to show it is a morphism of G-bundles.

To see that f is a morphism of bundles, we must check that $\psi_i \circ f \circ \varphi_j^{-1}$ has the appropriate form, where this is defined on $U_i \cap U_j \times F$. Since $U_i \cap U_j \times F \subset U_j \times F$ and since $f|\,p^{-1}(U_j) = \psi_j^{-1} \circ f_j \circ \varphi_j$, we have that

$$\psi_i \circ f \circ \varphi_j^{-1} = \psi_i \circ \psi_j^{-1} \circ f_j \circ \varphi_j \varphi_j^{-1} = \psi_{ij} \circ f_j \qquad \text{on } U_i \cap U_j \times F.$$

By the same reasoning applied to f_i, we may also write

$$\psi_i \circ f \circ \varphi_j^{-1} = \psi_i \circ \psi_i^{-1} \circ f_i \circ \varphi_i \circ \varphi_j^{-1} = f_i \circ \varphi_{ij}.$$

For $(b, v) \in (U_i \cap U_j) \times F$,

$$\psi_i\psi_j^{-1}f_j(b,v) = \psi_i\psi_j^{-1}(b, g_j^{-1}(b)\cdot v) = (b, g'_{ij}(b)g_j^{-1}(b)v)$$
$$f_i\varphi_i\varphi_j^{-1}(b,v) = f_i(b, g_{ij}(b)v) = (b, g_i^{-1}(b)g_{ij}(b)v)$$

so the desired function can be taken to be either

$$\overline{g}_{ij} = g'_{ij}g_j^{-1}$$

or

$$\overline{g}_{ij} = g_i^{-1}g_{ij}.$$

Since G acts effectively on F, these two choices are equal.

(⇒) Next time!

1.4 (∗) Exercises: 1

***Exercise** ((∗), ⋆).* *Fix a G-bundle ξ, show that the maps g_{ij} are in fact unique.* [Hint: Use that the left G action is effective.]

***Exercise** ((∗), ⋆⋆).* *Let ξ be a fiber bundle.*

(a) *Show that any G-atlas for ξ is contained in a unique maximal G-atlas.* [Hint: Look up the proof that any atlas for a manifold is contained in a unique maximal atlas and try to repeat the argument.]

(b) *Define a relation on the set of G-atlases for ξ by saying that two G-atlases $\mathscr{A}$ and $\mathscr{B}$ are equivalent and write $\mathscr{A} \sim \mathscr{B}$ if $\mathscr{A}$ and $\mathscr{B}$ are contained in the same maximal atlas. Show that $\sim$ is an equivalence relation on G-atlases for ξ.*

(c) *Suppose $\mathscr{A}$ and $\mathscr{B}$ are inequivalent G-atlases for the fiber bundle ξ. Show that there exists G-atlases $\mathscr{A}'$ and $\mathscr{B}'$ with $\mathscr{A} \sim \mathscr{A}'$ and $\mathscr{B} \sim \mathscr{B}'$ such that the trivializing open nbhds of $\mathscr{A}'$ and $\mathscr{B}'$ are the same. In other words, there is an index set I such that $\mathscr{A}' = \left\{(\varphi_i^A, U_i)\right\}_{i\in I}$ and $\mathscr{B}' = \left\{(\varphi_i^B, U_i)\right\}_{i\in I}$.* [Hint: Intersect the trivializing nbhds of $\mathscr{A}$ and $\mathscr{B}$.]

***Exercise** ((∗), ⋆).* *Show that the $\overline{g}_{ki}$ are unique if they exist.* [Hint: Use that the action of G on F is effective.]

***Exercise** ((∗), ⋆).* *Let $\xi = (E, p, B, G, F)$ and $\xi' = (E', p', B', G, F)$ be two objects in Bun_G^F. Given a morphism $(\widetilde{f}, f)\colon \xi \to \xi'$ in Bun_G^F, show that for each $b \in B$ the restriction $\widetilde{f}\big|p^{-1}(b)$ is an isomorphism from $p^{-1}(b)$ to $(p')^{-1}(f(b))$.*

Lecture 2

2.1 Examples

TO BE FILLED OUT

2.2 End of Proof of First Theorem

Recall the statement of the theorem.

Theorem. *Fix a choice for* CAT *to work in. Let* $\xi = (E, p, B, G, F, \mathscr{A})$ *and* $\xi' = (E', q, B, G, F, \mathscr{A}')$ *be two* G*-bundles in* Bun_G^F *over* B *with commonly refined atlases* $\mathscr{A} = \{(U_i, \varphi_i)\}$ *and* $\mathscr{A}' = \{(U_i, \psi_i)\}$*. Denote the transition functions* $\{g_{ij}\}$ *for* ξ *and* $\{g'_{ij}\}$ *for* ξ'*.*

(a) *ξ and ξ' are isomorphic as G-bundles over B **iff** there are functions $g_i\colon U_i \to G$ such that for all i, j, for all $(b, v) \in (U_i \cap U_j) \times F$, we have*
$$g'_{ij}(b)v = g_i^{-1}(b)g_{ij}(b)g_j(b)v.$$

(b) *The conclusion of **(a)** is independent of the choice of transition functions for ξ and ξ'.*

(c) *In particular, the isomorphism of **(a)** is given by defining $f_i\colon U_i \times F \to U_i \times F$ as $f_i(b, v) = (b, g_i^{-1}(b)v)$ and then setting f to be $\psi_i^{-1} \circ f_i \circ \varphi_i$ on $p^{-1}(U_i)$.*

Proof. ($\Rightarrow$) Suppose $f\colon E \to E'$ is an isomorphism of G-bundles over B with the same typical fiber F. Recall that this means the morphism on the base space is the identity. Then we know that $\psi_i \circ f \circ \varphi_j^{-1}$ has the form $(b, v) \mapsto (b, \overline{g}_{ij}(b)v)$ for some CAT map $\overline{g}_{ij}\colon U_{ij} \overset{\text{def}}{=} U_i \cap U_j \to G$. First, let us make a claim.

Claim 2. Let $\{\overline{g}_{ij}\}$ be the set of CAT maps witnessing that f is a G-bundle morphism per the definition. Then $f^{-1}\colon E' \to E$ has the set of CAT maps $\{\overline{h}_{ij}\}$ where $\overline{h}_{ij} = \overline{g}_{ji}^{-1}$ witnessing that f^{-1} it is a G-bundle morphism. Here, the inverse on $\overline{g}_{ji}$ indicates pointwise inversion in G.

To see this, observe that $\psi_i \circ f \circ \varphi_j^{-1}$ has the form $(x, v) \mapsto (x, \overline{g}_{ij}(x) \cdot v)$ and so since f is an isomorphism, the inverse $\varphi_j \circ f^{-1} \circ \psi_i^{-1}$ has the form $(x, v) \mapsto (x, \overline{g}_{ij}^{-1}(x) \cdot v)$ and so with $\overline{h}_{ji} = \overline{g}_{ij}^{-1}$, this does it once we know $\overline{g}_{ij}^{-1}$ is CAT; in the topological case, this is because inversion in G is continuous and in the smooth case this is because inversion in G is smooth.

Taking our cue from the end of the ($\Leftarrow$) implication, we define
$$\boxed{g_j = \overline{g}_{ij}^{-1} g'_{ij}.}$$

Then the claim is that
$$g'_{ij} = g_i^{-1} g_{ij} g_j.$$

Equivalently,
$$g'_{ij} = (g'_{ji})^{-1} \overline{g}_{ji} g_{ij} \overline{g}_{ij}^{-1} g'_{ij}.$$

Multiplying on the right by $(g'_{ij})^{-1}$, it suffices to show that
$$(g'_{ji})^{-1} \overline{g}_{ji} g_{ij} \overline{g}_{ij}^{-1} \equiv e.$$

Towards this end, we inspect what mappings give this combination of transition functions and $\overline{g}_{ij}$'s. Consider
$$(\psi_{ij})(\psi_j f \varphi_i^{-1})(\varphi_{ij})(\varphi_j f^{-1} \psi_i^{-1})(\psi_{ij}) = (\psi_i \psi_j^{-1})(\psi_j f \varphi_i^{-1})(\varphi_i \varphi_j^{-1})(\varphi_j f^{-1} \psi_i^{-1})(\psi_i \psi_j^{-1}).$$

As a consequence of the claim, we see that this composite eliminates to

$$\psi_i \psi_j^{-1}$$

and so sends $(b, v) \mapsto (b, g'_{ij}(b)v)$. On the other hand, the original composite sends

$$\begin{aligned}(b,v) &\mapsto (b, g'_{ij}(b)v) \mapsto (b, (\overline{g}_{ij}^{-1} g'_{ij})(b)v) \mapsto (b, (g_{ij}\overline{g}_{ij}^{-1} g'_{ij})(b)v) \\ &\mapsto (b, (\overline{g}_{ji} g_{ij}\overline{g}_{ij}^{-1} g'_{ij})(b)v) \mapsto (b, (g'_{ij}\overline{g}_{ji} g_{ij}\overline{g}_{ij}^{-1} g'_{ij})(b)v),\end{aligned}$$

and so we conclude that

$$(b, g'_{ij}(b)v) = (b, ((g'_{ji})^{-1}\overline{g}_{ji} g_{ij}\overline{g}_{ij}^{-1} g'_{ij})(b)v).$$

Since the action of G on F is effective, we conclude that

$$g'_{ij} = (g'_{ji})^{-1}\overline{g}_{ji} g_{ij}\overline{g}_{ij}^{-1} g'_{ij} = g_i^{-1} g_{ij} g_j,$$

as desired. ■

2.3 (∗) Exercises: 2

Exercise 11 **((∗), ⋆).** *Show that the conclusion of **Theorem 1** is independent of the* equivalence class *of the bundle atlas on ξ and ξ'.*

Exercise 12 **((∗), ⋆⋆).** *Prove the following generalization of **Theorem 1**.*

Let $\xi = (E, p, B, G, F, \mathscr{A})$ and $\xi' = (E', q, B', G, F, \mathscr{A}')$ be two G-bundles in Bun_G^F. Denote the transition functions $\{g_{\beta\beta'}\}$ for ξ and $\{g_{\alpha\alpha'}\}$ for ξ' respectively.

(a) *Given a bundle morphism $(\widetilde{f}, f)\colon \xi \to \xi'$*

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ \big\downarrow & & \big\downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

the following relation holds for all suitable $\alpha, \alpha', \beta, \beta', b$ and v for which the expression makes sense:

$$\overline{g}_{\alpha'\beta'}(b)v = g_{\alpha'\alpha}(f(b))\overline{g}_{\alpha\beta}(b) g_{\beta\beta'}(b)v. \qquad (*)$$

(b) *Given a map $f\colon B \to B'$, a morphism of bundles $(\widetilde{f}, f)\colon \xi \to \xi'$ exists **iff** there exist* CAT *morphisms $\overline{g}_{\alpha\beta}$ satisfying (∗).*

Lecture 3

3.1 Pullback Theorem

At this point we collect a small lemma by technically useful lemma.

Lemma 3.1.1. *Suppose $\pi\colon E \to B$ is a fiber bundle with typical fiber F. Then π is an open map.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of B by trivializing nbhds and let $U \subset E$ be an open set. The projection map π is an open map since it is locally an open map, we claim. Indeed, the local trivializations give commutative diagrams

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\cong} & U_i \times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U_i & = & U_i \end{array}$$

and the projections maps off of any product are open maps essentially by definition of the product topology. Hence, $\pi|\,\pi^{-1}(U_i)$ is an open map. In general, for U as above, we may write $U = \bigcup(U\cap\pi^{-1}(U_i))$ and then, since image commutes with union, $\pi(U) = \bigcup\pi(U\cap\pi^{-1}(U_i))$ and $\pi|\,\pi^{-1}(U_i)$ is an open map by the reasoning just given so that since $U\cap\pi^{-1}(U_i)$ is open in E and therefore an open subset of $\pi^{-1}(U_i)$, $\pi(U\cap\pi^{-1}(U_i))$ is open. Hence, $\pi(U)$ is open, being a union of open sets. ■

Before we present the main theorem of this section, we collect a corollary to the preceding theorem and an exercise.

Corollary 3.1.2. *Let $\xi,\xi' \in \mathsf{Bun}_G^F(B)$ be two G-bundles over B. Suppose ξ and ξ' admit G-atlases $\mathscr{A} = \{(\varphi_i, U_i)\}_{i\in I}$ and $\mathscr{A}' = \{(\varphi_i', U_i)\}_{i\in I}$ over the same collection of trivializing open sets and suppose that the transition functions associated to these two atlases are the same (i.e., $g_{ij} = g_{ij}'$). Then $\xi \cong \xi'$ over B.*

Proof. Take $g_i \equiv e$ in **Theorem 1**. ■

We will need another fact, we leave as an exercise.

Exercise 13 ((∗), ⋆). *Every morphism of $\mathsf{Bun}_G^F(B)$ is an isomorphism. In other words, a morphism of G-bundles with typical fiber F over B is necessarily and isomorphism. Hence, the category $\mathsf{Bun}_G^F(B)$ is a **groupoid**.*

For the next theorem, we will leave certain details as exercises. As stated in our conventions, we will not assume that the total spaces are paracompact in these cases.

Theorem 3.1.3 (Pullback Theorem). *Fix a choice of* CAT. *Let $\xi' = (P', p', B', G, F, \mathscr{A}') \in \mathsf{Bun}_G^F$ and let $f\colon B \to B'$. Denote $\mathscr{A}' = \{(\psi_i, U_i)\}_{i\in I}$ the G-atlas with transition functions $g_{ij}'\colon U_i \cap U_j \to G$.*

(a) *The pullback*

$$\begin{array}{ccc} f^*P' & \longrightarrow & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

exists and $f^*\xi' = (f^*P', B, G, F)$ can be given the structure of a G-bundle with

$$\textit{trivializing open sets } \left\{f^{-1}(U_i)\right\}_{i\in I} \textit{ and transition functions } g_{ij} \overset{def}{=} g_{ij}' \circ f.$$

Furthermore, with respect to this structure, the diagram above is a morphism $f^*\xi' \to \xi'$ in Bun_G^F.

(b) *A morphism $(\varphi, \widetilde{f}, f)\colon \xi \to \xi'$ in Bun_G^F, where $\xi = (P, p, B, G, F)$ and $\xi' = (P', p', B', G, F)$ is a pullback in the sense that Bun_G^F morphism*

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

is a pullback in CAT*. Moreover, there is an isomorphism of G-bundles $f^*\xi' \cong \xi$.*

(c) *The analogous statements are true when P and P' are rank n vector bundles.*

Before giving the proof, we give two remarks.

Remark. While we have already seen that for $G = \mathrm{GL}_n(\mathbf{R})$ and $F = \mathbf{R}^n$, the category Bun_G^F is equivalent to the category of rank n vector bundles. However, it is worth pointing out how this argument goes through for vector bundles as they are usually presented. We will only need to make a single comment about this and we therefore prove **(c)** along with **(a)**) and **(b)**.

Remark. Part **(a)** of this theorem says that the pullback in CAT exists in CAT and that, moreover, there is a natural way to give the pullback the structure of G-bundle such that the resulting pullback diagram constitutes a morphism in Bun_G^F. Part **(b)** is the converse—it says that to even give a morphism of G-bundles $(\widetilde{f}, f): \xi \to \xi'$, we must have had that $\xi \cong f^*\xi'$ to begin with. Looking ahead, what Milnor and Stasheff call the *naturality* condition of characteristic classes is really *functorality* as a consequence of this theorem.

Proof. **(a)** Let $f: B \to B'$ and $\xi' = (P', p', B', G, F) \in \mathsf{Bun}_G^F$. The pullback in the underlying category of *spaces* is

$$f^*P' = \{(b, e) \in B \times P' : f(b) = p'(e)\}$$

topologized as a *subspace* of $B \times P'$. Let $\pi = \mathrm{pr}_1 : f^*P' \to B$ be the evident projection. Then, at least on the level of the full category of topological spaces (and on the level of sets), we have the following pullback square

$$\begin{array}{ccc} f^*P' & \xrightarrow{\mathrm{pr}_2} & P' \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

In the vector bundle case, the vector space structure is defined by

$$r_1(b, e_1) + r_2(b, e_2) \stackrel{\text{def}}{=} (b, r_1 e_1 + r_2 e_2).$$

To show that this is a CAT pullback, there is more to do.

When CAT is paracompact Hausdorff spaces, there are no constraints placed on the total space f^*P', be we note that it is Hausdorff as it is a subspace of a Hausdorff space. We now show it is a G-bundle over B. Recall that we have denoted the G-bundle atlas as

$$\mathscr{A}' = \{(\psi_i, U_i)\} \qquad \text{with associated transition functions} \qquad g'_{ij}: U_i \cap U_j \to G.$$

We let

$$V_i = f^{-1}(U_i) \qquad \text{and define} \qquad \varphi_i : \pi^{-1}(V_i) \to V_i \times F$$

by defining its inverse $\varphi_i^{-1} : V_i \times F \to f^*P'$, which is psychologically easier; namely, using the model above for f^*P', we let

$$\boxed{\varphi_i^{-1}(b, x) = (b, \psi_i^{-1}(f(b), x)).}$$

This has set-theoretic inverse (recalling that $(b, e) \in f^*P$ is a valid point)

$$\boxed{\varphi_i(b, e) = (b, \mathrm{pr}_2\, \psi_i(e)).}$$

Indeed, by definition of f^*P', $\psi(e) = (f(b), \mathrm{pr}_2\, \psi_i(e))$. To justify this notation, we must show that φ_i and φ_i^{-1} are really inverse to each other, that they are homeomorphisms, and that they fit into the relevant trivialization diagram. We leave this as the following exercise.

Exercise 14 ((∗∗), ⋆⋆). *Show that φ_i and φ_i^{-1} as defined really constitute set-theoretic inverses and that the following diagram commutes on the level of sets*

$$\begin{array}{ccc} \pi^{-1}(V_i) & \longrightarrow & V_i \times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ V_i & = & V_i \end{array}$$

Then show that φ_i and φ_i^{-1} are continuous and conclude that they are homeomorphisms. [Hint: Recall that f^*P' is a subspace and $B \times P'$. Consider that the open subspace $V_i \times (p')^{-1}(U_i) = f^{-1}(U_i) \times (p')^{-1}(U_i)$ is homeomorphic to $f^{-1}(U_i) \times U_i \times F$.]

As a consequence of this exercise, $f^*P' \xrightarrow{\pi} B$ is at least a fiber bundle over B with typical fiber F.

Observe that on overlaps,

$$\varphi_{ij}(b, x) = (b, \mathrm{pr}_2\, \psi_i(\psi_j^{-1}(f(b), x))) = (b, g'_{ij}(f(b)) \cdot x)$$

which is certainly CAT in either order of composition. Setting

$$\boxed{g_{ij} \overset{\text{def}}{=} g'_{ij} \circ f.}$$

Then g_{ij} is certainly CAT as a composite of CAT maps and in particular we have that

$$\varphi_{ij}(b, x) = (b, g_{ij}(b) \cdot x)$$

which is the right form. This shows that $f^*P' \to B$ is indeed an object in Bun_G^F, at least when CAT is the category of paracompact Hausdorff spaces.

At this point, there are three things left to do. The first is to show that f^*P' is a topological manifold and then equip it with a smooth structure and show that the bundle projection is smooth with respect to this. The second part is to prove the categorical part of **(a)**. The third and last part is to show that we get a morphism of G-bundles $f^*\xi' \to \xi'$.

In the smooth case, as a consequence of what has been done up to this point, we already know that f^*P' is Hausdorff and locally Euclidean (our assumptions imply $U_i \times F$ is locally Euclidean in this case). We must show that it is also second-countable to show that f^*P' is at least a topological manifold. For this, simply note that since B' is second-countable, we may suppose the open cover of B' by bundle atlases was countable since second-countability allows us to pass to a countable subcover. Hence, f^*P' is a countable union of second-countable subspaces and is therefore second-countable.

That f^*P' is locally Euclidean and admits a smooth structure is left as an exercise. In fact, it is enough to show that the natural candidate charts that show f^*P' is locally Euclidean are actually smoothly compatible.

Exercise 15 ($(*)$, $\star\star$). *Show that f^*P' admits the structure of smooth manifold with an atlas of charts given by $(x \times y) \circ \varphi_i$ where $x \times y$ is a chart of $U_i \times F$ and thus of $B \times F$. Conclude that the φ_i are themselves diffeomorphisms with respect to this smooth structure.* [Hint: Regardless of whether or not φ_i and φ_j^{-1} are smooth in any differentiable structure on f^*P', the preceding computation shows that φ_{ij} is smooth.]

With this smooth structure, the projection map is smooth, we claim. Indeed, for a chart x of U_i and thus of B and chart y of F, it is enough to check that $x \circ \pi \circ \varphi_i^{-1} \circ x^{-1} \times y^{-1}$ is smooth. Of course, since the φ_i are bundle trivializations, $\pi \circ \varphi_i^{-1}$ is the projection onto the first coordinate. We therefore have

$$x \circ \pi \circ \varphi_i^{-1} \circ x^{-1} \times y^{-1} = \mathrm{pr}_1 \colon x(U) \times y(V) \to x(U),$$

which is certainly smooth.

Let us turn to the categorical part of **(a)**. Observe that the pullback in spaces is also the pullback of underlying sets. Hence, given CAT maps $v \colon X \to P'$ and $u \colon X \to B$, there is a unique *function of sets* $(u, v) \colon X \to f^*P$ making the obvious diagram commutes. In the non-smooth case, we already know that (u, v) is continuous because f^*P' is the pullback in the category of topological spaces. We must consider the smooth case.

Give f^*P' the smooth structure above. The above considerations mean it suffices to show that (u, v) is smooth. Since smoothness is a local property, we may as well suppose $X = \mathbf{R}^m$. Since $(u, v)(p) = (u(v), v(p))$, $\varphi_i \circ (u, v)(p) = (u(p), \mathrm{pr}_2\, \psi_i(v(p)))$ and this is CAT as a composite of CAT functions on each component and, hence, by **Exercise 15** we conclude that (u, v) is smooth. Thus, f^*P' is a pullback in CAT for either choice of CAT.

Remark. Needless to say, everything here goes through with rank n vector bundles as well.

Thus, for either choice of CAT, we have a square

$$\begin{array}{ccc} f^*P' & \xrightarrow{\mathrm{pr}_2} & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

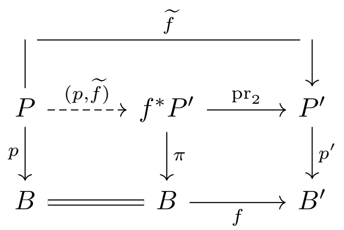
and this diagram constitutes a morphism of Bun_G^F since if we set $g_{ij} = g'_{ij} \circ f$, then

$$\psi_i \circ \mathrm{pr}_2 \circ \varphi_j^{-1}(b, x) = \psi_i p_2(b, \psi_j^{-1}(f(b), x)) = \psi_i \psi_j^{-1}(f(b), x) = (f(b), g'_{ij}(f(b)) \cdot x) = (b, g_{ij}(b) \cdot x)$$

which has the the right form to be a morphism of G-bundles and is certainly CAT because everything in sight is CAT.

Remark. Notice that this pullback exists even when we are considering manifolds with corners. We do not get so lucky in general when considering pullbacks between manifolds with corners.

(b) Define $P \to f^*P'$ by $x \mapsto (p(x), \widetilde{f}(x))$ and observe that this is well-defined and is a morphism over B by universal properties (namely, the pullback exists in CAT so this morphism is CAT).



Since this is a morphism of the total spaces of G-bundles with fiber F over a fixed base space, it suffices by **Exercise 13** to show that the map on total spaces is indeed a morphism of G-bundles, in which case the stated exercise implies that it must be an isomorphism of G-bundles. Thus, all we have to do is verify the bundle atlas compatibility of this map.

We have a CAT morphism from universal properties $(p, \widetilde{f})\colon P \to f^*P'$ as remarked above. Now, for trivializations η_j of ξ and φ_i of $f^*\xi'$,

$$\varphi_i \circ (p, \widetilde{f}) \circ \eta_j^{-1}(b, v) = \varphi_i(b, \widetilde{f}\eta_j^{-1}(b, v)) = (b, \mathrm{pr}_2\, \psi_i \widetilde{f}\eta_j^{-1}(b, v)).$$

But since $(\widetilde{f}, f)$ is a morphism of G-bundles $\xi \to \xi'$, we know that $\mathrm{pr}_2\, \psi_i \widetilde{f}\eta_j^{-1}(b, x) = \overline{g}_{ij}(b) \cdot x$ and so

$$\varphi_i \circ (p, \widetilde{f}) \circ \eta_j^{-1}(b, v) = (b, \overline{g}_{ij}(b) \cdot v)$$

which is the correct form and so $(p, \widetilde{f})$ is a morphism of G-bundles over B and is therefore an isomorphism.

(c) Mutatis-mutandis. ■

Remark. It follows that if $P \to X \times G$ is a morphism of principal G-bundles, then P must be trivial. Indeed, one can check by universal properties that the pullback must be $B \times G$.

3.2 Fiber Bundle Construction Theorem

Theorem 3.2.1 (Fiber Bundle Construction Theorem). *Fix B a base space, G a group and a G-space F, the desired fiber space. For any given open cover $\{U_i\}_{i\in I}$ and CAT maps $g_{ij}\colon U_i \cap U_j \to G$ satisfying the **cocycle conditions***

(1) $g_{ij} = g_{ik}g_{kj}$ *on* $U_i \cap U_k \cap U_j$*;*
(2) $g_{ii} \equiv e$*;*
(3) $g_{ij} = g_{ji}^{-1}$*, where the inverse means the inverse group element.*

there exists a G-bundle $\xi = (E, p, B, F, G)$ trivializable over the U_i and with transition functions g_{ij} and, furthermore, ξ is unique up to isomorphism over B.

More specifically, we will construct $(E, p, B, F, G, \mathscr{A})$ with a naturally occurring G-bundle atlas where $\mathscr{A} = \{(\varphi_i, U_i)\}_{i\in I}$ where

$$E = \coprod_{i\in I} U_i \times F / \{(j, b, f) \sim (i, x, g_{ij}(b) \cdot f) : x \in U_i \cap U_j,\ \ f \in F\}$$

with the quotient topology and where

$$\varphi_i\colon p^{-1}(U_i) \to U_i \times F \qquad [(i, b, v)] \mapsto (b, v).$$

The bundle map $p\colon E \to B$ is the evident projection.

Proof. It is easy to see that E as defined above will be Hausdorff. In the smooth case, we leave it as an exercise to the reader to show that E will be second-countable—the idea is the same as the one used in the preceding theorem.

The proffered bundle map $p\colon E \to B$ is induced by the projections onto the first coordinates of each summand of the disjoint union $\sum_{i\in I} \mathrm{pr}_1\colon \coprod_{i\in I} U_i \times F \to B$. This map is constant on the fibers of the quotient map

$$q\colon \coprod_{i\in I} U_i \times F \to E$$

and so descends to the map $p\colon : E \to B$ as claimed by the universal property of the quotient. In particular, $p\circ q = \sum_{i\in I} \mathrm{pr}_1$.

We now turn to the bundle structure.

Exercise 16. *Show that the φ_i are well-defined as functions of sets.*

We should like to show that, additionally, the φ_i are homeomorphisms, since the inverse $(b, v) \mapsto [(i, b, v)]$ is clear.

Claim 3. The restriction of the quotient map to q to $q^{-1}(p^{-1}(U_i)) = \coprod_{j\in I} U_i \cap U_j \times F$ induces a quotient map $q\colon \coprod_{j\in I} U_i \cap U_j \times F \to p^{-1}(U_i)$.

Of course, $q^{-1}(p^{-1}(U_i)) = (p\circ q)^{-1}(U_i)$ and we have seen that $p\circ q$ is simply the projections off of each piece of the disjoint union, so $q^{-1}(p^{-1}(U_i)) = \coprod_{j\in I} U_i \cap U_j \times F$ follows immediately. It is immediate that $q\colon \coprod_{j\in I} U_i \cap U_j \times F \to p^{-1}(U_i)$ is surjective and continuous so let us show that it is a quotient map by showing that for any set $V \subset p^{-1}(U_i)$ with $q^{-1}(V)$ open is itself open. If $V \subset p^{-1}(U_i)$ has preimage open in the open subspace $\coprod_{j\in I} U_i \cap U_j \times F \subset \coprod_{j\in I} U_j \times F$, then it must be open in $\coprod_{j\in I} U_j \times F$. But also the preimage of V in $\coprod_{j\in I} U_i \cap U_j \times F$ *is the same* as its preimage in $\coprod_{j\in I} U_j \times F$. Since $q\colon \coprod_{j\in I} U_j \times F \to E$ is a quotient map, this means that V is open in E and hence since $p^{-1}(U_i) \subset E$ is open, it is open in $p^{-1}(U_i)$.

With this claim in hand, observe that the diagonal

$$\begin{array}{ccc} \coprod_{j\in I} U_i \cap U_j \times F & & \\ {\scriptstyle q}\downarrow & \searrow & \\ p^{-1}(U_i) & \xrightarrow[\varphi_i]{} & U_i \times F \end{array}$$

is precisely the inclusion of each factor into $U_i \times F$ and is therefore continuous and, hence, by the universal property of the quotient map, φ_i must be continuous.

Conversely, φ_i^{-1} arises as the dashed arrow in

$$\begin{array}{ccc} U_i \cap U_i \times F & \xrightarrow{\text{in}} & \coprod_{j\in I} U_i \cap U_j \times F \\ {\scriptstyle \mathrm{id}}\uparrow & & \downarrow{\scriptstyle q} \\ U_i \times F & \dashrightarrow & p^{-1}(U_i) \end{array}$$

and is therefore continuous. In addition,

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{\varphi_i} & U_i \times F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U_i & = & U_i \end{array}$$

commutes since, chasing a typical element $[(i, b, v)]$,

$$\begin{array}{ccc} [(i, b, v)] & \longmapsto & (b, v) \\ \downarrow & & \downarrow \\ b & = & b \end{array}$$

Finally, $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ has the form

$$(b, v) \mapsto [(j, b, v)] = [(i, b, g_{ij}(b)\cdot v)] \mapsto (b, g_{ij}(b)\cdot v)$$

and so the transition functions for the trivializations are as claimed.

This shows that $E \to B$ is a G-bundle with fiber F trivializable over $\{U_i\}$ with trivializations φ_i and associated transition functions g_{ij}. It only remains to show that E can be given a smooth structure for which the φ_i are smooth.

We know that the φ_i are well-defined and homeomorphisms. The smooth structure on E will be determined by the collection of charts $(x \times y) \circ \varphi_i$ where $x \times y$ is a chart of $U_i \times F$. Since φ_{ij} has the form $(b, v) \mapsto (b, g_{ij}(b)\cdot v)$, it is smooth by our assumptions, so smooth compatibility of these charts is completely manifest as each component of this map is smooth. Moreover, with this smooth structure, the projection map p is smooth. Indeed, it is enough to check that $x \circ p \circ \varphi_i^{-1} \circ (x \times y)^{-1}$ is smooth. Writing this suggestively as $x \circ (p \circ \varphi_i^{-1}) \circ x^{-1} \times y^{-1}$, we observe that $p \circ \varphi_i^{-1} \colon U_i \times F \to U_i$ is the projection since φ_i is a bundle trivialization nd so

$$x \circ (p \circ \varphi_i^{-1}) \circ x^{-1} \times y^{-1} = \mathrm{pr}_1 \colon x(U) \times y(V) \to x(U),$$

which is certainly smooth.

The uniqueness statement is an immediate consequence of **Corollary 1** given at the beginning of this lecture. ■

3.3 The Associated Principal Bundle Functor

We give a preliminary definition of a principal G-bundle.

Definition (Preliminary). A ***principal G-bundle*** is an object of Bun_G^G where $G \curvearrowright G$ by left translation ($g \cdot g' = gg'$).

Remark. Normally one sees a principal G-bundle defined as a G-bundle $(P, p, B, G, F, \mathscr{A})$ along with a right G-action on the total space P such that the following diagram commutes for all $g \in G$

$$\begin{array}{ccc} P & \xrightarrow{-\cdot g} & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B & = & B \end{array}$$

and such that, with respect to this right G-action, all trivializations $(\varphi_i, U_i) \in \mathscr{A}$ are G-*equivariant*, by which we mean the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow{-\cdot g} & p^{-1}(U_i) \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \varphi_i} \\ U_i \times G & \xrightarrow{-\cdot g} & U_i \times G \end{array}$$

Here, $U_i \times G \curvearrowleft G$ by right translation (i.e., $(b, g') \cdot g = (b, g'g)$).

It will turn out that there is an essentially unique way to give $\xi \in \mathsf{Bun}_G^G$ these extra properties and we defer this analysis for a later lecture.

Definition. Let $\xi = (E, p, B, F, G, \mathscr{A})$ be a CAT G-bundle. Write $\mathscr{A} = \{(U_i, \varphi_i)\}$ and denote the transition functions for ξ by $g_{ij} \colon U_i \cap U_j \to G$.

Then the ***associated principal G-bundle*** over B is the CAT fiber bundle $\mathsf{P}(\xi)$ provided by the fiber bundle construction theorem for the data of

- the same base space B;
- typical fiber G where G acts on itself by translations (the obvious G-action);
- the open cover provided by the G-atlas $\mathscr{A}$;
- g_{ij} the associated transition functions for the G-bundle atlas.

This bundle is indeed a principal G-bundle since it has typical fiber G with action by translation and a G-atlas. Functoriality of this construction is a more delicate question. In fact, it is the only place where a more general theory breaks down if $G \curvearrowright F$ is *not* effective.

Reminder. It is useful to recall that $\mathsf{P}(\xi) = \coprod_{i \in I} U_i \times G / \{(j, x, g) \sim (i, x, g_{ij}(x) \cdot g) : x \in U_i \cap U_j,\ g \in G\}$.

It is worth reiterating that the following theorem is the **first place** the assumption that G acts faithfully on F is completely necessary.

Theorem 3.3.1. *The associated principal G-bundle construction extends to a functor* $\mathsf{P} \colon \mathsf{Bun}_G^F \to \mathsf{Bun}_G^G$. *In particular, given* $(\widetilde{f}, f) \colon \xi \to \xi'$ *in* Bun_G^F, *the morphism* $\mathsf{P}(\widetilde{f}, f)$ *has the same associated* $\overline{g}_{ij}$*'s.*

Remark. The details of this theorem are themselves not so important—it is mostly a tedious verification. What is important to know is that for a morphism in Bun_G^F, $(\widetilde{f}, f) \colon \xi \to \xi'$, for any indices i and k such that $U_i \cap f^{-1}(V_k) \neq \varnothing$, $\mathsf{P}(\widetilde{f}, f)$ sends $[(i, b, g)] \mapsto [(k, f(b), \overline{g}_{ki}(b) \cdot g)]$ where the $\overline{g}_{ki}$ come from the bundle morphism $(\widetilde{f}, f)$.

Proof. Using the explicit construction given in **Theorem 3**, P is defined on objects. It only remains to show that we may define it on morphisms functorially. This means that $\mathsf{P}(\mathrm{id}) = \mathrm{id}$ and $\mathsf{P}(\widetilde{f} \circ \widetilde{g}, f \circ g) = \mathsf{P}(\widetilde{f}, f) \circ \mathsf{P}(\widetilde{g}, g)$.

Suppose we are given a morphism $(\widetilde{f}, f) \colon \xi \to \xi'$ of G-bundles.

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{f}} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

Denote

$$\mathscr{A} = \{(\varphi_i, U_i)\}_{i\in I} \qquad \text{and} \qquad \mathscr{A}' = \{(\psi_j, V_j)\}_{j\in J}$$

the G-atlases for ξ and ξ', respectively. Then this means there are CAT morphisms $\overline{g}_{ji}\colon U_i \cap f^{-1}(V_j') \to G$ such that $\psi_j \widetilde{f} \varphi_i^{-1}(b, v) = (f(b), \overline{g}_{ji}(b) \cdot v)$ for all trivializations for which this expression makes sense.

Notation. To avoid clutter, we will use *the same* roman letters for the transition functions of $\mathscr{A}$ and $\mathscr{A}'$ and the maps $\overline{g}_{ji}$. We will differentiate between the transition functions of $\mathscr{A}$ and $\mathscr{A}'$ by putting a prime on those of $\mathscr{A}'$—in other words, we will write them as g_{ij} and g'_{ij} with the understanding that in the first case, $i, j \in I$ whereas in the second case $i, j \in J$. For $\overline{g}_{ji}$, we must understand that the first index (in this case, j) is an element of J and the second (in this case i) is an element of I.

By definition,

$$\mathsf{P}(\xi) = \coprod_{i\in I} U_i \times G / \{(j, b, g) \sim (i, b, g_{ij}(b) \cdot g) : i, j \in I,\ b \in U_i \cap U_j,\ g \in G\},$$

and

$$\mathsf{P}(\xi') = \coprod_{i\in J} V_i \times G / \{(j, b', g) \sim (i, b', g'_{ij}(b') \cdot g) : i, j \in J,\ b' \in V_i \cap V_j,\ g \in G\}.$$

There is an evident candidate for a map $\mathsf{P}(\widetilde{f}, f)\colon \mathsf{P}(\xi) \to \mathsf{P}(\xi')$—namely, whenever $U_i \cap f^{-1}(V_j) \neq \varnothing$,

$$[(i, b, g)] \mapsto [(j, f(b), \overline{g}_{ji}(b) \cdot g)].$$

To see that this is well-defined, we must show it is independent of the equivalence classes chosen. Towards this end, we prove the following claim.

Claim 4. For each $b \in (U_i \cap U_j) \cap f^{-1}(V_k)$,

$$\overline{g}_{ki}(b) g_{ij}(b) = \overline{g}_{kj}(b).$$

Similarly, for each $b \in (U_j \cap f^{-1}(V_i)) \cap f^{-1}(V_k)$,

$$g'_{ki}(f(b)) \overline{g}_{ij}(b) = \overline{g}_{kj}(b).$$

In particular, on the relevant domains,

$$\overline{g}_{ki} g_{ij} = \overline{g}_{kj} \qquad \text{and} \qquad (g'_{ki} \circ f) \overline{g}_{ij} = \overline{g}_{kj}.$$

In other words, we can "cancel" adjacent indices that are the same. Note that this makes sense because the adjacent indices appearing above depend on the same G-atlas—either both depend on $\mathscr{A}$ or both depend on $\mathscr{A}'$.

For the bundles ξ and ξ', we know that $\psi_k \widetilde{f} \varphi_j^{-1}(b, v) = (b, \overline{g}_{kj}(b) \cdot v)$ and so

$$(b, \overline{g}_{kj}(b) \cdot v) = \psi_k \widetilde{f} \varphi_j(b, v) = (\psi_k \widetilde{f} \varphi_i^{-1})(\varphi_i \varphi_j^{-1})(b, v) = (b, \overline{g}_{ki}(b) g_{ij}(b) \cdot v).$$

Since the action of G on F is effective, $G \to \mathrm{Aut}_{\mathsf{CAT}}(F)$ is injective and so it must be that $\overline{g}_{kj}(b) = \overline{g}_{ki}(b) g_{ij}(b)$. Similarly, for the bundles ξ and ξ', we know that $\psi_k \widetilde{f} \varphi_j^{-1}(b, v) = (b, \overline{g}_{kj}(b) \cdot v)$ and so

$$(b, \overline{g}_{kj}(b) \cdot v) = \psi_k \widetilde{f} \varphi_j(b, v) = (\psi_k \psi_i^{-1})(\psi_i \widetilde{f} \varphi_j^{-1})(b, v) = (b, g'_{ki}(f(b)) \overline{g}_{ij}(b) \cdot v).$$

Since the action of G on F is effective, $G \to \mathrm{Aut}_{\mathsf{CAT}}(F)$ is injective and so it must be that $g'_{ki}(f(b)) \overline{g}_{ij}(b) = \overline{g}_{kj}(b)$. In each of the two cases just considered, $b \in B$ was any element for which the expression makes sense, and hence

$$\overline{g}_{ki} g_{ij} = \overline{g}_{kj} \qquad \text{and} \qquad (g'_{ki} \circ f) \overline{g}_{ij} = \overline{g}_{kj}$$

where these expressions are defined. This proves the claim.

With the notation as in the claim, this shows that the assignment is well-defined since we may conclude that the following diagram commutes up to the quotient relations.

$$\begin{array}{ccccc}
(j,b,g) & \longmapsto & (k, f(b), \overline{g}_{kj}(b)\cdot g) & & \\
\| & & \wr & & \\
(j,b,g) & \overset{(**)}{\longmapsto} & (k', f(b), g'_{k'k}(f(b))\overline{g}_{kj}(b)\cdot g) & \overset{(*)}{=\!=} & (k', f(b), \overline{g}_{k'j}(b)\cdot g) \\
\wr & & \| & & \|(*) \\
(i,b,g_{ij}(b)\cdot g) & \underset{(***)}{\longmapsto} & (k', f(b), g'_{k'i}(f(b))\overline{g}_{ij}(b)\cdot g) & \underset{(*)}{=\!=} & (k', f(b), \overline{g}_{k'j}(b)\cdot g)
\end{array}$$

Each equality labeled $(*)$ follows from the claim. For $(**)$, we note that $(j,b,g)\mapsto(k', f(b), \overline{g}_{k'j}(b)\cdot g)$ and apply the claim to conclude that

$$(k', f(b), \overline{g}_{k'j}(b)\cdot g) = (k', f(b), g'_{k'k}(f(b))\overline{g}_{kj}(b)\cdot g).$$

The case of $(***)$ follows by observing that $(i,b,g_{ij}(b)\cdot g)\mapsto(k', b, \overline{g}_{k'i}(b)g_{ij}(b)\cdot g)$ and by the claim,

$$(k', b, \overline{g}_{k'i}(b)g_{ij}(b)\cdot g) = (k', b, \overline{g}_{k'j}(b)\cdot g).$$

The way to read the above diagram is as follows. If we choose a different equivalence for $(k, f(b), \overline{g}_j(b)\cdot g)$, then the square with bottom leg $(*)$ shows that the map is still well-defined since for the pair of indices j,k, $(j,b,g)\mapsto (k', f(b), g_{k'k}(f(b))\overline{g}_{kj}(b)\cdot g)$ and the target is identified with $(k, f(b), \overline{g}_{kj}(b)\cdot g)$. On the other hand, if we alter the equivalence class of (j,b,g), then the same reasoning applied to the bottom left square shows this association is well-defined since $(k', f(b), g_{k'k}(f(b))\overline{g}_{kj}(b)\cdot g)$ is identified with $(k, f(b), \overline{g}_{kj}(b)\cdot g)$.

This defines our map $\mathsf{P}(\widetilde{f}, f)$. To see that this map covers $f\colon B\to B'$ in the sense that the following diagram commutes

$$\begin{array}{ccc}
\mathsf{P}(\xi) & \xrightarrow{\mathsf{P}(\widetilde{f},f)} & \mathsf{P}(\xi') \\
\downarrow & & \downarrow \\
B & \xrightarrow[f]{} & B'
\end{array}$$

we chase elements. Taking $[(i,b,g)]\in\mathsf{P}(\xi)$,

$$\begin{array}{ccc}
[(i,b,g)] & \longmapsto & [(k, f(b), \overline{g}_{kj}(b)\cdot g)] \\
\downarrow & & \downarrow \\
b & \longmapsto & f(b)
\end{array}$$

and so we have an honest morphism of fiber bundles

To check that this is a CAT morphism and a morphism of G-bundles, it suffices to check that $\psi_k\mathsf{P}(\widetilde{f}, f)\varphi_i^{-1}$ is CAT and has the appropriate form. Taking $(b,g)\in U_i\times G$ and recalling how the trivializations behave the specific construction given in the fiber bundle construction theorem, we see that this map sends

$$(b,g)\overset{\varphi_i^{-1}}{\mapsto}[(i,b,g)]\overset{\mathsf{P}(\widetilde{f},f)}{\mapsto}[(k, f(b), \overline{g}_{ki}(b)\cdot g)]\overset{\psi_k}{\mapsto}(f(b), \overline{g}_{ki}(b)\cdot g).$$

This has the appropriate form to be a morphism of G-bundles and so $\mathsf{P}(\widetilde{f}, f)$ is a morphism of G-bundles. To see that this is sufficient to verify that the morphism is smooth when $\mathsf{CAT} = \mathsf{DIFF}$, recall that the trivializations φ_i and ψ_k are diffeomorphisms—hence, it suffices to check smoothness using these trivializations. From above computation, we see that this is certainly smooth since the association $(b,g)\mapsto(f(b), \overline{G}_{ki}(b)\cdot g)$ has smooth component functions from our assumptions.

The only thing left to check is that the association is functorial. For the identity map $\mathrm{id}\colon \xi\to\xi$, $\mathsf{P}(\mathrm{id})$ sends the equivalence class $[(i,b,g)]$ to the equivalence class $[(i,b,g)]$, so $\mathsf{P}(\mathrm{id}) = \mathrm{id}$.

The case of composites is more subtle and we must open the blackbox of **Claim 1** to make progress. For composites

$$\xi_1 \xrightarrow{(\widetilde{g},g)} \xi_2 \xrightarrow{(\widetilde{f},f)} \xi_3$$

where ξ_j has bundle atlas $\mathscr{A}_j = \{(\varphi_{j_i}, U_{j_i})\}_{i\in I_j}$. The form of the composite map

$$\varphi_{3_k}\widetilde{f}\varphi_{2_j}^{-1}\varphi_{2_j}\widetilde{g}\varphi_{1_i}\colon U_{1_i}\cap g^{-1}(U_{2_j})\cap(f\circ g)^{-1}(U_{3_k})$$

is

$$[(i,b,g)] \mapsto [(j,g(b),\overline{g}_{2_j1_i}(b)\cdot g)] \mapsto [(k,f(g(b)),\overline{g}_{3_k2_j}(f(b))\overline{g}_{2_j1_i}(b)\cdot g)].$$

From the analysis of **Claim 1**, we know that while $(\overline{g}_{3_k2_j}\circ f)\overline{g}_{2_j1_i}$ is only defined on $U_{1_i}\cap g^{-1}(U_{2_j})\cap (f\circ g)^{-1}(U_{3_k})$ it is admits a CAT extension to $U_{1_i}\cap (g\circ f)^{-1}(U_{3_k})$ and, hence, there is a CAT map $\overline{g}_{3_k1_i}\colon U_{1_i}\cap (g\circ f)^{-1}(U_{3_k})\to G$ for which

$$\varphi_{3_k}\widetilde{f}\circ\widetilde{g}\varphi_{1_i}\colon (U_{1_i}\cap (f\circ g)^{-1}(U_{3_k}))\times F\to U_{3_k}\times F$$

has the appropriate form

$$(b,g)\mapsto (b,\overline{g}_{3_k1_i}(b)\cdot g).$$

Thus, from how we defined P on morphisms, we conclude that $\mathsf{P}(\widetilde{f}\circ\widetilde{g}, f\circ g)$ sends

$$[(1_i,b,g)]\mapsto [(3_k,f(g(b)),\overline{g}_{3_k1_i}(b)\cdot g)].$$

On the other hand, fixing an appropriate index 2_j, $\mathsf{P}(\widetilde{f},f)\circ\mathsf{P}(\widetilde{g},g)$ is defined by

$$[(1_i,b,g)]\mapsto [(2_j,g(b),\overline{g}_{2_j1_i}(b)\cdot g)]\mapsto [(3_k,f(g(b)),\overline{g}_{3_k2_j}(g(b))\overline{g}_{2_j1_i}(b)\cdot g)].$$

Cracking open the blackbox of **Claim 1** yet again, we observe that $\overline{g}_{3_k1_i}$ was defined in such a way that it agrees with $(\overline{g}_{3_k2_j}\circ g)\overline{g}_{2_j1_i}$ on $U_{1_i}\cap g^{-1}(U_{2_j})\cap (f\circ g)^{-1}(U_{3_k})$ and so these two composites are locally equal and therefore equal.

We conclude that

$$\mathsf{P}(\widetilde{f}\circ\widetilde{g}, f\circ g) = \mathsf{P}(\widetilde{f},f)\circ\mathsf{P}(\widetilde{g},g),$$

completing the proof. ■

Remark. One way to make most things go through when $G \curvearrowright F$ is *not* effective is to replace all equalities between transition functions g_{ij} and the functions $\overline{g}_{ij}$ that appear in the definition of a morphism of G-bundles and replace them by equality in $\mathrm{Aut}(F)$. For instance, instead of requiring that $g_{ij}g_{jk}=g_{ik}$ in the cocycle conditions, we could only require that $g_{ij}(b)g_{jk}(b)\cdot g = g_{ik}(b)\cdot g$ for all $b\in B$ and $g\in G$ for which this expression makes sense.

Working with these definitions, what fails above when $G\curvearrowright F$ is not effective is that **Claim 4** would only be true in terms of the action of F—that is, $\overline{g}_{ki}(b)g_{ij}(b)\cdot v = \overline{g}_{kj}(b)\cdot v$. When the action of G on F is not effective, we cannot conclude from this that $\overline{g}_{ki}(b)g_{ij}(b)\cdot g = \overline{g}_{kj}(b)\cdot g$ and so the whole proof breaks down there.

3.4 (*) Exercises: 3

***Exercise** ((*), ⋆⋆). Every morphism of $\mathsf{Bun}_G^F(B)$ is an isomorphism. In other words, a morphism of G-bundles with typical fiber F over B is necessarily and isomorphism.*

***Exercise** ((**), ⋆⋆). Show that φ_i and φ_i^{-1} as defined really constitute set-theoretic inverses and that the following diagram commutes on the level of sets*

$$\begin{array}{ccc} \pi^{-1}(V_i) & \longrightarrow & V_i\times F \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ V_i & = & V_i \end{array}$$

Then show that φ_i and φ_i^{-1} are continuous and conclude that they are homeomorphisms. [Hint: Recall that f^*P' is a subspace and $B\times P'$. Consider that the open subspace $V_i\times (p')^{-1}(U_i) = f^{-1}(U_i)\times (p')^{-1}(U_i)$ is homeomorphic to $f^{-1}(U_i)\times U_i\times F$.]

***Exercise** ((*), ⋆⋆). Show that f^*P' admits the structure of smooth manifold with an atlas of charts given by $(x\times y)\circ\varphi_i$ where $x\times y$ is a chart of $U_i\times F$ and thus of $B\times F$. Conclude that the φ_i are themselves diffeomorphisms with respect to this smooth structure.* [Hint: Regardless of whether or not φ_i and φ_j^{-1} are smooth in any differentiable structure on f^*P', the preceding computation shows that φ_{ij} is smooth.]

Lecture 4

4.1 The Category Prin_G of Principal G-Bundles

We need a more workable definition of principal G-bundles. We construct such a definition in this section.

Lemma 4.1.1. *Fix a choice of* CAT*. Let* $\xi = (E, p, B, G, F, \mathscr{A}) \in \mathsf{Bun}_G^F$*. Let* $H \leq G$ *be a subgroup for which there is a right action* $F \curvearrowleft H$ *compatible with the* G *action making* F *into a* (G, H)*-space.*

(a) *There is a unique way to define a right action* $E \curvearrowleft H$ *such that*

  (i) *The right action of* H *on* E *is a fiberwise isomorphism and thus covers the projection to the base space;*
  (ii) *For every trivialization* $(\varphi, U) \in \mathscr{A}$*,* $\varphi \colon p^{-1}(U) \to U \times G$ *is right* H*-equivariant where the right action of* H *on* $U \times G$ *is defined by* $(b, g) \cdot h = (b, gh)$*.*

  This action is, moreover, independent of the equivalence class of the atlas for ξ*.*

(b) F *always admits a compatible action of* H *when* H *is contained in the center of* G*.*

Warning. It one tries to use the say left action of G on F, then equivariance will fail in general because we will want $\varphi_{ij}(x, gv) = g\varphi_{ij}(x, v)$ and $g\varphi_{ij}(x, v) = g(x, g_{ij}(x)v) = (x, gg_{ij}(x)v)$ whereas $\varphi_{ij}(x, gv) = (x, g_{ij}(x)gv)$ so in general these will not be equal unless G is abelian.

Proof. **(a)** It suffices to consider the case that $G = H$, since the proof for a more general subgroup of G will simply follow by changing letters where appropriate to indicate which elements belong to G and which elements belong to H.

For each trivialization $(\varphi, U) \in \mathscr{A}$, we assert that the following

$$x \cdot g = x \cdot_\varphi g \stackrel{\text{def}}{=} \varphi^{-1}(\varphi(x) \cdot g)$$

defines a right action of G on $p^{-1}(U)$. To see that $(x \cdot g) \cdot h = x \cdot gh$, we push symbols around and find

$$(x \cdot g) \cdot h = \varphi^{-1}(\varphi(\varphi^{-1}(\varphi(x) \cdot g)) \cdot h) = \varphi^{-1}((\varphi(x) \cdot g) \cdot h) = \varphi^{-1}(\varphi(x) \cdot gh)$$

as desired. Furthermore, $x \cdot e = \varphi^{-1}(\varphi(x) \cdot e) = x$, clearly. So this is indeed a group action and it is a CAT action since everything in sight is assumed to be CAT. Doing this for every trivialization in $\mathscr{A}$, we claim that the resulting action is well-defined, CAT and is such that every trivialization in $\mathscr{A}$ is right G-equivariant with respect to this action. If it is well-defined, it will certainly be CAT since continuity and smoothness are local conditions and the definition just given is smooth over each trivializing nbhd.

Thus, it suffices to show that this action makes all trivializations G-equivariant and that for any trivializations (φ, U) and (ψ, V) and for each $x \in p^{-1}(U \cap V)$,

$$x \cdot_\varphi g = x \cdot_\psi g.$$

That is, we would like $\varphi^{-1}(\varphi(x) \cdot g) = \psi^{-1}(\psi(x) \cdot g)$. We consider the latter first.

Since ψ is an isomorphism, we could just as well ask that $\psi(x) \cdot g = (\psi\varphi^{-1})(\varphi(x) \cdot g)$. In the coordinates of the trivialization, write

$$\varphi(x) = (z, f).$$

Then this looks like a map sending

$$(z, f) \cdot g = (z, f \cdot g) \mapsto (z, g_{V,U}(z)f \cdot g) = (z, g_{V,U}(z)f) \cdot g$$

where $g_{V,U} \colon U \cap V \to G$. Of course, when $g = e$, we know by cancelling the φ's that

$$(z, g_{V,U}(z)f) = (\psi\varphi^{-1})(\varphi(x)) = \psi(\varphi^{-1}\varphi(x)) = \psi(x).$$

This shows that we recover $\psi(x)\cdot g = (z, g_{V,U}(z)h)\cdot g$ as this is equal to $(\psi\varphi^{-1})(\varphi(x)\cdot g)$ as desired. We used the fact that the actions were compatible to write $g_{V,U}(z)(fg) = (g_{V,U}(z)f)g$.

Having shown that $x \cdot_\varphi g = x \cdot_\psi g$, equivariance of all trivializations is now immediate. Indeed, we have

$$\varphi(x\cdot g) = \varphi(x\cdot_\varphi g) = \varphi(x \cdot_\psi g) = \varphi(\varphi^{-1}(\varphi(x)\cdot g)) = \varphi(x)\cdot g.$$

As for uniqueness, given any other such action with $\varphi(x\cdot g) = \varphi(x)\cdot g$, it follows that $x\cdot g = \varphi^{-1}(\varphi(x)\cdot g)$ and that is uniqueness. Now, suppose $\mathscr{A}' \sim \mathscr{A}$ is an equivalent atlas. Then $\mathscr{A}'' = \mathscr{A}\cup\mathscr{A}'$ is also an equivalent atlas by **Exercise 4(c)**. Hence, the above construction shows that the action produced by $\mathscr{A}''$ is independent of the charts used and, hence, we can use either the charts of $\mathscr{A}$ or the charts of $\mathscr{A}'$ to produce this action.

(b) This is a modification of the proof above, since nothing goes wrong if the elements of H commute with everything in G and we may simply define the right action in this case to be $v\cdot h \overset{\text{def}}{=} h^{-1}\cdot F$ (left and right actions are the same for abelian groups). ■

Remark. For example, if $F = \mathbf{R}^n$, $G = \mathrm{GL}_n(\mathbf{R})$ and $H = \{rI\}_{r\in\mathbf{R}\setminus\{0\}}$ the subgroup of matrices that are a non-zero multiple of the identity matrix, then we recover the structure of a vector bundle on $\mathsf{Bun}^{\mathbf{R}^n}_{\mathrm{GL}_n(\mathbf{R})}$.

Example 3. For a principal CAT G-bundle $\xi\in\mathsf{Bun}^G_G$, there is an evident right G-action on each trivialization $U\times G$ by translation (i.e. $(x,g)\cdot g' = (x, gg')$). Hence, for a principal G-bundle ξ, there is unique way to give ξ a fiber preserving right G-action on the total space for which all trivializations are right G-equivariant by the preceding lemma.

Exercise 17.

(a) *Show that for $\mathsf{CAT} = \mathsf{TOP}$, if $P \xrightarrow{p} B$ is a principal G-bundle then there is an isomorphism of principal G-bundles*

$$\begin{array}{ccc} P & = & P \\ \downarrow & & \downarrow \\ P/G & \xrightarrow[\cong]{} & B \end{array}$$

It turns out this is true in DIFF too, as we will see in the next subsection.

(b) *Show that a CAT morphism of principal G-bundles $(\widetilde{f}, f)\colon \xi\to\xi'$ is the same as providing a pair of maps $\widetilde{f}\colon P\to P'$ and $f\colon B\to B'$ for which TFDC*

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & P' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

and such that $\widetilde{f}$ is right G-equivariant. Conversely, show that such a pair of maps determine a morphism of bundles in Bun^G_G. [Hint: Reduce to the case of trivial principal G-bundles.]

(c) *Given principal G-bundle ξ and ξ', show that a G-equivariant map on total spaces is the same as a morphism $\xi\to\xi'$ in Prin_G. In other words, a G-equivariant map of total spaces completely determines the map on base spaces.*

(d) *Show that there is at most one principal G-bundle structure on any* fiber bundle $E\to B$ *with fiber G and a free transitive right G action covering the projection to B.* [Hint: The identity map is equivariant. Apply a preceding result.]

(e) *For a right G-space E, let $E^* = \{(x,y)\in E\times E : xG = yG\}$ consist of all pairs of element in the same orbit. Suppose $E \curvearrowleft G$ is a free G-action. Show that $\pi\colon E\to E/G$ is a principal G-bundle **iff***

(1) *for each $p\in E/G$, there is an open nbhd U of p and a section $s_U\colon U\to E$ of π (naturally, we require the section to be continuous);*

(2) *the **translation function** $\tau\colon E^*\to G$ defined by*

$$\tau(x,y) = g \text{ where } x\cdot g = y$$

is continuous.

[Hint: First show ($\Rightarrow$). For ($\Leftarrow$), define (abusing notation) $\varphi_U^{-1}\colon U\times G\to\pi^{-1}(U)$ by $\varphi_U^{-1}(p,g) = s_U(p)\cdot g$. Show this is an honest a G-equivariant isomorphism by constructing an inverse using τ and then show that the associated transition functions are continuous.]

Remark. **(e)** of this exercise may be upgraded to the smooth category by equipping E/G with a compatible smooth structure for the fiber bundle and its topology and requiring all maps in sight to be smooth. We will say more about this smooth structure in **Lecture 4** but for the purposes of seeing how this may be done, you can just as well assume that E/G has a compatible smooth structure for the fiber bundle and its topology.

Warning. **(d)** only gives a criteria for recognizing when the quotient map is the projection of a principal G-bundle when the total space already has some fixed right G-action, but it does not tell us what structure it can be equipped with if we forget the G-action. Some of the following terminology used in this warning will be explained in **Lecture 4**.

For example, in the case $G = \mathrm{GL}_n(\mathbf{R})$, Ethan Atkin shows in "$K$-theory doesn't exist" (don't worry, it does) that if $p\colon P \to B$ is a principal G-bundle, then the composite projection $P \times G \to P \to B$ can be made into a principal $G \times G$-bundle in at two different ways by writing it as a pullback in two different ways. In the first case, as a pullback

$$\begin{array}{ccc} P \times G & \xrightarrow{\mu} & P \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ P & \longrightarrow & B \end{array}$$

and in the second case as a pullback

$$\begin{array}{ccc} P \times G & \xrightarrow{p\times 1} & B \times G \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow \\ P & \longrightarrow & B \end{array}$$

In the first case, the resulting bundle $P \times G \to P \to B$ is a principal $G \times G$-bundle whose associated $\mathbf{R}^{2n}$-bundle (i.e., a rank $2n$ vector bundle) is the Whitney sum of the associated $\mathbf{R}^n$-bundles to $P \to B$, whereas the latter has associated $\mathbf{R}^{2n}$-bundle the Whitney sum of the trivial $\mathbf{R}^n$-bundle with the associated $\mathbf{R}^n$-bundles to $P \to B$. These will not in general be equivalent!

Hence, by the equivalence of categories we will establish in **Lecture 4**, it must be that the two principal $G \times G$-bundle structures on $P \times G$ are non-isomorphic. Hence, this is an example of a bundle with fiber $G \times G$ for which we can find two different right $G \times G$-actions on the total space for which the projection can be made into a principal $G \times G$-bundle!

We are now ready to provide our improved definition of the category of principal G-bundles.

Definition. Fix a choice of CAT. Let Prin_G be the category whose objects are those Bun_G^G equipped with a fiberwise right G-action on the total space such that all trivializations of the bundle are G-equivariant where $U \times G$ is given the right G-action defined by $(x, g) \cdot g' = (x, gg')$. The morphisms of Prin_G are pairs of maps $\widetilde{f}\colon P \to P'$ and $f\colon B \to B'$ for which TFDC

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & P' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

and such that $\widetilde{f}$ is right G-equivariant.

Remark. It turns out that for $P \to B$ a principal G-bundle, the action $P \curvearrowleft G$ is necessarily free and faithful. Faithfulness follows from freeness since $G \neq \emptyset$ as it must have an identity element. For freeness, equivariance of the trivializations implies that it suffices to show that the action $X \times G \curvearrowleft G$ is free. This follows since $(x, g_0) \cdot g = (y, g_1) \cdot g$ if and only if $x = y$ and $g_0 = g_1$, visibly.

Exercise 18. *Show that there is an equivalence of categories* $\mathsf{Bun}_G^G \simeq \mathsf{Prin}_G$.

4.2 The Associated Bundle Construction

4.2.1 Digression on Quotients in DIFF

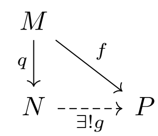
During the question period of **Lecture 3**, Min asked a very good question about the associated bundle construction—namely, what is the natural smooth structure to equip the space $P \times_G F$ with?

In this subsection, we will address Min's question by giving an exercise that provides a partial answer to the following related question.

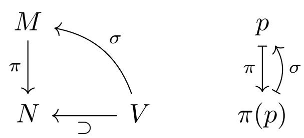
Question 1. When is a smooth map $M \to N$ between manifolds with corners a quotient map in DIFF?

Remark. When DIFF only includes manifolds without boundary, this question has a good partial answer—all surjective submersions between two manifolds without boundary are quotient maps in the category of smooth manifolds without boundary.

By a ***quotient map*** in DIFF, we mean a map $q\colon M \to N$ having the following universal property. If $f\colon M \to P$ is smooth and constant on the fibers of q[1], there is a unique smooth map $g\colon N \to P$ making TFDC



Definition. Fix a choice of CAT to work in. Let $\pi\colon M \to N$ be a CAT map. We will say that f satisfies the CAT ***local section condition*** if for every $p \in M$, there is a nbhd V of $\pi(p)$ and a CAT section



such that $\sigma(\pi(p)) = p$. We call σ a CAT ***local section of*** π.

We now establish some preliminaries.

Lemma 4.2.1. *Every fiber bundle satisfies the local section condition. In the smooth category, the sections may be take to be smooth.*

Proof. In a trivializing nbhd U, for each $v \in F$, there is a section $U \to U \times F$ given by $p \mapsto (p, v)$ which shows this. In the smooth category where manifolds have corners, we must take U so small that it is a coordinate nbhd, in which case we may suppose $U \subset \mathbf{R}^m_k$ WLOG. Then the very same section $U \to U \times F$ sending $p \mapsto (p, v)$ is the restriction of the smooth map $\mathbf{R}^m \to \mathbf{R}^m \times F$ having the very same form and so is smooth. ■

Lemma 4.2.2. *If $\pi\colon E \to B$ is a* CAT *fiber bundle with typical fiber F, then π is a* topological *quotient map—in particular, it is an open map. If $E \to B$ is simply a* CAT *map satisfying that each point $p \in B$ has an open nbhd U and at least one section, then it is surjective and a* topological *quotient map.*

Remark. The immediate corollary of this proposition is that maps satisfying the stronger local section condition are topological quotient maps.

Proof. Pick a base $\mathscr{B}$ for the topology of B consisting of trivializing open sets. This can be done by picking any base $\mathscr{B}'$ for B and a covering $\mathscr{U}$ of B by trivializing open nbhds and then letting $\mathscr{B} = \{U \cap B' : U \in \mathscr{U},\ B' \in \mathscr{B}'\}$. Then one easily checks that $\mathscr{E} = \left\{\pi^{-1}(U) : U \in \mathscr{B}\right\}$ is a base for the topology of E by thinking of it as $\{U \times F : U \in \mathscr{B}\}$. Hence, if V is any open set in E, then it is the union of sets in $\mathscr{E}$ and for each set $\pi^{-1}(U) \in \mathscr{E}$, $\pi(\pi^{-1}(U)) = U$ is open, so if we express $V = \bigcup_{i \in I} U_i$, then $\pi(V) = \bigcup_{i \in I} \pi(U_i)$ since image commutes with unions. This shows that π is an open map and surjective continuous open maps are quotient maps.

Let $\pi\colon E \to B$ satisfy the local section condition. Surjectivity follows essentially by the definition. To see that it is a quotient map, it suffices to show that if $\pi^{-1}(V)$ is open, then V is open in B. Cover B by open nbhds satisfying the local section condition, say $\{W_i\}_{i \in I}$ and let $\sigma_i\colon W_i \to E$ be the section that is guaranteed to exist by the hypotheses. By assumption, $\pi^{-1}(V)$ is open and, hence, $\sigma_i^{-1}\pi^{-1}(V) = (\pi \circ \sigma_i)^{-1}(V)$ is open in W_i and thus open in B. It now suffices to show that $\bigcup_{i \in I} (\pi \circ \sigma_i)^{-1}(V) = V$ since this is a union of open sets in B and, of course, this is true since the W_i cover B, completing the proof. ■

Lemma 4.2.3. *If $\pi\colon M \to N$ is a smooth map satisfying the local section condition, then π is a smooth submersion.*

Proof. This is a purely local problem, so by picking charts we may suppose $\pi\colon \mathbf{R}^{m-\ell} \times \mathbf{R}^{\ell}_{+} \to \mathbf{R}^{n-k} \times \mathbf{R}^{k}_{+}$ is smooth. Fix $p \in \mathbf{R}^m_\ell$ and let $V \subset \mathbf{R}^{n-k} \times \mathbf{R}^k_+$ be an open subset admitting a smooth local section $\sigma\colon V \to \mathbf{R}^m_\ell$ mapping $\pi(p)$ to p.

Now, $\pi\sigma = \mathrm{id}$ on V of $\pi(p)$ in $\mathbf{R}^n_k$ and therefore $\pi_* \circ \sigma_*$ is the identity on tangent vectors over W and so, in particular, π_{*p} is surjective. Of course, p was not fixed, so π_{*p} is surjective for all p. ■

[1] This means that the map $f|\, q^{-1}(p)$ is the constant map for all $p \in N$.

Remark. Even though $V \subset \mathbf{R}^n_k$, our definition of smoothness for domains and codomains of this sort implies that the differential of the function is well-defined on V—indeed, we can extend everything in sight appropriately and then simply use smoothness in the usual case.

Exercise 19. *In this exercise, all manifolds are assumed to have corners. Let $\pi\colon M \to N$ be a smooth* surjective *map satisfying the local section condition.*

(a) *A function $f\colon N \to P$ is smooth **iff** $f \circ \pi$ is smooth.* [Hint: Use surjectivity and the local section condition.]
(b) *For any smooth $f\colon M \to P$ constant on the fibers of π, there is a unique smooth map $\widetilde{f}\colon N \to P$ such that $\widetilde{f} \circ \pi = f$.* [Hint: Use one of the preliminary results.]
(c) *Conclude that that π is a quotient map in DIFF and therefore the smooth structure on N is the unique one, up to diffeomorphism, for which **(b)** is true for the map π.* [Hint: Make categorically flavored argument.]
(d) *Suppose $\pi'\colon M \to N'$ is another smooth* surjective *map satisfying the local section condition. Suppose that π and π' are constant on each other's fibers. Show that N and N' are diffeomorphic.* [Hint: This is a small generalization of **(c)**.]
(e) *If $P \to B$ is a principal G-bundle in $\mathsf{CAT} = \mathsf{DIFF}$, show that there is an isomorphism of DIFF principal G-bundles*

$$\begin{array}{ccc} P & = & P \\ \downarrow & & \downarrow \\ P/G & \xrightarrow[\cong]{} & B \end{array}$$

Here is a question that seems interesting.

Question 2. Consider the category of smooth manifolds *without* boundary. Are all smooth quotient maps smooth submersions?

4.2.2 Associated Bundle Construction

Lemma 4.2.4. *Let $\xi \ni \mathsf{Prin}_G$ be a CAT principal G-bundle and let $G \curvearrowright F$ by $\ell\colon G \times F \to F$. Denote the **associated bundle** with fiber F by $P[F, \ell] = P[F] \overset{def}{=} P \times_G F$, where $P \times_G F$ is the colimit of*

$$P \times G \times F \underset{\mathrm{id}\times\mathrm{act}}{\overset{\mathrm{act}\times\mathrm{id}}{\rightrightarrows}} P \times F$$

In other words,

$$P \times_G F = (P \times F)/\left\{(pg, v) \sim (p, gv)\right\}$$

with the quotient topology.

(a) *The projection $P \times F \to P \xrightarrow{p} B$ induces a CAT map $\widetilde{p}\colon P \times_G F \to B$.*
(b) *With this map $\widetilde{p}$, $P \times_G F$ is the total space of a CAT bundle $\xi \times_G F \in \mathsf{Bun}_G^F$ trivializable over the same open sets as ξ and having the same transition functions as ξ and whose trivializations are*

$$\widetilde{\varphi}_i\colon \widetilde{p}^{-1}(U_i) = p^{-1}(U_i) \times_G F \xrightarrow{\varphi_i \times_G \mathrm{id}} U_i \times G \times_G F \cong U_i \times F,$$

where the isomorphism $U \times G \times_G F \cong U \times F$ may be chosen naturally and φ_i is a trivialization of ξ.
(c) *When $\mathsf{CAT} = \mathsf{DIFF}$, the natural map $P \times F \to P \times_G F$ is a smooth quotient map.*

Proof. Let $\mathscr{A} = \{(\varphi_i, U_i)\}_{i \in I}$ be the G-atlas for ξ. Each φ_i is then G-equivariant by the definition of Prin_G. We have

$$\begin{array}{ccccc} P \times G \times F & \underset{\mathrm{id}\times\mathrm{act}}{\overset{\mathrm{act}\times\mathrm{id}}{\rightrightarrows}} & P \times F & \dashrightarrow & P \times_G F \\ \downarrow & & \downarrow & & \\ P & & P & & \Big\downarrow{\scriptstyle \widetilde{p}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} & & \\ B & = & B & = & B \end{array}$$

which yields $\widetilde{p}$ by universal properties. Note that $\widetilde{p}^{-1}(U_i)$ is open in $P \times_G F$ since $p^{-1}(U_i) \times F$ is an open saturated set for this topological quotient map. Moreover, $\widetilde{p}^{-1}(U_i) = p^{-1}(U_i) \times_G F$.

In the smooth category, we can define a smooth structure on $P \times_G F$ by requesting that each of the maps

$$\widetilde{\varphi}_i \colon \widetilde{p}^{-1}(U_i) = p^{-1}(U_i) \times_G F \xrightarrow{\varphi_i \times_G \mathrm{id}} U_i \times G \times_G F \cong U_i \times F$$

be part of the smooth structure. This is well-defined since φ_i is G-equivariant. The isomorphism $U_i \times G \times_G F \cong U_i \times F$ is given naturally by $(x, g, v) \mapsto (x, gv)$. Note that $\widetilde{\varphi}_i$ is defined on $[(x, v)] \in p^{-1}(U_i) \times_G F$ by

$$[(x, v)] \mapsto [(\varphi_i(x), v)] = [(p(x), g_0, v)] \mapsto (p(x), g_0 v)$$

This respects the group action since φ_i is G-equivariant, so

$$[(xg, g^{-1}v)] \mapsto [(p(x), g_0 g, g^{-1}v)] = \mapsto (p(x), g_0 v).$$

Smooth compatibility follows by observing that $\widetilde{\varphi}_{ij}$ has the form

$$(x, v) \mapsto [(x, e, v)] \mapsto [(\varphi_{ij}(x, e), v)] = [(x, g_{ij}(x), v)] \mapsto (x, g_{ij}(x)v)$$

where brackets denote equivalence class. This is a CAT function

$$U_i \cap U_j \times F \to U_i \cap U_j \times F$$

since g_{ij} was assumed CAT as the transition function for the bundle ξ. It is easy to verify that TFDC:

$$\begin{array}{ccc} \widetilde{p}^{-1}(U_i) & \xrightarrow{\widetilde{\varphi}_i} & U_i \times F \\ \downarrow & & \downarrow \\ U_i & = & U_i \end{array}$$

Thus, these are also trivializations for a fiber bundle.

This is the natural smooth structure on $P \times_G F$ since, with it, $P \times F \to P \times_G F$ satisfies the smooth local section condition and, hence, by **Exercise 17**, it is the unique smooth structure for which this is true and thus $P \times F \to P \times_G F$ is moreover a smooth quotient map. Indeed, pick $[(x, v)] \in P \times_G F$. The fiber over this point in $P \times F$ consists of all elements of the form (xg^{-1}, gv) with $g \in G$. Fix one such (pg^{-1}, gv). Working in a trivialization, $\widetilde{p}^{-1}(U_i) \cong U_i \times F$ and $p^{-1}(U_i) \cong U_i \times G$, suppose $\varphi_i(x) = (p(x), g_0)$. Consider the smooth map $U_i \times F \to U_i \times G \times F$ sending

$$(z, v) \mapsto (z, g_0 g, g^{-1} g_0^{-1} v) = (z, g_0 g, (g_0 g)^{-1} v).$$

This is smooth because the G-action is smooth. Using this map, we let the desired section be given by the dashed arrow in the following diagram

$$\begin{array}{ccc} U_i \times F & \longrightarrow & U_i \times G \times F \\ \downarrow\cong & & \downarrow\cong \\ \widetilde{p}^{-1}(U_i) & \dashrightarrow & p^{-1}(U_i) \times F \end{array}$$

By equivariance of the trivializations for ξ, we know that φ_i is a G-equivariant isomorphism, so since $\varphi_i(x) = (p(x), g_0)$,

$$\begin{array}{ccccc} [(p(x), g_0 v)] & \longmapsto & (p(x), g_0 g, g^{-1} g_0^{-1} g_0 v) & = & (p(x), g_0 g, g^{-1} v) \\ \uparrow & & & & \downarrow \\ [(x, v)] & & \longmapsto & & (\varphi_i^{-1}(p(x), g_0 g), g^{-1} v) \end{array}$$

and $(\varphi_i^{-1}(p(x), g_0 g), g^{-1}v) = (\varphi_i^{-1}(p(x), g_0)g, g^{-1}v) = (xg, g^{-1}v)$ as desired. ■

Theorem 4.2.5. *For each $G \curvearrowright F$, $- \times_G F \colon \mathsf{Prin}_G \to \mathsf{Bun}_G^F$ is a functor. For a morphism $(\widetilde{f}, f)$ of Prin_G considered as a morphism of Bun_G^G, the morphism $(\widetilde{f}, f) \times_G F$ has the same associated $\overline{g}_{ij}$'s and thus is a morphism in Bun_G^F.*

Proof. Given

$$\begin{array}{ccc} P & \xrightarrow{\widetilde{f}} & P' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

we wish to show that

$$\begin{array}{ccc} P \times_G F & \xrightarrow{\widetilde{f} \times_G F} & P' \times_G F \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

is a morphism in Bun_G^F. Since $\widetilde{f}$ is right G-equivariant, there is at least a map $\widetilde{f} \times_G F \colon P \times_G F \to P' \times_G F$ that is continuous. Clearly, when $(\widetilde{f}, f) = \mathrm{id}_\xi$, this construction will preserve the identity.

To see that it is a morphism of G-bundles, observe that since $(\widetilde{f}, f)$ is a morphism of G-bundles, it has the form

$$\psi_i \widetilde{f} \varphi_j^{-1}(b, g) = (f(b), \overline{g}_{ij}(b) \cdot g).$$

Since ψ_i and φ_j are G-equivariant isomorphisms, it follows that

$$\widetilde{\psi}_i \widetilde{f} \times_G F \widetilde{\varphi}_j^{-1}(b, v) = (f(b), \overline{g}_{ij}(b) \cdot v)$$

and so $\widetilde{f} \times_G F$ is a morphism of G-bundles. In particular, this shows that when $\mathsf{CAT} = \mathsf{DIFF}$, $\widetilde{f} \times_G F$ is smooth by definition of the smooth structures involved and the fact that $\overline{g}_{ij}$ is smooth. (Alternatively, smoothness of $\widetilde{f} \times_G F$ follows from the description of the relevant spaces as smooth quotients.)

It only remains to check that composites behave well. For this, it suffices to observe that $(\widetilde{f} \circ \widetilde{g}) \times_G F = \widetilde{f} \times_G F \circ \widetilde{g} \times_G F$ since $- \times_G F$ is a functor $\mathsf{Top}^G \to \mathsf{Top}$ being a colimit construction. ■

4.3 The Equivalence Between Prin_G and Bun_G^F

Theorem 4.3.1. *Fix a choice of* CAT. *If G acts effectively on F, then there is an equivalence of categories*

$$- \times_G F : \mathsf{Prin}_G \rightleftarrows \mathsf{Bun}_G^F : \mathsf{P}$$

Proof. Given $\xi = (P, p, B) \in \mathsf{Bun}_F^G$, $\mathsf{P}(\xi) \times_G F$ is the unique bundle having the same transition functions as ξ associated to the same open sets as the G-atlas for ξ. By **Corollary 1** and **Theorem 1(c)**, there is an isomorphism of bundles

$$\eta_\xi \colon \mathsf{P}(\xi) \times_G F \xrightarrow{\cong} \xi$$

where this isomorphism is defined on a trivialization nbhd U_i by

$$\underbrace{\psi_i^{-1}}_{\text{for } \xi} \circ \underbrace{\widetilde{\varphi}_i}_{\text{for } \mathsf{P}(\xi) \times_G F}$$

Hence, in the bundle coordinates provided by these two trivializations, the map $\mathsf{P}(\xi) \times_G F \to \xi$ looks like the identity map $U_i \times F \xrightarrow{\mathrm{id}} U_i \times F$.

We assert that this isomorphism is natural. Let $(\widetilde{f}, f) \colon \xi \to \xi'$. The naturality diagram we wish to consider is

$$\begin{array}{ccc} \mathsf{P}(\xi) \times_G F & \xrightarrow{(\widetilde{f}, f) \times_G F} & \mathsf{P}(\xi') \times_G F \\ \downarrow{\scriptstyle \eta_\xi} & & \downarrow{\scriptstyle \eta_{\xi'}} \\ \xi & \xrightarrow{(\widetilde{f}, f)} & \xi' \end{array}$$

The base space version of this diagram obviously commutes so the interesting action is on total spaces. In the evident well-chosen trivializations and using the definition for a morphism in Bun_G^F, the naturality diagram looks like

$$\begin{array}{ccc} U_i \cap f^{-1}(U'_k) \times F & \longrightarrow & U'_k \times F \\ \Big\downarrow \text{id} & & \Big\downarrow \text{id} \\ U_i \cap f^{-1}(U'_k) \times F & \longrightarrow & U'_k \times F \end{array}$$

where the horizontal arrows are both of the form $(b, v) \mapsto (f(b), \overline{g}_{ki}(b) \cdot v)$—for the top horizontal arrow, this is a consequence of **Theorem 4** and **Theorem 5**.

Now consider the case of $\mathsf{P}(\xi \times_G F)$. Once again, this has the same trivializing open sets as ξ and the same transition functions. **Theorem 1(c)** and **Corollary 1** once again furnish isomorphism we claim are natural. The argument is the same, mutatis mutandis. ■

4.4 Exercise

Exercise 20. *If $P \to B$ is a principal G-bundle, show that the orbit space P/G is isomorphic to B. In particular, show that this can be done in both the smooth and topological categories.* [Hint: Use **Exercise 19** to show that $P \to P/G$ is a smooth quotient and that $P \to B$ is a smooth quotient.]

Lecture 5

5.1 Homotopy Invariance

5.1.1 Preliminaries

The goal is the prove the following theorem in the smooth case.

Theorem 5.1.1 (Homotopy Invariance Theorem). *Fix a choice of* CAT*. Let F be an effective G-space.*

(a) *Given a principal G-bundle $\xi\colon G \to P \xrightarrow{\pi} B \times I$, there is an isomorphism $\xi_0 = \xi|\, B \times \{0\} \cong \xi|\, B \times \{1\} = \xi_1$ of principal G-bundles (so in particular the ξ_i are principal G-bundles).*

(b) *Given a G-bundle $\xi : F \to E \xrightarrow{p} B \times I$, there is an isomorphism $\xi_0 = \xi|\, B \times \{0\} \cong \xi|\, B \times \{1\} = \xi_1$.*

(c) *Given a pullback*

$$\begin{array}{ccc} f^*P' & \longrightarrow & P' \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

*the bundle f^*P' over B depends up to isomorphism only on the homotopy class of f. In the smooth case, when B' has boundary but no corners, then $f \simeq g$ smoothly **iff** $f \simeq g$ continuously.*

When CAT $=$ TOP, this is still true, but annoying to prove, so we restrict to the smooth case.

Remark. To prove this in the case CAT $=$ DIFF, it is useful to introduce connections on principal G-bundles. The idea is that a bundle ξ over $B \times I$ should look like instructions for flowing from $\xi|\, B \times \{0\}$ to $\xi|\, B \times \{1\}$. We will use the principal G-connection as crutch to construct the desired flow.

Definition (Whitney Sum). The ***Whitney sum*** of two vector bundles $E_1 \xrightarrow{p_1} B$ and $E_2 \xrightarrow{p_2} B$ over a base space B is the vector bundle over B with total space denoted by $E_1 \oplus E_2$ fitting into a pullback diagram

$$\begin{array}{ccc} E_1 \oplus E_2 & \longrightarrow & E_1 \times E_2 \\ \downarrow & & \downarrow{\scriptstyle p_1 \times p_2} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array}$$

where Δ is the diagonal map $b \mapsto (b, b)$.

Exercise 21. *Fix a choice of* CAT *and suppose $E_1 \xrightarrow{p_1} B$ and $E_2 \xrightarrow{p_2} B$ are vector bundles of rank k_1 and k_2, respectively, and with bundle atlases $\mathscr{A}_1 = \{(\varphi_{i,1}, U_{i,1})\}_{i \in I}$ and $\mathscr{A}_2 = \{(\varphi_{j,2}, U_{j,2})\}_{j \in J}$, respectively.*

(a) *Show that $E_1 \times E_2 \xrightarrow{p_1 \times p_2} B \times B$ is a vector bundle with bundle atlas $\mathscr{A}_1 \times \mathscr{A}_2$ and associated transition functions $g_{ii'} \times g_{jj'} \colon (U_{i,1} \times U_{j,2}) \cap (U_{i',1} \times U_{j',2}) \to \mathrm{GL}_{k_1}(\mathbf{R}) \times \mathrm{GL}_{k_2}(\mathbf{R}) \subset \mathrm{GL}_{k_1+k_2}(\mathbf{R})$ are the block diagonal matrices*

$$\begin{pmatrix} g_{ii'} & 0 \\ 0 & g_{jj'} \end{pmatrix}.$$

(b) *If $E_1 \to B$ has rank k_1 and $E_2 \to B$ has rank k_2, show that $E_1 \oplus E_2 \to B$ has rank $k_1 + k_2$.*

(c) *Using the description of the transition functions given in the pullback theorem, characterize the trivializations and transition functions for the vector bundle $E_1 \oplus E_2 \to B$.*

(d) *For $b \in B$, let E_{1b} and E_{2b} denote the fibers over b in E_1 and E_2, respectively. Let $\pi\colon \coprod_{b\in B} E_{1b} \oplus E_{2b} \to B$ be the evident projection. Show that π can be given the structure of a rank $k_1 + k_2$ vector bundle as follows. The topology on $\coprod_{b\in B} E_{1b} \oplus E_{2b}$ is generated by*

$$\left\{\pi^{-1}(V_{(ij)}) : \exists i \in I,\ \exists j \in J,\ V_{(ij)} \overset{open}{\subset} U_{i,1} \cap U_{j,2}\right\}$$

with trivializations

$$\varphi_i \times_B \varphi_j \colon \coprod_{b\in Ui,1\cap Uj,2} E_{1b} \oplus E_{2b} \to (U_{i,1} \cap U_{j,2}) \times (\mathbf{R}^{k_1} \oplus \mathbf{R}^{k_2})$$

sending $v_1 \oplus v_2 \in E_{1b} \oplus E_{2b}$ to $(b, \varphi_i(v_1) \oplus \varphi_j(v_2))$. When $\mathsf{CAT} = \mathsf{DIFF}$, equip this bundle with the structure of a smooth manifold and show that π is smooth.

Remark. The above exercise will be vastly generalized in a later exercise when we consider continuous functors and smooth functors on categories of finite dimensional vector spaces. This exercise should be considered as a warm-up.

Definitions. Fix $\mathsf{CAT} = \mathsf{DIFF}$ and let $\xi = (P, p, B, G, G)$ be a principal G-bundle.

(a) The subbundle $V \overset{\text{def}}{=} \operatorname{Ker} p_* \subset TP$ is called the ***vertical subbundle***. A ***principal G-connection*** is a choice of complement $H \subset TP$, the ***horizontal subbundle*** such that $V \oplus H \cong TP$ and $H_{p\cdot g} = (R_g)_{*p}(H_p)$, where $R_g\colon G \to G$ is right multiplication by g.

(b) Consider $\mathfrak{g} = T_e G$ topologized as $\mathbf{R}$. Each $X \in \mathfrak{g}$ determines the ***fundamental vector field*** $X_p^* = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX)$. In this way, $\mathfrak{g}$ acts on the total space of a principal G-bundle by $\sigma\colon P \times \mathfrak{g} \to TP$ sending $(p, X) \mapsto X_p^*$.

Remark. Any such splitting of TP is equivalent to a section a section $TP/V \to TP$ of the quotient map $TP \to V$ with the image determining H.

Proposition 5.1.2. *Let $G \to P \xrightarrow{\pi} B$ be a smooth principal G-bundle and fix a principal G-connection on P.*

(a) *Given a vector field X on B, there is a unique horizontal lift X^* of X. The lift X^* is invariant under G (i.e., $(R_g)_{*p} X_p^* = X_{pg}^*$ for all $p \in P$ and $g \in G$). Conversely, every horizontal vector field X^* on P invariant under G is the lift of a vector field X on B.*

(b) *For every smooth lift $\widetilde{\gamma}\colon I \to P$, every $g \in G$ and every vector field X along γ, there exists a unique horizontal lift $\widetilde{X}\colon I \to TP$ such that $\widetilde{X}_t \in T_{\widetilde{\gamma(t)}g} P$. For any fixed lift $\widetilde{\gamma}$ of γ, the collection of all such horizontal lifts of X assemble into a smooth map $I \times G \to TP$ that is G-equivariant.*

By a ***horizontal lift*** of a vector field, we mean a vector field for which $\pi_*(X_p^*) = X_{\pi(p)}$. By a ***vector field along a curve***, we mean a smooth curve $\gamma\colon I \to B$ and a commutative diagram of smooth maps

$$\begin{array}{ccc} & & TB \\ & \nearrow & \downarrow \\ I & \xrightarrow[\gamma]{} & B \end{array}$$

so equivalently a section of $\gamma^* TB$.

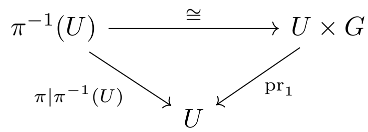
Warning. Throughout this proof, we implicitly rely upon the fact that the trivializations of a principal G-bundle are all G-equivariant. This is critical for passing from the local formulation provided by bundle trivializations back to the non-local situation.

Proof. **(a)** Write $TP = V \oplus H$. Note that π_* collapses V and induces a fiberwise isomorphism $\pi_{*p}\colon H_p \cong T_{\pi(p)} M$. Uniqueness is clear since we can and must take $X_p^* = \pi_{*p}^{-1}(X_{\pi(p)}) \in H_p$. To see invariance under G, observe that $(R_g)_{*p} X_p^* = X_p^*([- \circ R_g]_{pg})$ as a derivation of germs of smooth functions at pg, $\mathfrak{G}_{pg}^\infty$. To verify that this is X_{pg}^*, it suffices by uniqueness to check two things—we must verify that $(R_g)_{*p} X_p^* \in H_{pg}$ has trivial vertical component and we must verify that $\pi_{*pg}(R_g)_{*p} X_p^* = X_{\pi(pg)} = X_{\pi(p)}$. The first part follows since $X_p^* \in H_p$ with trivial vertical component and we assumed that $(R_g)_{*p} H_p = H_{pg}$ so this part is fine; for the second part, we observe that $\pi \circ R_g = \pi$ and therefore

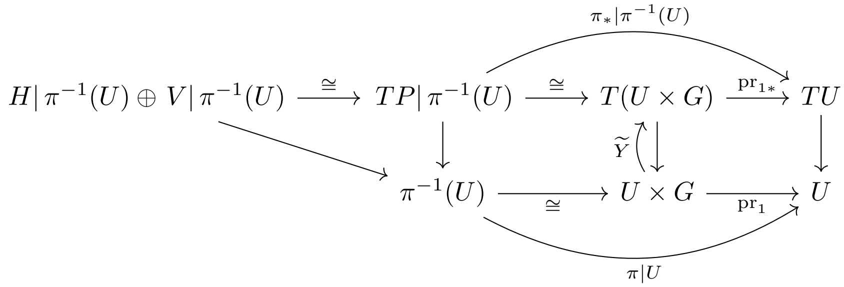
$$\pi_{*pg}(R_g)_{*p} X_p^* = (\pi \circ R_g)_{*p} X_p^* = \pi_{*p} X_p^*,$$

which is known to be equal to $X_{\pi(p)}$ as desired.

To check that this is smooth, we can take a nbhd U of $x \in B$ such that $\pi^{-1}(U) \cong U \times G$ and then using this isomorphism we obtain a smooth vector field Y on $\pi^{-1}(U)$ such that $\pi_* Y_p = X_{\pi(p)}$ by setting, for instance, $\widetilde{Y}_{x,g} = (X_x, 0) \in T_{(x,g)}(U \times G)$, which is certainly smooth, and then using the indicated isomorphism to produce Y. This checks out since naturality of tangent bundles and commutativity of the bundle projections over trivializations



yields the following commutative diagram with horizontal arrows the evident ones



We now see that X^* must be the horizontal component of Y, and since $TP = V \oplus H$, the projection $TP \to H$ is smooth and so we see that X^* is locally smooth and hence globally smooth.

Conversely, if given a horizontal vector field X^* on P which is invariant under the action of G, then for every $b \in B$ we pick $p \in \pi^{-1}(b)$ and set

$$X_b = \pi_{*,p} X^*_p.$$

This is independent of the choice of p since any other $p' \in \pi^{-1}(x)$ is related to p by $pg = p'$ for some g and so by invariance,

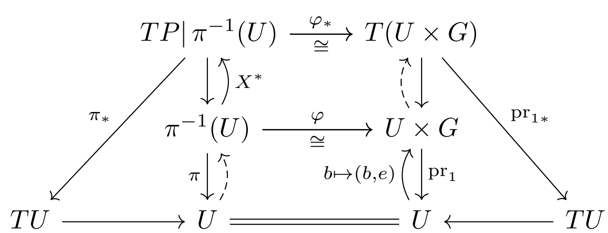
$$\pi_{*,pg}(X^*_{pg}) = \pi_{*,pg}((R_g)_{*p} X^*_p)$$

and the same calculation we did above shows that

$$\pi_{*,pg}(X^*_{pg}) = \pi_{*,pg}((R_g)_{*,p} X^*_p) = \pi_{*,p} X^*_p = X_b.$$

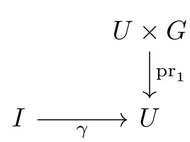
If X so-defined is smooth, then X^* is clearly its lift so we must show X is smooth.

Pick a trivializing open set for $\pi\colon P \to B$, say U with trivialization $\varphi\colon \pi^{-1}(U) \to U \times G$. In the following diagram, the straight-arrow part is commutative and the bent arrows are sections of the adjacent vertical arrows. The dashed bent arrows are induced by the solid for the corresponding square.



Then $X|U$ is the composite $U \to TP|\,\pi^{-1}(U) \to TU$ since for any $p \in \pi^{-1}(b)$, $X_b = \pi_{*,p} X^*_p$. This composite is smooth because the dashed section $U \to \pi^{-1}(U)$ is the composite of smooth functions defined by $b \mapsto \varphi^{-1}(b, e)$. Thus, X is smooth.

(b) Now consider the vector field along a curve case. First consider the case that $\mathrm{Im}(\gamma) \subset U$ where U is a trivializing open set. Since the trivializations of a principal G-bundle are G-equivariant, this suffices. WLOG we may suppose U is also the domain of a coordinate chart. We may then reduce to the case of a trivial principal G-bundle



Any lift $\widetilde{\gamma}\colon I \to U \times G$ of γ is thus given by $t \mapsto (\gamma(t), c(t))$ where $c\colon I \to G$ is smooth, so pick any smooth $c\colon I \to G$ (for example, $c \equiv e$). Note that $\mathrm{pr}_{1*} \dot{\widetilde{\gamma}}(t) = \dot{\gamma}(t)$ since on derivations of germs of smooth functions, this is

$$\frac{d}{ds} - \circ \mathrm{pr}_1 \circ \widetilde{\gamma}\Big|_{s=t} = \frac{d}{ds} - \circ \gamma\Big|_{s=t} = \dot{\gamma}(t)$$

as $\mathrm{pr}_1 \circ \widetilde{\gamma} = \gamma$.

Define $\widetilde{X}\colon I \to T(U\times G)$ by letting $\widetilde{X}_t \in T_{\widetilde{\gamma}(t)}(U\times G)$ be the unique horizontal vector projecting to $X_t \in T_{\gamma(t)}(U)$. To see that this is smooth, we note the smooth vector field $\widetilde{Y}\colon I \to T(U\times G)$ along $\widetilde{\gamma}$ sending $t \mapsto \dot{\widetilde{\gamma}}(t)$ satisfies that $\mathrm{pr}_{1*}\dot{\widetilde{\gamma}}(t) = \dot{\gamma}(t)$. So we may consider its horizontal component by the smooth projection $T(U\times G) \cong H \oplus V \to H$ onto the horizontal subbundle. Hence, $\widetilde{X}$ is smooth because it is the horizontal component of the smooth vector $\widetilde{Y}$ along $\widetilde{\gamma}$. Uniqueness is the same argument as before.

For the smooth vector field constructed above, define a smooth map $\Gamma\colon I\times G \to T(U\times G)$ by $(t,g) \mapsto R_{g*\widetilde{\gamma}(t)}\widetilde{X}_t$. The same computation as before shows that the vector field $I \to T(U\times G)$ sending $t \mapsto R_{g*}\widetilde{X}_t$ is smooth along $R_g \circ \widetilde{\gamma}$ as a composite of smooth functions and is horizontal as well. In fact, by uniqueness, $R_{g*\widetilde{\gamma}(t)}\widetilde{X}_t$ is the horizontal component of the smooth lift of γ given by $R_g \circ \widetilde{\gamma}$.

The same idea as above works to show Γ is smooth—the function $\widetilde{\Gamma}\colon I\times G \to T(U\times G)$ sending $(t,g) \mapsto \dfrac{d}{dt}R_g\circ\widetilde{\gamma}$ is smooth since if $\mu\colon P\times G \to P$ is the action map, then this is $(t,g) \mapsto \dfrac{d}{dt}\mu(\widetilde{\gamma}(t),g)$ and $\mu(\widetilde{\gamma}(t),g)$ is a composite of smooth functions. Thus, Γ is the horizontal component of $\widetilde{\Gamma}$ by the above and so is smooth. As for equivariance, $(t,g)\cdot g' = (t,gg')$ maps under Γ to (note the contravariance of the right action!)

$$\Gamma(t,gg') = R_{gg'*\widetilde{\gamma}(t)}\widetilde{X}_t = R_{g'*\widetilde{\gamma}(t)\cdot g} \circ R_{g*\widetilde{\gamma}(t)}\widetilde{X}_t = \Gamma(t,g)\cdot g'.$$

In the general case, one takes a covering of $\operatorname{Im}\gamma$ by finitely many open trivializable open sets that are also domains of manifold charts by compactness, say $U_1,\dots,U_n$, and when $n>1$, one finds numbers $0 < w_1 < t_1 < w_2 < t_2 < \cdots < w_n = t_n = 1$ such that $[0,t_1) \in \gamma^{-1}(U_1)$, $(w_1,t_2) \in \gamma^{-1}(U_2)$ and in general for $i \neq 1,n$, $\gamma^{-1}(U_i) = (w_{i-1},t_i)$ and $\gamma^{-1}(U_n) = (w_{n-1},1]$.

The base case of the induction suffices to see how the argument goes, so suppose $n=2$. Then by the above we can construct the lift of $\gamma|\,[0,t_1)$, say $\widetilde{\gamma}|\,[0,t_1)$ by abuse of notation. For the lift of $\gamma|\,(w_1,1]$, we associate to it in coordinates the smooth curve c_1 and for the latter the smooth curve c_2 and we require these to glue appropriately to give a global lift by requiring that they match on overlaps which we call $\widetilde{\gamma}$. The local case above shows that we may construct $\widetilde{X}$ on $[0,t_1)$ and on $(w_1,1]$ such that the two pieces agree on the common time domain and satisfies that $\widetilde{X}_t$ is a vector field along $\widetilde{\gamma}$ covering X.

Exercise 22. *Show that this covering may be arranged and fill in the details.* [Hint: This is done in the proof of **Theorem 9**.]

As before, we set $\Gamma(t,g) = R_{g*\widetilde{\gamma}(t)}\widetilde{X}_t$ and show it is smooth since it is the horizontal component of $(t,g) \mapsto \dfrac{d}{dt}\mu(\widetilde{\gamma}(t),g)$. This completes the proof. ■

Remark. An important case of the second part of this proposition is when $X\colon I \to TB$ is velocity field v of γ. By abuse of notation, we will consider the resulting collection of horizontal lifts of this vector field $v^*\colon I\times G \to TP$ the horizontal lift of the velocity field.

The analogue of parallel transport is constructed using the following theorem. We need to recall a basic ODE result first. See, for instance, Theorem 2.2 here.

Theorem 5.1.3 (Picard-Lindelöf Theorem). *Fix $\Omega \subset \mathbf{R}\times\mathbf{R}^n$ open and $(t_0,y_0)\in\Omega$. Let $F\colon \Omega \to \mathbf{R}^n$ be of class C^k $(0 \le k \le \infty)$. Consider the IVP*

$$\dot{y}(t) = F(t,y), \qquad y(t_0) = y_0. \tag{$*$}$$

Suppose F is locally Lipschitz continuous in the second argument and uniformly continuous with respect to the first. Then for any $a,b>0$ with

$$[t_0-a,t_0+a]\times\{x\in\mathbf{R}^n : d(x,y_0)\le b\} \subset \Omega \qquad \textit{and} \qquad M = \sup\{|F(t,y)| : (t,y)\in[t_0-a,t_0+a]\times\{z\in\mathbf{R}^n : d(z,y_0)\le b\}\},$$

there exists a unique solution to $(*)$ on the interval

$$[t_0 - \min\{a,b/M\}, t_0 + \min\{a,b/M\}]$$

of class C^k on $(t_0 - \min\{a,b/M\}, t_0 + \min\{a,b/M\})$ and left (resp. right) differentiable at the appropriate endpoints.

Remark. In order to streamline the presentation, we defer the proof of the next theorem to another subsection.

Theorem 5.1.4. *Let $G \to P \xrightarrow{\pi} B$ be a principal G-bundle and suppose we chosen a principal G-bundle connection. If $\gamma\colon I \to B$ is smooth, then for each $p \in \pi^{-1}\gamma(0)$, there is a unique smooth horizontal lift $\gamma^*\colon I \to P$ covering γ and starting at p.*

For a lift to be horizontal, we mean that all tangent vectors lie in the specified horizontal subspace.

Corollary 5.1.5. *Let $G \to P \xrightarrow{\pi} B$ be a principal G-bundle and suppose we chosen a principal G-bundle connection. Let $J \subset \mathbf{R}$ be one of $\mathbf{R}$, $\mathbf{R}_+$ or $\mathbf{R}_-$. If $\gamma\colon J \to B$ is smooth, then for each $p \in \pi^{-1}\gamma(0)$, there is a unique smooth horizontal lift $\gamma^*\colon J \to P$ covering γ and starting at p.*

Proof. The case of $\mathbf{R}_+ = [0, \infty)$ suffices. First, we observe that the restriction to I is smooth and so has a smooth lift. We can repeat this at $[1, 2]$ as well and so on and so forth lifting at each endpoint in the obvious way. Call the thus constructed lift γ^*. To verify that this is indeed smooth, note that the only possible issue occurs at integers n. But since the lift over an interval $[n - \frac{1}{2}, n + \frac{1}{2}]$ starting at $\gamma^*(n - \frac{1}{2})$ is smooth by the theorem and, hence, by uniqueness, the lift must be smooth at n. ■

5.1.2 Proof of Smooth Homotopy Invariance

We can now prove **Theorem 5**.

*Proof (**Theorem 5**).* **(a)** Fix once and for all a principal G-connection on the principal G-bundle ξ.

Let X be a vector field on $B \times I$ given by $X_{(p,s)} = \left.\dfrac{d}{dt}\right|_{t=s}$ where we think of $T_{(p,s)}(B \times I)$ as having derivations of germs of smooth functions as its elements. Let X^* be a horizontal lift of this vector field. As we have seen, X^* is π-related to X in that TFDC:

$$\begin{array}{ccc} TP & \xrightarrow{\pi_*} & T(B \times I) \\ {\scriptstyle X^*}\uparrow & & \uparrow{\scriptstyle X} \\ P & \xrightarrow[\pi]{} & B \times I \end{array}$$

Notice that the integral curve of X through (p, s_0), say $\gamma = \gamma^{(b,s_0)}$ is the solution to $X\gamma(s) = \dot{\gamma}(s)$. In other words, if we think of $\gamma\colon I \to B \times \mathbf{R}$ having components (γ_1, γ_2), then

$$\left.\frac{d}{dt}\right|_{t=\gamma(s)} = \dot{\gamma}(s)$$

which forces $\dot{\gamma}_1 \equiv 0$ and forces $\dot{\gamma}_2(s) = 1$ so that $\dot{\gamma}_2 \equiv 1$ and so $\gamma_2(s) = s + C$. In particular, subject to $\gamma(0) = (b, s_0)$, $\gamma(s) = (b, s + s_0)$. Hence, the flow is given by

$$\Phi^X(t, b, s) = \gamma^{(b,s)}(t) = (b, t + s).$$

Since we are allowing the base manifold to have boundary or even corners, the usual results showing the existence and smoothness of flows do not hold. However, Φ^X is smooth and clearly exists by virtue of our just having described it explicitly. The flow domain is the subset

$$A_X = \{(t, b, s) \in \mathbf{R} \times B \times I : 0 \le t + s \le 1,\ 0 \le s \le 1\}.$$

Observe that the flow is the restriction of the smooth map $\mathbf{R} \times B \times \mathbf{R} \to B \times \mathbf{R}$ sending $(t, b, s) \mapsto (b, t + s)$ to the subset A_X. Thus, the map restricted to $\mathbf{R} \times B \times I$ is smooth and so to show the flow Φ^X is smooth, it suffices by the universal property of submanifolds (even with corners) to show that A_X is a submanifold of $\mathbf{R} \times B \times I$.

Note that $\{(t, s) \in \mathbf{R} \times I : 0 \le t + s \le 1,\ 0 \le s \le 1\}$ is a submanifold of $\mathbf{R} \times I$. This is because the map $(t, s) \mapsto (t+s, s)$ is a diffeomorphism onto $I \times I$ and so for a chart of B about b, say $x\colon U \to \mathbf{R}^{n-k} \times \mathbf{R}^k_+$, the map $(t, b, s) \mapsto (s, x(b), t - s)$ is a diffeomorphism into $I \times x(U) \times I \subset \mathbf{R} \times \mathbf{R}^n_k \times \mathbf{R}$ and it is clear from here how to construct submanifold charts.

Now we wish to construct the flow for the horizontal lift Φ^{X^*} and show that it is smooth. By the preceding corollary, for each $p \in \pi^{-1}((b, s))$, there is a unique lift of the entire integral curve $\gamma^{(b,s)}$ to a curve $\widetilde{\gamma}^p$ and $\widetilde{\gamma}^p$ will be an integral curve of X^*. We note that $X^*_p \neq 0$ for any p since X is never 0.

Claim 5. The lifts $\widetilde{\gamma}^p$ assemble into a smooth flow Φ^{X^*} for X^*.

The flow certainly exists. The only possible problem is smoothness at boundary points. Since X^* is smooth, it admits a smooth extension in nbhds U of boundary points in coordinates. Since flows exist, are unique and are smooth locally in Euclidean space, the restriction of the smooth local flow generated by this extension of X^* to U is consequently smooth and so Φ^{X^*} is smooth.

Claim 6. The maximal flow domain D_* of Φ^{X^*} is the preimage of the maximal flow domain A_X for Φ^X under the map $\pi\colon \mathbf{R}\times P \to \mathbf{R}\times (B\times I)$.

Indeed, if we could extend $\widetilde{\gamma}^p$ in time further than $\gamma^{\pi(p)}$, then $\widetilde{\gamma}^p$ would project down to an extension of $\gamma^{\pi(p)}$ satisfying all relevant properties which contradicts the fact that $\gamma^{\pi(p)}$ manifestly cannot be extended further without shooting off the manifold.

Pick $p\in\pi^{-1}(x,t)$. Notice that $R_g\widetilde{\gamma}^p = \widetilde{\gamma}^{pg}$. This is because $R_g\widetilde{\gamma}^p$ covers $\gamma^{(x,t)}$, $R_g\widetilde{\gamma}^p(0)=\widetilde{\gamma}^p(0)g$ and

$$R_{g*}(\widetilde{\gamma}^p)'(t) = R_{g*}X^*_{\widetilde{\gamma}(t)} = X^*_{\widetilde{\gamma}(t)g}$$

since X^* is a horizontal lift and $\widetilde{\gamma}^p$ is an integral curve of X^*. Together, this means that $R_g\widetilde{\gamma}^p$ is a horizontal lift of γ^p satisfying $R_g\widetilde{\gamma}^p(0)=\widetilde{\gamma}^p(0)g$. By uniqueness of horizontal lifts, $R_g\widetilde{\gamma}^p=\widetilde{\gamma}^{pg}$. Thus, Φ^{X^*} is G-equivariant. In other words,

$$R_g\Phi^{X^*}(t,p) = \Phi^{X^*}(t,R_gp) = \Phi^{X^*}(t,pg).$$

Define $f\colon \xi\to\xi_1\times I$ by $p\mapsto \big(\Phi^{X^*}(1-\mathrm{pr}_2\,\pi(p),p),\mathrm{pr}_2\,\pi(p)\big)$. Observe that $\xi_1=\xi|\,B\times\{1\}$ is an honest submanifold of ξ so since $\Phi^{X^*}(1-\mathrm{pr}_2\,\pi(p),p)$ has image in ξ_1, we may understand it as a smooth map into ξ_1. f is clearly a smooth map and we are forced to take as its smooth inverse $(p,t)\mapsto\Phi^{X^*}(t-1,p)$. We know both of these exist by the preceding claim. Note that since $p\in\xi_1$, it can only flow backwards and thus its flow domain can only consist of non-positive numbers by analyzing the flow domain of X.

Indeed, supposing we have access to the group law, we would have

$$\Phi^{X^*}\Big(\mathrm{pr}_2\,\pi(p)-1,\Phi^{X^*}(1-\mathrm{pr}_2\,\pi(p),p)\Big) = \Phi^{X^*}(\mathrm{pr}_2\,\pi(p)-1+1-\mathrm{pr}_2\,\pi(p),p)=p$$

and similarly in the other direction. Since both composites are always defined, where we consider them, this checks out.

The only thing left to show is that the diffeomorphism so constructed is G-equivariant. Since we have shown that Φ^{X^*} is G-equivariant, this follows. Hence, this is indeed an isomorphism of principal G-bundles.

(b) Given a G-bundle ξ over $B\times I$ with fiber F, we take its associated principal G-bundle $\mathsf{P}(\xi)$ and apply **(a)** to conclude that $\mathsf{P}(\xi)\cong\mathsf{P}(\xi)_1\times I$. We must show that $\mathsf{P}(\xi)_1$ is the associated principal G-bundle to ξ_1 and that $\mathsf{P}(\xi)_1\times I\times_G F\cong\xi_1\times I$. This first thing follows simply by observing that ξ_1 has a G-bundle atlas afforded by restrictions of G-bundle charts for ξ and, hence, similarly for $\mathsf{P}(\xi)$. Thus, $\mathsf{P}(\xi)_1$ is built in the same way and from the same restrictions of trivializations with the same transition functions, as in the definition of the associated principal bundle construction. Thus, $\mathsf{P}(\xi_1)\cong\mathsf{P}(\xi)_1$ by **Theorem 1**. Finally, we wish to check that $\mathsf{P}(\xi)_1\times I\times_G F\cong(\mathsf{P}(\xi)_1\times_G F)\times I$. Since the G-action only intertwines with $\mathsf{P}(\xi)_1$, this is essentially automatic.

(c) When B' has boundary but no corners, the ***Whitney approximation theorem*** allows us to deduce that $f\simeq g$ in the smooth category **iff** $f\simeq g$ in the topological category.

If there is a smooth homotopy $h\colon f\simeq g$, where $f\simeq g\colon B\to B'$, then we may pullback the bundle by $h\colon B\times I\to B'$ and apply the preceding. ■

Remark. This shows that homotopic maps induce equivalent principal G-bundles in the smooth category. One might wonder whether there is a principal G-bundle $P'\to B'$ for which $f^*P'\cong g^*P'$ if and only if $f\simeq g$. This question does not have an answer in DIFF, but for topological (paracompact Hausdorff) spaces, there does exist an answer to this question and the bundle is called a ***universal bundle***—it turns out that there are multiple choices for this bundle.

5.1.3 Proof of Unique Horizontal Path Lifting

Warning. Do not get bogged down in this proof. This argument is *purely a proof of concept*—that is, one can prove the unique horizontal path lifting statement by a geometric/analytic argument I asserted in lecture. For a vastly slicker proof using the connection form see Kobayashi & Nomizu Volume I. Their argument actually works in our setting of manifolds with corners—their proof can simply be inserted following the proof of **Claim 8** below. A different proof may be found in Morita's *Geometry of Differential Forms*.

We begin with the following observation.

Lemma 5.1.6. *For a trivial principal G-bundle $B\times G\to B$, a principal G-connection is completely characterized by a choice of splitting of $T(B\times G)|\,B\times\{e\}\cong H\oplus V|\,B\times\{e\}$.*

Proof. Certainly if we are given a principal G-connection, then $R_{g*}H_{(b,e)}=H_{(b,g)}$ is required. Conversely, given a complement H of $V|\,B\times\{e\}$, extend H by defining $H_{(b,g)}=R_{g*}H_{(b,e)}$, by abuse of notation. Note that the action of G on V

sends vertical vectors to vertical vectors since if $\pi_*(v) = 0$, then $\pi_* \circ R_{g*}(v) = (\pi \circ R_g)_* = \pi_*(v) = 0$. In particular, the G action on V determines a fiberwise automorphism of V. Hence,

$$R_{g*}H_{(b,e)} \cap V_{(b,g)} = R_{g*}H_{(b,e)} \cap R_{g*}V_{(b,g)} = R_{g*}(H_{(b,e)} \cap V_{(b,e)}) = R_{g*}0 = 0.$$

Dimension constraints then force $T_{(b,g)}(B \times G) = H_{(b,g)} \oplus V_{(b,g)}$. Moreover,

$$R_{g'*}H_{(b,g)} = R_{g'*}R_{g*}H_{(b,e)} = R_{gg'*}H_{(b,e)} = H_{(b,gg')}$$

so H is invariant under the right G action on $T(B \times G)$.

Finally, to see that H so defined is a smooth subbundle, pick a smooth trivializating frame of sections $s_1, \dots, s_n$ of $H|\, B \times \{e\}$ in an open set U of $B \times \{e\}$ and extend this by setting $s_1(b,g) = R_{g*}s_1(b,e)$ or, in other words, if $\mu\colon T(B \times G) \times G \to T(B \times G)$ is the action, $s_1(b,g) = R_{g*}s_1(b,e) = \mu(s_1(b,e), g)$. Then we claim that the collection so defined is a trivialization of H over $U \times G$. Indeed, each s_i is smooth as a composite of smooth functions and since R_{g*} defines an isomorphism from $H_{(b,g')}$ to $H_{(b,g'g)}$, the collection of s_i remain linearly independent. Thus, they are a smooth trivialization over $U \times G$.

Since $H \cap V = 0$ fiberwise and $H + V = T(B \times G)$ fiberwise, $T(B \times G) = H \oplus V$. For instance, one constructs a retract in the SES of vector bundles

$$0 \to V \to T(B \times G) \to T(B \times G)/V \to 0$$

by collapsing the subbundle H. ■

Remark. Since all Lie groups are parallelizable, it should not come as a surprise that the major obstruction to trivializing the bundle H comes from B. In fact, the idea to trivialize TG is roughly what we have done above.

We are now ready to give a proof of **Theorem 9**.

*Proof (**Theorem 9**).* We begin with two reductions that will take us all the way to the case of trivial G-bundle of the form $\mathbf{R}^n \times G \to \mathbf{R}^n$.

Claim 7. If the assertion is true for trivial principal G-bundles, then it is true for all principal G-bundles.

Indeed, fix $\gamma\colon I \to B$ smooth. As before, we take a covering of $\operatorname{Im}\gamma$ by finitely many trivializable open sets by compactness, say $U_1, \dots, U_n$, and find numbers $0 < w_1 < t_1 < w_2 < t_2 < \cdots < w_n = t_n = 1$ such that, when $n = 1$, $[0,1] = \gamma^{-1}(U_1)$ and, in general for $n > 1$, $[0, t_1) \in \gamma^{-1}(U_1)$, $(w_1, t_2) \in \gamma^{-1}(U_2)$ and in general for $i \neq 1, n$, $\gamma^{-1}(U_i) = (w_{i-1}, t_i)$ and $\gamma^{-1}(U_n) = (w_{n-1}, 1]$. Inducting on n, when $n = 1$, the claim furnishes the assertion.

The case of $n = 2$ illustrates the general case and induction step, so we consider it with $0 < w_1 < t_1 < t_2 = 1$. Pick numbers $r_1 < r_2$ such that $0 < w_1 < r_1 < r_2 < t_1 < t_2 = 1$. We first construct the horizontal lift on $[0, r_2]$, call it $\widetilde{\gamma}_1$. We then construct a lift $\widetilde{\gamma}_2$ of γ over the interval $[r_1, 1]$ by requiring that $\widetilde{\gamma}_2(r_1) = \widetilde{\gamma}_1(r_1)$. Then $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are smooth and by local uniqueness agree on $[0, r_2] \cap [r_1, 1] = [r_1, r_2]$ and so they agree on the open set (r_1, r_2). Define

$$\widetilde{\gamma}(t) = \begin{cases} \widetilde{\gamma}_1(t) & t \in [0, r_2) \\ \widetilde{\gamma}_2(t) & t \in (r_1, 1]. \end{cases}$$

This is well-defined since for any $t \in (r_1, r_2)$, $\widetilde{\gamma}_1(t) = \widetilde{\gamma}_2(t)$ and it is smooth since the two pieces are smooth and agree on an open set. For general n, one proceeds by choosing numbers $r_{1,1} < r_{1,2} < r_{2,1} < r_{2,2} < \cdots < r_{n-1,1} < r_{n-1,2}$ partitioning I as

$$0 = w_0 = t_0 < w_1 < r_{1,1} < r_{1,2} < t_1 < w_2 < r_{2,1} < r_{2,2} < t_2 < w_3 < \cdots < w_{n-1} < r_{n-1,1} < r_{n-1,2} < w_n = t_n = 1$$

and repeating the same argument.

We may therefore suppose the principal G-bundle is trivial. Indeed, given a bundle trivialization for P, say $\varphi\colon p^{-1}(U) \xrightarrow{\cong} U \times G$, since φ is G-equivariant, φ_* sends the horizontal subbundle $H|\,U$ to a horizontal subbundle for $U \times G$ that satisfies the relevant properties to be such and we thereby obtain a principal G-connection on the trivial principal G-bundle $\mathbf{R}^n_k \times G \to \mathbf{R}^n_k$.

Claim 8. We may further suppose the base space $B = \mathbf{R}^n$.

By the same sort of argument as above except with charts instead of bundle trivializations, we may reduce to assuming that $B = \mathbf{R}^{n-k} \times \mathbf{R}^k_+$ (using a boundary chart). However, there is no harm in replacing $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ in these cases by $\mathbf{R}^n$ as the principal G-bundle over $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ is trivial. However, we must check that the principal G-connection extends as well.

Idea. We will equip H and V over $\mathbf{R}^n_k$ with a smooth bundle metrics by the standard partition of unity argument such that $V^\perp = H$. Smoothness and triviality of all metrics and bundles in sight will allow us to smoothly extend the metrics while preserving $V^\perp = H$ over $\mathbf{R}^n_k$.

As a consequence of the preceding lemma, it suffices to pick a (smooth, as always here) complement of $V|\,\mathbf{R}^n \times \{e\}$ that extends the complement of $V|\,\mathbf{R}^n_k \times \{e\}$ determining the principal G-connection on $\mathbf{R}^n_k \times G \to \mathbf{R}^n_k$. We will define this complement by constructing a suitable Riemannian metric on $\mathbf{R}^n \times G$.

At this point, it is convenient to identify

$$T^*(\mathbf{R}^n \times G)|\,\mathbf{R}^n \times \{e\} \cong T(\mathbf{R}^n) \times \mathbf{R}^m \cong \mathbf{R}^n \times (\mathbf{R}^n \times \mathbf{R}^m)$$

and

$$T^*(\mathbf{R}^n_k \times G)|\,\mathbf{R}^n_k \times \{e\} \cong \mathbf{R}^n_k \times (\mathbf{R}^n \times \mathbf{R}^m).$$

One sees these identifications respect the inclusion of $\mathbf{R}^n_k \times G$ into $\mathbf{R}^n \times G$ using the global standard coordinate systems for Euclidean space and its standard submanifolds with corners. The same is true for all codimension 0 submanifolds of $\mathbf{R}^n$, in fact.

We will build metrics on H and V in a particular way. We illustrate how we do this in the case of H. We then remark that the same can be done for V.

Take a locally finite open cover of $\mathbf{R}^n_k$ by trivializing open sets for the bundle $H^{*\otimes 2}\big|\,\mathbf{R}^n_k \times \{e\}$—these may be taken to be the same as those for H^* so say

$$\mathscr{U} = \{U_i\}_{i \in I}$$

are trivializing for both. By shrinking, we may further suppose it is trivializing for both V and V^* as well, so we do this WLOG. Let $\{\lambda_i\}$ be a partition of unity subordinate to this open cover which—*in particular*—is such that whenever U_i is a boundary chart, λ_i is the restriction of a smooth function $\widetilde{\lambda}_i$ with support contained in a nbhd $\widetilde{U}_i$ of U_i in $\mathbf{R}^n$ where $\widetilde{U}_i$ is a trivializing open set for the projection $V \to \mathbf{R}^n$ and, furthermore, $\left\{\widetilde{\lambda}_i\right\}$ satisfies that $\operatorname{supp}\widetilde{\lambda}_i \subset \widetilde{U}_i$ where, additionally, this open cover is locally finite and where there is an open nbhd W of $\mathbf{R}^n_k$ contained in $\widetilde{U} = \bigcup \widetilde{U}_i$ with $W \subset \overline{W} \subset \widetilde{U}$ and $\sum \widetilde{\lambda}_i\Big|\,\overline{W} \equiv 1$.

To see that this can always be arranged, we proceed as in the usual argument that smooth partitions of unity exist. Take an open cover $\mathscr{V} = \{V_j\}_{j \in J}$ of $\mathbf{R}^n_k$ *in the space* $\mathbf{R}^n$ with the property that $V_j \cap \mathbf{R}^n_k$ is trivializing for both bundles over $\mathbf{R}^n_k$ and V_j is trivializing for V over $\mathbf{R}^n$. We may further suppose they are trivializing for the dual bundles. By paracompactness, this has a locally finite refinement—WLOG call this $\mathscr{V}$ as well. Let $\mathcal{V} = \bigcup \mathscr{V}$ and let $\mathbf{R}^n_k \subset W \subset \overline{W} \subset \mathcal{V}$ be an open subset of $\mathbf{R}^n_k$ with closure contained in $\mathcal{V}$. Such a W exists say by taking a locally finite collection of open subsets contained in $\mathcal{V}$ whose closures are also a subset of $\mathcal{V}$—the closure operator commutes with unions of locally finite collections of sets. Then $\mathscr{W} = \{W_j\}_{j \in J}$ where $W_j = V_j \cap W$ is a locally finite open cover of W. Let φ_j be a smooth function with $0 \le \varphi_j \le 1$, $\varphi_j|\,\overline{W}_j \equiv 1$ and $\operatorname{supp}\varphi_j \subset V_j$ and set $\varphi = \sum \varphi_j$. This is smooth since the cover is locally finite. Set $\rho_j = \varphi_j/\varphi$ and observe that $\sum \rho_j \equiv 1$ so that each λ_j satisfies $0 \le \rho_j \le 1$ and $\operatorname{supp}\rho_j \subset V_j$. It is not hard to see that these ρ_j form the desired partition of unity. As for existence of the functions φ_j, this is a consequence of the strong form of the smooth Urysohn lemma.

Extend the open cover $\left\{\widetilde{U}_i\right\}_{i \in I}$ by appending trivializing open sets for the bundle V and V^* over $\mathbf{R}^n$ which do not intersect $\overline{W}$ and suppose that this is locally finite WLOG. By using the shrinking lemma along with the same procedure as above, we can construct a partition of unity subordinate to this open cover whose ρ_j's restrict to the original ones on W.

Choose a trivializing frame $s_1, \ldots, s_m$ for $T^*(\mathbf{R}^n \times G)|\,U_i$ such that the first n elements of the frame span H pointwise and the last $m - n$ elements span V pointwise. Let $\omega_i \colon U_i \to T^{*\otimes 2}(\mathbf{R}^n \times G)\big|\,U_i \cong U_i \times \mathbf{R}^{m^2}$ be the map sending $x \mapsto \sum_i s_i(x) \otimes s_i(x)$. Then ω_i is positive-definite and symmetric. WLOG we may suppose that on boundary charts everything extends to $\widetilde{U}_i$ in such a way that the s_i remain linearly independent—linear independence is an open condition. The argument proceeds essentially vebatim what is done for Riemannian metrics. It is not hard to see that it is smooth. Call this metric g. Then one sees that $H|\,\mathbf{R}^n_k = (V|\,\mathbf{R}^n_k)^\perp$. Now we should like to smoothly extend this without perturbing it over $\mathbf{R}^n_k$.

By what has already been done, we may do this by constructing the metric however we desire over the new elements of our open cover. Since the ρ_j restrict to the original ones and since the new open sets added to the cover do not intersect $\overline{W}$, the metric over $\mathbf{R}^n_k$ (and indeed W) is the same as the one originally constructed for which we had $H = V^\perp$. We thus declare, globally, $H = V^\perp$.

We are thus free to declare globally $H = V^\perp$ over $\mathbf{R}^n$, furnishing the smooth complement. The preceding lemma now extends this to all of $\mathbf{R}^n \times G$. We are thus reduced to considering the trivial case of $\mathbf{R}^n \times G \to \mathbf{R}^n$.

Observe that a smooth lift

$$\begin{array}{ccc} & & \mathbf{R}^n \times G \\ & \overset{\widetilde{\gamma}}{\nearrow} & \downarrow \\ I & \xrightarrow[\gamma]{} & \mathbf{R}^n \end{array}$$

must have the form $\widetilde{\gamma} = (\gamma, c)$ for some smooth $c\colon I \to G$. Hence, we should like that $\dot{c}$ is the unique vector in $T_{c(t)}G$ such that $\dot{\widetilde{\gamma}}(t) \in H_{(\gamma(t),c(t))}$. Smoothness of γ means that we may replace I by an open interval J around I and thus consider

$$\begin{array}{ccc} & & \mathbf{R}^n \times G \\ & \overset{\widetilde{\gamma}}{\nearrow} & \downarrow \\ J & \xrightarrow[\gamma]{} & \mathbf{R}^n \end{array}$$

For the sake of the following claim, we work in the general non-local case.

Claim 9. There is a unique vertical vector $w \in T_gG$ such that, in a given trivialization (φ, U), $\varphi\colon \pi^{-1}U \cong U \times G$, $(\dot{\gamma}(t), w) \in \varphi_*(H)_{(\gamma(t),g)}$.

By abuse of notation, we denote the induced horizontal subbundle of $T(U \times G)$ coming from $\varphi_*|\,(H|\,\pi^{-1}(U))$ by H as well and similarly denote the vertical subbundle by V, thereby fully reducing to the local case. Uniqueness follows since if $(0, w - w') \in H_{(\gamma(t),g)}$ while $(0, w - w') \in V_{(\gamma(t),g)}$ and hence $w = w'$. For existence, note that every vector not in the vertical subbundle V is the sum of a unique vector in H and a unique vector in V—hence, we may write $(\dot{\gamma}(0), 0) = h + v$ for some horizontal and vertical vectors h and v and therefore $(\dot{\gamma}(t), -v) = h$ is horizontal.

The goal now is to assemble the collection of all such w's into something usable. Again, for the sake of the following claim, we consider the general non-local case.

Claim 10. This unique vector w varies smoothly and assembles into a smooth map into the vertical subbundle $W\colon J \times G \to V \subset TP$. In fact, $W_{(t,g)} = R_{g*}W_{(t,e)}$.

Indeed, consider the function $\Gamma\colon I \times G \to TP$ which in a trivialization has the form $\Gamma(t, g) = (\dot{\gamma}(t), 0) \in T_{(\gamma(t),g)}(\mathbf{R}^n \times G)$. This is smooth by extending $I = [0, 1]$ to some larger open interval J using smoothness of γ. It follows that $W = \Gamma - v^*$ from the above reasoning and the fact that v^* is obtained from Γ by post-composition with the projection onto the horizontal subbundle. To see that W is smooth, it suffices to observe that it is actually the negative of the projection of Γ onto the vertical subbundle.

To see the last part, observe that by the preceding lemma, $R_{g*}\Gamma(t, e) = \Gamma(t, g)$ and so since $\Gamma(t, e) + W_{(t,e)} \in H_{(t,e)}$,

$$\Gamma(t, g) + R_{g*}W_{(t,e)} = R_{g*}(\Gamma(t, e) + W_{(t,e)}) \in R_{g*}H_{(t,e)} = H_{(t,g)}.$$

Exercise 23.

(a) *Show that $\Gamma + W$ is smooth by working in local coordinates.*

(b) *Show that $\Gamma + W$ is smooth in a coordinate free way. Namely, consider the* ***Whitney sum*** *of bundles $TP \oplus TP$ which is, equivalently, the pullback bundle in the diagram*

$$\begin{array}{ccc} TP \oplus TP & \longrightarrow & TP \times TP \\ \downarrow & & \downarrow \\ P & \xrightarrow[\Delta]{} & P \times P \end{array}$$

where Δ is the diagonal map. This is the bundle over P whose fiber over $p \in P$ is $T_pP \oplus T_pP$. Construct a fiberwise linear map $TP \oplus TP \to TP$ sending $(v, w) \in T_pP \oplus T_pP$ to $v + w \in T_pP$. Show that this is smooth using the smooth structure constructed in the pullback theorem and conclude that $\Gamma + W$ is smooth.

We have therefore reduced to the trivial case. Fix $g \in G$. Our set up is

$$\begin{array}{ccc} & & \mathbf{R}^n \times G \\ & & \downarrow \\ J & \longrightarrow & \mathbf{R}^n \end{array}$$

where wish to solve the ODE on the tangent bundle of $\mathbf{R}^n \times G$ given by

$$\dot{y}(t) = W(t, y(t)), \qquad y(0) = g. \tag{$*$}$$

Picking a coordinate nbhd about g, say $(x, \widetilde{U})$, we may assume this is completely Euclidean with $x(\widetilde{U}) = \mathbf{R}^m$. By naturality, we have the following commutative diagram

$$\begin{array}{ccccc} & & \mathbf{R}^n \times \widetilde{U} & \xrightarrow{\mathrm{id} \times x} & \mathbf{R}^n \times \mathbf{R}^m \\ & & \downarrow & & \downarrow \\ J & \longrightarrow & \mathbf{R}^n & = & \mathbf{R}^n \end{array}$$

It follows that the vertical subbundle of $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is the set of vectors at each point of $\mathbf{R}^n \times \mathbf{R}^m$ of the form $(0, v)$ with $v \in \mathbf{R}^m$. Let $W_0 = (\mathrm{id}_{\mathbf{R}^n} \times x_*) \circ W \circ (\mathrm{id}_J \times x^{-1}) \colon J \times \mathbf{R}^m \to T(\mathbf{R}^n \times \mathbf{R}^m)$. Since W_0 lands in the vertical subbundle V of $T(\mathbf{R}^n \times \mathbf{R}^n)$, we may assume that $W_0 \colon J \times \mathbf{R}^m \to T(\mathbf{R}^n \times \mathbf{R}^m)$ lands in the set of vectors of the form $(0, v)$.

Since the tangent bundle is trivial over such a coordinate system, we have a canonical identification of $T(\mathbf{R}^n \times \mathbf{R}^m)$ with $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^m$ and by triviality we may forget "basepoints" and thus we let $W_1 \colon J \times \mathbf{R}^m \to \mathbf{R}^m$ be the projection of W_0 onto the $\mathbf{R}^m$ vector component. Thus, by the usual reasoning we may find a local solution to the ODE

$$\dot{y}(t) = W_1(t, y(t)), \qquad y(0) = x(g)$$

Passing back in our coordinate system, we may assemble this into a local smooth lift $\widetilde{\gamma}$ landing in $\mathbf{R}^n \times G$ that covers γ. We must now extend this solution.

Let $\mathscr{S}$ be the set of all *smooth* (i.e., admitting extensions if necessary) curves η defined on a subinterval of J of the form $[0, c]$ or $[0, c)$ $(c > 0)$ that are solutions to $(*)$. Order this set by the relation $\gamma \leq \gamma'$ if γ' extends γ. Then $\mathscr{S} \neq \emptyset$ by the above. For any chain in $\mathscr{S}$, say $\{\gamma_i\}_{i \in I}$, the union $\gamma = \bigcup \gamma_i$ is a solution since the union of intervals of the form $[0, c]$ or $[0, c)$ is an interval of one the same two types and since for any $t \in \mathrm{dom}(\gamma)$, $\gamma(t) = \gamma_i(t)$ for some $i \in I$, γ satisfies $\dot{\gamma}(t) = W(t, \gamma(t))$ wherever this makes sense as γ_i satisfies this and similarly $\gamma(0) = g$ since $\gamma(0) = \gamma_i(0) = g$ for *each* $i \in I$. Hence, $\mathscr{S}$ contains a maximal element by Zorn's lemma.

The uniqueness property Picard-Lindelöf theorem, implies that all such extensions are unique. Indeed, the local application above shows that in a nbhd of 0 the extension is unique so any further extension extends this local solution.

Exercise 24. *Show that $\mathscr{S}$ is in fact totally ordered.* [Hint: The Picard-Lindelöf Theorem actually gives uniqueness and existence on a closed interval, where smoothness at the endpoints is taken in a one-sided manner. Use this and smoothness of W to show that $\mathscr{S}$ is totally ordered by extending from endpoints.]

It follows that $\mathscr{S}$ contains a greatest element, say c. Write

$$\mathrm{dom}\, c = (a, b) \subset J.$$

Note that J necessarily contains 0, so suppose $b < 1 + \varepsilon$, where $J = (-\varepsilon, 1 + \varepsilon)$.

Idea. The next part of the proof is a cute Riemannian and metric geometry argument. Roughly, the idea is that we have to ***trap*** some segment $c|\,(t_0, b)$ in a compact set. On the metric side, we need to exclude something like the topologists' sine curve. This is where the Riemannian geodesic distance metric comes in. The idea is that W ought to be bounded since it is obtained from right multiplication. Indeed, we are saved because Lie groups are highly symmetric spaces and so we are able to equip the vertical bundle with the right invariant metric in a suitable way.

Claim 11. Assuming $b < 1$, $\mathrm{Im}\, c|\,[0, b)$ is contained in a compact subset of G.

Give G a ***right invariant Riemannian metric***, g_G^R. One can always find such a thing by right translation. Explicitly, define

$$g_{G,g}^R(v, w) = g_{G,e}^R(R_{g^{-1}*} v, R_{g^{-1}*} w)$$

for any inner product on $T_e G = \mathfrak{g}$. When G is compact, we can even construct a bi-invariant metric.

Exercise 25. *Let G be a Lie group.*

(a) *If G is compact, then G admits a bi-invariant metric for which it is complete.* [Hint: The ***Hopf-Rinow Theorem*** says that it is enough to show the exponential map arising from the metric is defined on the entire tangent space $T_g G$.]

(b) *Every left (resp. right) invariant metric on G is such that left (resp. right) multiplication is an isometry of G.*

Thus, when G is compact, it will follow by the the ***Hopf-Rinow Theorem*** that every closed and bounded subset of G in this metric is compact.

We have seen above that

$$W_{(t,g)} = R_{g*} W_{(t,e)}.$$

Perhaps by shrinking J (we only care that it is open about $I = [0, 1]$), we may assume by a compactness argument that $t \mapsto \|W_{(t,e)}\|_{g_G^R}$ is bounded above by some $M > 0$, say. But by right invariance of the metric,

$$\begin{aligned}\|W_{(t,g)}\|_{g_G^R} &= \|R_{g*}W_{(t,e)}\|_{g_G^R} = \sqrt{g_{G,e}^R(R_{g^{-1}*}R_{g*}W_{(t,e)}, R_{g^{-1}*}R_{g*}W_{(t,e)})}\\ &= \sqrt{g_{G,e}^R(W_{(t,e)}, W_{(t,e)})} = \|W_{(t,e)}\|_{g_G^R}.\end{aligned}$$

It follows that $\|W\|$ is bounded, say $\|W\| \leq M$. By assumption, $\dot{c} = W(t, c(t))$ and we may suppose $\|W(t, c(t))\| \leq M$ by the above. Then, with d_g the geodesic distance metric,

$$\begin{aligned}\lim_{s\to b} d_g(c(0), c(s)) &= \lim_{s\to b} \inf_{\text{smooth curves } \eta\colon c(0)\to c(s)} \int_0^1 \|\dot{\eta}(t)\|\, dt\\ &\leq \lim_{s\to b} \int_0^1 \|\frac{d}{dt}c(st)\|\, dt = \lim_{s\to b} \int_0^s \|\dot{c}(t)\|\, dt \leq Mb.\end{aligned}$$

In the penultimate step, we have used u-substitution. By the ***monotone convergence theorem***, the limit

$$\lim_{s\to b} \int_0^s \|\dot{c}(t)\|\, dt$$

exists since the assignment $s \mapsto \int_0^s \|\dot{c}(t)\|\, dt$ is increasing and bounded above.

This shows that, in the equivalent topology induced by the geodesic distance metric, the distance between $c(0)$ and each point $c(t)$ is bounded. When G is compact, the argument from here may proceed by the Hopf-Rinow theorem since in that case c does not escape all compact sets in G as the closed ball of radius Mb about $c(0)$ is compact. We will offer a different approach suitable for arbitrary Lie groups G that only relies upon symmetry properties..

Consider an open coordinate system (x, U) centered at e in G satisfying the following.

(i) U consists of all points a distance less than some $\varepsilon > 0$ from e in the right-invariant metric.
(ii) U has compact closure and (x, U) is the restriction of a larger chart (y, U').
(iii) $x(U)$ is a coordinate ball of finite radius in $\mathbf{R}^m$ and hence has compact closure in $\mathbf{R}^m$ with $\overline{x(U)} = y(\overline{U})$.

We can arrange for this to exist by suitably shrinking things and using a local compactness argument. It follows that every closed subset $F \subset U$ for which $d_g(F, \partial U) \neq 0$ is compact in G since it is a closed subset of $\overline{U}$ which is compact and compactness transcends subspace inclusion.

Now, for each $g \in G$, the coordinate system $(x \circ R_{g^{-1}}, U \cdot g)$ about g satisfies the analogous property—this is because $R_{g^{-1}}$ is an isometry in our right-invariant metric. Pick $0 < t_0 < b$ such that for each $t_0 < s < b$,

$$d_g(c(t_0), c(s)) < \varepsilon/2.$$

To see that this exists, note that the function $[t_0, b] \to \mathbf{R}$ defined by

$$s \mapsto \begin{cases} d_g(c(t_0), c(s)) & s \neq b\\ \lim_{t\to b} d_g(c(t_0), c(t)) & s = b\end{cases}$$

is continuous and thus has compact image.

Then $\operatorname{Im} c|\,[0, t_0]$ is compact and admits a covering by finitely many open trivializing coordinate balls. We may append to this covering the open coordinate ball $(x \circ R_{c(t_0)^{-1}}, U \cdot c(t_0))$ which has the property that $\overline{\operatorname{Im} c|\,[t_0, b)} \subset U \cdot c(t_0)$ since $U \cdot c(t_0)$ consists of all points a distance less than ε from $c(t_0)$.

For ease of notation, call $V = U \cdot c(t_0)$ and $C = \overline{\operatorname{Im} c|\,[t_0, b)}$. Observe that the preceding remarks imply that C is compact, being a suitable closed subspace of V which has compact closure.

Now, WLOG, we may suppose we are considering the Euclidean case as before by using the translated coordinate ball about $c(t_0)$ and by considering a coordinate nbhd of $\gamma(b)$ in the base space say containing $(t_0 - \delta, b + \delta) \subset [t_0 - \delta, b + \delta] \subset J$. At this point, we note that only topological properties are preserved, not metric properties, but this will not end up mattering. Since C is compact and since V has compact closure, there is some point $p \in \overline{\operatorname{Im}(c|\,[t_0, b))}$ such that $d_{\mathbf{R}^m}(p, \partial V) = d_{\mathbf{R}^m}(\overline{\operatorname{Im}(c|\,[t_0, b))}, \partial V)$. Say

$$r_1 = d_{\mathbf{R}^m}(p, \partial V).$$

We have implicitly used all conditions on the chart (x, U) above to argue up to this point.

The ODE we wish to consider is

$$\dot{y}(t) = W(t, y(t)) \qquad y(t_1) = c(t_1),$$

where we understand this to be taken with respect to the coordinate system of V. Perhaps by shrinking, we may suppose $W\colon (t_0 - \delta, b + \delta) \times V \to \mathbf{R}^m$ is bounded above by $M \geq 0$, say. Pick $t_1 < b$ so close that $b - t_1 < \min\left\{\delta, \dfrac{r_1}{2M}\right\}$. We can do this since the initial condition for t_1 is $y(t_1) = c(t_1)$ and $d(c(t_1), \partial V) \leq r_1$. Then the Picard-Lindelöf theorem guarantees us an extension of c past b. This means that c was not maximal; this is a contradiction. Hence, c is smooth on $[0, 1+\delta)$ and thus on $[0,1]$ as desired. This shows existence and uniqueness in the local case which suffices. ■

5.2 Additional Content

Lemma 5.2.1. *For a principal G-bundle $G \to P \xrightarrow{\pi} B$, let $\mathcal{O}_p \colon G \to P$ be the p-orbit map $g \mapsto p \cdot g$.*

(a) *$\mathcal{O}_p$ is smooth.*
(b) *For $X \in \mathfrak{g} = T_eG$, $(\mathcal{O}_p)_{*e}(X) = X^*_p$.*
(c) *The fundamental vector field is always vertical.*

Proof. **(a)** $\{p\} \times G \subset P \times G$ is a submanifold, so since the action map is smooth, so too is its restriction to this submanifold.

(b) Thinking of a tangent vector as a derivation of germs of smooth functions at e, given $X \in T_eG$,

$$(\mathcal{O}_p)_{*e}(X) \overset{\text{def}}{=} X([- \circ \mathcal{O}_p]_e)$$

whereas

$$\left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(tX) \overset{\text{def}}{=} (p \cdot \exp(tX))_*(\left.\frac{d}{dt}\right|_{t=0})$$

which is

$$\left.\frac{d}{dt}(- \circ R_{\exp(tX)}(p)\right|_{t=0}$$

and comparing the two on coordinate functions x^i, we have by the chain rule and a small computation,

$$\left.\frac{d}{dt}((R_{\exp(tX)}(p))^i\right|_{t=0} = (R_e(p))_*(X) = X([- \circ R_{-}(p)]_e)$$

which is the same derivation obtained above.

(c) This is the same sort of computation with $\pi_*(\mathcal{O}_p)_{*e}(X) = X([- \circ \pi \circ \mathcal{O}_p]) = 0$ since $\pi\mathcal{O}_p \equiv p$. ■

Example 4. A G-invariant Riemannian metric on P yields a horizontal distribution by letting $H = V^{\perp}$.

Exercise 26. *Let $\xi = (P, p, B)$ be a smooth principal G-bundle over B and let $H \subset TP$ be a principal G-connection.*

(a) *Show that there is a canonical isomorphism between T_eG and the vector space of all* ***left invariant*** *vector fields on G—a left invariant vector field X is one for which $L_{g*}X = X \circ L_g$. We call either of these the* ***Lie algebra*** *$\mathfrak{g}$ of the Lie group G.*
(b) *Show that $\mathfrak{g}^*$ is naturally isomorphic to the space of left invariant one-forms on G—that is, one-forms ω such that $L_g^*\omega = \omega \circ L_g$ for all $g \in G$.*
(c) *For each $g \in G$, define the conjugation map $\mathrm{Ad}_g \colon G \to G$ by $a \mapsto gag^{-1}$. Show that $\mathrm{Ad} \colon G \times G \to G$ defined by $(g,h) \mapsto \mathrm{Ad}_g(h)$ is smooth and has full rank and that $(\mathrm{Ad}_g)_*$ is a linear automorphism of $\mathfrak{g}$.*
(d) *Define $\mathrm{ad} \colon G \to \mathrm{GL}(\mathfrak{g})$ the* ***adjoint representation*** *by $\mathrm{ad}(g) \overset{def}{=} (\mathrm{Ad}_g)_*$. Show that ad is smooth. Hence, there is a smooth vector bundle given by the associated bundle construction $\mathrm{Ad}(P) \overset{def}{=} P \times_G \mathfrak{g}$.*
(e) *Say a* ***Lie algebra-valued*** ***k-form*** *on a smooth manifold M is a smooth section of $(M \times \mathfrak{g}) \otimes \bigwedge^k T^*M$ and denote these by $\Omega^k(M, \mathfrak{g})$. Show that $(M \times \mathfrak{g}) \otimes \bigwedge^k T^*M \cong \bigwedge^k(T^*M \otimes M \times \mathfrak{g})$. Conclude that sections of $(M \times \mathfrak{g}) \otimes \bigwedge^k T^*M$ are effectively what would be called Lie algebra-valued k-forms in the vernacular.*
(f) *Say the* ***connection form*** *of the given principal G-connection on ξ is a Lie algebra valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ satisfying*

(i) *For each $X \in \mathfrak{g}$, $\omega(X^*) = X$.*
(ii) *$(R_g)^*\omega = \mathrm{ad}(g^{-1}) \circ \omega$ for all $g \in G$.*

Then such an ω exists and is unique. Conversely, any such Lie algebra-valued 1-form defines a unique principal G-connection.

5.3 Exercises: Construction of Bundles

Definition. Let C and D be topologically enriched categories[1]. Then $\prod_{i=1}^{k} \mathsf{C}$ is a topologically enriched category in the obvious way. An ordinary functor $F\colon \prod_{i=1}^{k} \mathsf{C} \to \mathsf{D}$ (which we say is k-ary) is said to be a ***continuous functor*** if the evident maps $\prod_{i=1}^{k} \operatorname{Hom}(a_i, b_i) \to \prod_{i=1}^{k} \operatorname{Hom}(Fa_i, Fb_i)$ are continuous.

If, in addition, we can equip the mapping spaces with a smooth structure such that composition is smooth, then we say C and D are ***smooth***. Then an ordinary functor $F\colon \prod_{i=1}^{k} \mathsf{C} \to \mathsf{D}$ is said to be a ***smooth functor*** if the evident maps $\prod_{i=1}^{k} \operatorname{Hom}(a_i, b_i) \to \prod_{i=1}^{k} \operatorname{Hom}(Fa_i, Fb_i)$ are smooth.

Remark. The opposite of a topologically enriched category C^{op} has hom-spaces $\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(a, b) = \operatorname{Hom}_{\mathsf{C}}(b, a)$.

Let us collect a classical fact of analysis.

Theorem. *Let k be one of $\mathbf{C}$ or $\mathbf{R}$ with its usual topology. For any finite dimensional vector space E over k, there is a unique Hausdorff topology on E under which E is a TVS and thus any norms on E are equivalent—when $k = \mathbb{H}$, this topology is still unique and is induced by viewing E as an $\mathbf{R}$-vector space.*

The second part follows from the first since it says that identity map between the two topologies is a linear homeomorphism and is therefore bounded and similarly its inverse is a linear homeomorphism and is bounded.

Lemma. *Let k be $\mathbf{C}$ or $\mathbf{R}$ and let fdVect_k be the category of finite-dimensional vector spaces over k. Then fdVect_k is naturally a topologically enriched category and a smooth category.*

Proof. By the preceding theorem since $\operatorname{Hom}_{\mathsf{Vect}}(V, W)$ is itself a finite-dimensional vector space, it admits a unique Hausdorff topology making it a TVS. By identification with k^n and therefore admit a smooth structure inherited from k^n. Composition will be smooth since, after choosing bases, it corresponds to matrix multiplication which is smooth. ■

Exercise 27. *Let k be $\mathbf{C}$ or $\mathbf{R}$. The following functors are smooth.*

(a) $\bigoplus\colon \mathsf{fdVect}_k^{\times n} \to \mathsf{fdVect}_k$.
(b) $\bigotimes\colon \mathsf{fdVect}_k^{\times n} \to \mathsf{fdVect}_k$.
(c) $\operatorname{Hom}(-,-)\colon \mathsf{fdVect}_k^{\mathrm{op}} \times \mathsf{fdVect}_k \to \mathsf{fdVect}_k$.
(d) $\bigwedge^n\colon \mathsf{fdVect}_k \to \mathsf{fdVect}_k$.
(e) $\operatorname{Sym}^n\colon \mathsf{fdVect}_k \to \mathsf{fdVect}_k$.
(f) $(-)^*\colon \mathsf{fdVect}_k^{\mathrm{op}} \to \mathsf{fdVect}_k$.
(g) $\Delta\colon \mathsf{fdVect}_k \to \mathsf{fdVect}_k^{\times n}$.

Here, $\bigwedge_{i=1}^{n} E$ is generated by the alternating tensors and $\operatorname{Sym}^n E$ is generating by the symmetric tensors, at least in characteristic 0.

Exercise 28. *Given a smooth (resp. continuous) functor $T\colon \mathsf{fdVect}_k^{\times n} \to \mathsf{fdVect}_k$, there is a unique (up to natural isomorphism) functor $\underline{T}\colon \mathsf{fdVectBun}_{k/M}^{\times n} \to \mathsf{fdVectBun}_{k/M}$ on the category of finite rank vector bundles over the smooth manifold (resp. space) M such that $\underline{T}(E_1, \ldots, E_n)_p = T(E_{1,p}, \ldots, E_{n,p})$, $\underline{T}(f_1, \ldots, f_n)$ is fiberwise $T(f_1(p), \ldots, f_n(p))$ and the transition functions of $\underline{T}(E_1, \ldots, E_n)$ are obtained pointwise as $T(g^1(p), \ldots, g^n(p))$ where g^i is a transition function of E_i.* [Hint: For uniqueness, note that a bundle is recovered up to isomorphism by its transition functions.]

[1] The ordinary notion of a category has hom-sets and composition functions $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ along with units in each $\hom(a, a)$. A topologically enriched category has hom-spaces and continuous composition functions $\operatorname{Hom}(a, b) \times \operatorname{Hom}(b, c) \to \operatorname{Hom}(a, c)$ along with units in each $\operatorname{Hom}(a, a)$. One can safely understand a topological enrichment to mean "hom-sets have a topology and composition is continuous."

Lecture 6

6.1 Classifying Spaces

6.1.1 Numerable Bundles

Definition. Say a fiber bundle $p\colon E \to B$ is ***numerable*** if it admits an open cover by trivializing open sets $\mathscr{U} = \{U_i\}_{i\in I}$ for which there exists a partition of unity subordinate to $\mathscr{U}$. Equivalently, there is a family of continuous maps $\rho_i : B \to [0,1]$ for which $\operatorname{supp} \rho_i \subset U_i$ and $\sum \rho_i \equiv 1$. Say an open cover of a space is ***numerable*** if it admits a subordinate partition of unity.

We will say a principal G-bundle $p\colon E \to B$ is ***numerable*** if it is equivariantly trivializable over a numerable covering. Note that this means that there are right G-equivariant isomorphism

$$p^{-1}(U_i) = p^{-1}(\rho_i^{-1}((0,1])) \cong \rho_i^{-1}((0,1]) \times G = U_i \times G$$

for all $i \in I$, not simply an isomorphism.

Remark. Equivalently, a principal G-bundle is ***numerable*** if there is a locally finite collection of trivializing open sets trivializing open sets $\mathscr{U} = \{U_i\}_{i\in I}$ and a partition of unity $\{\rho_i\}_{i\in I}$ such that $\rho_i^{-1}((0,1]) = U_i$ and, furthermore, such that each $U_i \subset V_i$ where V_i is also a trivializing open set for the principal G-bundle such that $\operatorname{supp} \rho_i \subset V_i$.

Exercise 29. *If the base space B in the definition above is paracompact Hausdorff, then all fiber bundles are numerable.* [Hint: Use a partition of unity.]

Lemma 6.1.1. *A numerable cover a space determines a partition of unity.*

Proof. For a numerable cover with functions ρ_i, we let

$$\eta_j = \frac{\rho_j}{\sum_{i\in I} \rho_i}$$

for each $j \in I$. This is well-defined because the cover is locally finite and so at any one point the sum is finite and has finite value. The η_i then determines a partition of unity. ∎

Proposition 6.1.2 (Husemöller, 7.2.1.2). *If $E \to B$ is a numerable principal G-bundle, then there is a* countable *partition of unity $\{\rho_i\}_{i\in \mathbf{N}}$ such that E is trivial over each $\rho_i^{-1}((0,1])$ and, hence, admits a G-equivariant trivialization.*

Proof. Take a partition of unity $\{\xi_i\}_{i\in I}$ and let $I(b) = \{i \in I : \xi_i(b) > 0\}$ and for each $J \subset I$ with$\#J < \infty$, set

$$V(J) = \{b \in B : \xi_j(b) > \xi_i(b) \text{ for all } j \in J \text{ and } i \in I \setminus J\}$$

Then $V(J)$ is open. For such J, let

$$\xi_J(b) = \max\left\{0, \min_{j\in J, i \in I\setminus J} (\xi_j(b) - \xi_i(b))\right\}.$$

Then $\xi_J^{-1}((0,1]) = V(J)$.

Then if $\#J' = \#J''$ for two finite subsets of I and $J' \neq J''$, then $V(J') \cap V(J'') = \emptyset$ as we cannot have both $\xi_{j'}(b) > \xi_{j''}(b)$ and $\xi_{j''}(b) > \xi_{j'}(b)$.

Thus, we may set $V_m = \bigcup_{J\subset I,\ \#J=m} V(J)$ a disjoint union and let $\xi_m = \sum_{J\subset I,\ \#J=m} \xi_J$. Then $\xi_m^{-1}((0,1]) = V_m$ and $P|\,V_m$ is trivial because it is trivial over each set in the disjoint union for V_m. Then the desired partition of unity is given by

$$\rho_m = \frac{\xi_m}{\sum_{n\geq 0} \xi_n}$$

where $\rho_m^{1}((0,1]) = V_m$. G-equivariance is clear since we are considering disjoint unions of open trivializing sets. ∎

Exercise 30. *In this exercise, you will define a category* $\mathsf{Bun}^F_{G,\mathrm{num}}$ *to be the **category of numerable** G**-bundles with fiber** F and establish variants of theorems in the first five lectures.*

(a) *The objects of numerable* G*-bundles with fiber* F $\xi = (E, p, B, G, F, \mathscr{A})$ *where* $E, B, G, F \in \mathsf{Top}$ *and where* $\mathscr{A}$ *is a* G*-atlas. Show that, with morphisms defined as usual,* $\mathsf{Bun}^F_{G,\mathrm{num}}$ *is a category.* [Hint: Numerability does not play a role here. Adapt the proof of **Claim 1**.]

(b) *Show that the pullback theorem holds for* $\mathsf{Bun}^F_{G,\mathrm{num}}$. [Hint: Numerability matters here. Note that while the G-atlas need not contain any numerable open covering, there always exists such a covering (hence, G-atlas) that is compatible with it.]

(c) *Show that there is an equivalence of categories* $\mathsf{Bun}^G_{G,\mathrm{num}} \simeq \mathsf{Prin}_{G,\mathrm{num}}$ *where on the right-hand side the morphisms of numerable principal* G*-bundles are morphisms of fiber bundles that are* G*-equivariant on the total space.*

(d) *Prove topological homotopy invariance by looking up the proof in Dold's paper Partitions of unity in the theory of fibrations or adapt Dan Freed's writeup here.*

Remark. This allows us to assume all numerable principal G-bundles are covered by a countable locally finite collection of trivializing open sets.

6.1.2 Universal Bundles & Milnor's Construction

6.1.2.1 The General Case

Remark. The following is taken from Milnor, Dold and Husemöller. We will not provide full details for the sake of brevity and leave them to the reader. We will make the construction in a series of claims .

Lemma 6.1.3. *Let* X *be a space. The isomorphism classes of (numerable) principal* G*-bundles over* X *form a set.*

Proof. From **Theorem 1.3.1**, it follows that the isomorphism classes of (numerable) principal G-bundles can be mapped injectively into the set consisting of all open covers $\mathscr{U}$ of X and maps $U_i \cap U_j \to G$. This is a set since if $\tau(X)$ is the set of opens for X, then this set has size at most $\tau(X) \times \{f : U \to G : U \in \tau(X)\}$ which is a set. ■

Definition. For each topological group G, define a functor

$$k_G \colon \mathrm{Ho}(\mathsf{Top})^{\mathrm{op}} \to \mathsf{Set}$$

sending a space X to the set $[\mathsf{Prin}_{G,\mathrm{num}}(X)]$ of isomorphism classes of numerable principal G-bundles over X. A numerable principal G-bundle $P \to B$ is called a ***universal bundle*** and B is called a ***classifying space*** of G if there is a natural isomorphism $[-, B] \cong k_G$ given by sending a homotopy class $f \colon X \to B$ to the principal G-bundle f^*P.

Remark. This says that k_G is representable and that, moreover, the natural isomorphism $[-, BG] \cong k_G$ has a particularly nice description.

Remark. The content of this subsection is that universal bundles exist.

Idea. The basic idea of the Milnor construction is to repackage the data of the second component of the trivializations $w_n = (\pi, u_n) \colon\ P|\, \rho_n^{-1}((0,1]) \xrightarrow{\cong} \rho_n^{-1}((0,1]) \times G$ and the functions ρ_n into transition functions.

Conventions. We fix a principal G-bundle $\pi \colon P \to B$ with a countable locally finite *trivializing* open cover $\left\{\rho_i^{-1}((0,1])\right\}_{i\in\mathbf{N}}$ determined by a corresponding partition of unity $\{\rho_i\}_{i\in\mathbf{N}}$ such that $P|\,\rho_i^{-1}((0,1])$ is trivial throughout. For each $n \in \mathbf{N}$, fix

$$w_n = (\pi, u_n) \colon\ P|\, \rho_n^{-1}((0,1]) \xrightarrow{\cong} \rho_n^{-1}((0,1]) \times G$$

a choice of trivialization.

Definition. Let $\Delta^n \subset \mathbf{R}^{n+1}$ be all $(n+1)$-tuples of points $(s_0, \dots, s_n)$ with $s_i \geq 0$ such that $\sum s_i = 1$. This is the topological n-simplex. It has vertices $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ the i-th standard basis vector.

Notation. Let $W_n = \bigcup_{i=0}^n \rho_i^{-1}((0,1])$ and define *as a set* (not as a space)

$$E_n G = \Delta^n \times G^{n+1} / \sim$$

where

$$(s_0, \ldots, s_n, g_0, \ldots, g_n) \sim (s'_0, \ldots, s'_n, g'_0, \ldots, g'_n)$$

if and only if $s_i = s'_i$ for $i = 0, \ldots, n$ and if $s_i = s'_i > 0$ then we also require $g_i = g'_i$. We will denote points of E_nG using a semicolon $(s_0, \ldots, s_n : g_0, \ldots, g_n)$.

As a *space*, E_nG has the finest topology for which the coordinate functions

$$t_i \colon E_nG \to [0,1], \qquad \mathrm{pr}_i \colon t_i^{-1}((0,1]) \to G \qquad (0 \le i \le n)$$

are continuous. The function $t_i \colon E_nG \to [0,1]$ projects onto s_i and $\mathrm{pr}_i \colon t_i^{-1}((0,1]) \to G$ projects onto g_i.

Remark. Hence, when $s_i = s'_i = 0$, for the purposes of the equivalence relation, the values of g_i and g'_i are irrelevant. One way to think about this is that when a coordinate of Δ^n is 0, we forget the corresponding i-th G-coordinate but remember that it used to be there.

The construction of E_nG is a special case of a slightly more general construction called the ***Milnor join*** of spaces.

Definitions. Let $X_0, \ldots, X_n$ be a finite collection of spaces.

(i) The ***join*** of the spaces X_i is the quotient space

$$*_{i=0}^n X_i = X_0 * X_1 * \cdots * X_n \overset{\text{def}}{=} (\Delta^n \times X_1 \times \cdots \times X_n) / \sim$$

where $(s_0, \ldots, s_0, x_1, \ldots, x_n) \sim (r_0, \ldots, r_n, y_0, \ldots, y_n)$ if and only if for each $0 \le i \le n$, $s_i = r_i$ and when $s_i = r_i > 0$, we also require $x_i = y_i$. We will denote points of this space as $(s_0x_0, \ldots, s_nx_n)$ or as $(s_0, \ldots, s_n : x_0, \ldots, x_n)$, where $(s_0, \ldots, s_n) \in \Delta^n$ and $(x_0, \ldots, x_n) \in \prod_{i=0}^n X_i$.

(ii) The ***Milnor join*** of a the spaces X_i, denoted

$$\bigstar_{i=0}^n X_i = X_1 \bigstar X_2 \bigstar \cdots \bigstar X_n$$

has *the same underlying set* as the join of the X_i, but is equipped with the finest topology for which the coordinate functions

$$t_i \colon \bigstar_{i=0}^n X_i \to [0,1], \qquad \mathrm{pr}_i \colon t_i^{-1}((0,1]) \to X_i \qquad (0 \le i \le n)$$

are continuous, where

$$t_i \colon \bigstar_{i=0}^n X_i \to [0,1] (s_0, \ldots, s_n : x_0, \ldots, x_n) \mapsto s_i$$
$$\mathrm{pr}_i \colon t_i^{-1}((0,1]) \to X_i (s_0, \ldots, s_n : x_0, \ldots, x_n) \mapsto x_i.$$

Warning. You should attempt the next two exercises, or at least digest them.

Exercise 31 ($\star$). *Consider E_nG equipped with the topology constructed above.*

(a) *Show that the topology on E_nG satisfies the following universal property. A function $f \colon X \to E_nG$ is continuous **iff** for each $0 \le i \le n$ the functions*

$$t_i \circ f \colon E_nG \to [0,1] \qquad \mathrm{pr}_i \circ f \colon f^{-1}t_i^{-1}((0,1]) \to G$$

are continuous. Show that any Milnor join of spaces satisfies the analogous universal property.

(b) *Show that a subbase for the topology on E_nG consists of all sets of the following two types:*

$$t_j^{-1}((\alpha, \beta)) = \{(s_0, \ldots, s_n : g_0, \ldots, g_n) \in E_nG : \alpha < s_j < \beta\} \text{ where } 0 \le j \le n \text{ and } \alpha < \beta, \alpha, \beta \in \mathbf{R}. \tag{I}$$

$$\mathrm{pr}_i^{-1}(U) = \{(s_0, \ldots, s_n : g_0, \ldots, g_n) \in E_nG : s_j \neq 0 \text{ and } g_j \in U \text{ where } U \subset G \text{ is open}\} \text{ where } 0 \le j \le n. \tag{II}$$

Show that this subbase contains a set which is the whole space E_nG. Show that any Milnor join of spaces has an analogous subbase.

Exercise 32 ($\star$). *Show that the Milnor join is associative.*

Define a free right action of G on E_nG by

$$(s_0, \ldots, s_n : g_0, \ldots, g_n)g = (s_0, \ldots, s_n : g_0g, \ldots, g_ng).$$

This is easily seen to be continuous using the above exercise. It is furthermore clearly free. Define

$$B_nG = E_nG/G.$$

Claim 12. There is a G-equivariant closed map and, in particular, closed embedding

$$i\colon E_nG \hookrightarrow E_{n+1}G$$

where i sends $(s_0, \ldots, s_n : g_0, \ldots, g_n) \mapsto (s_0, \ldots, s_n, 0 : g_0, \ldots, g_n, e)$.

Equivariant is obvious. The complement of $i(E_nG)$ consists of all points $(s_0, \ldots, s_{n+1} : g_0, \ldots, g_{n+1})$ such that $s_{n+1} > 0$. This set is open essentially by definition of the topology. Note that the topology on the subspace of $E_{n+1}G$ consisting of all points of the form $(s_0, \ldots, s_n, 0 : g_0, \ldots, g_n, e)$ satisfies the same universal property as E_nG; from this it is easy to see that $E_nG \to E_{n+1}G$ is an embedding and it is furthermore a closed map since if $F \subset E_nG$ is closed, then $i(F) \subset E_{n+1}G$ is closed in $i(E_nG)$ and $i(E_nG)$ is closed—it follows that there is a closed $F' \subset E_{n+1}G$ such that $i(F) = F'' \cap i(E_nG)$ which is an intersection of closed sets in $E_{n+1}G$ and is therefore closed. ⫽

Lemma 6.1.4. *Let*

$$E'G = \operatorname{colim}_n E_nG$$

with $E_nG \to E_{n+1}G$ the closed embedding and closed map described above and let

$$EG = \operatorname{colim}_n E_nG \qquad \textit{as a set,}$$

*but equipped with the following strong topology, satisfying the following universal property. A function $f\colon X \to EG$ is continuous **iff** for each $0 \leq i < \infty$ the functions*

$$t_i \circ f\colon EG \to [0,1] \qquad \operatorname{pr}_i \circ f\colon f^{-1}t_i^{-1}((0,1]) \to G$$

are continuous. In other words, the topology of EG is the finest one for which the coordinate functions

$$t_i\colon E_nG \to [0,1], \qquad \operatorname{pr}_i\colon t_i^{-1}((0,1]) \to G \qquad (0 \leq i < \infty)$$

are continuous. It has the corresponding subbase as in the previous exercise.

(a) *If G is locally compact Hausdorff, then $E'G$ has a free right G-action induced from those of E_nG.*
(b) *EG has a free right G-action induced from those of E_nG.*
(c) *There is a natural set-theoretic bijection $E'G \to EG$ which is furthermore continuous. In other words, the colimit topology on $E'G$ contains the topology generated by the two classes of projection functions.*

Proof. **(a)** This follows easily using the fact that $\operatorname{colim}(E_nG \times G) \cong \operatorname{colim}(E_nG) \times G$.

(b) The G-action induced by those of E_nG is obtained by defining $EG \curvearrowleft G$ by

$$((s_i) : (g_i)) \cdot g = ((s_i) : (g_i g))$$

This is clearly a free action. Call this map $A\colon EG \times G \to EG$ and denote $\mu\colon G \times G \to G$ the group multiplication map. To check A is continuous, observe that for $U \subset G$ open, $(\operatorname{pr}_i \circ A)^{-1}(U) = \big\{(((s_i) : (g_i)), g) : (g_i, g) \in \mu^{-1}(U)\big\}$ which is open because $\mu^{-1}(U)$ is open in $G \times G$ and so can be written as a union of open rectangles—one verifies more precisely by using type (II) sets. Similarly, if $J \subset [0,1]$ is open, then one easily checks $(t_i \circ A)^{-1}(J) = t_i^{-1}(J)$ which is open.

(c) The bijection is the evident one (the identity map by our construction). The projections t_i and pr_i on $E'G$ are continuous because they are continuous on each $E_{i+k}G$ for $k \geq 0$ (essentially using the universal property of the colimit). Thus, if $f\colon X \to EG$ is continuous then $t_i \circ f$ and $\operatorname{pr}_i \circ f$ are continuous for all i. Thus, the colimit topology on EG at least *contains* the topology generated by these functions. ■

Exercise 33. *Is it true that $EG \cong \operatorname{colim} E_nG$ in the category Top_G of spaces with a continuous right G-action and continuous equivariant maps regardless of assumptions on G?*

Claim 13. The projection $p\colon EG \to EG/G = BG$ is a numerable principal G-bundle.

Recall that EG has natural projections

$$t_k\colon : EG \to [0,1]$$

sending $((s_i)_{i\in\mathbf{N}} : (g_i)_{i\in\mathbf{N}})$ to s_k. These respect G-orbits and hence descend to maps

$$\tau_k\colon BG \to [0,1]$$

defined by the same formula. Let $U_k = t_k^{-1}((0,1])$. This is an open saturated set for the quotient map $p\colon EG \to EG/G \cong BG$. Hence, its image

$$U_k/G = V_k = \tau_k^{-1}((0,1])$$

is open in BG.

The collection $\{V_i\}_{i\geq 0}$ is a locally finite open cover of BG since every representative of a point $[((s_i),(g_i))] \in BG$ has the same $(s_i)_{i\in\mathbf{N}}$ coordinates and by definition of EG, each point has only finitely many $s_i \neq 0$. We can furthermore trivialize $EG \to BG$ over each V_i G-equivariantly. Indeed, $EG|\, V_i = U_i$ is the set of $((s_i) : (g_i))$ with $s_i > 0$. Define $\varphi_i \colon U_i \to V_i \times G$ by

$$((s_j) : (g_j)) \mapsto (p((s_j) : (g_j)), \mathrm{pr}_i((s_j) : (g_j))) = ([(s_j) : (g_j)], g_i).$$

This is continuous because its two components are continuous functions restricted to the open subset U_i. G-equivariance of the association is immediate. For an inverse, pick a representative of a point $[(s_j) : (g_j)]$ such that $g_i = e$ and call this representative $((s_j) : (g'_j))$. Then define its inverse $\varphi_i^{-1} \colon V_i \times G \to U_i$ by

$$([(s_j) : (g_j)], g) \mapsto ((s_j) : (g'_j g))$$

or, equivalently,

$$(p((s_j) : (g_j)), g) \mapsto ((s_j) : (g_j g_i^{-1} g)).$$

This association is manifestly G-equivariant and an inverse to the previous map. It is well-defined because there is one and only one representative of the class $[(s_j) : (g_j)]$ with $g_i = e$. If $([(s_i) : (g_i)], g) \mapsto ((s_i) : (g'_i g))$ is continuous, then we are done because the transitions $\varphi_{ij} \colon V_i \cap V_j \times G \to V_i \cap V_j \times G$ will have associated transition functions

$$g_{ij}(p((s_j) : (g_j))) = g_{ij}([((s_j) : (g_j))]) = g_i g_j^{-1}$$

or, equivalently,

$$g_{ij}([(s_j) : (g_j)]) = \mathrm{pr}_i\, \varphi_j^{-1}([(s_j) : (g_j)], e)$$

where pr_i is the coordinate function $EG \to G$ and so

$$g_{ij} = \mathrm{pr}_i \circ \varphi_j^{-1}(-, e) \colon V_i \cap V_j \to G$$

which is certainly continuous being a composite of continuous functions (note that $\varphi_j^{-1}(-, e)$ is continuous). //

Exercise 34. *Show that this association* $([(s_j) : (g_j)], g) \mapsto ((s_j) : (g'_j g))$ *is continuous and equivalent to* $(p((s_j) : (g_j)), g) \mapsto ((s_j) : (g_j g_i^{-1} g))$. [Hint: Check that the subspace topology on U_i induced from the coordinate function topology on EG satisfies a similar universal property to EG. Then simply check on subbase elements.]

Theorem 6.1.5. *The principal G-bundle $p \colon EG \to BG$ is universal.*

Proof. Let $\pi \colon E \to B$ be a numerable principal G-bundle. We may assume there is a countable (locally finite) partition of unity $\rho_i \colon B \to [0,1]$ such that $E \to B$ is (equivariantly) trivializable over $U_i = \rho_i^{-1}((0,1])$ say with trivializations $\varphi_i = (\pi, u_i) \colon \pi^{-1}(U_i) \to U_i \times G$.

Given such data, we can define a map $E \to EG$ given by $\widetilde{f}(x) = ((\rho_i(\pi(x)) : (u_i(x)))$. This is well-defined since only finitely many $\rho_i(x) \neq 0$ and since where $\rho_i(x) \neq 0$, $u_i(x)$ makes sense and where $\rho_i(x) = 0$, the values of $u_i(x)$ are irrelevant and can therefore be set to e. It is G-equivariant because the u_i are G-equivariant. It is continuous because post-composition with the coordinate functions yields a continuous function essentially by design. This shows that every numerable principal G-bundle has a map to $EG \to G$. We must show homotopy invariance now—that is, $[-, BG] \cong \mathsf{Prin}_{G,\mathrm{num}}(-)/\mathrm{iso}$ as in the definition of the universal bundle. Let us begin by showing that pulling back the universal bundle establishes a bijection $[X, BG] \cong \mathsf{Prin}_{G,\mathrm{num}}(-)/\mathrm{iso}$.

The following was an exercise but we give the proof here anyways.

Claim 14. Pulling back a map $B \to BG$ yields a numerable principal G-bundle over B.

By the pullback theorem, it suffices to check that for a given a map of principal G-bundles $\widetilde{f} \colon E \to EG$, the bundle $E \to B$ is numerable. Indeed, we obtain a G-equivariant map by composition $t_i \circ \widetilde{f} \colon E \to [0,1]$ where $[0,1] \curvearrowleft G$ is the trivial action. Hence $t_i \circ \widetilde{f}$ descends to a map $\rho_i \colon B \to [0,1]$. Notice that the covering $\rho_i^{-1}((0,1])$ is a locally finite open cover and indeed a partition of unity using the ρ_i—this is because, necessarily, $\sum t_i \equiv 1$ (check how the simplices Δ^n are defined and note that points of EG are just points lying in some $E_n G$) and this cover is locally finite since for any $x \in EG$, only finitely many $t_i(x)$ are non-zero and the G-action preserves their values in that $t_i(x \cdot g) = t_i(x)$.

It follows that

$$E|\, \rho_i^{-1}((0,1]) \xrightarrow{(\rho_i \circ \pi, \mathrm{pr}_i \circ \widetilde{f})} [0,1] \times G$$

is a morphism of G-bundles over $\rho_i^{-1}((0,1]) \xrightarrow{\rho_i} [0,1]$ since $\widetilde{f}$ and pr_i are both G-equivariant. Hence, $E|\,\rho_i^{-1}((0,1])$ is the pullback of $\rho_i \colon \rho_i^{-1}((0,1]) \to [0,1]$ by the pullback theorem. But the pullback of a trivial G-bundle is trivial so we conclude, so there is an isomorphism of principal G-bundles $E|\,\rho_i^{-1}((0,1]) \cong \rho_i^{-1}((0,1]) \times G$.

Now we establish the desired bijection $[B, BG] \cong \mathsf{Prin}_{G,\mathrm{num}}(B)/\mathrm{iso}$ by pulling back the universal bundle. We have already seen that every principal G-bundle has a bundle morphism into $EG \to BG$. By the pullback theorem and homotopy invariance, this shows that pulling back the universal bundle is a surjective map $[B, BG] \to \mathsf{Prin}_{G,\mathrm{num}}(B)/\mathrm{iso}$. Now we must show that it is an injective correspondence and this is the more subtle part.

Suppose $\pi \colon E \to B$ is pulled back (up to isomorphism) from two maps $f, g \colon B \to BG$. If we can show that the two induced G-equivariant maps $\widetilde{f}, \widetilde{g} \colon E \to B$ are G-equivariantly homotopic, then this homotopy will descend to a homotopy on the base space between f and g. In a little more detail, given $\widetilde{f}, \widetilde{g} \colon E \to EG$ G-equivariant, we must show that there is a homotopy $\widetilde{H} \colon E \times I \to EG$ from f to g satisfying that $\widetilde{H}(x \cdot g, t) = \widetilde{H}(x,t) \cdot g$ as G-equivariance of the homotopy (and since I is locally compact Hausdorff, an annoying point-set technicality we generally do not have to worry about) means it will descend to a homotopy $H \colon B \times I \to BG$ between f and g, where $f, g \colon B \to BG$ are also induced by passage to the quotient.

Write $\widetilde{f}, \widetilde{g}$ as $\widetilde{f}(x) = ((u_i(x)) : F_i(x))$ and $\widetilde{g}(x) = ((v_i(x)) : G_i(x))$. For $1 \leq k < \infty$. We will construct a homotopy that inserts 0s between all s_i's in the coordinate expression for $\widetilde{f}$ and $\widetilde{g}$, thereby showing that they are both homotopic some other maps which are themselves homotopic, and therefore $\widetilde{f} \simeq \widetilde{g}$. The two cases are similar so let us start with $\widetilde{f}$. Define maps $\widetilde{H}^k \colon E \times I \to EG$ by

$$\widetilde{H}^k(x,t) = ((u_0(x), \ldots, u_{k-1}(x), tu_k(x), (1-t)u_k(x), tu_{k+1}(x), (1-t)u_{k+1}(x), \ldots) : (F_0(x), \ldots, F_{k-1}(x), F_k(x), F_k(x), F_{k+1}(x), F_{k+1}(x$$

with the pattern continuing. This is continuous, well-defined (note that the coordinates all sum to 1 with only finitely many non-zero) and G-equivariant. As in the usual trick, we define $\widetilde{H} \colon E \times I \to EG$ by

$$\widetilde{H}(x,t) = \begin{cases} \widetilde{H}(x,1) = \widetilde{f}(x) & \\ \widetilde{H}^1(x,2t) & t \in [0,1/2] \\ \vdots & \\ \widetilde{H}^k(x, 2t - (1 - 2^{-(k-1)})) & t \in \left[1 - 2^{-(k-1)}, 1 - 2^{-k}\right] \\ \vdots & \end{cases}$$

Observe that $\widetilde{H}(x,0) = (u_0(x), 0, u_1(x), 0, \ldots : F_0(x), e, F_1(x), e, \ldots)$ (recall that this is really an equivalence class so we can replace the e terms on the right by any other group element). This is continuous because each $\widetilde{H}^k$ is continuous, $\widetilde{H}^k(x,0) = \widetilde{H}^{k-1}(x,1)$ and since each coordinate only plays a role in finitely many of the steps $\widetilde{H}^k$; it is clearly G-equivariant. This gives a G-equivariant homotopy from $(u_0(x), 0, u_1(x), 0, \ldots : F_0(x), e, F_1(x), e, \ldots)$ to $\widetilde{f}$. We can do the same for $\widetilde{g}$ in such a way that we obtain a G-equivariant homotopy from $(0, v_0(x), 0, v_1(x), 0, v_2(x), \ldots : e, G_0(x), e, G_1(x), e, G_2(x), \ldots)$. Finally, we can connect these two starting points of the homotopy by another G-equivariant homotopy

$$H \colon E \times I \to EG$$

defined by

$$H(x,t) = ((1-t)u_0(x), tv_0(x), (1-t)u_1(x), tv_1(x), \ldots : F_0(x), G_0(x), F_1(x), G_1(x), \ldots).$$

This is certainly continuous as can be verified using the coordinate functions and G-equivariant. It is well-defined since the relation on EG makes identifications $((s_j) : (g_j)) \sim ((s_j') : (g_j'))$ when $s_j = s_j'$ for all j and whenever $s_j > 0$ we demand $g_j = g_j'$ (but when $s_j = 0$ we make no such stipulation). Similarly, one can check that only finitely many terms of the left-hand side of the coordinate expression for H are non-zero and that these terms all sum to 1. ■

Exercise 35. *Fill in the details that $\widetilde{H}$ as constructed above is continuous.* [Hint: It may help to look at the proof Hatcher gives for Whitehead's Theorem showing that weak equivalences between CW-complexes are actually homotopy equivalences.]

Exercise 36. *Verify that the association so defined by pulling back the universal bundle is natural. We only verified that it provides a bijection $[B, BG] \to \mathsf{Prin}_{G,\mathrm{num}}(B)/iso$ for each space $B \in \mathrm{Ho}(\mathsf{Top})$.*

Exercise 37. *Using the ideas above, show the total space of the Milnor construction is contractible: $EG \simeq *$.*

Lemma 6.1.6. *The Milnor construction is functorial in the group G as a numerable structured fiber bundle. In particular, the Milnor construction on the base space assembles into a functor $B \colon \mathsf{Top} - \mathsf{Grp} \to \mathsf{Top}$.*

Proof. Given a continuous homomorphism $\varphi\colon G \to G'$, $E\varphi\colon EG \to EG'$ is defined by $((s_i) : (g_i)) \mapsto ((s_i) : (\varphi(g_i)))$. This is certainly well-defined and continuous, as can be checked using the coordinate projection maps. This map satisfies that $((s_i) : (g_i)) \cdot g \mapsto ((s_i) : (\varphi(g_i)) \cdot \varphi(g)$ and so it is a fiberwise map and, in particular, descends to a map on quotients because φ is a group-homomorphism and thus respects the quotient relation. We call this $B\varphi$.

$$\begin{array}{ccc} EG & \xrightarrow{E\varphi} & EG' \\ \downarrow & & \downarrow \\ BG & \underset{B\varphi}{\dashrightarrow} & BG' \end{array}$$

Functorality follows quickly from this. ■

6.1.2.2 Specializations and Simplifications: $- \times G$ Commutes With Colimits

We now investigate what can be said about a model for EG when G is locally compact Hausdorff or we are working in some convenient category of spaces.

Convention. We henceforth assume G is locally compact Hausdorff and we furthermore now denote $E'G$ by EG and correspondingly denote $E'G/G$ by BG. Alternatively, we may suppose we are working in a convenient category of spaces.

Of course, since we saw that the topology on EG contains the topology generated by the projection functions, everything from before carries over verbatim. We include one nice fact about the case we are considering—namely, we obtain a filtration of BG as well.

Claim 15. Let $BG = EG/G$. Then $BG \cong \operatorname{colim}_n B_nG = \operatorname{colim}_n E_nG/G$. Moreover, the maps $B_nG \to B_{n+1}G$ induced by the closed maps and embeddings $E_nG \to E_{n+1}G$ are themselves closed maps and embeddings.

Category theory makes the first part trivial. Let G_0 be the underlying discrete version of the topological group G and let $\mathsf{B}G_0$ be the category with one object $\bullet$ and $\operatorname{End}_{\mathbf{B}G_0}(\bullet, \bullet) = G_0$ as a group. We saw that EG is the colimit in right G-spaces and it follows immediately that EG is the colimit in the category of right G_0 spaces as well. We wish to compute the colimit of the functor $F\colon \mathbf{B}G_0 \times (\mathbf{N}, \leq) \to \mathsf{Top}$ where $F(\bullet, n) = E_nG$ as a space and with G-action given by $F(g, n) = - \cdot g\colon E_nG \to E_nG$ the right G-action on E_nG. This is a functor since we saw that $E_nG \to E_{n+1}G$ is G-equivariant. Since colimits commute with colimits,

$$\begin{aligned} \operatorname{colim}_{\mathbf{B}G\times(\mathbf{N},\leq)} F &\cong \operatorname{colim}_{(\mathbf{N},\leq)} \operatorname{colim}_{\mathbf{B}G_0} F \cong \operatorname{colim}_n B_nG = \operatorname{colim}_n E_nG/G \\ \operatorname{colim}_{\mathbf{B}G\times(\mathbf{N},\leq)} F &\cong \operatorname{colim}_{\mathbf{B}G_0} \operatorname{colim}_{(\mathbf{N},\leq)} F \cong \operatorname{colim}_{\mathbf{B}G_0} EG = EG/G. \end{aligned}$$

Hence, we have a zig-zag of (natural!) isomorphisms connecting the two constructions, as desired.

As for the second part, let $F \subset B_nG$ be closed and let $\overline{F}$ be its preimage in E_nG and note that $\overline{F}$ is itself a right G-space. Hence, $\overline{F} \subset E_{n+1}G$ is closed and a right G-subspace because $E_nG \subset E_{n+1}G$ is a closed subspace and $E_nG \to E_{n+1}G$ is G-equivariant. The projection $E_{n+1}G \to E_{n+1}G/G$ sends this to a closed subspace since $\overline{F}$ already contains all G-orbits of points in it and hence is a closed saturated set for the quotient map—it follows that $B_nG \to B_{n+1}G$ is a closed map. Since it is injective and continuous, it is a closed embedding. //

Remark. As noted before, the point-set headache that immediately precludes us from considering $EG = \operatorname{colim} E_nG$ as a possible model vanishes when we insist that G is locally compact Hausdorff. We can weaken this assumption further by declaring that spaces are all compactly generated and weak Hausdorff (i.e., work in a convenient category of spaces). Many papers in the literature do this implicitly and thereby identify Milnor's construction with $\operatorname{colim} *^n G$.

Claim 16. In fact, when G is locally compact Hausdorff, we may take $EG = \operatorname{colim} E_nG$. Then $EG \to EG/G = BG$ is a numerable principal G-bundle that is universal.

The action on EG is described in **Lemma 6.1.4(a)** as it respects the inclusions $E_nG \to E_{n+1}G$ defined by $(t_1, \ldots, t_n, g_1, \ldots, g_n) \mapsto (t_1, \ldots, t_n, 0, g_1, \ldots, g_n, e)$. On the level of sets, this action agrees with the one constructed previously and is thus likewise clearly free.

Define $BG = EG/G$. The projection $EG \to EG/G = BG$ is then a numerable principal G-bundle; this follows verbatim from **Claim 13** since by **Lemma 6.1.4(c)** the coordinate functions on EG are continuous, which is all that is used in **Claim 13**. Rather than arguing as in **Theorem 6.1.5** that this bundle is universal, we will appeal to **Theorem 6.2.2** (the classification theorem) in the next section—it suffices to show that EG is contractible. In other words, it suffices to show that $\operatorname{colim} E_nG$ is contractible. We will proceed similarly to the latter part of the proof of **Theorem 6.1.5**.

Consider the homotopy $\widetilde{H}$ constructed in the proof of **Theorem 6.1.5** but apply it to $\widetilde{f} = \mathrm{id}_{EG}$. Note that $EG \times I \cong \mathrm{colim}(E_nG \times I)$. We claim $\widetilde{H} \colon EG \times I \to EG$ so defined is continuous. To see this, observe that $\widetilde{H}\Big| E_nG \times I$ is a homotopy $E_nG \times I \to E_nG$ which is continuous for the same reason $\widetilde{H}$ is continuous in **Theorem 6.1.5** (use the coordinate functions). It therefore induces a map $EG \times I \cong \mathrm{colim}(E_nG \times I) \to EG$ which is a homotopy from the map

$$((s_i) : (g_i)) \mapsto (s_0, 0, s_1, 0, \ldots : g_0, e, g_1, e, \ldots)$$

to the identity map on EG. Similarly, the map $EG \to EG$ sending $((s_i) : (g_i)) \mapsto (1, 0, \ldots, : e, e, \ldots)$ is homotopic to $(0, 1, 0, 0, \ldots : e, e, e, e, \ldots)$. The only thing left to check is that the homotopy $H \colon EG \times I \to EG$ defined by

$$H(((s_i) : g(_i)), t) = ((1-t)s_0, t, (1-t)s_1, 0, (1-t)s_2, 0, \ldots : g_0, e, g_1, e, g_2, e, \ldots)$$

is continuous. To see this, we observe once again that H restricts to a continuous function $E_nG \times I \to E_nG$ on each $E_nG \times I \subset EG \times I \cong \mathrm{colim}(E_nG \times I)$. This gives a contracting homotopy as desired. //

6.1.2.3 Specializations and Simplifications: Restricting to CW-Complexes

If we restrict bundles whose base space is a CW-complex, then the restriction that EG be contractible can be weakened to the assumption that it is *weakly* contractible. In this case, supposing G is locally compact Hausdorff, we may take $EG = \mathrm{colim}(*^n G)$ and, in particular, restricting to the full subcategory of CW-complexes in $\mathrm{Ho}(\mathsf{Top})^{\mathrm{op}}$, there is a natural isomorphism $\kappa \colon [B, EG/G] \cong k_G(B)$ sending f to $f^*(EG \to EG/G)$. Let us spell out how this goes.

Let ξ be the bundle $EG \to EG/G$. Given a principal G-bundle $P \to B$, the associated bundle $P \times_G EG \to B$ (where $G \curvearrowright EG$ by $g \cdot x = x \cdot g^{-1}$) has fiber EG which is weakly contractible. Hence, by basic obstruction theory, working cell by cell, there is a section of $P \times_G EG \to B$ and by **Lemma 6.8.1** this corresponds to a map $P \to EG$ of principal G-bundles. More precisely, working inductively, suppose we have constructed sections on the $(n-1)$-skeleton of B and consider a cell $D^n \to B$ with attaching map $S^{n-1} \xrightarrow{\phi} B$. Then the bundle $P \times_G EG$ pulls back to a bundle over a trivial space and thus the question of extending a section over the image of D^n is equivalent to construction a section of $D^n \times EG \to D^n$ that extends the pulled back section $S^{n-1} \to D^n \times EG$ and thus this is equivalent to extending $S^{n-1} \to EG$ to $D^n \to EG$ and there is no obstruction to this since $\pi_{n-1}(EG) = *$ so there is a null-homotopy of $S^{n-1} \to EG$ realized by a map $D^n \to EG$.

For injectivity, suppose we have an isomorphism $\varphi \colon f^*\xi \cong g^*\xi$ of principal G-bundles over B. We will produce a homotopy $f \simeq g$. It suffices to produce a map of principal G-bundles (hence, a right G-equivariant map) $f^*\xi I \to \xi$ such that, upon passage to G-orbits, this map descends to a homotopy $B \times I \to EG/G$ between f and g. By **Lemma 6.8.1**, giving a right G-equivariant map $f^*\xi I \to \xi$ is the same as giving a section of $(f^*\xi \times I) \times_G EG \to B \times I$. First construct a section over $B \times \partial I$ such that the section $B \times \{0\} \to ((f^*\xi \times I) \times_G EG)_0 \cong f^*\xi \times_G EG$ corresponds to the induced map on total spaces of $f^*\xi \to \xi$ and the section $B \times \{1\} \to ((f^*\xi \times I) \times_G EG)_1 \cong f^*\xi \times_G EG$ corresponds to the map on total spaces of $f^*\xi \xrightarrow{\varphi} g^*\xi \to \xi$ with $g^*\xi \to \xi$ the induced morphism of principal G-bundles. By similar reasoning, since the fiber of $(f^*\xi \times I) \times_G EG \to B \times I$ is EG is weakly contractible, this section extends to all of $B \times I$ and thus furnishes a G-equivariant map $f^*\xi \times I \to EG$ which restricts over $B \times \{0\}$ to f and restricts of $B \times \{1\}$ to g—this is because a G-equivariant morphism of the total spaces of principal bundles completely determines what happens on the base by passing to orbits, as we have seen—and so, in particular, passing to orbits we obtain the desired homotopy

$$f^*\xi \times I/G \cong B \times I \to EG/G.$$

Thus, $EG \to EG/G$ is a universal bundle for all bundles over CW-complexes—note that CW-complexes are paracompact Hausdorff so there is no question about the bundles being numerable.

6.1.3 Bar Constructions in Homotopy Theory and Another Model for The Universal Bundle

Idea. In algebra, one learns that the relative tensor product $A \otimes_R B$ of a right R-module A with a left R-module B is given by the following ***reflexive coequalizer***, where the tensors are taken over $\mathbf{Z}$

$$A \otimes B \; \underset{\xrightarrow{\;s_0\;}}{\overset{d_0}{\underset{d_1}{\leftleftarrows}}} \; A \otimes R \otimes B$$

where

$$\begin{aligned} d_0 &= \mathrm{act} \otimes B \\ d_1 &= A \otimes \mathrm{act} \\ s_0 &= \text{insertion of multiplicative identity } 1_R \end{aligned}$$

To be a reflexive coequalizer means the underlying diagram

$$\bullet \; \underset{\xrightarrow{\;s_0\;}}{\overset{d_0}{\underset{d_1}{\leftleftarrows}}} \; \bullet \qquad (*)$$

satisfies $d_0 s_0 = \mathrm{id} = d_1 s_0$. A basic theorem is that the colimit of this diagram is the coequalizer of d_0 and d_1.

Such an object of this sort is fine for regular algebra, but in homotopy theory our objects are only associative, unital and commutative up to higher coherences. To get better behavior in such a situation, we must extend this diagram to capture the relations of homotopy theory. There are many ways to do this, but the following is the basic idea.

Suppose you have a space X and you wish to consider how its points are related to each other. The first basic obstruction to relating two points of a space is if they lie in the same path-component of X. But it is not always the case that any two paths connecting two points are themselves homotopic with the homotopy fixing the endpoints. For instance, consider two paths with opposite orientation connecting any two points in S^1. We can similarly ask for homotopy relations between these homotopies and so on and so forth. The combinatorics of this question is completely captured by the ***cube category*** $\square$. The category $\square$ has objects $I^{\times n}$ for $n \geq 0$ where $I = *0 \leq 1$ is a poset. The morphisms of $\square$ are the morphisms of posets (i.e., order non-reversing maps) $I^{\times n} \to I^{\times m}$ where $I^{\times n}$ is a poset by using $0 \leq 1$ on each coordinate.

In the particular setup we are considering, the object that will track all of the relevant homotopies and relations is the ***cubical set***

$$\hom_{\mathsf{Top}}(I^{-}, X) \colon \square^{\mathrm{op}} \to \mathsf{Set}$$

where, here, $I = [0,1]$ and the morphisms $I^n \to I^m$ are the maps corresponding to those in $\square$. Of course, the combinatorics of the category $\square$ is complicated. A simpler option is subdivide each cube into simplices. The relevant category to consider which will still capture the desired homotopy data is the simplex category $\mathbf{\Delta}$. It has objects the posets $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ and its morphisms are morphisms of posets. The object to consider is then the ***simplicial set***

$$\hom_{\mathsf{Top}}(\Delta^{-}, X) \colon \mathbf{\Delta} \to \mathsf{Set}.$$

Notice as well that the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ on $[0]$ and $[1]$ is precisely the reflexive coequalizer diagram $(*)$.

To see that we've done a good job, we should like that this object captures the homotopy type of the space X.

Theorem 6.1.7. *The geometric realization of the simplicial set* $\hom_{\mathsf{Top}}(\Delta^{-}, X)$ *for any space* X *is naturally weakly equivalent to* X.

Proof. The functor $\hom_{\mathsf{Top}}(\Delta^{-}, -) \colon \mathsf{Top} \to \mathsf{sSet}$ is the singular simplicial set functor. It is a right Quillen functor and geometric realization $|-| \colon \mathsf{sSet} \to \mathsf{Top}$ is its left adjoint and also a left Quillen functor. These functors form a Quillen equivalence. Since every object in fibrant in Top and every object is cofibrant in sSet, model category theory implies that the $|\hom_{\mathsf{Top}}(\Delta^{-}, X)| \to X$ is a weak equivalence. ■

This all suggests that the relative tensor product in homotopy theory should be a resolution by the *relations* of homotopy theory if it is to capture the higher coherence data. To make this perfectly precise, we should also replace "colimit" by "homotopy colimit" but on the point-set level it will pay to distinguish between them. Let us write $* \times_G *$ for the realization of the simplicial object which in degree n is $G^{\times n}$ and with face and degeneracy maps given by multiplication/deletion and insertion of identities, respectively. One can check that when $e \to G$ is a cofibration, this is a cofibrant simplicial diagram in spaces and therefore its realization is already derived in a suitable sense. It will turn out that $BG \simeq * \times_G *$. Similarly, $EG \simeq G \times_G *$ where $G \times_G *$ denotes the realization of the simplicial object which in degree n in $G \times G^{\times n}$ with face and degeneracy maps given by multiplication/deletion and insertion of identities, respectively.

Let us now define the ***two-sided simplicial bar construction***, which one often simply calls the ***bar construction*** in homotopy theory. We shall define it in a fully functorial manner.

Exercise 38. *Show that there is a functor* $\mathbf{\Delta} \to \mathsf{Top}$ *(resp.* $\mathbf{\Delta} \to \mathsf{Top}_*$*) sending* $[n] \mapsto \Delta^n$ *(resp.* $[n] \mapsto \Delta^n_+$*) and sending an order preserving map* $\theta \colon [n] \to [m]$ *to*

$$\theta_*(t_0, \ldots, t_n) = (s_0, \ldots, s_n)$$

$$s_i = \begin{cases} 0 & \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \theta^{-1}(i) \neq \emptyset. \end{cases}$$

Definition. Let $\mathsf{C} = \mathsf{Top}$ or Top_*. These categories are symmetric monoidal under the $\times$ and $\wedge$, respectively—note that when we replace them by their "convenient" analogues, they are closed symmetric monoidal. We denote these by $\otimes$ for convenience. By abuse of notation, let $\mathsf{Bar}(\mathsf{C})$ be the following category.

1. The objects of $\mathsf{Bar}(\mathsf{C})$ are triples (A, M, B) where M is a monoid in C and A and B are right and left modules over M, respectively.
2. The morphisms of $\mathsf{Bar}(\mathsf{C})$ are triples $(f, \varphi, g)\colon (A, M, B) \to (A', M', B')$ such that φ is a morphism of monoids, $f\colon A \to \varphi^* A'$ and $g\colon B \to \varphi^* B'$ are morphisms of right (resp. left) modules over M where φ^* indicates restriction of scalars.

The ***two-sided simplicial bar construction*** or simply the ***bar construction*** is the functor

$$\mathsf{Bar}\colon \mathsf{Bar}(\mathsf{C}) \to \mathsf{sC} = \mathsf{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$$

defined as follows. Levelwise, $\mathsf{Bar}(A, M, B)_n = A \otimes M^{\otimes n} \otimes B$. Fixing an element $a_0 \otimes m_1 \otimes \ldots \otimes m_n \otimes b_{n+1} \in \mathsf{Bar}(A, M, B)_n$, the face maps $d_i\colon \mathsf{Bar}(A, M, B)_n \to \mathsf{Bar}(A, M, B)_{n-1}$ $(0 \le i \le n)$ multiply the i-th and $i+1$-st elements of $a_0 \otimes m_1 \otimes \ldots \otimes m_n \otimes b_{n+1}$ and the degeneracy maps $s_i\colon \mathsf{Bar}(A, M, B)_n \to \mathsf{Bar}(A, M, B)_{n+1}$ $(0 \le i \le n)$ inserts a unit after the i-th element in $a_0 \otimes m_1 \otimes \ldots \otimes m_n \otimes b_{n+1}$.

Definition. Let $\mathsf{C} = \mathsf{Top}$ or Top_* and denote the monoidal product of C by $\otimes$ as above.
Let $X \in \mathsf{C}^{\boldsymbol{\Delta}^{\mathrm{op}}}$. Define the ***geometric realization*** $|X|$ of X to be the following ***coend***

$$|X| = X \otimes_{\boldsymbol{\Delta}} \Delta \overset{\text{def}}{=} \begin{cases} \int^{[m]\in\boldsymbol{\Delta}} X_n \times \Delta^n & \mathsf{C} = \mathsf{Top} \\ \int^{[m]\in\boldsymbol{\Delta}} X_n \wedge \Delta^n_+ & \mathsf{C} = \mathsf{Top}_*. \end{cases}$$

Here, $(-)_+$ is the functor from spaces to pointed spaces adding a disjoint basepoint.

Notation. We shall denote the functor $|\mathsf{Bar}| : \mathsf{Bar}(\mathsf{C}) \to \mathsf{C}$ simply by B.

Proposition 6.1.8. *The functor* $|-| : \mathsf{sC} \to \mathsf{C}$ *is a left adjoint. When* $\mathsf{C} = \mathsf{Top}$*, its right adjoint is the functor* $X \mapsto \mathbf{Hom}_{\mathsf{Top}}(\Delta^-, X)$ *and when* $\mathsf{C} = \mathsf{Top}_*$ *its right adjoint is the functor* $X \mapsto \mathbf{Hom}_{\mathsf{Top}_*}(\Delta^-_+, X)$*.*

Remark. Above, we do not need to restrict ourselves to convenient subcategories of spaces. Just as is done in Quillen's manuscript *Homotopical Algebra*, we only need closure of the evident simplicial enrichment over *finite* simplicial sets and hence only over *finite* simplicial complexes. In other words, while the adjunction of mapping spaces $\mathrm{Hom}_{\mathsf{Top}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{Top}}(X, \mathrm{Hom}(Y, Z))$ *fails* in general, it is true when Y is a locally compact Hausdorff space and all finite simplicial complexes are such.

Remark. As far as the author knows, the following construction is due to Peter May but the idea goes back further to Dold, Lashof and Steenrod. This next theorem is **Theorem 8.2** here, among other items.
The non-degeneracy of the basepoint is essential for homotopical reasons—one should hope that $\mathsf{B}(*, G, G)$ is already derived in a suitable sense and this condition makes this possible because it allows $\mathsf{Bar}(*, G, G)$ to be a cofibrant simplicial object in the Hurewicz model structure on compactly generated weak Hausdorff spaces. In particular, call a cofibration in this category a closed Hurewicz cofibration (note in subcategory of compactly generated weak Hausdorff spaces all Hurewicz cofibrations are already closed embeddings). We shall show that for such G, the simplicial object $\mathsf{Bar}(*, G, *)$ is a cofibrant simplicial object.
Fix a simplicial object $X\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{Top}_{\mathrm{CGWH}}$ in the convenient category of compactly generated weak Hausdorff spaces. Let $\boldsymbol{\Delta}^{\mathrm{op}}_+$ denote the full subcategory generated by all degeneracies s_i—that is, all maps in $\boldsymbol{\Delta}^{\mathrm{op}}_+$ are composites of the degeneracies—in other words, all the maps of $\boldsymbol{\Delta}^{\mathrm{op}}_+$ have underlying map in $\boldsymbol{\Delta}$ a surjective map. For each $[n] \in \boldsymbol{\Delta}^{\mathrm{op}}$, define the $\partial(\boldsymbol{\Delta}^{\mathrm{op}}_{+/[n]})_+$ category whose objects are the maps $[m] \to [n]$ in $\boldsymbol{\Delta}^{\mathrm{op}}_+$ *excluding the identity map* $[n] \xrightarrow{id} [n]$ in $\boldsymbol{\Delta}$ and whose morphisms the maps in $\boldsymbol{\Delta}^{\mathrm{op}}_+$ making TFDC:

$$\begin{array}{ccc} [m_1] & \xrightarrow{s} & [m_2] \\ \downarrow & & \downarrow \\ [n] & = & [n] \end{array}$$

Define the ***n-th latching object*** $L_n X$ of the simplicial object X to be

$$L_n X = \mathrm{colim}_{[m]\to[n]\in\partial(\boldsymbol{\Delta}^{\mathrm{op}}_{/[n]})_+} X_m.$$

Note that $L_0 X = \emptyset$ and $L_1 X = X_0$. We say $X\colon \boldsymbol{\Delta}^{\mathrm{op}} \to \mathsf{Top}_{\mathrm{CGWH}}$ is a *cofibrant* if the natural map $L_n X \to X_n$ is a closed Hurewicz cofibration.

Now, for each n, one can show that $L_n\mathsf{Bar}(*, G, *) \cong \bigvee_{i\in\{1,\dots,n\}} G$ with the wedge point taken to be the identity element $e \in G$. The natural map $\bigvee_{i=1}^n G \to \prod_{i=1}^n G$ with image the "coordinate axes" of the product "with origin $e = (e, \dots, e)$" is a closed Hurewicz cofibration in the full category of topological space. Since the basepoint $e \to G$ is a cofibration, (G, e) is an NDR-pair in the full category of spaces and so there are continuous functions

$$H : G \times I \to G \qquad u : G \to I$$

such that $H(e, -) \equiv e$, $H(-, 0) = \mathrm{id}_G$ and for all $g \in u^{-1}([0, 1))$, $H(g, 1) = e$ and $u^{-1}(0) = e$. Let $\widetilde{U} = u^{-1}([0, 1))$ and let $V \subset \prod_i G$ to be

$$\bigcup_{j=1}^{n} \widetilde{U} \times \cdots \times \widetilde{U} \times G \times \widetilde{U} \times \cdots \times \widetilde{U}$$

where the G appears in the j-th factor of the product. This is a "tubular" nbhd of the coordinate axes of the product. Define a function $t\colon \prod_i G_i \to I$ by

$$v(g_1, \dots, g_n) = \min\{1, \inf\{t \in I : H(g_i, t) = e \text{ for some } 1 \le i \le n\}\}.$$

Note that when $\{t \in I : H(g_i, t) = e \text{ for some } 1 \le i \le n\} \neq \emptyset$, the infimum is actually a minimum since it is the minimum of over the sets $H(g_i, -)^{-1}(e)$ which are all compact subspaces of I and at least one of which is non-empty. Note that $\inf \emptyset = +\infty$. One may easily check this function is continuous and thus the function $\widetilde{H}\colon \prod_i G \times I \to \prod_i G$ defined by

$$\widetilde{H}(g_1, \dots, g_n, t) = \begin{cases} (H(g_1, t), \dots, H(g_n, t)) & t \le v(g_1, \dots, g_n) \\ (H(g_1, v(g_1, \dots, g_n)), \dots, H(g_n, v(g_1, \dots, g_n))) & t \ge v(g_1, \dots, g_n). \end{cases}$$

is continuous.

This shows that $(\prod_i G, \bigvee_i G)$ is an NDR-pair for the functions $(\widetilde{H}, v)$. Hence, $\bigvee_i G \to \prod_i G$ is a closed cofibration. //

Theorem 6.1.9. *Suppose* Top *is a convenient category of spaces and let G be a group for which $e \hookrightarrow G$ is a closed cofibration. Then $EG = \mathsf{B}(*, G, G)$ is a right G-space and $EG \to EG/G \cong \mathsf{B}(*, G, *)$ is a numerable principal G-bundle.*

Remark. While will not use this construction, it has superb functorial properties as long as we restrict our attention to groups as in the statement of theorem and work in a convenient category of spaces.

6.2 The Classification Theorems: Consequences and Applications

6.2.1 The Classification Theorems

Recall the following from the subsection *Universal Bundles & Milnor's Construction.*

Lemma. *Let X be a space. The isomorphism classes of (numerable) principal G-bundles over X form a set.*

Definition. For each topological group G, define a functor $k_G\colon \mathrm{Ho}(\mathsf{Top})^{\mathrm{op}} \to \mathsf{Set}$ sending a space X to the set $[\mathsf{Prin}_{G,\mathrm{num}}(X)]$ of isomorphism classes of numerable principal G-bundles over X. A numerable principal G-bundle $P \to B$ is called a ***universal bundle*** and B is called a ***classifying space*** of G if there is a natural isomorphism $[-, B] \cong k_G$ given by sending a homotopy class $f\colon X \to B$ to the principal G-bundle f^*P.

Remark. This says that k_G is representable and that, moreover, the natural isomorphism $[-, BG] \cong k_G$ has a particularly nice description.

We refer to the two following theorems as the ***Classification Theorem***. We will prove the latter of the two at the end of the lecture.

Theorem 6.2.1. *Fix a topological group G. The functor $k_G\colon \mathrm{Ho}(\mathsf{Top})^{\mathrm{op}} \to \mathsf{Set}$ is representable. In particular, there is a natural isomorphism $k_G \cong [-, BG]$ which associates to an isomorphism class of a principal G-bundle $E \in k_G(B)$ the homotopy class of the map $f\colon B \to BG$ such that $E \cong f^*EG$ where EG and BG are given as in Milnor's construction.*

Proof. In our construction of Milnor's universal bundle, we saw that a numerable principal G-bundle $E \to B$ is exactly the same (up to isomorphism) as a morphism of principal G-bundles $\widetilde{f}E \to EG$. The pullback theorem implies that $\widetilde{f}\colon E \to EG$ is determined by the map it induces on base spaces $f\colon B \to BG$. Topological homotopy invariance now implies that the association $f \mapsto [f]$ is well-defined and thus it is a bijection. Naturality follows immediately by pasting pullbacks (i.e., two pullback squares paste to a single pullback square). ■

Theorem 6.2.2 (Dold, 7.5). *A numerable principal G-bundle ξ is a universal bundle **iff** the total space E of ξ is contractible. In particular, any two universal (numerable) bundles are G-equivariantly homotopy equivalent.*

6.2.2 The Closed Subgroup Theorem & Principal Bundles as Quotient Maps $G \to G/H$

Before continuing, we collect some technical result. We defer the proof of the closed subgroup theorem to the end of the section.

We leave the results about quotient maps to the reader as exercises. Note that when $\mathsf{CAT} = \mathsf{DIFF}$, we are not guaranteed that G/H is Lie group in general unless H is a closed normal subgroup.

Exercise 39. *Let G be a CAT group and $H \leq G$ a CAT subgroup.*

(a) *Show that the quotient map $p\colon G \to G/H$ admits a CAT principal H-bundle **iff** there is a nbhd U of the basepoint $e = eH \in G/H$ along with a local CAT section $s\colon U \to G$ of p.* [Hint: Define $\psi_g^{-1}\colon gU \times H \to p^{-1}(gU)$ by $\psi_g^{-1}(gu, h) = g \cdot s(u) \cdot h$. Show these are H-equivariant homeomorphisms (diffeomorphisms in the smooth case) whence the inverse makes sense.]
(b) *Does every principal H-bundle structure on $G \to G/H$ arises from translations of a trivialization about the identity $e \in G/H$?* [Hint: **Exercise 17(d)** will be useful.]

Exercise 40. *Fix a choice of CAT. Let G be a CAT group, let $H \leq G$ be a CAT subgroup which is closed when $\mathsf{CAT} = \mathsf{DIFF}$. Suppose there exist open sets $e \in E \subset G$ and $e \in U \subset H$ such that the group multiplication $E \times U \to E \cdot U \subset G$ gives an isomorphism onto its image. Show that the projection $q\colon G \to G/H$ can be made into a CAT principal H-bundle.* [Hints/Steps:

(1) By the quotient manifold theorem, G/H is a smooth manifold if $H \leq G$ is a closed Lie subgroup.
(2) Write $U = H \cap V$ for an open $V \subset G$ by definition of the subspace topology. Show there is an open set $W \subset V$ such that $W^{-1}W \subset V$.
(3) Set $T = E \cap W \subset G$ and show that $T \times H \to TH \subset G$ is an isomorphism onto its image.
(4) Show TH is open in G and that its image in G/H is open as well. Denote the image of TH in G/H by TH as well.
(5) Show this gives an H-equivariant CAT isomorphism $q^{-1}(TH) \to T \times H$. In particular $T \times H \to TH = q^{-1}(TH)$ is such and so its inverse is as well.
(6) By left translation, show that G/H is covered by such open sets with corresponding trivializations. Verify that transitions functions for these are CAT and land in H.
(7) Put everything together to conclude that $G \to G/H$ is a CAT principal H-bundle.]

Remark. The structure supplied as a principal H-bundle by the two procedures outlined above is independent of the choice of E or U. To see this, deploy **Exercise 17(d)**.

Theorem 6.2.3 (Closed Subgroup Theorem). *Let G be a Lie group of dimension n and $H \leq G$ a closed topological subgroup. Define $\mathfrak{h} = \stackrel{def}{=} \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \mathbf{R}\}$.*

(a) *$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$.*
(b) *H is in fact an embedded submanifold and Lie subgroup of G.*
(c) *$G \to G/H$ is a principal H-bundle.*

Remark. In general, G/H will only be a a Lie group when H is a normal subgroup as well. For $H \leq G$ a closed subgroup, G/H is a smooth manifold for which the projection $G \to G/H$ is smooth.

6.2.3 Consequences and Applications

Corollary 6.2.4. *Any two base spaces of a universal bundle are homotopy equivalent.*

Proof. If $E \to B$ and $E' \to B'$ are universal bundles, then $[-, B]$ and $[-, B']$ both represent the functor k_G. Hence, there is a natural isomorphism $[-, B] \cong [-, B']$ and hence, by Yoneda, this is represented by a homotopy equivalence (i.e., an isomorphism in $\mathrm{Ho}(\mathsf{Top})$) in $[B, B']$. ■

Exercise 41. *Given a homotopy equivalence $f\colon B \simeq BG$, show that $f^*EG \to B$ is a numerable principal G-bundle and that, moreover, $f^*EG \to B$ is a universal principal G-bundle.*

Proposition 6.2.5. *There is a homotopy equivalence* $B(G_1 \times G_2) \simeq BG_1 \times BG_2$.

Proof. Certainly $EG_1 \times EG_2 \to BG_1 \times BG_2$ is a numerable principal $G_1 \times G_2$-bundle and the product of two contractible spaces is a contractible space. Hence, $EG_1 \times EG_2 \to BG_1 \times BG_2$ is a universal bundle. By the corollary, there exists a homotopy equivalence $B(G_1 \times G_2) \simeq BG_1 \times BG_2$. ■

Proposition 6.2.6. *There is a weak equivalence* $G \xrightarrow{\sim} \Omega BG$.

Proof. Fix a basepoint $e \in EG$ and let b be the image of e in BG. We treat these as basepoints. Note that the bundle projection $EG \xrightarrow{p} BG$ is a quotient map and therefore surjects so we loose nothing by doing this—every point $b \in BG$ is the image of some point $e \in EG$.

By **Theorem E.1.3**, fiber bundles are Serre fibrations, so the fiber sequence $G \to EG \to BG$ is a (homotopy) fiber sequence. Similarly, $\mathsf{Map}_{\mathsf{Top}_*}((I,0),(BG,b)) = PBG \xrightarrow{\mathrm{ev}_1} BG$ be the path-space fibration having basepoint the constant map at b. Since this is a Hurewicz fibration, its fiber ΩBG is its homotopy fiber.

Let $h\colon EG \times I \to EG$ be a contracting homotopy with $h(x,0) = e$ the basepoint and $h(x,1) = x$. Define $\widetilde{h}\colon EG \to PBG$ by $x \mapsto (p \circ h)(x,t)$. This is a pointed map. Let $\alpha = \widetilde{h}\Big| p^{-1}(b)$. We then have a morphism of (homotopy) fiber sequences

$$\begin{array}{ccccc} G & \longrightarrow & EG & \twoheadrightarrow & BG \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widetilde{h}} & & \| \\ \Omega BG & \longrightarrow & PBG & \twoheadrightarrow & BG \end{array}$$

Since $EG \simeq * \simeq PBG$, the long exact sequence in π_* for a fibration along with the 5-lemma furnishes the result for basepoints as chosen. Any other choices of basepoint work out the same so α is an honest weak equivalence. ■

Remark. Because of this, we say that BG is a ***delooping*** of G.

Corollary 6.2.7. $\pi_0(BG) = *$. *If G is path-connected, then BG is simply connected.*

Proof. There is an adjunction for homotopy classes of pointed spaces $[\Sigma X, Y]_* \cong [X, \Omega Y]_*$ and $\Sigma S^n = S^{n+1}$. Hence, $\pi_n(\Omega Y) = \pi_{n+1} Y$. The result now follows since $G \to \Omega BG$ is a weak equivalence. ■

Definition (Coinduced Bundle). Let $\varphi\colon G \to G'$ be a CAT homomorphism of CAT groups and let $\xi = (E, B, p, G)$ be a CAT principal G-bundle. Then the ***coinduced bundle*** $\varphi_*\xi$ is the CAT principal G'-bundle $\xi \times_G G'$ of the construction in **Lemma 4.2.4**. Here, G acts on the right of the total space through the principal bundle structure and G acts on the left of G' by $g \cdot g' = \varphi(g)g'$. In this case, we identify the transition functions of $\xi \times_G G'$ as those of ξ post-composed with φ. In other words, $\varphi \circ g_{ij}$. Since φ is CAT, this remains a CAT principal G'-bundle.

Lemma 6.2.8. *Let us denote ξ_G and $\xi_{G'}$ the universal principal bundles for G and G' respectively. Let $\varphi\colon G \to G'$ be continuous group-homomorphism. This gives a continuous action $G \curvearrowright G'$ by $g \cdot g' = \varphi(g) \cdot g'$.*

TFAE up to homotopy:

(a) *$B\varphi$ as in the Milnor construction.*
(b) *If there exists a continuous map $\widetilde{f}\colon EG \to EG'$ such that $\widetilde{f}(x \cdot g) = \widetilde{f}(x) \cdot \varphi(g)$, then passage to quotients induces $B\varphi$.*
(c) *$B\varphi$ is the map $BG \to BG'$ classifying the numerable principal G'-bundle $\varphi_*\xi_G$ which is the bundle $EG \times_G G' \to BG$.*

Remark. Even though $G \to G'$ specifies an action $G \curvearrowright G'$, it may not be an effective action, but this is alright. The only place where effectiveness is required is in obtaining functorality of the associated principal G-bundle construction and from there the equivalence $\mathsf{Prin}_G \simeq \mathsf{Bun}_G^F$.

In **(b)**, we only require that $\widetilde{f}$ be continuous. The fact that $EG \curvearrowleft G$ and $EG' \curvearrowleft G'$ are free on each fiber implies that such a $\widetilde{f}$ is a fiber-preserving map.

Proof. It suffices to work with the objects constructed in the Milnor construction throughout by the forgeoing considerations.

(**c**) $\Leftrightarrow$ (**a**) Given a map classifying the principal G'-bundle $EG \times_G G' \to BG$, we would like to show that it is homotopic to the map $B\varphi$ of the Milnor construction and, conversely, we would like to show that Milnor $B\varphi$ classifies $EG \times_G G' \to BG$. By the classification theorem, **Theorems 6.2.1 & 6.2.2**, and the pullback theorem, it suffices to show that the $B\varphi$ of the Milnor construction classifies $EG \times_G G' \to BG$ where $EG \times_G G'$ has transition functions the same as EG except we post-compose them with φ (i.e., $\varphi \circ g_{ij}$). This pops out of the associated bundle construction since $G \curvearrowright G'$ through φ.

Let $p\colon EG \to BG$ and $q\colon EG' \to BG'$ be the projections of the Milnor construction. The Milnor construction's $B\varphi$ sends $p((s_i) : (g_i)) \mapsto q((s_i) : (\varphi(g_i)))$. We can define $EG \times_G G' \to EG' \times_{G'} G' \cong EG'$ by $E\varphi \times_\varphi G'$. We thus have the following diagram

$$\begin{array}{ccc} EG \times_G G' & \xrightarrow{E\varphi \times_\varphi G'} & EG' \times_{G'} G' \\ \downarrow & & \downarrow \\ BG & \xrightarrow[B\varphi]{} & BG' \end{array}$$

and it is easy to check this commutes

$$\begin{array}{ccc} [((s_i) : (g_i)), g] & \longmapsto & [(((s_i) : (\varphi(g_i))), \varphi(g))] \\ \downarrow & & \downarrow \\ p(((s_i) : (g_i))) & \longmapsto & q(((s_i) : (\varphi(g_i)))) \end{array}$$

If we can show that $E\varphi \times_\varphi G'$ is a morphism of principal G'-bundles, then the pullback theorem tells us that this must be a pullback diagram and so we can conclude. For this, it is easy to see that the coordinate form of $E\varphi$ (perhaps shrinking V first) has the form

$$V \times G \to U \times G' \qquad (x, g) \mapsto (B\varphi(x), \overline{g}_{UV}(x)\varphi(g))$$

where $\overline{g}_{UV}(x) \in G'$ is the image of (x, e). As in **Theorem 4.2.5**, it follows that upon taking associated bundles, the form of this map on the same trivializing open sets is

$$V \times G' \to U \times G' \qquad (x, g') \mapsto (B\varphi(x), \overline{g}_{UV}(x)g')$$

which is therefore a morphism of principal G'-bundles and hence so we may conclude.

(**b**) $\Leftrightarrow$ (**c**) Given a morphism of principal G'-bundles

$$\begin{array}{ccc} EG \times_G G' & \xrightarrow{F} & EG' \\ \downarrow & & \downarrow \\ BG & \longrightarrow & BG' \end{array}$$

define $EG \to EG \times_G G'$ to be the composite

$$EG \xrightarrow{\mathrm{id} \times e} EG \times G' \to EG \times_G G'$$

sending $x \mapsto [(x, e)]$. Then the induced map

$$EG \to EG'$$

sends $x \mapsto F([(x, e)])$ but also

$$x \cdot g \mapsto F([(x \cdot g, e)]) = F([(x, \varphi(g))]) = F([(x, e)] \cdot \varphi(g)) = F([(x, e)]) \cdot \varphi(g)$$

so that $EG \to EG'$ satisfies the hypotheses sought for $\widetilde{f}$ and, moreover, the following diagram commutes

$$\begin{array}{ccccc} EG & \longrightarrow & EG \times_G G' & \longrightarrow & EG' \\ \downarrow & & \downarrow & & \downarrow \\ BG & = & BG & \longrightarrow & BG' \end{array}$$

so that $B\varphi$ is described as in **(b)**.

Conversely, given a map $\widetilde{f}\colon EG \to EG'$ satisfying $\widetilde{f}(x \cdot g) = \widetilde{f}(x) \cdot \varphi(g)$, we know $\widetilde{f}$ is a fiber preserving map from the remark above. This furnishes a morphism of principal G'-bundles $\widetilde{f} \times_\varphi G'\colon EG \times_G G' \to EG' \times_{G'} G' \cong EG'$. More explicitly, in the evident morphism of coequalizer diagrams, this arrows arises as the dashed one

$$\begin{array}{ccccc} EG \times G \times G' & \rightrightarrows & EG \times G' & \longrightarrow & EG \times_G G' \\ \downarrow{\scriptstyle \widetilde{f} \times \varphi \times 1} & & \downarrow{\scriptstyle \widetilde{f} \times 1} & & \vdots \\ EG' \times G' \times G' & \rightrightarrows & EG' \times G' & \longrightarrow & EG' \times_G G' \end{array}$$

This diagram commutes because $\widetilde{f}(x \cdot g) = \widetilde{f}(x) \cdot \varphi(g')$.

To see that this map is G'-equivariant, we must unravel the associated bundle construction once more. The bundle $q = p \times_G G'\colon EG \times_G G' \to BG$ has the same trivializing open sets and for $p\colon EG \to BG$, $q^{-1}(U) = p^{-1}(U) \times_G G'$ with trivialization $\varphi \times_G G'\colon p^{-1}(U) \times_G G' \to U \times G \times_G G' \cong U \times G'$. For $[(x, g')] \in q^{-1}(U)$, the action of $g'' \in G'$ on this element is defined by

$$[(x,g')]\cdot g'' = (\varphi \times_G G')^{-1}((\varphi \times_G G')([(x,g')])\cdot g'')$$

as we have seen before. This can easily be computed to be $[(x,g')]\cdot g'' = [(x,g'g'')]$. In particular, this means the action on $EG \times_G G'$ is induced by the right action of G' on itself. On the other hand, the bundle $EG' \times_{G'} G' \to BG'$ is isomorphic to the bundle $EG' \to BG'$, so with the former model the map $\widetilde{f} \times_G G' \colon EG \times_G G' \to EG' \times_G G'$ is manifestly right G-equivariant since $\widetilde{f}([(x,g'g'')]) = [(\widetilde{f}(x), g'g'')] = \widetilde{f}([(x,g')])\cdot g''$.

Thus, passing to the quotient, this construction yields a map $BG \to BG'$. Thus, $\widetilde{f}\times_\varphi G'$ classifies the bundle $EG\times_G G' \to BG$ by the pullback theorem, homotopy invariance and the classification theorem, **Theorems 6.2.1 & 6.2.2**. It is easy to see these two procedures are inverse to one another up to homotopy and this finishes the proof. ■

Reminder. The following lemma is most useful when paired with **Exercise 40**: $G \to G/H$ admits the structure of a principal H-bundle **iff** it admits a section in a nbhd of $e \in G/H$. If $H \le G$ is a closed subgroup of Lie group, then we have constructed a numerable principal H-bundle structure on $G \to G/H$.

Lemma 6.2.9. *Let $H \le G$ and let $p\colon G \to G/H$ be the quotient map.*

(a) *Suppose $p\colon G \to G/H$ is a principal H-bundle. Then $q\colon EG \to EG/H$ is a principal H-bundle and, more generally, for any* CAT *principal G-bundle $E \to B$ the analogous statement that $E \to E/H$ is a* CAT *principal H-bundle is true.*
(b) *The natural map $EG/H \to EG/G$ is a numerable fiber bundle with typical fiber G/H and structure group G. More generally, for any (numerable)* CAT *principal G-bundle $E \to B$ the analogous statement that $E/H \to B$ is a (numerable)* CAT *fiber bundle with typical fiber G/H and structure group G is true. In particular, $E/H \cong E \times_G (G/H)$.*
(c) *Suppose $p\colon G \to G/H$ is a numerable principal H-bundle. Then $EG \to EG/H$ is a universal principal H-bundle and, more generally, for any numerable* CAT *principal G-bundle $E \to B$, $E \to E/H$ is a numerable* CAT *principal H-bundle.*
(d) *There is a homotopy equivalence $EG/H \simeq BH$.*
(e) *There is a model for $Bi\colon BH \to BG$ which is a numerable fiber bundle with fiber G/H and structure group G. In particular, when $G \curvearrowright G/H$ effectively by translation, this is universal among all numerable bundles with structure group G and fiber G/H in the sense that homotopy classes of maps $X \to BG$ are in natural bijection with numerable such bundles over X by pullback.*

Remarks. The CAT = DIFF statements follow easily from the spaces version mutatis-mutandis, so we only address the continuous case. By the closed subgroup theorem, when CAT = DIFF and $H \le G$ is a closed subgroup, then $G \to G/H$ is a principal H-bundle.

Proof. **(a)** Fix a nbhd U of $eH \in G/H$ as in the preceding exercise. Fix a G-bundle atlas for $\pi\colon E \to B$, say $\{U_i, \varphi_i\}$. Let $W_{i,g} = q\varphi_i^{-1}(U_i \times p^{-1}(gU)) \subset E/H$. This is open because $\varphi_i^{-1}(U_i \times p^{-1}(gU))$ is an open saturated set for q and they cover E/H. Define $\phi_{i,g}\colon q^{-1}(W_{i,g}) \to W_{i,g} \times H$ by

$$\phi_{i,g}(\varphi_i^{-1}(x, gs(u)h)) = \phi_{i,g}(\varphi_i^{-1}(x, \psi_g(gu,h))) = (q\varphi_i^{-1}(x,\psi_g(gu,e)),h) = (q\varphi_i^{-1}(x,gs(u)),h).$$

It is not hard to see that this is well-defined, a homeomorphism and right H-equivariant since the φ_i are G-equivariant. As for transitions functions, the transition map

$$\phi_{i,g} \circ \phi_{j,g'}^{-1} \colon W_{i,g} \cap W_{j,g'} \times H \to W_{i,g} \cap W_{j,g'} \times H$$

is a homeomorphism and right H-equivariant—this means for fixed $x \in W_{i,g} \cap W_{j,g'}$, the function $\phi_{i,g} \circ \phi_{j,g'}^{-1}(x,h) = \phi_{i,g} \circ \phi_{j,g'}^{-1}(x,e)h$ and so we extract a proposed transition function

$$h_{(i,g),(j,g')} = \mathrm{pr}_H \circ \phi_{i,g} \circ \phi_{j,g'}^{-1}(-,e).$$

This is clearly continuous and this shows $E \to E/H$ is a principal H-bundle for any principal G-bundle $E \to B$.

Remark. It is worth pointing out at this point that, in general, the above argument should fail but we get away with it here because the structure group *is* the typical fiber.

(b) Suppose $p\colon G \to G/H$ is a numerable principal H-bundle. It is not hard to see by hand that $E/H \cong E \times_G G/H$ and some thought shows by the associated bundle construction that $E/H = E \times_G G/H \to B$ is a (numerable) fiber bundle since $E \to B$ is a (numerable) bundle—it is easily verified that the two projection maps $E/H \to B$ and $E \times_G G/H \to B$ are the same and we have seen that this construction preserves the trivializing open sets. It follows that $E/H \to B$ locally has the form $U \times G/H$ and over the same open sets for which $E \to B$ has the form $U \times G$. The statement about structure groups follows what has been shown about the associated bundle construction. Recall again that the only place where effectiveness of actions was needed for the various bundle constructions was in proving functorality of the associated principal bundle functor.

(c) Suppose $p\colon G \to G/H$ is a numerable principal H-bundle. We consider the case of $q\colon EG \to EG/H$ because the proof for $E \to E/H$ is the same mutatis-mutandis. We know that $q\colon EG \to EG/G$ is a numerable bundle and so we may assume it has a locally finite and countable open cover given by a partition of unity witnessing this say $\mathscr{U} = \{(U_i, \rho_i)\}$ where $\rho_i\colon BG \to [0,1]$ with $U_i = \rho_i^{-1}((0,1])$ and $\varphi_i\colon q^{-1}(U_i) \cong U_i \times G$ equivariantly. Let us continue to identify $EG/H \cong EG \times_G G/H$ as in **(b)**.

On trivializations of $EG/G = BG$, $EG \to EG/H$ looks like the projection $(\mathrm{id}, p)\colon U_i \times G \to U_i \times G/H$. One easily verifies this using the description of the trivializations for the associated bundle construction. We also know that $U_i \times G \to * \times G \cong G$ is a numerable bundle. Since $G \to G/H$ is numerable, it has some covering $\mathscr{V} = \{(V_j, \lambda_j)\}_{j\in J}$ as $EG \to BG$ does. Define a new partition of unity for $EG \to EG/H$ by

$$\mathscr{W} = \left\{((\rho_i\lambda_j)^{-1}((0,1]), \rho_i\lambda_j)\right\}_{i\in I, j\in J}, \qquad W_{ij} \stackrel{\text{def}}{=} (\rho_i\lambda_j)^{-1}((0,1])$$

(understood appropriately). We have abused notation. Here, we understand $\rho_i\lambda_j$ as follows. By **(b)**, $q_0\colon : EG/H \to EG/G$ is a numerable fiber bundle with fiber G/H and, in particular, it is numerated by the same collection $\mathscr{U}$ as $EG \to EG/G$. Let $\psi_i\colon q_0^{-1}(U_i) \to U_i \times G/H$ be trivializations for this fiber bundle over U_i. Then $\rho_i\lambda_j$ is defined to be the function

$$(\rho_i\lambda_j)(x) = \begin{cases} \rho_i(q_0(x))\lambda_j(\mathrm{pr}_2\, \psi_i(x)) & x \in q_0^{-1}(U_i) \\ 0 & \text{else.} \end{cases}$$

Recall from definition of numerability we may assume each U_i is actually contained in a larger trivializing open set U_i' (see the remark following the definition of a numerability) such that $\operatorname{supp}\rho_i \subset U_i'$. The analogous assertion holds for the λ_j. Thus, suppose WLOG that

$$\psi_i\colon q_0^{-1}(U_i') \to U_i' \times G/H$$

To see that this function is continuous, it helps to rewrite it. Let $\rho_i \cdot \lambda_j\colon EG/G \times G/H \to [0,1]$ be the evident composite

$$EG/G \times G/H \to [0,1] \times [0,1] \to [0,1]$$

which is continuous as a composite of continuous functions. Then $\rho_i\lambda_j|\, q_0^{-1}(U_i')$ is the composite

$$\rho_i \cdot \lambda_j \circ \psi_i$$

where we consider $\psi_i\colon q_0^{-1}(U_i') \to U_i' \times G/H \subset EG/G \times G/H$. This is continuous as a composite of continuous functions. Since $\rho_i\lambda_j \equiv 0$ outside of $q_0^{-1}(U_i')$ and since $\operatorname{supp}(\rho_i\lambda_j) \subset q_0^{-1}(U_i')$, this is sufficient to conclude that $\rho_i\lambda_j$ is continuous with some thought. (For instance, writing the preimage of an interval $[0,a)$ as the union of the complement of the support of $\rho_i\lambda_j$, which is open, with a set known to be open since $\rho_i\lambda_j|\, q_0^{-1}(U_i')$ is continuous. With some thought, one can see that this argument is sufficient.)

As for the partition of unity, simply observe that for fixed $x \in W_{ij} \subset EG/H$

$$\sum_{i,j} \rho_i(x)\lambda_j(x) = \sum_i \rho_i(x) \sum_j \lambda_j(x) = \sum_i \rho_i(x) = 1$$

where we can move the sum as we have indicated by local finiteness—we have always suppressed dependence of λ_j on i but the dependence forces us to take the sum over the λ_j first. This shows the bundle is numerable.

(d) This follows from preceding considerations.

(e) The first part follows from preceding considerations. For the second, we simply observe that $BH \simeq EG \times_G G/H$ and from here we may appeal to the equivalence of categories we constructed. ∎

Example 5. Later, we will see that $O(n)/O(n-1) \cong SO(n)/SO(n-1) \cong S^{n-1}$. The natural action of $O(n)$ (resp. $SO(n)$) on S^{n-1} induced under this isomorphism is effective by an easy geometric argument. More precisely, one can compute that $\bigcap_{T\in O(n)} T^{-1}O(n-1)T = e$ which implies that the action of $O(n)$ on $O(n)/O(n-1)$ is effective.

Corollary 6.2.10. *If $H \leq G$ is a subgroup for which $G \to G/H$ is a principal H-bundle, then for any subgroup $K \leq H$, $H/K \to G/K \to G/H$ is a fiber bundle with structure group H. If $H \leq G$ is a closed subgroup of a Lie group G and $K \leq H$ is any Lie subgroup, then $H/K \to G/K \to G/H$ is a smooth fiber bundle with structure group H.*

Proof. In the smooth case, since $H \leq G$ is a closed subgroup, the closed subgroup theorem tells us that $G \to G/H$ is a smooth principal H-bundle. By **Lemma 6.2.9(b)** applied to the quotient map $H \to H/K$, it follows that $G/K \to G/H$ is a smooth fiber bundle with fiber H/K and structure group H. The non-smooth case is analogous. ∎

6.2.4 Proof of The Closed Subgroup Theorem

Proof (Closed Subgroup Theorem). **(b)** Suppose **(a)** is true. Let $\mathfrak{h}^\perp$ be any complement to $\mathfrak{h}$ in $\mathfrak{g}$. We claim there is a nbhd V' of 0 in $\mathfrak{h}^\perp$ such that if $0 \neq X' \in V'$, then $\exp X' \notin H$.

To see, this first consider the following. Let $X_i \in \mathfrak{g}$ with $X_i \to X$ and let $t_i \to 0$ with $t_i \neq 0$ for all i. If $\exp t_i X_i \in H$ for all i, then $X \in \mathfrak{h}$. WLOG, we may assume each $t_i > 0$ since $\exp(-X) = (\exp(X))^{-1}$; put $k_i(t) = \lfloor t/t_i \rfloor$ for $t > 0$ so that $t/t_i - 1 < k_i(t) \leq t/t_i$ and so $t_i k_i(t) \to t$. Then $\exp(k_i(t) t_i X_i) = (\exp(t_i X_i))^{k_i(t)} \in H$ by the definition of H and, furthermore, $k_i(t) t_i X_i \to tX$ and thus $\exp tX \in H$ since H is closed and exp continuous. Since analogously we have $\exp tX \in H$ for $t < 0$, we have that $X \in \mathfrak{h}$ by the definition of $\mathfrak{h}$.

Giving $\mathfrak{h}^\perp$ any inner product, let $K \subset \mathfrak{h}^\perp$ be the compact set of all $X' \in \mathfrak{h}^\perp$ with $1 \leq |X'| \leq 2$. Suppose for a contradiction no such V' exists. Then for every nbhd V' of 0 in $\mathfrak{h}^\perp$, if $X' \in V'$, then $\exp X' \in H$. Thus, there would be a sequence $0 \neq X'_i \in \mathfrak{h}'$ with $X'_i \to 0$ and $\exp X'_i \in H$. Choosing $n_i \in \mathbf{Z}_{>0}$ with $n_i X'_i \in K$ and passing to a subsequence using compactness of K, we may suppose WLOG that $X'_i \to X \in K$. Since $1/n_i \to 0$, setting $X_i = n_i X'_i$ we have that for all $i \in \mathbf{N}$, $\exp(1/n_i X_i) = \exp(X'_i) \in H$ so that $X \in \mathfrak{h}$ by the above which is impossible unless $X = 0$ which is likewise impossible and thus our contradiction.

We now claim that the map $\phi\colon \mathfrak{h} \oplus \mathfrak{h}^\perp \to G$ defined by $\phi(X, X') = \exp(X)\exp(X')$ is a diffeomorphism on a nbhd of 0. To see this choose a basis of $\mathfrak{g}$, say $X_1, \ldots, X_n$ such that $X_1, \ldots, X_k$ is a basis for $\mathfrak{h}$ Then ϕ is given by $\phi \sum a_i X_i = \exp(\sum_{i=1}^k a_i X_i)\exp(\sum_{i=k+1}^n a_i X_i)$. It suffices to show that $\psi\colon \mathbf{R}^n \to G$ defined by

$$\psi(a_1, \ldots, a_n) = \exp(\sum_{i=1}^k a_i X_i)\exp(\sum_{i=k+1}^n a_i X_i)$$

is a diffeomorphism in a nbhd of $0 \in \mathbf{R}^n$ and this follows since its differential at 0 clearly satisfies

$$\psi_{*0}(\left.\frac{\partial}{\partial x^i}\right|_0) = X_i.$$

Choose now a nbhd $U = W \times W'$ (perhaps by shrinking) of $\mathfrak{g}$ on which exp is a diffeomorphism with W a nbhd of $0 \in \mathfrak{h}$ and W' a nbhd of $0 \in \mathfrak{h}^\perp$ such that W' is contained in the nbhd V' above and such that $W \times W'$ is a nbhd of 0 upon which the map ϕ above is a diffeomorphism. Then

$$\exp(\mathfrak{h} \cap U) \subset H \cap \exp(U)$$

and also, we claim, the reverse inclusion. Indeed, if $h \in H \cap \exp(U)$, then $h = \exp X \exp X'$ by the above and since $a, \exp X \in H$, it must be that $\exp X' \in H$ since $\exp X' = (\exp X)^{-1} a = \exp(-X) a$ and this is only possible if $X' = 0$ and thus $a \in \exp(\mathfrak{h} \cap U)$.

This shows that $(\exp(U), \log)$ is a submanifold chart for H about e. By translation, this gives submanifold charts covering H showing that H is a submanifold.

(c) Suppose **(b)** is true. Write $\mathfrak{h} \oplus \mathfrak{h}^\perp = \mathfrak{g}$ where by $\mathfrak{h}^\perp$ we mean any complement of $\mathfrak{h} \subset \mathfrak{g}$. Define $F\colon \mathfrak{h}^\perp \times H \to G$ by

$$F(v, h) = \exp(v) \cdot h.$$

To compute $F_{*,(0,e)}(w, w')$, we note that this is

$$F_{*,0}(-, e)(w) + F_{*,e}(0, -)(w').$$

Now, $F_{*,0}(-, e)(w) = \exp(-)_{*,0}(w) = w$ under the usual identification of $T_0\mathfrak{h}^\perp \cong \mathfrak{h}^\perp$ and similarly since $F_{*,e}(0, -) = e \cdot - = \mathrm{id}_H$, $F_{*,e}(0, -)(w') = w'$. Thus,

$$F_{*,(e,0)}(w, w') = w + w'$$

This gives a map $T_0\mathfrak{h}^\perp \times \mathfrak{h} \cong \mathfrak{h}^\perp \times \mathfrak{h} \to \mathfrak{g}$ and so is an isomorphism. Hence, this is a local diffeomorphism about $(0, e)$. Hence, there exist open sets $E_1 \subset \mathfrak{h}^\perp$ and $U \subset H$ each containing $e \in G$ such that $f| E_1 \times U$ is a diffeomorphism onto its image. In particular, $E = \exp(E_1)$ is open. Then for E and U as chosen, the preceding exercise shows that $G \to G/H$ is a smooth principal H-bundle.

(a) Given $X, Y \in \mathfrak{h}$, $\exp(tX)\exp(tY) = \exp(t(X+Y) + tZ(t))$ where $Z(t) \to 0$ as $t \to 0$. Set $X_i = X + Y + Z(t_i)$ and let $t_i \to 0$ with $t_i > 0$. Then by part of what we have shown in the proof of **(b)** (which is independent of **(a)**), we must have $X + Y \in \mathfrak{h}$. To see that this $Z(t)$ exists, first note that for small t, by abuse of notation, $\exp(tX)\exp(tY) = \exp Z(t)$ for some smooth $Z(t)$. This is because the exponential map is a local diffeomorphism at $t = 0$ and thus for small t, $\exp(tX)\exp(tY) = \exp(Z(t))$ for some unique $Z(t)$. Taylor expanding $Z(t)$, we have that $Z(t) = tZ_1 + t^2 Z_2 + O(t^3)$ where

$Z_1, Z_2 \in \mathfrak{g}$. If f is a coordinate function in some sufficiently small chart, then $f(\exp Z(t))$ Taylor expands as

$$f(\exp(tZ_1 + t^2 Z_2)) + O(t^3) = t(\widetilde{Z}_1 f)(e) + t^2(\widetilde{Z}_2 f)(e) + (t^2/2)\widetilde{Z}_1(\widetilde{Z}_1 f)(e) + O(t^3).$$

Taylor expanding $f(\exp(tX)\exp(tY))$, we find that it equals

$$t((\widetilde{X} + \widetilde{Y})f)(e) + t^2(\widetilde{X}\widetilde{X}f/2 + \widetilde{X}\widetilde{Y}f + \widetilde{Y}\widetilde{Y}f/2)(e) + O(t^3).$$

Comparing these, we must have by consideration of the invariance properties of left-invariant vector fields and matching the derivative powers at e, that $\widetilde{X} + \widetilde{Y} = \widetilde{Z}_1$ and $\widetilde{Z}_2 = \frac{1}{2}[X, Y]$. One now puts $Z(t)$ in the expression $\exp(tX)\exp(tY) = \exp(t(X + Y) + tZ(t))$ to be the terms $t^2/2[X, Y] + O(t^3)$. ■

6.3 Characteristic Classes: Definition

Definition. Fix a cohomology theory h. A ***characteristic class*** for numerable principal G-bundles is a natural transformation $c\colon k_G \to h^*$ of functors $\mathrm{Ho}(\mathsf{Top}) \to \mathsf{Set}$ where $k_G(X) = \mathsf{Prin}_{G,\mathrm{num}}(X)/\mathrm{iso}$ where $* \in \mathbf{Z}$ is fixed.

Theorem 6.3.1. *All characteristic classes of principal G-bundles are pullbacks of cohomology classes under classifying maps $X \to BG$. In particular,* $\mathrm{Nat}(k_G, h^*) \cong h^*(BG)$.

Proof. We have seen by the classification theorem that $k_G \cong [-, BG] = \hom_{\mathrm{Ho}(\mathsf{Top})}(-, BG)$ and hence by the Yoneda lemma, there is a natural isomorphism $\mathrm{Nat}([-, BG], h^*) \cong h^*BG$. Given a characteristic class $c \in h^*BG$ and $f\colon X \to BG$, naturality of the correspondence gives $f^*c \in h^*X$. ■

Definition. Fix a cohomology theory h. A ***characteristic class*** for principal $\Bbbk$-vector bundles of rank n are natural transformations $c\colon k_n^{\Bbbk} \to h^*$ of functors $\mathrm{Ho}(\mathsf{Top}) \to \mathsf{Set}$ where $k_n^{\Bbbk}(X) = \mathsf{Vect}^n_{\Bbbk,\mathrm{num}}(X)/\mathrm{iso}$ are numerable $\Bbbk$-vector bundles of rank n up to isomorphism and where $* \in \mathbf{Z}$ is fixed.

Exercise 42. *Let $\Bbbk$ be either* $\mathbf{R}$ *or* $\mathbf{C}$.

(a) *Let $G \curvearrowright F$ effectively. Show that the associated principal G-bundle functor and the associated bundle functors constitute an equivalence of categories* $\mathsf{P} : \mathsf{Bun}^F_{G,\mathrm{num}} \simeq \mathsf{Prin}_{G,\mathrm{num}} : - \times_G F$ *between the subcategories of numerable bundles.*
(b) *Show that all characteristic classes of numerable $\Bbbk$-vector bundles of rank n are pullbacks of cohomology classes under classifying maps $X \to BG$. In particular,* $\mathrm{Nat}(k_n^{\Bbbk}, h^*) \cong h^*(BG)$. [Hint: The associated $\Bbbk^n$-bundle of the universal bundle is universal in the same sense for vector bundles by **(a)**.]

Remark. This definition of characteristic classes suggests an intensive algebraic study of classifying spaces in order to determine what algebraic properties they should have. There are various ways to go about this. Our path of inquiry will be through spectral sequences and algebraic topology and we will begin with some explicit constructions of classifying spaces for classical groups. A more geometric approach to this inquiry is taken in Milnor and Stasheff's book.

6.4 Universal Bundles For Some Classical Groups and the Cohomology of Their Classifying Spaces

6.4.1 Explicit Universal Bundles

Almost everything in this section goes through with G any of the classical groups defined below. We will mostly ignore the symplectic case and leave the details to the interested reader. (E.g., the quaternionic Stiefel manifold $V_{k,n}(\mathbb{H})$ may be defined as $U(n, \mathbb{H})/U(n-k, \mathbb{H})$.) Let us explain this notation.

Notation. In order to avoid confusion with notation used later, we reserve $\mathbf{H}^n = [0, \infty)^{\times n}$ for the upper half-plane and use blackboard $\mathbb{H}$ for the quaternion field (i.e., the quaternion algebra over $\mathbf{R}$).

Definitions (Classical Groups). Consider the following subgroups of the Lie group $\mathrm{GL}_n(\Bbbk)$ for $\Bbbk = \mathbf{R}$ or $\mathbf{C}$.

(a) $O(n) \le \mathrm{GL}_n(\mathbf{R})$ consists of the matrices A with $AA^t = I$ (i.e., those matrices preserving the standard inner product on $\mathbf{R}^n$) the ***orthogonal group***.

(b) $SO(n) \le O(n) \le \mathrm{GL}_n(\mathbf{R})$ is the connected component of $O(n)$ consisting of all $A \in O(n)$ with $\det A = 1$ the ***special orthogonal group***.
(c) $U(n) \le \mathrm{GL}_n(\mathbf{C})$ consists of all matrices with $AA^* = I$ (i.e., those matrices preserving the standard inner product on $\mathbf{C}^n$) the ***unitary group***
(d) $SU(n) \le U(n) \le \mathrm{GL}_n(\mathbf{C})$ the subgroup consisting of all matrices $A \in U(n)$ with $|AA^*| = 1$ the ***special unitary group***.
(e) Let $\mathrm{Sp}_{2n}(\mathbf{k}) \le \mathrm{GL}_{2n}(\mathbf{k})$ be the subgroup of matrices preserving the canonical symplectic form on $\mathbf{k}^{2n}$ which is given in the standard basis as the block matrix

$$-J_{2n} = \begin{pmatrix} 0 & \mathrm{id}_{n\times n} \\ -\mathrm{id}_{n\times n} & 0 \end{pmatrix}.$$

This is the ***non-compact symplectic group***. It turns out this consists of block matrixes $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$ such that $B = B^t$ and $C = C^t$ where we mean transpose and *not* conjugate transpose.
(f) Let $\mathrm{Sp}(n) \overset{\mathrm{def}}{=} \mathrm{Sp}_{2n}(\mathbf{C}) \cap U(2n) = \mathrm{Sp}_{2n}(\mathbf{C}) \cap SU(2n)$ be the ***compact symplectic group***. This can also be defined as follows. Let $\mathbb{H}$ be the real quaternions and $\mathrm{GL}_n(\mathbb{H})$ the group of invertible $n \times n$ quaternionic matrices. Then $\mathrm{Sp}(n) = U(n, \mathbb{H})$ where $U(n,\mathbb{H}) \le \mathrm{GL}_n(\mathbb{H})$ is the subgroup of matrices preserving the standard hermitian form on $\mathbb{H}^n$ defined by $\langle x \mid y \rangle = \sum \overline{x}_i y_i$ where $\overline{a + bi + cj + dk} = a - bi - cj - dk$.

If V is a $\mathbb{k}$-vector space, then we can define $G(V)$ with G one of GL, O, SO, U and SU in the manner we expect. For the symplectic groups, it is convenient to change notation. If (V, ω) is a symplectic $\mathbb{k}$-vector space, then we denote the ***non-compact symplectic group*** on V by

$$\mathrm{Sp}(V,\omega) \overset{\mathrm{def}}{=} \{L \in \mathrm{GL}(V) : L^*\omega = \omega\} \le \mathrm{GL}(V),$$

that is, the subgroup of $\mathrm{GL}(V)$ preserving the symplectic form. If V is a quaternionic inner product space, then

$$\mathrm{USp}(V) \overset{\mathrm{def}}{=} U(V)$$

the subgroup of $\mathrm{GL}(V)$ consisting of all quaternionic marices preserving the inner product on V.

Remark. Note that since symplectic vector spaces are necessarily even-dimensional this does indeed accord with the previous definitions.

Reminder. It is useful to recall that the columns of a matrix in $O(n)$ have standard norm 1.

Theorem 6.4.1 (E. Cartan, Malcev, Iwasawa). *Every connected Lie group G is homotopy equivalent to any of its maximal compact subgroups (these are all necessarily connected and exist).*

Proof. Omitted.

Lemma 6.4.2. *The maximal compact subgroup of a Lie group is itself a Lie group.*

Proof. By the closed subgroup theorem, every closed subgroup of a Lie group is itself a Lie group. A compact subset of a Hausdorff space is always closed. ■

Corollary 6.4.3. *There is a homotopy equivalence $\mathrm{Sp}_{2n}(\mathbf{R}) \simeq U(n)$; in particular $U(n)$ is a (strong) deformation retract of $\mathrm{Sp}_{2n}(\mathbf{R})$ and, in particular, $U(n)$ is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbf{R})$. Hence, $B\,\mathrm{Sp}_{2n}(\mathbf{R}) \simeq BU(n)$.*

Remark. Part of this is **Proposition 2.2.4** in the third edition of *Introduction to Symplectic Topology* by Dusa McDuff and Dietmar Salamon. There is an error in the second and first editions for this argument. They explicitly construct the deformation retract.

It is worth pointing out that the stronger version of the theorem we gave above allows us to conclude that any Lie group deformation retracts onto its maximal compact subgroup.

Proof. The remark allows us to only consider the homotopy equivalence part. This is still subtle. We would like to deloop the inclusion homomorphism $U(n) \to \mathrm{Sp}_{2n}(\mathbf{R})$, which is already a homotopy equivalence, to a homotopy equivalence. Since the identity element here is manifestly non-degenerate and all groups in question certainly "convenient" (say compactly generated and weakly Hausdorff) May's bar construction **Theorem 6.1.9** affords this for us since it preserves weak equivalences (and, assuming we use cellular models, homotopy equivalences) between such groups. ■

Remark. It follows that even dimensional manifolds have a series of obstructions to having a symplectic structure arising from their symplectic classes, which are analogous to Chern classes but for real vector bundles. In particular, this means that $H^*(B\,\mathrm{Sp}_{2n}(\mathbf{R}); \mathbf{Z}) \cong \mathbf{Z}[c_1, \ldots, c_n]$ with $|c_i| = 2i$ from what we shall see later.

We now construct explicit models for $BO(n)$ and $BU(n)$ and their special counterparts in a series of exercises along with some additional assertions. Everything done here carries over for G any of the classical groups above.

Remark. For the next exercises, we understand $O(n)\times O(m)\le O(n+m)$ by identifying $(A,B)\in O(n)\times O(m)$ as the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

and we will understand $O(n-k)\le O(n)$ by identifying $A\in O(n-k)$ as the matrix

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Exercise 43. *Fix $\Bbbk=\mathbf{R}$ or $\mathbf{C}$. Let $k\le n$ and let $\kappa=\dim_{\mathbf{R}}\Bbbk$.*

(a) *Let $V_{k,n}(\Bbbk)$ be the* ***Stiefel manifold*** *of orthonormal k-frames in $\Bbbk^n$ with respect to the standard inner product on $\Bbbk^n$—that is, $V_{k,n}(\Bbbk)\subset\Bbbk^{nk}$ is a k-tuple of vectors in $\Bbbk^n$ that are all mutually orthogonal. Show that there are homeomorphisms*

$$V_{k,n}(\mathbf{R})\cong O(n)/O(n-k)\qquad V_{k,n}(\mathbf{C})\cong U(n)/U(n-k).$$

Conclude that $V_{k,n}$ is compact and can be given a smooth structure via these homeomorphisms. [Hint: A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.]

(b) *Let $\mathrm{Gr}_{k,n}(\Bbbk)$ be the* ***Grassmannian manifold*** *whose points are the set of all k-dimensional planes through the origin of $\Bbbk^n$. Topologize $\mathrm{Gr}_{k,n}(\Bbbk)$ as a quotient of $V_{k,n}(\Bbbk)\to\mathrm{Gr}_{k,n}(\Bbbk)$ via the map $V_{k,n}(\Bbbk)\to\mathrm{Gr}_{k,n}(\Bbbk)$ sending a k-frame to the plane it spans. Show that $\mathrm{Gr}_{k,n}(\Bbbk)$ so topologized is a compact smooth manifold of dimension $\kappa k(n-k)$ by exhibiting a homeomorphism*

$$\mathrm{Gr}_{k,n}(\mathbf{R})\cong O(n)/(O(n-k)\times O(k))\qquad \mathrm{Gr}_{k,n}(\mathbf{C})\cong U(n)/(U(n-k)\times U(k)).$$

(c) *Identifying $V_{k,n}(\Bbbk)$ as orthonormal k-frames in $\Bbbk^n$, show that $V_{k,n}(\Bbbk)\to\mathrm{Gr}_{k,n}(\Bbbk)$ can be identified with row-reduction of the matrix associated to the k-frames.*

(d) *Show that the projection $V_{k,n}(\Bbbk)\to\mathrm{Gr}_{k,n}(\Bbbk)$ is a principal $O(k)$-bundle when $\Bbbk=\mathbf{R}$ and is a principal $U(n)$-bundle when $\Bbbk=\mathbf{C}$.* [Hint: Use **Lemma 6.2.10** and carefully consider where the subgroups lie in the larger group.]

[Other Hinst: Consider the map $U(n)/U(n-k)\to V_{k,n}(\mathbf{C})$ sending a coset $U\cdot U(n-k)\mapsto(U\mathbf{e}_{n-k+1},\dots,U\mathbf{e}_n)$ where $\mathbf{e}_i$ is the i-th standard basis vector of $\mathbf{C}^n$. Do the analogous thing for the real Stiefel manifold.]

Exercise 44. *Repeat the above exercise after replacing the word “orthogonal” by “linear independent.” This exercise is easier because the Stiefel manifold will be an open submanifold of $\mathbf{R}^{nk}$.*

Exercise 45. *Let $\Bbbk=\mathbf{R}$ or $\mathbf{C}$.*

(a) *Show that if $0<k<n$, then there is a transitive and smooth action $SO(n)\curvearrowright V_{k,n}(\mathbf{R})$ with stabilizer for any point $x\in V_{k,n}(\mathbf{R})$ diffeomorphic to the subgroup $SO(n-k)$. Conclude that there is a diffeomorphism*

$$V_{k,n}(\mathbf{R})\cong SO(n)/SO(n-k)$$

and similarly, show that for $k<n$ there is a diffeomorphism

$$V_{k,n}(\mathbf{C})\cong SU(n)/SU(n-k).$$

(b) *Let $\mathrm{Gr}^+_{k,n}(\Bbbk)$ be the* ***Grassmannian*** *of oriented k-planes in $\Bbbk^n$ where*

$$\mathrm{Gr}^+_{k,n}(\mathbf{R})\overset{def}{=}O(n)/(O(n-k)\times SO(k))\qquad \mathrm{Gr}^+_{k,n}(\mathbf{C})\overset{def}{=}U(n)/(U(n-k)\times SU(k)).$$

Show that $\mathrm{Gr}^+_{k,n}(\Bbbk)$ is a smooth quotient of $V_{k,n}(\Bbbk)$ and that for $k<n$, the quotient map $V_{k,n}(\Bbbk)\to\mathrm{Gr}^+_{k,n}(\Bbbk)$ is a principal $SO(k)$ bundle when $\Bbbk=\mathbf{R}$ and is a principal $SU(k)$-bundle when $\Bbbk=\mathbf{C}$.

[Hint: Define an action $V_{n,n+k}(\Bbbk)\curvearrowleft G(n)$ by embedding $G(n)\hookrightarrow G(n+k)$ as the subgroup of matrices of the form $\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$ where $A\in G(n)$. To show the relevant things are principal bundles, appeal to **Theorem 6.2.5**.]

Remark. Note that for all $0<k<n$, $\mathrm{Gr}^+_{k,n}(\Bbbk)$ is the same as $SU(n)/(SU(n-k)\times SU(k))$ in the complex case and the same as $SO(n)/(SO(n-k)\times SO(k))$ as a consequence of **(a)**.

Exercise 46. *Show that there are fiber sequences* $O(n-1) \to O(n) \to O(n)/O(n-1) \cong S^{n-1}$ *and* $U(n-1) \to U(n) \to U(n)/U(n-1) \cong S^{2n-1}$. *Use this to determine the connectivity of the inclusions* $O(n-1) \to O(n)$ *and* $U(n-1) \to U(n)$ *via the long exact sequence in homotopy groups and conclude that* $V_{k,n}(\Bbbk)$ *increases in connectivity as* $n \to \infty$. [Hint: $U(1) \cong S^1$ and $O(1) \cong S^0$.]

For the following exercise, you will need **(a)** and **(b)** of the following point-set lemma.

Lemma 6.4.4.

(a) *If* $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ *is a sequence of closed embeddings between paracompact Hausdorff spaces, then* $X = \operatorname{colim} X_n$ *is paracompact Hausdorff.*
(b) *If* $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ *and* $B_1 \hookrightarrow B_2 \hookrightarrow \cdots$ *are two sequences of embeddings where all* A_i *and* B_i *are locally compact, then* $(\operatorname{colim} A) \times (\operatorname{colim} B) \cong \operatorname{colim}(A_i \times B_i)$.
(c) *If* $A_1 \hookrightarrow A_2 \hookrightarrow \cdots$ *and* $B_1 \hookrightarrow B_2 \hookrightarrow \cdots$ *are two sequences of embeddings where all* A_i *and* B_i *are locally compact and Hausdorff, then* $(\operatorname{colim} A) \times (\operatorname{colim} B) \cong \operatorname{colim}(A_i \times B_i)$. *In particular,* $\operatorname{colim}(A_i \times B_i)$ *is Hausdorff.*
(d) *Fix sequences* $X_1^i \to X_2^i \to X_3^i \to \cdots$ *for* $i = 1, 2, 3$ *with morphisms of these sequences*

$$\{X_j^1\} \overset{\{f_j\}}{\to} \{X_j^2\} \overset{\{g_j\}}{\leftarrow} \{X_j^3\}$$

Suppose the sequences satisfy the following conditions:

(i) *the maps* $X_j^1 \to X_{j+1}^1$ *are closed embeddings for all* j*;*
(ii) *for all* j*, the space* X_j^2 *is locally compact Hausdorff;*
(iii) *for all* j*, the space* X_j^3 *is Hausdorff.*

Then the natural map $\operatorname{colim}_n(X_n^1 \times_{X_n^2} X_n^3) \to \operatorname{colim}_n(X_n^1) \times_{\operatorname{colim}_n(X_n^2)} \operatorname{colim}_n(X_n^3)$ *is a homeomorphism. That is, the colimit commutes with the pullback.*
(e) *If for each* $i = 1, \ldots, n$, $X_1^i \to X_2^i \to \cdots$ *is a sequence of closed embeddings between Hausdorff spaces, then* $\operatorname{colim}_n(\prod_i X_n^i) \cong \prod_i \operatorname{colim}_n(X_n^i)$.

Proof. **(a)** A proof of this may be found on the nLab, reproduced from a paper of Ernest Michael.

(b) This is **Lemma 5.5** of Milnor-Stasheff.

(c) This is follows from **(b)** along with the fact that Hausdorff spaces are closed under products. ■

(d) This is a type of general categorical fact for the category of topological spaces. See, for instance, Yonatan Harpaz's answer here.

(e) The product in the category of spaces is equivalently $X \times Y \cong X \times_* Y$ the pullback over the unique maps to the point. Thus, this is a special case of **(d)** (along with a suitable but evident) induction, where each pullback is taken over a point. A more straightforward way to see this is to observe that $i \times j\colon X_1 \times Y_1 \to X_2 \times Y_2$ is a closed embedding if i and j are closed embeddings. ■

Exercise 47. *For convenience, denote by* $G = G(n)$ *any one of* $O(n)$, $U(n)$, $SO(n)$ *and* $SU(n)$.

(a) *Using the standard inclusions* $\Bbbk^n \cong \{\mathbf{0}\} \times \Bbbk^n \hookrightarrow \Bbbk^{n+1}$, *induce the horizontal maps in the following diagram and show that they are smooth and make the following diagram commute. We have suggestively indicated what the maps do to the right.*

$$\begin{array}{ccc} V_{n,n+k}(\Bbbk) & \dashrightarrow & V_{n,n+k+1}(\Bbbk) \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{n,n+k}(\Bbbk) & \dashrightarrow & \mathrm{Gr}_{n,n+k+1}(\Bbbk) \end{array} \qquad \begin{array}{c} O(n+k)/O(k) \to O(1+n+k)/O(1+k) \\ \\ O(n+k)/(O(k)\times O(n)) \to O(1+n+k)/(O(1+k)\times O(n)). \end{array}$$

(To make sense of this, recall that the second coordinate refers to the dimension $\mathbf{R}^{n+k}$ *and the first subscript refers to* n*-frames (resp.* n*-planes) in* $\mathbf{R}^{n+k}$*.) Describe these maps in terms of the homeomorphisms of* ***Exercise 43*** *and* ***Exercise 41****.*
(b) *Show that the horizontal maps in* ***(a)*** *are closed embeddings.*
(c) *Define*

$$\begin{aligned} EO(n) &\overset{def}{=} \operatorname{colim}_k V_{n,n+k}(\mathbf{R}) &\quad \text{and} \quad BO(n) &\overset{def}{=} \operatorname{colim}_k \mathrm{Gr}_{n,n+k}(\mathbf{R}) \\ EU(n) &\overset{def}{=} \operatorname{colim}_k V_{n,n+k}(\mathbf{C}) &\quad \text{and} \quad BU(n) &\overset{def}{=} \operatorname{colim}_k \mathrm{Gr}_{n,n+k}(\mathbf{C}) \\ ESO(n) &\overset{def}{=} \operatorname{colim}_k V_{n,n+k}(\mathbf{R}) &\quad \text{and} \quad BSO(n) &\overset{def}{=} \operatorname{colim}_k \mathrm{Gr}^+_{n,n+k}(\mathbf{R}) \\ ESU(n) &\overset{def}{=} \operatorname{colim}_k V_{n,n+k}(\mathbf{C}) &\quad \text{and} \quad BSU(n) &\overset{def}{=} \operatorname{colim}_k \mathrm{Gr}^+_{n,n+k}(\mathbf{C}) \end{aligned}$$

and let

$$\gamma^n \colon EG \to BG$$

be the map induced by the universal property of the colimit by way of part ***(a)***. *Call the space (+ suppressed if necessary)*

$$\mathrm{Gr}_n(\Bbbk^\infty) \overset{def}{=} BG(n)$$

the ***infinite Grassmannian***. *Show that the naive definition*[1] *for $ESO(n)$ and $ESU(n)$ agrees with the definition given above* [Hint: Use **Exercise 43(a)** and appeal to the fact that $\mathrm{colim}(X_0 \to X_1 \to X_2 \to \cdots) \cong \mathrm{colim}(X_n \to X_{n+1} \to X_{n+2} \to \cdots)$.]

Remark. For the some of the colimit manipulations below, it will be useful to know that the open subsets of a sequence of embeddings $\mathrm{colim}(X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots)$ are the subsets $U \subset \bigcup X_i$ such that $U \cap X_i$ is open in X_i for all i.

Exercise 48. *For convenience, denote by $G = G(n)$ any one of $O(n)$, $U(n)$, $SO(n)$ and $SU(n)$. Let $\Bbbk^\infty \overset{def}{=} \bigoplus_{n\in\mathbf{N}} \Bbbk$ as a vector space*

(a) *Show that $\Bbbk^\infty$ is a topological $\Bbbk$-vector space when topologized as the colimit $\Bbbk \subset \Bbbk^2 \subset \Bbbk^3 \subset \cdots$ where the identification $\Bbbk^n \subset \Bbbk^{m+n}$ is either*

$$\Bbbk^n \cong \mathbf{0} \times \Bbbk^n \subset \Bbbk^{m+n} \qquad or \qquad \Bbbk^n \cong \Bbbk^n \times \mathbf{0} \subset \Bbbk^{m+n}$$

and show that the result is independent of which identification scheme we choose. [Hint: Construct a ladder of isomorphisms by suitable permutation between the two sequences of inclusions to show that their colimits are isomorphic—construct these by induction and observe that the permutation will get more convoluted the farther you go out but with only "finite complexity" at each step. Use **Lemma 6.4.4** to show continuity of the addition $\Bbbk^\infty \times \Bbbk^\infty \to \Bbbk$ and the scalar multiplication $\Bbbk \times \Bbbk^\infty \to \Bbbk^\infty$.]

(b) *Show that $\Bbbk^\infty$ is paracompact and locally compact Hausdorff and show that it inherits a normed structure from each $\Bbbk^n$.* [Hint: Use **Lemma 6.4.4(a)** to show $\Bbbk^\infty$ is paracompact locally compact Hausdorff. To show that it is a normed linear space, note that any two elements $v, w \in \Bbbk^\infty$ lie in the image of some $\Bbbk^n$ and $\Bbbk^m$ in $\Bbbk^\infty$, respectively. Setting, $N = \max\{m, n\}$, define $\langle v \mid w\rangle$ by embedding $v, w \in \Bbbk^N \subset \Bbbk^\infty$ and using the standard (hermitian) inner product for $\Bbbk^N$. Note that this standard (hermitian) inner product induces the topology on $\Bbbk^N$ under the induced norm metric and use this to show the same is true for $\Bbbk^\infty$.]

(c) *Show that $EG(n)$ is homeomorphic to the subspace of $(\Bbbk^\infty)^n$ of orthonormal n-frames where orthogonality is taken with respect to the standard (hermitian) inner product constructed in* ***(b)***—*more blithely, $EG(n) \cong V_{n,\infty}$. Show that there is bijection between $\mathrm{Gr}_k(\Bbbk^\infty)$ (resp. $\mathrm{Gr}_k^+(\Bbbk^\infty)$) and the set of k-planes (resp. oriented k-planes) in $\Bbbk^\infty$.* [Hint: For the second part, use the definition of $V_{n,n+k}(\Bbbk)$ as a subspace of $\Bbbk^{(n+k)n} \cong (\Bbbk^{n+k})^n$. Take the colimit in k and appeal to **Lemma 6.4.4(e)**.]

(d) *For $G(n) = O(n)$ or $U(n)$, that $BG(n)$ may be identified with the set of n-dimensional subspaces of $\Bbbk^\infty$. For $G(n) = SO(n)$ or $SU(n)$, show that $BG(n)$ may be identified with the set of n-dimensional subspaces of $\Bbbk^\infty$ with an orientation (i.e., $\pm$).* [Hint: This about *sets*, not about spaces.]

(e) *Show that $EG(n)$ is contractible and has a free fiberwise right $G(n)$-action for which $\gamma^n \colon EG(n) \to BG(n)$ is a principal $G(n)$-bundle. Show that $BG(n)$ is paracompact Hausdorff and conclude that γ^n is a universal (hence, numerable) principal $G(n)$-bundle.*

(f) *Show that there is a double cover $\mathrm{Gr}_k^+(\Bbbk^\infty) \to \mathrm{Gr}_k(\Bbbk^\infty)$ by forgetting orientation. Show that this is the universal cover.* [Hint: Show that $\pi_0 SO(m) = *$ for all $m \geq 1$. Show that $SO(n-k) \times SO(k) \to SO(n) \to SO(n)/(SO(n-k) \times SO(k))$ is a fiber sequence by **Theorem 6.2.5** and **Theorem E.1.3**. Show that for sufficiently large $n \geq m > N$, $\pi_1 SO(m) \to \pi_1 SO(n)$ is surjective and use the long exact sequence in homotopy groups for a fibration.]

[Other hints:

(i) For geometric reasoning, it is best to think of the Stiefel and Grassmannian manifolds as the manifolds of frames and planes, respectively.

(ii) Since $G(n)$ is locally compact Hausdorff, $- \times G(n)$ commutes with colimits in the category of spaces. Use this to define the action on $EG(n)$.

(iii) Use the preceding lemma to show $BG(n)$ is paracompact Hausdorff.

(iv) Using this action and the fact that colimits commute with colimits, show that $BG \cong EG/G$ and that the projection $EG \to BG$ is isomorphic to the quotient map $EG \to EG/G$.

(v) Define the orthogonal projection of v onto w by $\mathrm{proj}_w(v) \overset{\mathrm{def}}{=} \dfrac{\langle w \mid v\rangle}{\|w\|^2} w$ where $\langle - \mid - \rangle$ skew linear in the first entry. For each fixed (oriented) plane $V \in BG$, let $U_V \subset BG$ be the set of (oriented) planes whose image under the orthogonal

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[1] That is, setting $ESO(n) = \mathrm{colim}_k\, SO(n+k)/SO(n)$ and $ESU(n) = \mathrm{colim}_k\, SU(n+k)/SU(n)$.

projection $\Bbbk^\infty \to V$ is surjective (and orientation preserving). Show that U_V is open by showing that $U_V \cap \mathrm{Gr}_{n,k+n}(\Bbbk)$ is open (resp. $U_V \cap \mathrm{Gr}^+_{n,k+n}$ is open) for all $k \geq 0$. Use the fact that $V_{n,n+k} \to \mathrm{Gr}_{n,n+k}$ is a quotient map.

(vi) Show that $EG \to BG$ is equivariantly trivializable over the open subsets U_V for each $V \in BG$.

]

Remark. It turns out that the homotopy type of $BG(n)$ can be identified with an infinite dimensional manifold. The proof of the homotopy equivalence is subtle, however.

Proposition 6.4.5. *Let $i = i_{n,m} : G(n) \to G(n+m)$ be the standard inclusion $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. Let $\varphi : G(m) \times G(m) \to G(n+m)$ be the smooth map*

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

(a) *The maps $i_{n,m}$ induce maps $V_{k,n+k} \to V_{k+m,n+k+m}$ and $\mathrm{Gr}_{k,n+k} \to \mathrm{Gr}_{k+m,n+k+m}$ (+ and field suppressed) for all $k \geq 0$ making TFDsC*

$$\begin{array}{ccc} \mathrm{Gr}_{n,n+k} & \longrightarrow & \mathrm{Gr}_{n,n+k+1} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{n+m,n+k+m} & \longrightarrow & \mathrm{Gr}_{n+m,n+k+1+m} \end{array} \qquad \begin{array}{ccc} V_{n,n+k} & \longrightarrow & V_{n,n+k+1} \\ \downarrow & & \downarrow \\ V_{n+m,n+k+m} & \longrightarrow & V_{n+m,n+k+1+m} \end{array}$$

In particular, upon taking colimits and invoking universal properties, we obtain a commutative diagram

$$\begin{array}{ccc} EG(n) & \longrightarrow & EG(n+m) \\ \downarrow & & \downarrow \\ BG(n) & \longrightarrow & BG(n+m) \end{array}$$

(b) *The induced map $BG(n) \to BG(n+m)$ in the diagram above is, up to homotopy, $Bi_{n,m}$. In particular, the map $EG(n) \to EG(n+m)$ (call it j) respects the identification of $i_{n,m} : G(n) \to G(n+m)$ in the sense that $j(x \cdot g) = j(x) \cdot i(g)$.*

(c) *The map $Bi : BG(n) \to BG(n+m)$ classifies the bundle over $BG(n)$ given by the coinduced bundle*

$$\varphi_*(\gamma^n \times G(m)) \overset{def}{=} (\gamma^n \times G(m)) \times_{G(n) \times G(m)} G(n+m).$$

Here, $\gamma^n \times G(m)$ denotes the principal $G(n) \times G(m)$-bundle $EG(n) \times G(m) \to BG(n) \times \{*\} \cong BG(n)$.

Proof. **(a)** This is straightforward.

(b) By **Lemma 6.2.8**, it suffices to show that the map $j : EG(n) \to EG(n+m)$ induced in **(a)** satisfies $j(x \cdot g_n) = j(x) \cdot g_n$ where we identify $G(n) \subset G(n+m)$ via the inclusion $i_{n,m} : G(n) \to G(n+m)$ of **(a)**. For this, note that the map $V_{k,n} = O(n)/O(n-k) \to O(n+m)/O(n-k) = V_{k+m,n+m}$ already satisfies this under our identification scheme. From this, it follows that the map on the colimit must as well by how the $O(k)$ action is defined (see **(b)** of **Exercise 46** and the hint **(ii)**).

(c) For convenience, let us denote $G(n \times m) \overset{\text{def}}{=} G(n) \times G(m)$. By **Lemma 6.2.8**, we know that $Bi = Bi_{n,m}$ classifies the bundle $EG(n) \times_{G(n)} G(n+m)$. This allows us to reduce this to a categorical argument since it now suffices to show that there is a *fiberwise* (i.e., equivariant) isomorphism $(EG(n) \times G(m)) \times_{G(n \times m)} G(n+m) \cong (EG(n)) \times_{G(n)} G(n+m)$.

Claim 17. Suppose $H \times K \leq G$ are subgroups and suppose we have a free action $X \curvearrowleft H$. Then with the canonical action of $H \times K \curvearrowright G$, there is an isomorphism of right G-spaces $(X \times K) \times_{H \times K} G \cong X \times_H G$.

Fix G and let $\mathrm{Ind}_H^{H \times K} : \mathsf{Top}_H \to \mathsf{Top}_{H \times K}$ be the functor $X \mapsto \mathrm{Ind}_H^{H \times K}(X) = X \times K$. Now, the left-hand side is the composite $\mathrm{Ind}_{H \times K}^G \circ \mathrm{Ind}_H^{H \times K} \cong \mathrm{Ind}_H^G$ and the right-hand side is induction

$$\mathrm{Ind}_{H \times K}^G(X \times K) = (X \times K) \times_{H \times K} G \cong X \times_H G = \mathrm{Ind}_H^G(X),$$

as claimed. //

It follows that there is natural $G(n+m)$-equivariant isomorphism $\widetilde{f} : (EG(n) \times G(m)) \times_{G(n \times m)} G(n+m) \to (EG(n)) \times_{G(n)} G(n+m)$. To see that this is a morphism of principal $G(n+m)$-bundles over B, note that equivariance implies that this map descends to the quotient

$$f : (EG(n) \times G(m)) \times_{G(n \times m)} G(n+m)/G(n+m) \to (EG(n)) \times_{G(n)} G(n+m)/G(n+m)$$

and since $\widetilde{f}$ is an isomorphism, so too is f. We may now conclude by **(a)** and **(b)** of **Exercise 17**. ■

Proposition 6.4.6. *The map $Bi = Bi_{n,m} : BG(n) \to BG(n+m)$ classifies the vector bundle $(\gamma^n \times_{G(n)} \Bbbk^n) \oplus \underline{\Bbbk^m}$.*

Here, $\underline{\Bbbk^m}$ means the trivial rank m vector bundle over $BG(n)$.

Proof. We know that Bi classifies the principal $G(n+m)$-bundle $(\gamma^n) \times_{G(n)} G(n+m)$. Forming its associated vector bundle, we have isomorphisms

$$\begin{aligned}(\gamma^n \times_{G(n)} G(n+m)) \times_{G(n+m)} \Bbbk^{n+m} &\cong \gamma^n \times_{G(n)} (G(n+m) \times_{G(n+m)} \Bbbk^{n+m}) \\ &\cong \gamma^n \times_{G(n)} \Bbbk^{n+m} \cong \gamma^n \times_{G(n)} \Bbbk^n \times \Bbbk^m.\end{aligned}$$

From our identifications, the action of $G(n) \curvearrowright \Bbbk^n \times \Bbbk^m$ is seen to act on the $\Bbbk^n$ piece. Thus,

$$\gamma^n \times_{G(n)} \Bbbk^n \times \Bbbk^m \cong (\gamma^n \times_{G(n)} \Bbbk^n) \times \Bbbk^m$$

and this is easily seen to be

$$(\gamma^n \times_{G(n)} \Bbbk^n) \times \Bbbk^m \cong (\gamma^n \times_{G(n)} \Bbbk^n) \oplus \underline{\Bbbk^m}$$

as desired. ■

Exercise 49. *Fill in the missing detail that $(\gamma^n \times_{G(n)} \Bbbk^n) \times \Bbbk^m \cong (\gamma^n \times_{G(n)} \Bbbk^n) \oplus \underline{\Bbbk^m}$.*

Corollary 6.4.7. *If $X \to BG(n)$ classifies a vector bundle ξ of rank n, then the composite $X \to BG(n) \to BG(n+1)$ classifies the vector bundle $\xi \oplus \underline{\Bbbk}$.*

Proof. Pulling back the classified vector bundles and pasting pullbacks we have a diagram

$$\begin{array}{ccccc} ? & \longrightarrow & EG(n) \times_{G(n)} \Bbbk^n \oplus \underline{\Bbbk} & \longrightarrow & EG(n+1) \times_{G(n+1)} \Bbbk^{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & BG(n) & \longrightarrow & BG(n+1) \end{array}$$

Since ξ is obtained as the pullback

$$\begin{array}{ccc} E & \longrightarrow & EG(n) \times_{G(n)} \Bbbk^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & BG(n) \end{array}$$

one easily computes that ? in the diagram above must be $\xi \oplus \underline{\Bbbk}$. ■

Definition. Under the inclusion $i : G(n) \hookrightarrow G(n+1)$ defined above, let $G = G(\infty) = \operatorname{colim} G(n)$. When $G(n) = O(n)$, this is called the ***infinite orthogonal group*** and similarly for the other options for $G(n)$.

Exercise 50. *Show that $G = G(\infty) = \operatorname{colim} G(n)$ is a paracompact Hausdorff topological group.* [Hint: To define the multiplication, use **Lemma 6.4.4(b)**. Use **Lemma 6.4.4(a)** to show $G(\infty)$ is paracompact Hausdorff.]

Theorem 6.4.8. (a) *There is a cell structure on each $\mathrm{Gr}_{n,k+n}(\Bbbk)$ for which the inclusions of **Proposition 6.4.5***

$$\mathrm{Gr}_{k,n} \hookrightarrow \mathrm{Gr}_{k,n+m}, \qquad \mathrm{Gr}_{k,n} \hookrightarrow \mathrm{Gr}_{k+m,n+m}$$

are inclusions of subcomplexes for all $m \geq 1$.
*Similarly, there is a cell structure on each $\mathrm{Gr}^+_{n,k+n}(\Bbbk)$ for which the inclusions of **Proposition 6.4.5***

$$\mathrm{Gr}^+_{k,n} \hookrightarrow \mathrm{Gr}^+_{k,n+m}, \qquad \mathrm{Gr}^+_{k,n} \hookrightarrow \mathrm{Gr}^+_{k+m,n+m}$$

are inclusions of subcomplexes for all $m \geq 1$.

(b) *There is a cell structure on each $V_{k,n}(\Bbbk)$ for which the inclusions of **Proposition 6.4.5***

$$V_{k,n} \hookrightarrow V_{k,n+m}, \qquad V_{k,n} \hookrightarrow V_{k+m,n+m}$$

are inclusions of subcomplexes for all $m \geq 1$.

Proof. **(a)** Omitted. This can be found in Chapter 6 of Milnor and Stasheff's *Characteristic Classes* for the unoriented case. For quick overview of the main ideas, see **5.4C** of Fomenko and Fuchs' *Homotopical Topology.* The evident modification for quaternionic Grassmannians works as well.

(b) By **Exercise 43(a)**, $V_{k,n} \cong U(n,\Bbbk)/U(n-k,\Bbbk)$, where $U(n,\mathbf{R}) = SO(n)$ when $n > k$. Let us agree to denote

$$U(n/n-k,\Bbbk) = U(n,\Bbbk)/U(n-k,\Bbbk).$$

(b) Omitted. This can be deduced from the cellular description of $U(n,\mathbf{R})$ given in **Chapter 3.D** of Hatcher's *Algebraic Topology.* In the same chapter, Hatcher shows that the projection $U(n,\Bbbk) \to U(n/n-k,\Bbbk)$ (which by **Theorem 6.2.5** is a principal $U(n-k,\Bbbk)$-bundle) induces a cell structure on the target. For $\Bbbk = \mathbf{C}$ or $\mathbb{H}$ the quaternions, $V_{n,n}(\mathbf{k}) = U(n,\Bbbk)$ and the same cell structure Hatcher constructs in the case of $\Bbbk = \mathbb{R}$ introduces a compatible cell structure on $V_{n,n}$ after replacing $\mathbf{R}P^n$ by $\Bbbk P^n$ and replacing all other maps by their complex or quaternionic analogues. When $\Bbbk = \mathbf{R}$, $V_{n,n}(\mathbb{R}) = O(n) \cong SO(n) \coprod SO(n)$ and the CW structure constructed on $U(n,\mathbf{R})$ passes by disjoint union to a compatible CW structure on $O(n)$. //

Let us summarize in a remark the discussion mentioned in **5.4C** of Fomenko and Fuchs' *Homotopical Topology.*

Remark. The main idea is the Schubert cell decomposition of the (oriented) Grassmannian manifold of k-planes in $\Bbbk^n$. Fix integers k and n and consider $\mathrm{Gr}_{k,n}$ first. We associate to each finite and possibly empty list $(m_1,\ldots,m_s)$ of length $s \le n-k$ satisfying $k \ge m_1 \ge m_2 \ge \cdots \ge m_s$, called a ***Schubert symbol***, define the subspace $e(m_1,\ldots,m_s) \subset \mathrm{Gr}_{k,n}$, the $(m_1,\ldots,m_s)$ ***Schubert cell*** as follows.

Fix a Schubert symbol $(m_1,\ldots,m_s)$ for $\mathrm{Gr}_{k,n}$. Let $m_0 = k$ and for each $s < j \le n-k+1$, let $m_j = 0$. Then $e(m_1,\ldots,m_s)$ is defined to be thesubspace comprised of all k-dimensional subspaces Π of $\mathbf{R}^n$ satisfying that for each $0 \le j \le s$ and each $1 \le m \le n$,

$$\text{if } k - m_j + j \le m \le k - m_{j+1} + j, \text{ then } \dim(\Pi \cap \Bbbk^m \times \mathbf{0}_{\Bbbk}^{n-m}) = m - j.$$

The Schubert cells are pairwise disjoint and cover $\mathrm{Gr}_{k,n}$ and moreover satisfy that $e(m_1,\ldots,m_s) \cong \Bbbk^{\sum_{i=1}^{s} m_i}$. In the oriented case $\mathrm{Gr}_{k,n}^{+}$, the cells $e(m_1,\ldots,m_s)$ split into pairs $e_+(m_1,\ldots,m_s)$ and $e_-(m_1,\ldots,m_s)$ corresponding to the orientation class of the plane.

For example, in the case of the Schubert symbol $\emptyset$, $s = 0$, so $m_0 = k$, $m_1 = \cdots = m_{n-k+1} = 0$ and $0 \le j \le 0$ so that $j = 0$. Hence, the values of m fall in $k - m_0 \le m \le k - m_1$ so that $0 \le m \le k$. Thus, $e(\emptyset)$ consists of all k-dimensional subspace of $\Bbbk^n$ such that $\dim(\Pi \cap \Bbbk^m) = m$. From this, it is easy to see that $e(\emptyset) = \{\Bbbk^k \times \mathbf{0}_{\Bbbk}^{n-k}\}$ is a singleton.

The point is that the inclusions $e(m_1,\ldots,m_s) \hookrightarrow \mathrm{Gr}_{k,n}$ are embeddings that are the inclusions of the open cells of a CW decomposition of $\mathrm{Gr}_{k,n}$. The analogous thing is true in the oriented case. In each case, the embedding $\mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k,n+m}$ and $\mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k+m,n+m}$ sends Schubert cells with symbol $(m_1\ldots,m_s)$ onto the Schubert cell with the same symbol in the target.

Corollary 6.4.9. *Let $G = G(\infty) = \operatorname{colim} G(n)$ as above.*

- **(a)** *The induced inclusions $BG(n) \hookrightarrow BG(n+1)$ has a model for which $BG(n) \to BG(n+1)$ is the inclusion of a subcomplex and each $BG(n)$ is locally compact. Hence, $BG(\infty) = \operatorname{colim}_n BG(n)$ is naturally a CW-complex and is paracompact Hausdorff. In particular, $BG(\infty) \simeq \operatorname*{hocolim}_{n\to\infty} BG(n)$ is a homotopy colimit.*
- **(b)** *The induced inclusions $EG(n) \hookrightarrow EG(n+1)$ has a model for which $EG(n) \to EG(n+1)$ is the inclusion of a subcomplex and each $EG(n)$ is locally compact. Hence, $E'G(\infty) = \operatorname{colim}_n EG(n)$ is naturally a CW-complex and hence paracompact Hausdorff. In particular, $E'G(\infty) \simeq \operatorname*{hocolim}_{n\to\infty} EG(n)$ is a homotopy colimit.*
- **(c)** *The natural maps $EG(n) \to EG(n+m)$ and $BG(n) \to BG(n+m)$ are inclusions of subcomplexes for each $m \ge 1$.*
- **(d)** *All the inclusions above are closed embeddings.*
- **(e)** *Both $EG(n)$ and $BG(n)$ are further locally compact.*

Proof. Most of this follows more or less immediately from the preceding theorem. The colimit of a sequence of inclusions of subcomplexes is always a CW-complex. Hence, each $BG(n)$ and $EG(n)$ are CW-complexes and thus paracompact Hausdorff. The structure maps inducing the morphisms $EG(n) \to EG(n+m)$ and $BG(n) \to BG(n+m)$ are also inclusions of CW-complexes from which it follows that these maps are themselves inclusions of CW-complexes. Hence, $BG(\infty)$ and $E'G(\infty)$ are CW-complexes and so are paracompact Hausdorff. The homotopy colimit assertions follow by model categorical considerations. The last assertion follows from the general fact that a subcomplex of a CW-complex is a closed subspace.

Let us turn to last assertion **(e)**. Note that to say $EG(n)$ and $BG(n)$ are locally compact CW-complexes is equivalent to the statement that they are locally finite[2] CW-complexes. This suggests a proof different from the one we provide below

[2] A CW-complex X is ***locally finite*** if each $x \in X$ is contained in only finitely many cells of X.

based on careful analysis of Schubert cells for $BG(n)$ and similarly an analysis of the indicated cellular decomposition for $V_{n,n+k}$. We shall provide a proof independent of the niceness of these decompositions. Let us consider $EG(n)$ first.

Exercise 46(c) implies that it is the subspace of $\Bbbk^{\infty\times n}$ of orthonormal n-frames. By **(b)** of the same exercise, $\Bbbk^\infty$ is a normed linear space that is paracompact locally compact Hausdorff. In particular, this means that $\Bbbk^\infty$ is a metric space and hence $\Bbbk^{\infty\times n}$ is a metric space with the product metric and so is paracompact. The product of any finite number of spaces that are both locally compact and Hausdorff is again locally compact and Hausdorff, so $\Bbbk^{\infty\times n}$ is paracompact and locally compact Hausdorff. By the hint of **Exercise 46(c)**, $EG(n)$ is isomorphic to $\operatorname{colim}_k V_{n,n+k}(\Bbbk) \subset \operatorname{colim}_k \Bbbk^{\infty\times n}$ the subspace of orthonormal n-frames. Since each $V_{n,n+k}$ is compact by **Exercise 41(a)**, it is a compact and hence closed subspace of $\Bbbk^{(n+k)n}$. Taking colimits as in **Exercise 46(c)**, it follows that $EG(n)$ is, in particular, a closed subspace of $\Bbbk^{\infty\times n}$ since each of the structure maps inducing the embedding $EG(n) \to \Bbbk^{\infty\times n}$ is a closed embedding—one deduces this by noting that the topology on $EG(n)$ is such that a set F is closed if and only if $F \cap V_{n,n+k}$ is closed for all k and similarly a set $F \subset \Bbbk^{\infty\times n}$ is closed if and only if $F \cap \Bbbk^{(n+k)n}$ is closed for all k.

Thus, $EG(n)$ is a closed subspace of a locally compact space. It is a basic fact of point-set topology that such subspaces are themselves locally compact. See for instance, this math.stackexchange question for a proof. Thus, the total space of the universal $G(n)$-bundle $\gamma^n\colon EG(n) \to BG(n)$ is locally compact and we shall use this to show that $BG(n)$ must be locally compact.

Local triviality of the bundle $\gamma^n\colon EG(n) \to BG(n)$ implies that $BG(n)$ is covered by open sets $\{U_i\}_{i\in I}$ such that $(\gamma^n)^{-1}(U_i) \cong U_i \times G(n)$. It is similarly a basic fact of point-set topology that every open subspace of a locally compact and Hausdorff space is itself locally compact. See for instance, this math.stackexchange question for a proof. Hence, since $U_i \times G(n)$ is isomorphic to an open subspace of $EG(n)$, it locally compact. Hence, each point $(x, e) \in U_i \times G(n)$ has an open nbhd V contained in a compact set K. Since the projection map $\mathrm{pr}_1 \colon U_i \times G(n) \to U_i$ onto the factor of a product is open, $\mathrm{pr}_1(V)$ is an open nbhd of x in U_i and thus an open nbhd of x in $BG(n)$. Similarly, $\mathrm{pr}_1(K)$ is a compact subspace of U_i containing $\mathrm{pr}_1(V)$. Since compactness transcends subspace inclusion, x is contained in an open set which is itself contained in a compact set. Since this is true for every $i \in I$, this shows that $BG(n)$ is locally compact. ■

Remark. While one might hope the map $\gamma^\infty\colon E'G(\infty) \to BG(\infty)$ is a universal principal $G(\infty)$-bundle, it is not clear that it is (equivariantly) locally trivial and, in fact, upon passage to colimits, our models will not supply universal principal $G(\infty)$-bundle! It is, however, possible to show that $\operatorname{colim} BG(n)$ is the correct model for $BG(\infty)$. We shall use model categorical machinery to do this, but note that it is not strictly necessary.

We fix the convenient model category of spaces corresponding to the compactly generated weak Hausdorff Hurewicz model structure. We will be careful in what follows to distinguish between when a colimit in taken in our convenient category of spaces is the same as the one taken in the full category of spaces,. We will use May's bar construction functor **Theorem 6.1.9** and use the remark preceding that theorem.

We have shown that $BG(n)$ is a locally compact Hausdorff space and all such spaces are compactly generated and weak Hausdorff. Similarly, each $G(n)$ is locally compact Hausdorff and since each $G(n)$ is a manifold it necessarily has a non-degenerate basepoint—this includes the case of $n = \infty$ where one shows the identity element $e \in G(\infty)$ is a non-degenerate basepoint but providing it with a nbhd homeomorphic to $\mathbf{R}^\infty$.

Now, as a consequence of **Theorem 6.1.9** and the classification theorem, there is a homotopy equivalence $BG(n) \to \mathsf{B}(*, G(n), *)$—note that, technically, $\mathsf{B}(*, G(n), *)$ only exists in the category of compactly generated weak Hausdorff spaces and so we can only conclude it classifies numerable principal $G(n)$-bundles in that category. Since $BG(n)$ classifies numerable principal $G(n)$-bundles in the full category of spaces, it likewise classifies them in the restricted category of spaces. A Yoneda lemma style argument along with the classification theorem implies that there is a homotopy equivalence $BG(n) \to \mathsf{B}(*, G(n), *)$ in the category of compactly generated weak Hausdorff spaces and therefore also in the full category of spaces.

On the other hand,

$$\mathsf{B}(*, \operatorname{colim} G(n), *) = |\mathsf{Bar}(*, \operatorname{colim}_n G(n), *)| \cong |\operatorname{colim}_n \mathsf{Bar}(*, G(n), *)| \cong \operatorname{colim}_n |\mathsf{Bar}(*, G(n), *)|$$

since realization is a left adjoint—realization is a left adjoint for even the full category of spaces because the spaces Δ^n are locally compact Hausdorff and thus "exponentiable" objects. But this says that

$$\mathsf{B}(*, G(\infty), *) \cong \operatorname{colim}_n \mathsf{B}(*, G(n), *).$$

The maps $\mathsf{Bar}(*, G(n), *) \to \mathsf{Bar}(*, G(n+1), *)$ can be shown to be Reedy cofibrations by an argument similar to the one given in the remark preceding **Theorem 6.1.9** from which it follows that $\mathsf{B}(*, G(n), *) \to \mathsf{B}(*, G(n+1), *)$ are closed cofibrations. We have already seen that $BG(n) \to BG(n+1)$ are inclusions of subcomplexes and thus they too are closed cofibrations. Using the homotopy extension property of cofibrations, we may inductively modify the homotopy equivalences $BG(n) \to \mathsf{B}(*, G(n), *)$ within the homotopy class and thereby suppose we have vertical homotopy equivalences forming a commutative diagram

$$\begin{array}{ccccc} \mathsf{B}(*, G(1), *) & \longrightarrow & \mathsf{B}(*, G(2), *) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ BG(1) & \longrightarrow & BG(2) & \longrightarrow & \cdots \end{array}$$

It follows by model categorical considerations that the colimiting map $BG(\infty) \to \operatorname{colim} \mathsf{B}(*, G(n), *) \cong \mathsf{B}(*, G(\infty), *)$ is a homotopy equivalence, thereby establishing the desired equivalence. //

6.5 The Cohomology of Some Classifying Spaces

Remark. The author of these notes regrets not having the time to give these computations in session. Because we have motivated characteristic classes through classifying spaces, the next natural step is to investigate the cohomology of the classifying spaces. For a lack of room and in order to keep these notes self-contained, we will not find formulae for characteristic classes of certain bundle constructions through explicit computations in the cohomology of classifying spaces.

First, we supply a nice computation.

Proposition 6.5.1. *Fix $m \geq 2$ and $n \geq k \geq 1$. We fix $\mathbf{Z}/2$ coefficients throughout and suppress the coefficient group.*

(a) *There is an isomorphism of graded rings*

$$H^*(V_{k,n}(\mathbf{R})) \cong \mathbf{Z}/2[Y_{n-k}, \ldots, Y_{n-1}]/\sim \qquad |Y_i| = i$$

where $\Lambda_{\mathbf{Z}/2}$ indicates the free exterior algebra construction over $\mathbf{Z}/2$ on the terms in the brackets. Here, $\sim$ imposes some unspecified relations on the generators. In particular, $Y_i^2 \neq 0$ iff $H^{2i}(V_{k,n}) \neq 0$.

(b) *The maximal torus T of $SO(2)$ is $SO(2) \cong S^1$ itself. A maximal torus T for $SO(2n)$ consists of the block diagonal matrices with 2×2 blocks and hence is isomorphic to $SO(2)^{\times n}$; a maximal torus T for $SO(2n+1)$ consists of the block diagonal matrices with first n blocks 2×2 matrices and with the last entry upon the diagonal 1 and hence is isomorphic to $SO(2)^{\times n}$.*

(c) *If Z consists of the block diagonal matrices of $O(n)$ then Z is contained in the center of $O(n)$ and is is a normal subgroup. Moreover, there is an isomorphism $Z \simeq (\mathbf{Z}/2)^{\times} n$ and $H^*(BZ) \cong \mathbf{Z}/2[X_1, \ldots, X_n]$ with $|X_i| = 1$*

(d) *$H^*(SO(n)) \cong \mathbf{Z}/2[X_1, \ldots, X_m]/\sim$ where $\sim$ is some ideal.*

Proof. **(a)** The cohomology of

$$V_{1,n}(\mathbf{R}) \cong SO(n)/SO(n-1) \cong S^{n-1}$$

is known as a consequence of the indicated diffeomorphism. This is the base case of $k = 1$ for an induction k. For $n \geq k \geq 2$, we start by observing that there is a fiber bundle

$$S^{n-k} \to V_{k,n}(\mathbf{R}) \to V_{k-1,n}(\mathbf{R})$$

which, on the level of quotients of Lie groups is the map $SO(n)/SO(n-k) \to SO(n)/SO(n-k+1)$ induced by the standard identification of $SO(n-k) \subset SO(n-k+1)$ by

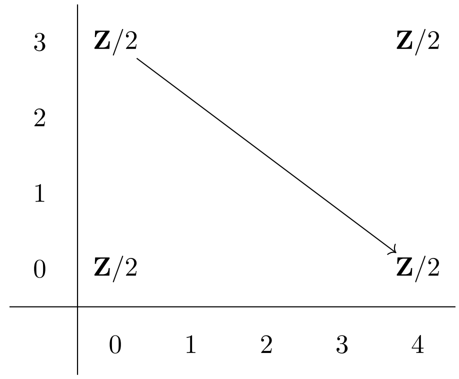
$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

having fiber $V_{1,n-k+1}(\mathbf{R}) \cong SO(n-k+1)/SO(n-k) \cong S^{n-k}$. This follows from an application of **Corollary 6.2.10**.

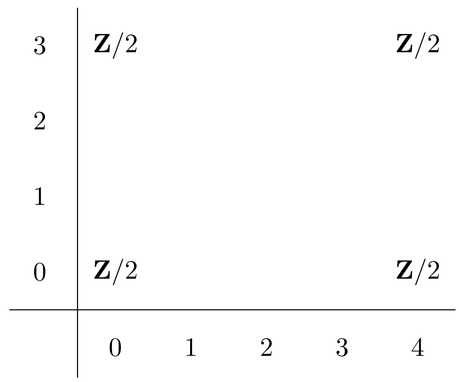
Any local coefficient system for this fibration taken with $\mathbf{Z}/2$ coefficients is trivial since there is only one automorphism of $H^*(S^{n-1}; \mathbf{Z}/2) \cong \mathbf{Z}/2$ so the Serre spectral sequence can be run as usual. The spectral sequence for this fibration has E_2 page

$$E_2^{p,q} = H^p(V_{k-1,n}; H^q(S^{n-k})) \cong \begin{cases} 0 & q \neq 0, n-k \\ H^p(V_{k-1,n}; \mathbf{Z}/2) & q = 0, n-k. \end{cases}$$

In the case of $k = 2$ and $n = 5$, this has E_4 page of the form



and so $E_5 = E_\infty$ at least. The differential is either 0 or an isomorphism. Hence, every element of $H^0(V_{k-1,n}; H^{n-2}(S^{n-2}; \mathbf{Z}/2)) \cong H^{n-2}(S^{n-2}; \mathbf{Z}/2)$ is transgressive and the differential $d_4 = \tau$ is the transgression. If the transgression was not zero, it would be an isomorphism, but since for $1 \leq k \leq n$, $V_{k,n}$ is $(n-k-1)$-connected with $\pi_{n-k}(V_{k,n}) \neq 0$, the Hurewicz theorem forces our hand by convergence of the spectral sequence since we would otherwise have $H^{n-k}(V_{k,n}; \mathbf{Z}/2) = 0$. (One deduces this π_* statement from the long exact sequence of the fibration.) Hence, $E_2 = E_\infty$ in fact and this page is



and the multiplicative statement follows from the coherence of the cup product on the E_∞ page with the cup product for the total space. In the general case, all differentials are zero except the transgression and the analysis is exactly the same that the transgression is non-zero. This finishes the induction step and the latter computation.

(b) It is obvious $SO(2)$ is the circle group, so consider $n \geq 3$. Let T denote the claimed maximal torus. If T is not maximal, it is necessarily contained in a larger torus. A short computation shows that any other matrix commuting with these matrices is necessarily of the same form, however.

(c) The diagonal matrices commute with everything so Z is contained in the center of $O(n)$. It is easy to see that $Z \cong (\mathbf{Z}/2)^{\times n}$. In fact, the diagonal matrices are a closed normal subgroup of $O(n)$ so $O(n)/Z$ is again a Lie group as a consequence of what we have shown before. We know that, up to homotopy equivalence, $B(\mathbf{Z}/2)^{\times n} \simeq (B\mathbf{Z}/2)^{\times n} \simeq (\mathbf{R}P^\infty)^{\times n}$. Let us take for granted that$H^*(\mathbf{R}P^\infty) \cong \mathbf{Z}/2[X]$ with $|X| = 1$. Thus, by the Künneth theorem for $\mathbf{Z}/2$ coefficients, $H^*(Z) \cong H^*(\mathbf{R}P^\infty)^{\otimes n}$ and one can easily verify that, when taken with $\mathbf{Z}/2$ coefficients, this ring is $\mathbf{Z}/2[X_1, \ldots, X_n]/(X_i^2)$ where $|X_i| = 1$.

(d) We proceed by induction since $SO(2) = S^1$. We use the spectral sequence for the fibration $SO(n-1) \to SO(n) \to V_{1,n} \cong S^{n-1}$ with $\mathbf{Z}/2$-coefficients. For lacunary reasons, the only relevant differentials are of the form (for $k \geq 0$) $H^{n-2+k}(SO(n-1)) \to H^k(SO(n))$ on the E_{n-1} page. An induction using the multiplicative structure of the page shows that these differentials all vanish except, perhaps, the differential (and, in fact, transgression) $\tau\colon H^{n-2}(SO(n-1)) \to H^0(SO(n))$ which on the level of the cohomology $\mathbf{Z}/2$-modules is $\mathbf{Z}/2 \to \mathbf{Z}/2$. Since $SO(n-1) \to SO(n)$ is an $(n-2)$-equivalence, it is a basic fact of algebraic topology that $H^*(SO(n)) \cong H^*(SO(n-2))$ for $0 \leq * \leq n-3$ and $H^{n-2}(SO(n)) \to H^{n-2}(SO(n-1))$ is injective—see, for instance, the proof of **Proposition 4.21** in Hatcher's book. Since $n-2 < n$, our induction hypothesis guarantees that these groups are both $\mathbf{Z}/2$ and hence an injective map is an isomorphism. Hence, if the differential were not 0, and so an isomorphism, then convergence would force us to conclude that $H^{n-2}(SO(n)) = 0$ which contradicts what we just observed. Hence, it follows that $E_2 = E_\infty$. Working with $\mathbf{Z}/2$ coefficients, all extension problems split, since $\mathbf{Z}/2$ is a field and the result follows easily from this. ■

This is easier for the other classical groups.

Exercise 51. *Fix coefficient group* $\mathbf{Z}$.

(a) *Show that*

$$H^*(V_{k,n+k}(\mathbf{C})) = H^*(U(n+k)/U(n)) \cong \Lambda(e_{2n+1}, e_{2n+3}, \ldots, e_{2(n+k)-1})$$
$$H^*(V_{k,n+k}(\mathbb{H})) = H^*(\mathrm{Sp}(n+k)/\,\mathrm{Sp}(n)) \cong \Lambda(e_{4n+3}, e_{4n+7}, \ldots, e_{4(n+k)-1}$$

where $|e_i| = i$ *and* Λ *indicates the free* $\mathbf{Z}$ *exterior algebra on the indicated generators.* [Hint: Let $m = \dim_{\mathbf{R}} \Bbbk$ where $\Bbbk = \mathbf{C}$ or $\mathbb{H}$. Show that there is a fiber bundle $S^{m(n+1)-1} \to V_{k,n+k}(\Bbbk) \to V_{k-1,n+k}(\Bbbk)$ and use the Gysin sequence for this by arguing that the base space is simply-connected to compute the cohomology rings of the relevant Stiefel manifolds by induction on k.*]*

(b) *Show that*

$$H^*(U(n)) \cong \Lambda(e_1, e_3, \ldots, e_{2n-1})$$
$$H^*(SU(n)) \cong \Lambda(e_3, \ldots, e_{2n-1})$$
$$H^*(\mathrm{Sp}(n)) \cong \Lambda(e_3, e_7, \ldots, e_{4n-1})$$

where $|e_i| = i$*.* [Hint: Show that there is a fiber bundle $U(n-k, \Bbbk) \to U(n, \Bbbk) \to V_{k,n}$ and proceed by induction on k, using **(a)** and the Serre spectral sequence. For the SU case, observe that for $n \geq 1$, $U(n+k)/U(n) \cong SU(n+k)/SU(n)$.]

Theorem 6.5.2. *There are natural isomorphisms of graded rings*

$$H^*(BSO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_2, \ldots, w_n], \qquad |w_i| = i$$
$$H^*(BSO; \mathbf{Z}/2) \cong \mathbf{Z}/2[w_2, , w_3, \ldots], \qquad |w_i| = i$$
$$H^*(BO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_1, \ldots, w_n], \qquad |w_i| = i$$
$$H^*(BO; \mathbf{Z}/2) \cong \mathbf{Z}/2[w_1, w_2 \ldots], \qquad |w_i| = i$$
$$H^*(U(n); \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_1, X_3, \ldots, X_{2n-1}], \qquad |X_i| = i$$
$$H^*(SU(n); \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_3, \ldots, X_{2n-1}], \qquad |X_i| = i$$
$$H^*(\mathrm{Sp}(n); \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_3, X_7, \ldots, X_{4n-1}], \qquad |X_i| = i$$
$$H^*(BU(n); \mathbf{Z}) \cong \mathbf{Z}[c_1, \ldots, c_n], \qquad |c_i| = 2i$$
$$H^*(BSU(n); \mathbf{Z}) \cong \mathbf{Z}[c_2, \ldots, c_n], \qquad |c_i| = 2i$$
$$H^*(BU; \mathbf{Z}) \cong \mathbf{Z}[c_1, c_2 \ldots], \qquad |c_i| = 2i$$
$$H^*(B\,\mathrm{Sp}(n); \mathbf{Z}) \cong \mathbf{Z}[q_1, \ldots, q_n], \qquad |q_i| = 4i$$
$$H^*(B\,\mathrm{Sp}; \mathbf{Z}) \cong \mathbf{Z}[q_1, q_2, \ldots], \qquad |q_i| = 4i$$

where $\Lambda_{\mathbf{Z}}[X_1, X_3, \ldots, X_{2n-1}]$ *indicates the free exterior algebra over* $\mathbf{Z}$ *on the indicated generators. Furthermore, for the deloopings of the canonical inclusions* $Bi\colon BG(n-1) \to BG(n)$ *for* $G(-)$ *any of the groups above, the generators may be chosen such that* Bi^* *maps the multiplicative generators to multiplicative generators whenever their degrees agree and maps the top multiplicative generator to* 0*.*

Remark. Really, one should write $H^*(BG(\infty))$ as an infinite power series on the generators by way of the $\lim^1$ argument below. However, we are choosing our convention to be that H^* is always the *direct sum* of the H^i and no the product of the H^i.

Warning. We will silently suppress coefficients in the proof below. They are always implied.

Proof. The proofs for H^* of $U(n)$, $SU(n)$ and $\mathrm{Sp}(n)$ follow the same pattern, so we only give the case of $U(n)$. There is a fiber bundle $U(n-1) \to U(n) \to U(n)/U(n-1) \cong S^{2n-1}$ for all $n \geq 1$; this is because $U(m)$ is a compact Lie group for all m and $U(n)$ acts transitively and properly on S^{2n-1}, which is also compact, and with stabilizer $U(n-1)$—so the quotient manifold theorem applies from which the isomorphism follows. Note that $H^*(S^{2n-1}; \mathbf{Z})$ is concentrated in degrees $2n-1$ and 0 with a copy of $\mathbf{Z}$ in each. Since $U(1) = S^1$, we know $H^*S^1 \cong \mathbf{Z}[X]/(X^2) = \Lambda_{\mathbf{Z}}[X]$. This furnishes the base case of an induction on n. For $n \geq 2$ the base space is simply-connected and so the Serre spectral sequence runs in its nicest form since the local coefficient system is trivial. In general, one can easily show that the product pairing on the E_2 page induces an isomorphism $E_2^{p,q} \cong H^p(S^{2n-1}; \mathbf{Z}) \otimes_{\mathbf{Z}} H^q(U(n); \mathbf{Z})$ either by direct computation or by appealing to **Theorem F.1.5(c)**. By induction hypothesis, $H^*(U(n-1))$ is generated in odd degrees by elements whose only relations are $X_iX_j = -X_jX_i$ (which implies $X_i^2 = 0$). The multiplicative structure then implies it is enough to understand the differentials on the E_2 page with domain the fiber terms $H^i(U(n))$ with i odd and in particular what these differential do to X_i. Since $n \geq 2$, $2n-1 \geq 3$, the E_2 page presents itself in the case of $n = 2$ and with only differentials having domain fiber cohomology groups containing a generator X_i displayed as

4	$\vdots$	$\vdots$	
3	$H^3(U(n-1))$	$H^3(U(n-1))$	
2	$H^2(U(n-1))$	$H^2(U(n-1))$	
1	$H^1(U(n-1))$	$H^1(U(n-1))$	
0	$\mathbf{Z}$	$\mathbf{Z}$	
	0	$2n-1$	$22n-1$

Thus, in particular, the only interesting page E_{2n-1} in which case the only possible non-zero differentials are $d_{2n-1}^{0,*}\colon H^*(U(n)) \to H^{*-n}(U(n))$ and in particular for $* \geq n$. The first possibly non-zero differential is $H^{2n-2}(U(n-1)) \to H^0(U(n-1)) \cong \mathbf{Z}$. By induction, $H^*(U(n-1))$ has generators living in odd degrees up to $2(n-1)-1 = 2n-3$ and has top dimensional cohomology group $\sum_{k=1}^{n-1} 2k-1 = (n-1)^2$. Thus, the multiplicative structure implies this differential is zero. All other possibly non-zero differentials are zero for the same reason. This means that $E_2 = E_\infty$ and we are thus relegated to resolving the extension problems. When $n \leq 3$ there are no extension problems so we may as well suppose $n \geq 4$. The only possible non-trivial extension problems are of the from

$$0 \to H^{2n-1+k}(U(n-1)) \to ? \to H^k(U(n-1)) \to 0$$

for $0 \leq k \leq (n-2)^2+2$ and $n \geq 4$—the $(n-2)^2+2$ comes from a simple algebra computation, solving for k in $(n-1)^2 = 2n-1+k$ and noting that this can be written as $((n-1)-1)^2+2 = (n-1))^2-2(n-1)+3 = (n-1)^2-2n+1 = k$. For such k this extension computes $H^{2n-2+k}(U(n))$—by induction hypothesis, each known term of this short exact sequence is a free $\mathbf{Z}$-module, from which it follows that $\mathsf{Ext}^1_{\mathbf{Z}}(H^{2n-2+k}(U(n-1)), H^k(U(n-1))) = 0$ which implies that the only extension is the split extension. Thus,

$$H^k(U(n)) \cong \begin{cases} H^k(U(n-1)) & k \leq 2n-2 \\ H^k(U(n-1)) \oplus H^{k-(2n-1)}(U(n-1)) & 2n-1 \leq k \leq (n-2)^2+2 \\ H^k(U(n-1)) & (n-2)^2+3 \leq k \leq (n-1)^2. \end{cases}$$

It only remains to compute the multiplicative structure and since all extension problems split this can be read off directly from the $E_2 = E_{2n-1} = E_\infty$ page with the understanding that the multiplicative structure on the $E_2 = E_{2n-1}$ page $E_{2n-1}^{p,q} \times E_{2n-1}^{s,t} \to E_{2n-1}^{p+s,q+t}$ is $(-1)^{qs}$ times the cup product on in $H^*(U(n-1))$ and the conclusion follows.

As for $BSU(n)$, $BU(n)$ and $B\operatorname{Sp}(n)$, the arguments follow the same pattern and so we only give the calculation of $BU(n)$. The interested reader can carry these out on their own.

We know $U(1) = S^1$ so that $BU(1) = \mathbf{C}P^\infty$ whose cohomology ring we know. This starts an induction argument. Using the fiber bundle $S^{2n-1} \cong U(n)/U(n-1) \to BU(n-1) \xrightarrow{Bi} BU(n)$, we use the Gysin sequence to obtain the exact sequence

$$\cdots \to H^i(BU(n-1)) \xrightarrow{\delta} H^{i-(2n-1)}(BU(n)) \xrightarrow{\gamma\smile -} H^{i+1}(BU(n)) \xrightarrow{Bi^*} H^{i+1}(BU(n-1)) \to \cdots$$

where $\gamma \in H^{2n}(BU(n))$. For each element c_i of $H^i(BU(n))$ where $1 \leq i \leq n-1$, $\delta(c_i) = 0$ because it lands in a trivial group and so by exactness $c_i \in \operatorname{Im} Bi^*$. This implies that Bi^* is surjective on cohomology as it hits all multiplicative generators. We thus extract a short exact sequence

$$0 \to H^{i-(2n-1)}(BU(n)) \xrightarrow{\gamma\smile -} H^{i+1}(BU(n)) \xrightarrow{Bi^*} H^{i+1}(BU(n-1)) \to 0$$

By abuse of notation, let $c_i = (Bi^*)^{-1}(c_i)$ in $H^i(BU(n))$ and let $c_n = \gamma$ (we shall see later this is the Euler class of the bundle). The foregoing considerations imply that $c_n \neq 0$.

We claim there is no polynomial relation among the c_i. Let $G \in \mathbf{Z}[X_1, \ldots, X_n]$ be non-zero and homogeneous where $|X|_i = 2i$, and suppose for a contradiction that $G(c_1, \ldots, c_n) = 0$. We induct on $n = \deg G \neq -\infty$ to obtain contradictions for all $n \geq 0$. When $0 \leq \deg G < 2n$, we have the contradiction for free since $G(c_1 \ldots, c_n) = G(c_1, \ldots, c_{n-1}, 0)$ by degree considerations so that $Bi^*G(c_1, \ldots, c_n) = G(c_1, \ldots, c_{n-1}, c_n) = G(c_1, \ldots, c_{n-1}, 0) = 0$ but since $G \neq 0$ and since there are no polynomial relations among the $c_1, \ldots, c_{n-1}$ in $H^*BU(n-1)$, this is a contradiction. When $\deg G = 2n$, either X_n either appears in G as mX_n for some $m \in \mathbf{Z} \setminus \{0\}$ or $m = 0$. In the latter case we obtain the same contradiction as before. In the former case we can write $G(X_1, \ldots, X_n) = F(X_1, \ldots, c_{n-1}) + m \cdot X_n$ and see that $Bi^*G(c_1, \ldots, c_n) = Bi^*F(c_1, \ldots, c_{n-1}) = 0$ so that $F = 0$ by our assumption that $H^*(BU(n-1)) \cong \mathbf{Z}[c_1, \ldots, c_{n-1}]$. Now suppose $\deg G = 2n + k$ where $k \geq 1$. Then since $Bi^*G(c_1, \ldots, c_n) = G(c_1, \ldots, c_{n-1}, 0) = 0$, we must be able to pull out a factor of X_n from $G(X_1, \ldots, X_n)$ and thereby write $G(X_1, \ldots, X_n) = X_n F(X_1, \ldots, X_n)$. By the exact above, $c_n \smile -$ is injective. Hence, $c_n \smile F(c_1, \ldots, c_n) = 0$ if and only if $F(c_1, \ldots, c_n) = 0$ and since $c_n \smile F(c_1, \ldots, c_n) = G(c_1, \ldots, c_n)$, it must be that $F(c_1, \ldots, c_n) = 0$. But now $\deg F = \deg G - 1$ and so by our induction hypothesis on the degree of the monomials it cannot be that $F \neq 0$ and this is a contradiction.

Hence, there are no polynomial relations among the $c_1, \ldots, c_n$ which shows that the subring they generate is isomorphic to $\mathbf{Z}[c_1, \ldots, c_n]$. We must show that this subring is in fact the whole ring. To see this, let $\varphi \in H^*(BU(n))$ be homogeneous and note that $Bi^*\varphi = F(c_1, \ldots, c_{n-1})$ for some possibly trivial polynomial $F \in \mathbf{Z}[X_1, \ldots, X_{n-1}]$. Hence, $Bi^*(\varphi - F(c_1, \ldots, c_{n-1})) = 0$ and so by exactness of the above sequence, it must be that $x = F(c_1, \ldots, c_{n-1}) + c_n \smile \psi$ for some $\psi \in H^{*-2n}(BU(n))$. We now induct on $|\psi|$. When $|\psi| = 0$, we are done, this element lies in the span of the $c_1, \ldots, c_n$. If $|\psi| > 0$, then the same argument we applied for φ shows that $\psi = G(c_1, \ldots, c_{n-1}) + c_n \smile \tau$ and now $|\tau| < |\psi|$ so that τ may be written as a polynomial in the $c_1, \ldots, c_n$ completing the induction step and thereby showing that $H^*(BU(n)) \cong \mathbf{Z}[c_1, \ldots, c_n]$.

Observation. In the above proof we chose the c_i in such a way that they satisfy $Bi^*(c_i) = c_i$ for $1 \leq i \leq n-1$ and $Bi^*c_n = 0$. This shows that we may choose the multiplicative generators in the manner asserted and this may be done coherently (i.e., compatibly for all n) by an induction argument.

We now consider the case of the orthogonal groups. There is a fiber sequence $SO(k) \to SO(n) \to SO(n)/SO(k)$ with $SO(k) \to SO(n)$ a cofibration. We know that $SO(n)/SO(k)$ is $(n-k-1)$-connected from homotopical reasonings and the fibration sequence $SO(n-1) \to SO(n) \to S^{n-1} \cong SO(n)/SO(n-1) \cong O(n)/O(n-1)$. But we have seen that $S^{n-1} \cong SO(n)/SO(n-1) \cong O(n)/O(n-1)$ is the fiber of the fiber bundle $Bi\colon BSO(n-1) \to BSO(n)$ (and also $BO(n-1) \to BO(n)$). We should like to apply the Gysin sequence to this fibration, but for the case of $BO(n)$, we do not know that the local coefficient system is trivial a priori since $\pi_1 BO(n) \neq 0$. If $n \geq 2$, then the fiber is path-connected and with $\mathbf{Z}/2$-coefficients one easily checks that this is indeed a trivial local coefficient system. For $n = 1$, we note that $O(1) = \mathbf{Z}/2$ and $B\mathbf{Z}/2$ is a $K(\mathbf{Z}/2, 1)$ up to weak equivalence and so is weakly equivalent to $\mathbf{R}P^\infty$. Hence, since weak equivalences preserve cohomology, and since $H^*(\mathbf{R}P^\infty; \mathbf{Z}/2) \cong \mathbf{Z}/2[X]$ with $|X| = 1$, this is agrees with what we asserted. Hence, we may suppose that $n \geq 2$.

For $n \geq 2$, we have seen the local coefficient system is trivial for $\mathbf{Z}/2$ coefficients and we may consider the Gysin sequence for the spherical bundle $S^{n-1} \to BO(n-1) \to BO(n)$. This gives a long exact sequence (with $\mathbf{Z}/2$ coefficients suppressed)

$$\cdots \to H^i(BO(n-1)) \xrightarrow{\delta} H^{i-(n-1)}BO(n) \xrightarrow{\gamma \smile -} H^{i+1}(BO(n)) \xrightarrow{Bi^*} H^{i+1}(BO(n-1)) \to \cdots$$

where $\gamma \in H^n(BO(n))$ is (as we shall see later) the $\mathbf{Z}/2$ reduction of the Euler class of this $O(n)$-bundle and thus its n-th Stiefel-Whitney class.

We note that for $0 \leq j \leq n-2$, $H^{j-(n-1)}(BO(n)) = 0$ which implies that Bi^* is surjective on H^j for $1 \leq i \leq n-1$ and it is an isomorphism for $1 \leq j \leq n-2$ by extending this long exact sequence. Since $BO(n)$ is path-connected for all n, we know that Bi^* is an isomorphism on H^0 as well.

The proof now proceeds by induction on n with the case of $n = 1$ known and done.

$$\text{For the general case, let us suppose that we know } \gamma \neq 0. \qquad (*)$$

We will show that $\gamma \neq 0$ after we indicate how this implies the result.

It follows that $\delta\colon H^{n-1}(BO(n-1)) \to H^0(BO(n)) \cong \mathbf{Z}/2$ is the 0 map since otherwise $\delta(w) = 1$ and so $\gamma \smile w = \gamma \smile 1 = \gamma \neq 0$ which contradicts exactness.

In particular, this implies that Bi^* is an isomorphism on H^{n-1} as well and thus Bi^* is an isomorphism on H^* for $* \leq n-1$. $(**)$

We claim this all implies that γ cannot be written as a product of lower degree terms in $H^*(BO(n))$. For a contradiction, suppose that it could be so written. Then since Bi^* is a map of cohomology rings and surjective on lower degree terms, $Bi^*\gamma$ is a product of terms known by induction to have non-zero product and this contradicts exactness since $Bi^*\gamma = Bi^*(\gamma \smile 1)$.

Observe that since $Bi\colon BO(n-1) \to BO(n)$ induces an isomorphism in degrees $0 \leq i \leq n-1$ on cohomology, it must send some element $H^i(BO(n))$ that is not a product of lower order terms to the generator $w_i \in H^i(BO(n-1))$; certainly w_i is in the image of Bi^* and if Bi^* mapped a product of lower order terms to w_i, then we could express w_i as a product of lower order terms in $H^*(BO(n-1))$ which is impossible by induction hypothesis on the cohomology of $BO(m)$. By abuse of notation, let us call this element $w_i \in H^i(BO(n))$. Since γ is not a product of elements, we somewhat abuse notation and write

$$\gamma = w_n.$$

In particular, this means that $Bi^*(w_i) = w_i$ and, moreover, on the subrings generated by the classes $w_1, \ldots, w_{n-1}$, Bi^* is an isomorphism—indeed, any relation among the w_i in $H^*(BO(n))$ would, by way of Bi^*, imply a relation among the w_i in $H^*(BO(n-1))$ and this is impossible by induction hypothesis. We can refine this observation further to see that Bi^* surjects everywhere; this is because, by induction hypothesis, we know $H^*(BO(n-1))$ and so $Bi^*(w_{i_1} \cdots w_{i_j}) = Bi^*(w_{i_1}) \cdots Bi^*(w_{i_j}) = w_{i_1} \cdots w_{i_j}$. Hence, by exactness of the Gysin sequences, $w_n \smile -$ is injective everywhere.

From this, we can also conclude that that there are no polynomial relations between w_n and the elements $w_1, \ldots, w_n$. Indeed, suppose G is a homogeneous polynomial in $X_1, \ldots, X_n$ with $|X_i| = i$ in $\mathbf{Z}/2$ of total degree $n + k$ for some $k \geq 0$ and so, in particular, is non-zero. Suppose further that $G(w_1, \ldots, w_n) = 0$. We will obtain contradictions for all $k \geq 0$ by induction implying the non-existence of polynomial relations among these elements. For the base case, either w_n is a summand of G or it is not—in the latter case, we know from the above that it must be that $G = 0$ which is a contradiction whereas in the former case, $G(w_1, \ldots, w_n)$ contains w_n as a summand and so w_n is a sum of lower degree terms but then since Bi^* sends those lower degree terms isomorphically into $H^*(BO(n-1))$ we would conclude that $Bi^* w_n \neq 0$ which is a contradiction. In general, it easy to see that $Bi^* G(w_1, \ldots, w_n) = G(w_1, \ldots, w_{n-1}, 0) = 0$ and so it must be that G has a common power of X_n everywhere and thus we may write $G(w_1, \ldots, w_n) = w_n G_1(w_1, \ldots, w_n) = 0$. Since $w_n \smile -$ is injective everywhere, we must have that $G_1(w_1, \ldots, w_n) = 0$ and so G_1 satisfies the hypotheses for which our induction hypothesis applies and so we conclude that $G_1 = 0$ which is a contradiction. It follows that the subring the elements $w_1, \ldots, w_n$ generate is isomorphic to $\mathbf{Z}/2[w_1, \ldots, w_n]$ where $|w_i| = i$. We must show that this is everything.

Let $\varphi \in H^*(BO(n))$ and write $Bi^*\varphi = F(w_1, \ldots, w_{n-1})$ a polynomial in the generators of $H^*BO(n-1)$. Then $Bi^*(\varphi - F(w_1, \ldots, w_{n-1})) = 0$ from what we just showed and therefore there exists an element $\psi \in H^*(BO(n))$ such that $\varphi = F(w_1, \ldots, w_{n-1}) + w_n \smile \psi$ by exactness of the Gysin sequence. We proceed by induction on $|\psi|$. If $|\psi| = 0$, it is easy to see this expression is unique from the above. the case of $\psi = 0$ is trivial and if $\psi = 1$ this is because there are no polynomial relations between w_n and the $w_1, \ldots, w_{n-1}$. For induction step, note that $|\psi| < |\varphi|$ and so we may write $\psi = G(w_1, \ldots, w_{n-1}) + w_n \smile \tau$ as above; it follows that $|\tau| < |\psi| < |\varphi|$ and so our induction hypothesis applies so that ψ may be written uniquely as a polynomial in the $w_1, \ldots, w_n$. It follows that $F(w_1, \ldots, w_{n-1}) + w_n \smile \psi$ then admits a unique expression as a polynomial in $w_1, \ldots, w_n$ because there are no relations between w_n and the $w_1, \ldots, w_{n-1}$.

It now only remains to show that $(*)$ holds—that is, $\gamma \neq 0$. In the section below, we will define the ***tautological line bundle*** γ_1^n over $\mathbf{R}P^n$, but we will need it now. Suppose for now we have shown that γ_1^1 is the Möbius bundle over the circle so that $\gamma_1^{1\times n}$ is a rank n vector bundle over $S^{1\times n}$. We will use this bundle and naturality of the Gysin exact sequence to conclude that the $\gamma \neq 0$ above.

The bundle $\gamma_1^{1\times n}$ naturally has structure group $O(1)^{\times n} \subset O(n)$, the subgroup of diagonal matrices with entries ± 1 and, hence, has structure group $O(n)$. We further take its associated spherical bundle

$$\xi = \mathsf{P}(\gamma_1^{1\times n}) \times_{O(n)} S^{n-1}.$$

Since $S^{n-1} \to BO(n-1) \to BO(n)$ is universal by **Lemma 6.2.9(d)** and **Example 5**, we obtain a morphism of fiber sequences with the map $f \colon S^{1\times n} \to BO(n)$ classifying the fiber bundle on the top row.

$$\begin{array}{ccccc} S^{n-1} & \longrightarrow & E\left(\mathsf{P}(\gamma_1^{1\times n}) \times_{O(n)} S^{n-1}\right) & \longrightarrow & S^{1\times n} \\ \| & & \downarrow & & \downarrow f \\ S^{n-1} & \longrightarrow & BO(n-1) & \longrightarrow & BO(n) \end{array}$$

The notation $E\left(\mathsf{P}(\gamma_1^{1\times n}) \times_{O(n)} S^{n-1}\right)$ means the total space of the bundle $\mathsf{P}(\gamma_1^{1\times n}) \times_{O(n)} S^{n-1}$.

We know that naturality of the Gysin sequence implies that f^* maps the element γ of the bottom row to γ of the top row. Now, by the Künneth isomorphism, the cross product affords an isomorphism of $H^*(S^1) \otimes \cdots \otimes H^*(S^1) \cong H^*(S^1 \times \cdots \times S^1)$ and since we are using $\mathbf{Z}/2$-coefficients this left-hand ring is isomorphic to $\mathbf{Z}/2[X_1, \ldots, X_n]/(X_i^2)$ with $|X|_i = 1$.

We claim the class γ of the top row is the n-fold cross product of the corresponding class for the $O(1) = S^0$-bundle $\mathsf{P}(\gamma_1^1)$ and we claim that $\mathsf{P}(\gamma_1^1)$ is isomorphic to the two-fold connected covering space $O(1) \to S^1 \xrightarrow{p} S^1 = \mathbf{R}P^1$ where $p = \times 2$ is the map $e^{i\theta} \mapsto e^{2i\theta}$ viewing $S^1 \subset \mathbf{C}^\times$ in the usual way. To see this latter thing, simply note that $\mathsf{P}(\gamma_1^1)$ is necessarily an $O(1)$-principal bundle over S^1 and since $O(1)$ is discrete this is a normal covering space having $O(1)$ as the group of deck transformations. There are precisely two such covering spaces—one that is disconnected and one that is path-connected. We may exclude this disconnected total space case since its associated bundle with fiber $\mathbf{R}$ is easily computed to be the trivial vector bundle and it is known (see below) that γ_1^1 is not trivial.

With $\mathbf{Z}/2$ coefficients, the local coefficient system for this bundle is trivial, and so its Gysin sequence proceeds as

$$\cdots \to H^i(S^1) \xrightarrow{\delta} H^i(S^1) \xrightarrow{\gamma \smile -} H^{i+1}(S^1) \xrightarrow{p^*} H^{i+1}(S^1) \to \cdots$$

then when $i = 0$ we can extract an exact sequence

$$\begin{array}{ccccccc}
H^0(S^1) & \xrightarrow{p^*} & H^0(S^1) & \xrightarrow{\delta} & H^0(S^1) & \xrightarrow{\gamma\smile -} & H^1(S^1) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & \mathbf{Z}/2
\end{array}$$

and since p^* is an isomorphism in degree 0, we conclude $\delta = 0$ and, hence $\gamma \neq 0$ since this would then contradict exactness of the remaining part of this exact sequence. This takes care of the case of $n = 1$ in the bundle ξ above.

For $n \geq 2$ in ξ, note that with $\mathbf{Z}/2$ coefficients the local coefficient system is still trivial. Notice that, in fact, $S^{1\times n} \to BO(n)$ factors through $S^{1\times n} \to BO(1)^{\times n} \to BO(n)$ since ξ has a reduction of structure group to $O(1)^{\times n}$—this follows since the transitions of ξ all lie in $O(1)^{\times n}$ with $S^{1\times n} \to BO(1)^{\times n}$ the map with i-component $S^1 \to BO(1)$ classifying the double cover $O(1) \to S^1 \to S^1$. It follows immediately on cohomology that γ_ξ is the n-fold cross product described by naturality of the Gysin sequence and this is non-zero in S^1. Hence, by naturality of the Gysin sequence, we further have that γ for the universal spherical bundle is non-zero as it maps to something non-zero. This concludes the analysis of $BO(n)$.

Observation. In the above proof we chose the w_i in such a way that they satisfy $Bi^*(w_i) = w_i$ for $1 \leq i \leq n-1$ and $Bi^*w_n = 0$. This shows that we may choose the multiplicative generators in the manner asserted and this may be done coherently (i.e., compatibly for all n) by an induction argument.

As for $BSO(n)$, $SO(1) = *$ and so $BSO(1) \simeq *$ and $SO(2) \cong S^1 \cong U(1)$ as Lie groups, the case of $BSO(2)$ follows from $BU(1)$ which we know. We consider the case of $n \geq 3$ now. Let $Bj\colon BSO(n) \to BO(n)$; this is part of the fiber bundle $O(n)/SO(n) \cong O(1) = S^0 \to BSO(n) \to BO(n)$ and with $\mathbf{Z}/2$ coefficients this clearly has a trivial local coefficient system so the Gysin sequence applies. We may run the Serre spectral sequence for the fibration $S^{n-1} \to BSO(n-1) \to BSO(n)$ with $\mathbf{Z}/2$ coefficients. Convergence and an easy induction starting at $n = 3$ shows that $H^1(BSO(n)) = 0$. Thus, in the Gysin sequence the element γ for the product morphism is non-zero and thus must be $w_1 \in H^1(BO(n))$ and from this we deduce that Bj^* is surjective with kernel anything with the term w_n in it. Since Bj^* is a ring map, the conclusion easily follows.

We now turn to the the infinite groups; the proof is a $\lim^1$ argument. The pattern of the argument is the same in all cases, so we show how to do this with BO. Let $Bi\colon BO(n-1) \to BO(n)$ be the delooping of the inclusion $O(n-1) \to O(n)$. Recall that we know this is the projection of a fiber bundle with structure group $O(n)$ and fiber $O(n)/O(n-1) \cong S^{n-1}$. With $\mathbf{Z}/2$ coefficients the local coefficient system for this fibration is trivial. We saw in ($**$) in the Gysin sequence for this bundle that Bi^* is an isomorphism in all degrees $*$ with $* \leq n-1$. Hence, may choose the w_i such that $Bi^*(w_i) = w_i$ for $i \leq n-1$. Thus, the Mittag-Leffler condition is satisfied in cohomology and so the $\lim^1$ term vanishes from which the conclusion follows.

The final assertion ■

Remark. Let $\xi \in \mathsf{Bun}_G^F$ with $G \curvearrowright F$ effective; denote $E(\xi)$ and $B(\xi)$ the total and base space, respectively. In the course of this proof, we used the following facts.

(i) If $\eta \in \mathsf{Bun}_H^T$ and $H^{\times n} \leq G$ then $\eta^{\times n}$ has structure group $H^{\times n} \leq G$.

(ii) If ξ admits a reduction of structure group to $H \leq G$, then $\mathsf{P}_G\xi$ admits a reduction of structure group to H.

(iii) If ξ admits a reduction of structure group to $H \leq G$, then the classifying map $B(\xi) \to BG$ factors through $Bi\colon BH \to BG$ where $i\colon H \to G$ is the inclusion of the subgroup.

(iv) If $\eta \in \mathsf{Bun}_H^T$, $H \curvearrowright T$ effectively and $H^{\times n} \leq G$, then $\eta^{\times n}$, then the classifying map for η as a bundle with structure group G factors through $Bi\colon BH^{\times n} \to BG$ where $i\colon H^{\times n} \to G$ is the inclusion.

These are not hard and will be exercises.

6.6 Reduction of Structure Group

We end with the topic of reduction of structure groups.

Definitions. Fix a choice of CAT and let Prin be the category whose objects are principal bundles and whose morphisms from a principal H-bundle ξ' to a principal G-bundle ξ are triples of maps $(\varphi, \widetilde{f}, f)\colon (H, P', B') \to (G, P, B)$ where $\varphi\colon H \to G$ is a CAT homomorphims of topological groups such that $\widetilde{f}(x \cdot h) = \widetilde{f}(x) \cdot \varphi(h)$ and such that TFDC:

$$\begin{array}{ccc}
P' & \xrightarrow{\widetilde{f}} & P \\
\downarrow & & \downarrow \\
B' & \xrightarrow[f]{} & B
\end{array}$$

When $f = \mathrm{id}_B$, such a morphism is called a ***reduction of the structure group of*** ξ ***to*** H or a ***reduction of the structure group from*** G ***to*** H of ξ. More precisely, a ***reduction of the structure group of*** ξ ***to*** H ***along*** φ is a principal H-bundle P', a CAT-homomorphism $\varphi\colon H \to G$ and a commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\widetilde{f}} & P \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

where $\widetilde{f}(x \cdot h) = \widetilde{f}(x) \cdot \varphi(h)$.

When H is a subgroup of G, as is almost always the case, we will mean by a ***reduction of the structure group from*** G ***to*** H a reduction of the structure group of G to H along the inclusion $i\colon H \hookrightarrow G$.

This is a more abstract version of the more concrete bundle atlas version. We shall see they are equivalent.

Proposition 6.6.1. *Fix a choice of* CAT*. The reduction of the structure group of a principal* G*-bundle* ξ *to* H *is equivalent to finding a* G*-subatlas* $\mathscr{A}'$ *of the maximal atlas* $\mathscr{A}$ *for* ξ *and* CAT *morphisms* $h_{ij}\colon U_i \cap U_j \to H$ *on intersections of elements* $\mathscr{A}'$ *satisfying the cocycle conditions such that the maps* $\{\varphi \circ h_{ij}\}$ *determine transition functions* ξ*—in other words, there are trivializations* $\psi_i \in \mathscr{A}$ *over* U_i *such that* ψ_{ij} *has transition function* $\varphi \circ h_{ij}$*.*

Before we give the proof of this proposition, we record a lemma, only the first part of which is new.

Lemma 6.6.2. *Fix a choice of* CAT *and let* ξ_H *and* ξ_G *be a principal* H*-bundle and a principal* G*-bundle over* B*, respectively. Let* $\varphi\colon H \to G$ *be a* CAT*-homomorphism.*

(a) *Each morphism of fiber bundles* $\widetilde{f}\colon \xi_H \to \xi_G$ *such that* $\widetilde{f}(x \cdot h) = \widetilde{f}(x) \cdot \varphi(x)$*,* $\widetilde{f}$ *can be considered as an* H*-equivariant morphism. Moreover, there is an open cover* $\{U_i\}_{i \in I}$ *of* B *and subatlases of* ξ_H *and* ξ_G *such that* $\varphi_i \widetilde{f} \psi_i^{-1} = 1 \times \varphi$ *for each* $i \in I$*.*

(b) *If two principal* G*-bundles over a base* B *has atlases defined over the same open cover of* B *and having the same transition functions, then they are isomorphic.*

Proof. **(a)** φ gives an action of H on ξ_G in the obvious way—namely, by

$$H \xrightarrow{\varphi} G \xrightarrow{\text{act}} \operatorname{Aut} \xi_G$$

where we must insert the usual caveats about smoothness or continuity when adjointing this over to the action map $\xi_G \times H \to \xi_G$. It is clear that any such morphism $\widetilde{f}$ is then H-equivariant with respect to this action.

(b) We have seen this before. ■

Proof (of proposition). $(\to)$ Suppose we are given a reduction of the structure group in the form of a commutative diagram (note that we do not mean the top square commutes)

$$\begin{array}{ccc} H & \overset{\varphi}{\dashrightarrow} & G \\ | & & | \\ P' & \xrightarrow{\widetilde{f}} & P \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

where $\widetilde{f}(x \cdot h) = \widetilde{f}(x) \cdot \varphi(h)$. Call the H-principal bundle ξ'. By **(e)** of **Exercise 17**, the collection $\{\varphi \circ h_{ij}\}$ are transition functions for P if and only if they determine a G-atlas which whose trivializations are G-equivariant. WLOG we may take $\mathscr{A}' = \{(\psi_i, U_i)\}_{i \in I}$ to be any H-subatlas of ξ' upon whose domains U_i we may also form a G-atlas $\{(\varphi_i, U_i)\}$ for ξ—such U_i may be found by taking intersections. For this atlas $\mathscr{A}'$ of ξ', let h_{ij} be the trivializations and consider the transition data $\{(U_i, h_{ij})\}_{i \in I}$ which, we note, satisfy the cocycle conditions, and one easily verifies that the transition data $\{(U_i, \varphi \circ h_{ij})\}_{i \in I}$ satisfy the cocycle conditions as well.

Thus, it follows by the fiber bundle construction theorem that there exists a G-bundle η over B having fiber G, structure group G, and a G-atlas on $\mathscr{A}'$ for which $\{(U_i, \varphi \circ h_{ij})\}_{i \in I}$ is the transition data. The goal is to show that there is an isomorphism over B between η and ξ, say $f_0\colon \eta \cong \xi$. With some thought, one sees that this suffices since it allows us to push the bundle atlas for η onto ξ. By **Theorem 1.3.1** (note that we are in the effective action regime), we may build this by finding CAT morphisms $g_i\colon U_i \to G$ such that $\varphi \circ h_{ij} = g_i^{-1} \cdot g_{ij} \cdot g_j$. We aim to discover these now.

Choose an index $i \in I$ (the same thing we do here works if we choose some ψ_j and φ_i generally but we do not need it). Since ξ' and ξ are principal bundles, φ_i is an H-equivariant trivialization and ψ_i is a G-equivariant trivialization. Thus, from the preceding lemma, $\widetilde{f}$ must have coordinate form an H-equivariant morphism $U_i \times H \to U_i \times G$ where H acts on G here on the right by $g \cdot h = g\varphi(h)$. It is easy to see that, with this H action, the H-equivariant maps $H \to G$ are maps of the form $h \mapsto g\varphi(h)$ for some g by chasing e, and every possible choice of g is gives an H-equivariant map. Hence, since φ_i is a CAT isomorphism, we may suppose we start with some (x, e) on the left, in which case this diagram must have the form

$$\begin{array}{ccc} (x,e) & \longrightarrow & (x,g_i(x)) \\ \big\downarrow & & \big\downarrow \\ (x,h) & \longrightarrow & (x,g_i(x)\varphi(h)) \end{array}$$

for some $g_i(x) \in G$. $g_i \colon U_i \to G$ is CAT be the usual considerations. Now write $U_{ij} = U_i \cap U_j$. Observe that the following diagram commutes because each square commutes

$$\begin{array}{ccc} (x,h) & \longrightarrow & (x,g_j(x)\cdot\varphi(h)) \\ {\scriptstyle \varphi_j^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_j^{-1}} \\ p & \longrightarrow & \widetilde{f}(p) \\ {\scriptstyle \varphi_i}\big\downarrow & & \big\downarrow{\scriptstyle \psi_i} \\ (x,h_{ij}(x)\cdot h) & \longrightarrow & (x,g_i(x)\cdot\varphi(h_{ij}(x))\cdot\varphi(h)) \end{array}$$

and since the the whole diagram commutes, it must be that

$$g_i(x) \cdot \varphi(h_{ij}(x)) \cdot \varphi(h) = g_{ij}(x) \cdot g_j(x) \cdot \varphi(h)$$

which by canceling the $\varphi(h)$ on both sides is equivalent to

$$\varphi(h_{ij}(x)) = g_i(x)^{-1} \cdot g_{ij}(x) \cdot g_j(x)$$

and so, over $U_i \cap U_j$,

$$\varphi \circ h_{ij} = g_i^{-1} \cdot g_{ij} \cdot g_j.$$

This does it by **Theorem 1.3.1**.

($\leftarrow$) Now suppose we are given a G-subatlas $\mathscr{A}'$ of the maximal atlas $\mathscr{A}$ for ξ and CAT morphisms $h_{ij} \colon U_i \cap U_j \to H$ on intersections of elements $\mathscr{A}' = \{U_i\}_{i\in I}$ satisfying the cocycle conditions for which we have trivializations $\psi_i \colon p^{-1}(U_i) \to U_i \times G$ such that $\{(\psi_i, U_i)\}_{i\in I} \subset \mathscr{A}$ have transition functions $\varphi \circ h_{ij}$ (these also satisfy the cocycle conditions, clearly). Build a principal H-bundle $\xi' = (P', p, B)$ having transition functions h_{ij} and trivializations φ_i by the fiber bundle construction theorem. We must build a map $\widetilde{f} \colon \xi' \to \xi$ that is a reduction of the structure group. We may suppose that we are working with the G-bundle atlas $\mathscr{A}'$ having transitions $\varphi \circ h_{ij}$ at this point.

One obvious way to do this is to take our cue from the above. Define $f_i \colon \xi'|U_i \to \xi|U_i$ in bundle coordinates by $(x, h) \mapsto (x, \varphi(h))$, $U_i \times H \to U_i \times G$. In other words, $f_i = \psi_i \circ (\mathrm{id}_{U_i} \times \varphi) \circ \varphi_i^{-1}$. By the pasting lemma, this will be CAT if each f_i is CAT and the f_i agree on overlaps. Showing this reduces to show $\varphi(h_{ij}(x))\varphi(h_{ji}(x) \cdot -) = \varphi$ and this follows since

$$\varphi(h_{ij}(x))\varphi(h_{ji}(x)) = \varphi(h_{ij}(x)h_{ji}(x) \cdot -) = \varphi(e \cdot -) = \varphi$$

from the cocycle conditions. So this is indeed well-defined. To see that it is CAT, recall that the smooth structure on ξ' is essentially given by the bundle atlas, and the map f_i in the evident well-chosen bundle coordinates is manifestly smooth. The same argument works, mutatis-mutandis, when we are working in the topological category.

($\Leftrightarrow$) Suppose we start with the data of the h_{ij} and build the bundle morphism above with the refined common trivializing open sets for their atlases. In this step, the new h_{ij}'s are obtained as the transition functions of our principal H-bundle, and thus we recover the h_{ij}'s we started with.

Conversely, if we start with $\widetilde{f} \colon \xi' \to \xi$ as in the hypotheses, after refining the atlases, we obtained the desired h_{ij}. The next step is to build a principal H-bundle from these h_{ij}, but by **(b)** of the lemma above, we must recover the principal H-bundle we started with up to isomorphism. Inspecting the ($\rightarrow$) direction shows that the morphism we recover is likewise the same one. ∎

Proposition 6.6.3. *Suppose $\xi_G = (P_G, p_G, B, G, G) \in \mathsf{Prin}_G$ admits a reduction of the structure group to $H \leq G$ and let $\xi_H = (P_H, p_H, B, H, H) \in \mathsf{Prin}_H$ be such that*

$$\begin{array}{ccc} P_H & \xrightarrow{\widetilde{f}} & P_G \\ \big\downarrow & & \big\downarrow \\ B & = & B \end{array}$$

exhibits this reduction of the structure group.

(a) *$\widetilde{f}$ is H-equivariant.*
(b) *The associated bundle $P_H \times_H G$ is isomorphic to P_G.*

Proof. **(a)** This is essentially part of the definition.

(b) Let $i\colon H \to G$ be the inclusion. As in the proof of **Lemma 6.2.8**, we obtain a G-equivariant map $\widetilde{f} \times_i G\colon P_H \times_H G \to P_G \times_G G \cong P_G$. Note that there is an isomorphism $\xi_G \times_G G \cong \xi_G$. Since $\widetilde{f} \times_i G$ is a G-equivariant map between principal G-bundles over B, it is an isomorphism by **Exercise 13** under the equivalence of Bun_G^G with Prin_G. ■

Remark. By the equivalence between Prin_G and Bun_G^F where $G \curvearrowright F$ is effective, the reduction of structure group for principal G-bundles affords the analogous results in the category Bun_G^F.

6.7 Exercises

Exercise 52. *Let $i\colon H \to G$ be the inclusion of a subgroup. Let $f\colon X \to BG$ be map classifying a principal G-bundle ξ. Show that a reduction of structure group of ξ from G to H (along i) is the same as a map $g\colon X \to BH$ such that the following diagram commutes up to homotopy*

$$\begin{array}{ccc} & & BH \\ & {\scriptstyle g}\nearrow & \downarrow{\scriptstyle B\varphi} \\ X & \xrightarrow[f]{} & BG \end{array}$$

[Hint: In one direction, topological homotopy invariance provides an isomorphism $(B\varphi \circ g)^*\xi_G \cong f^*\xi_G$ where ξ_G is the universal bundle. Build a map $g^*\xi_H \to (B\varphi \circ g)^*\xi_G$ exhibiting a reduction of structure group. In the other direction, write the principal bundles in question as pullbacks of the universal bundles and use universal properties to write the map as induced by maps into EG and B making the evident diagram commute. Show that the map to EG must factor through EH and the map to B must simply be the projection. Verify these procedures are inverse to each other.]

Exercise 53. *Let G and H be topological groups.*

(a) *Show that $B(G \times H) \simeq BG \times BH$. In particular, show that $EG \times EH \to BG \times BH$ is a universal $G \times H$-bundle.* [Hint: The only subtle part is verifying that the product of universal bundles remains numerable. After this, one needs only apply something we've shown before.]
(b) *Given $f\colon X \to BG \times BH$, let f_G and f_H be the components of f and let $\Delta\colon X \to X \times X$ be the diagonal map. Show that f classifies the bundle $\Delta^*(f_G^*EG \times f_H^*EH)$ and give an explicit description of $f^*(EG \times EH)$.* [Hint: Paste pullbacks.]

Exercise 54. *Let $B\rho_{nm}\colon BG(m) \times BG(n) \to BG(m+n)$ be induced from the block sum of matrices $\rho_{mn}\colon G(n) \times G(m) \to G(n+m)$ with identifications as above for our explicit models for classifying spaces.*

(a) *Show that $B\rho_{mn}$ classifies the coinduced principal $G(n+m)$-bundle $\rho_{mn*}(\gamma^n \times \gamma^m)$ over $BG(n) \times BG(m)$, where γ^n and γ^m are the universal $G(n)$ and $G(m)$ bundles, respectively.* [Hint: Appeal to **Lemma 6.2.8(c)**.]
(b) *Given $f\colon X \to BG(n) \times BG(m)$, show that, on the level of vector bundles, the bundle $f^*\rho_{mn*}(\gamma^n \times \gamma^m)$ is the Whitney sum of the vector bundles classified by $\mathrm{pr}_{BG(n)} f$ and $\mathrm{pr}_{BG(m)} f$. In other words, $\rho_{mn*}(\gamma^n \times \gamma^m)$ "classifies" Whitney sums of bundles.* [Hint: Do this in the following slick way: use the pullback description of the Whitney sum and use the preceding exercise.]

Exercise 55. *Let $\otimes_{mn}\colon G(m) \times G(n) \to G(mn)$ be the map sending an $(m \times m)$-matrix $A = (a_{ij})$ and an $(n \times n)$-matrix $B = (b_{ij})$ to their* ***Kronecker product*** *written in block form as*

$$A \otimes_{mn} B = (a_{ij}B)_{i,j}.$$

(a) *Fixing bases, show that $\otimes_{mn}$ is the tensor product $A \otimes B$ of the two linear maps A and B and that $A \otimes B \in G(mn)$.*
(b) *Show that $B\otimes_{mn}$ classifies the coinduced principal $G(mn)$-bundle $\otimes_{mn*}(\gamma^m \times \gamma^n)$ over $BG(n) \times BG(m)$, where γ^{mn} is the universal $G(mn)$ bundles.* [Hint: Appeal to Appeal to **Lemma 6.2.8(c)**.]
(c) *Given $f\colon X \to BG(m) \times BG(n)$, show that, on the level of vector bundles, the bundle $f^* \otimes_{mn*} (\gamma^m \times \gamma^n)$ is the tensor product of the vector bundles classified by $\mathrm{pr}_{BG(m)} f$ and $\mathrm{pr}_{BG(n)} f$. In other words, $\otimes_{mn*}(\gamma^m \times \gamma^n)$ "classifies" tensor products of bundles.* [Hint: Using the definition of the coinduced bundle, show that the transition functions of $\otimes_{mn*}\gamma^{mn}$ are of the form $g_{ij}^m \otimes g_{k\ell}^n$. Conclude that $f^* \otimes_{mn*} (\gamma^{mn})$ has transition functions the tensor products of the transition functions associated to $\mathrm{pr}_{BG(m)} f$ and $\mathrm{pr}_{BG(n)} f$ using the explicit description afforded by the pullback theorem.]

Exercise 56. *More generally, let* $T\colon (\mathsf{fdVect}_{\Bbbk}^{\mathrm{op}})^{\times p} \times \mathsf{fdVect}_{\Bbbk}^{\times q} \to \mathsf{fdVect}_{\Bbbk}$ *be any functor describing functorial operations on finite-dimensional vector spaces over* $\Bbbk = \mathbf{R}$ *or* $\mathbf{C}$*—for instance,* $\mathrm{Hom}(-,-)\colon \mathsf{fdVect}_{\Bbbk}^{\mathrm{op}} \times \mathsf{fdVect}_{\Bbbk} \to \mathsf{fdVect}_{\Bbbk}$*. Note that since* $\Bbbk$ *is a smooth manifold and topological field, the finite dimensional vector spaces comes with a unique smooth structure making them into smooth topological vector spaces. Similarly, the hom-sets of* $\mathsf{fdVect}_{\Bbbk}$ *come equipped with a linear structure over* $\Bbbk$ *and a smooth structure. Let* G *be any one of the classical lie group families defined for* $\Bbbk$*-vector spaces.*

(a) *Call a functor* T *as above a* ***smooth functor*** *(resp.* ***continuous functor****) if the maps it induces on the mapping vector spaces are smooth (resp. continuous) functions. Show that giving such a functor* T *induces maps* $\prod_{i=1}^{p}(G(V_i))^{\mathrm{op}} \times \prod_{i=p+1}^{q} G(V_i) \to G(T(V_1,\ldots,V_{p+q}))$*. (Continuous homomorphisms between Lie groups are always smooth.) Here,* $(G(V_i))^{\mathrm{op}}$ *is the* ***opposite group*** *of* $G(V_i)$*. It has the same underlying space but has its group operation* $\cdot'$ *defined by* $g_1 \cdot' g_2 = g_2 \cdot g_1$ *where* $\cdot$ *indicates the group operation for* $G(V_i)$*.*

(b) *By abuse of notation, call the map so induced on Lie groups* T*. Show that* BT *classifies the coinduced principal bundle* $T_*\gamma$ *where* $\gamma^{\operatorname{cod} T}$ *is the universal principal* $\operatorname{cod} T$*-bundle.* [Hint: Appeal to a previous result in the notes.]

(c) *Let* $\varphi\colon \prod_{i=1}^{p} G(V_i) \to \prod_{i=1}^{p}(G(V_i))^{\mathrm{op}}$ *be the product of the group-isomorphisms* $G(V_i) \to (G(V_i))^{\mathrm{op}}$ *sending* $g \mapsto g^{-1}$*. Recall that* B *commutes with products up to homotopy and let* $f\colon X \to \prod_{i=1}^{p+q} BG(V_i)$ *have components classifying vector bundles* ξ_i*. Show that, on the level of vector bundles*

$$X \to \prod_{i=1}^{p+q} B(G(V_i)) \xrightarrow{B(\varphi\times\mathrm{id})} \prod_{i=1}^{p} B((G(V_i))^{\mathrm{op}}) \times \prod_{i=p+1}^{p+q} BG(V_i) \xrightarrow{BT} BG(T(V_1,\ldots,V_{p+q}))$$

classifies the bundle $\underline{T}(\xi_1,\ldots,\xi_{p+q})$ *of* ***Exercise 28****.* [Hint: Recall that, up to homotopy, B commutes with products. Use the last part of the preceding exercise as a warm-up case.]

The following exercise provides the background necessary for the exercise following it in the symplectic cases.

Exercise 57. *Let* (W,ω) *be a symplectic* $\Bbbk$*-vector space.*

(a) *Say that a* ***symplectic linear map*** $T\colon (V,\omega') \to (W,\omega)$ *between symplectic* $\Bbbk$*-vector spaces is a linear map* T *that preserves the symplectic form:* $\omega' = T^*\omega = \omega(T-,T-)$*. Show that a symplectic linear map is injective and has image a* ***symplectic subspace*** *of* W *(i.e., a subspace* $S \le W$ *such that* $(S, \omega|_S)$ *is a symplectic vector space).*

(b) *Let* $V \le (W,\omega)$ *be a subspace. Define the* ***symplectic complement*** *of* V *to be* $V^\omega \stackrel{def}{=} \{w \in W : \omega(w,v) = 0 \text{ for all } v \in V\}$*. Show that* V *is a symplectic subspace of* W ***iff*** $V \cap V^\omega = 0$*.*

(c) *Let* $V \le (W,\omega)$ *be a subspace. Show that* $\dim V + \dim V^\omega = \dim W$*.*

(d) *Let* $V \le (W,\omega)$ *be a symplectic subspace. Show that* $W = V \oplus V^\omega$ *as symplectic vector spaces. In particular, show that the form on* $V \oplus V^\omega$ *defined by* $\omega|_V \oplus \omega|_{V^\omega}$ *is a symplectic form.*

Remark. In the following exercise, in all parts after **(a)** except for the last, the smooth structure (and thus also topological enrichment) on the domain categories comes from identifying the hom-spaces with Stiefel manifolds. There are complex and real symplectic Stiefel manifolds. The interested reader should be able to work this out themselves.

Exercise 58. *Let* G *stand for one of the classical groups* $\mathrm{GL}, O, SO, U, SU, \mathrm{Sp}$ *or* USp *as defined in* ***6.4.1****. Let* $\Bbbk$ *stand for* $\mathbf{R}$ *or* $\mathbf{C}$*. As usual, we will mean by a complex inner product space a complex vector space with a positive-definite hermitian form.*

(i) *Suppose* $G = \mathrm{GL}$*. Let* C *be the smooth category of finite-dimensional* $\Bbbk$*-vector spaces and linear maps. Show that* $V \mapsto \mathrm{GL}(V)$ *is a continuous functor* $\mathsf{C} \to \mathsf{Top}$ *in a natural way.*

(ii) *Suppose* $G = O$ *(resp.* U*). Let* C *be the smooth category of finite-dimensional real (resp. complex) inner product spaces and linear isometries. Show that* $V \mapsto O(V)$ *(resp.* $U(V)$*) can be made into a continuous functor* $\mathsf{C} \to \mathsf{Top}$ *in a natural way.* [Hint: A linear isometry $i\colon V \to W$ induces a splitting $W = i(V) \oplus i(V)^\perp$. If $X \in G(V)$, $G(i)(X)$ can be defined to act on $i(V)$ alone and in the appropriate way.]

(iii) *Suppose* $G = SO$ *(resp.* SU*). Let* C *be the smooth category of finite-dimensional real (resp. complex) inner product spaces and linear isometries. Show that* $V \mapsto SO(V)$ *(resp.* $SU(V)$*) can be made into a continuous functor* $\mathsf{C} \to \mathsf{Top}$ *in a natural way.* [Hint: The same hint as before applies.]

(iv) *Suppose* $G = \mathrm{Sp}$*. Let* C *be the smooth category of finite-dimensional symplectic* $\Bbbk$*-vector spaces and symplectic linear maps. Show that* $V \mapsto \mathrm{Sp}(V)$ *can be made into a continuous functor* $\mathsf{C} \to \mathsf{Top}$ *in a natural way.* [Hint: Use the preceeding exercise. The preceeding hints then become relevant in an altered but analogous form.]

(v) *Suppose* $G = \mathrm{USp}$*. Let* C *be the category of finite-dimensional* $\mathbb{H}$ *inner product spaces* $\mathbb{H}$*-linear isometries. Show that* $V \mapsto \mathrm{Sp}(V)$ *can be made into a functor* $\mathsf{C} \to \mathsf{Top}$ *in a natural way.* [Remark: The non-commutativity of the quaternions precludes us from making the hom-sets into $\mathbb{H}$-vector spaces so we have disregarded topologies in this part of the exercise.]

(vi) *Show that in all cases, the functors constructed above may be supposed to land in pointed spaces* Top_**.*

Remark. By delooping these functors, one obtains continuous functors $V \mapsto BG(V)$ landing in pointed spaces—some care must be used to show that with some functorial model of B this really can be assumed, but Milnor's model suffices with the choice of basepoint $(0,0,0,\dots : e,e,e,\dots)$. The study of functors such as $V \mapsto BO(V)$ is the study of Michael Weiss' ***orthogonal calculus***.

Exercise 59. *Suppose X admits a basepoint x_0 for which it is well-pointed*[3] *and Y is path-connected.*

(a) *Define an action of $\pi_1(Y, y_0)$ on the set of pointed homotopy classes of maps $[(X, x_0), (Y, y_0)]$.* [Hint: The easiest way to construct this is to use a model categorical blackbox, growing a whisker on X and including it into $X \times I$ and then lifting appropriately. Otherwise, you should look this up rather than think about it.]

(b) *Show that there is an isomorphism*

$$[(X, x_0), (Y, y_0)]/\pi_1(Y) \cong [X, Y]$$

where $[X, Y]$ denotes (free) homotopy classes of maps. [Hint: Use the homotopy extension property for closed Hurewicz cofibrations.]

(c) *Show that the action of $\pi_1(Y)$ on $[(S^n, s_0), (Y, y_0)]$ is the same as the usual action of $\pi_1(Y)$ on $\pi_n(Y)$.*

(d) *For any path-connected topological group G or any topological group such that $\pi_0(G)$ is an abelian group, show that $[S^n, BG] \cong \pi_n(BG)$.*

(e) *Show that $\pi_1 BO(n) \cong \mathbf{Z}/2$ for all $n \geq 1$. Conclude that $[S^n, BO(n)] \cong \pi_n(BO(n))$.* [Hint: It is enough to show $\pi_0 O(n) \cong \mathbf{Z}/2$.]

From **(c)**, it follows that we may understand principal G-bundles (with G path-connected) over CW-complexes by ***obstruction theory***.

Exercise 60. *Consider the attachment of an m-cell to a space X given by the pushout*

$$\begin{array}{ccc} S^{m-1} & \xrightarrow{\Phi} & X \\ \downarrow & & \downarrow \\ D^m & \longrightarrow & X \cup_\Phi D^m \end{array}$$

Suppose $g\colon X \to BG$ classifies a principal G-bundle over X.

(a) *Show that if $f\colon X \cup_\Phi D^m \to BG$ extends g, then there is an isomorphism of principal G-bundles over S^n, $f^*EG|X \cong g^*EG$.*

(b) *Show that such an extension g exists in this toy model* ***iff*** *$[g \circ \Phi] = 0$ in $\pi_{m-1}BG$.*

(c) *Show that if Y is an H****-space***[4] *then Y is a* ***simple space*** *(i.e., the action of $\pi_1(Y)$ on $\pi_n(Y)$ is trivial for all $n \geq 1$).*

(d) *Show that for G a discrete group, BG is weakly equivalent to an H-space.* [Hint: BG is an Eilenberg-Maclane space $K(G, 1)$ and loop spaces are H-spaces.]

(e) *Classify principal $\mathbf{Z}/2$-bundles over S^n for all $n \geq 1$.* [Hint: The only interesting case is $n = 1$.]

Exercise 61. *Let $\xi \in \mathsf{Bun}_G^F$ with $G \curvearrowright F$ effective; denote $E(\xi)$ and $B(\xi)$ the total and base space, respectively. In the course of this proof, we used the following facts.*

(i) *If $\eta \in \mathsf{Bun}_H^T$ and $H^{\times n} \leq G$ then $\eta^{\times n}$ has structure group $H^{\times n} \leq G$.*

(ii) *If ξ admits a reduction of structure group to $H \leq G$, then $\mathsf{P}_G\xi$ admits a reduction of structure group to H.*

(iii) *If ξ admits a reduction of structure group to $H \leq G$, then the classifying map $B(\xi) \to BG$ factors through $Bi\colon BH \to BG$ where $i\colon H \to G$ is the inclusion of the subgroup.* [Hint: By **Lemma 6.2.8** Bi classifies the bundle $EH \times_H G$.]

(iv) *If $\eta \in \mathsf{Bun}_H^T$, $H \curvearrowright T$ effectively and $H^{\times n} \leq G$, then $\eta^{\times n}$, then the classifying map for η as a bundle with structure group G factors through $Bi\colon BH^{\times n} \to BG$ where $i\colon H^{\times n} \to G$ is the inclusion.*

6.8 Proof of Classification Theorem

We need a lemma first.

Lemma 6.8.1. *Let ξ_i be principal G-bundles for $i = 0, 1$ with projections $p\colon P_1 \to B_1$ and $q\colon P_2 \to B_2$.*

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[3] This means that the inclusion $\{x_0\} \hookrightarrow X$ is a closed Hurewicz cofibration. This is always the case for CW-complexes and topological manifolds.

[4] An H-space is a pointed space Y with basepoint $*$ and a product $\mu\colon Y \times Y \to Y$ such that $y \mapsto * \cdot y$ and $y \mapsto y \cdot *$ are both homotopic to the identity map.

(a) *Let X be a right G-space and let $G \curvearrowright X$ by $g \cdot x = x \cdot g^{-1}$. Then $G \curvearrowright X$ is free and faithful if the right action is free and faithful.*

(b) *For X as above, there is a natural bijection*

$$\hom_G(P_1, X) \cong \Gamma(\xi_1 \times_G X)$$

where $\Gamma(\xi_1 \times_G X)$ is the set of continuous sections of the bundle $\xi_1 \times_G X \to B_1$.

(c) *There is a natural bijection*

$$\hom_{\mathsf{Prin}_G}(\xi_0, \xi_1) \cong \Gamma(\xi_0 \times_G P_2)$$

*with action on P_2 as described in **(a)***

Proof. **(a)** This is obvious.

(b) Let us establish the map $\hom_G(P_1, X) \to \Gamma(\xi_1 \times_G X)$. Given a G-equivariant $f\colon P_1 \to X$, the map $(\mathrm{id}, f)\colon P_1 \to P_1 \times X$ is G-equivariant where $(p, x) \cdot g = (p \cdot g, g^{-1} \cdot x) = (p \cdot g, x \cdot g)$. Hence, it descends by passage to G-orbits to a map $P_1/G \to P_1 \times_G X$ and since ξ_1 is a principal G-bundle, $P_1/G \cong B_1$.

Consider the trivial principal G-bundle. Then $\hom_{\mathsf{CAT}_G}(B \times G, X) \cong \hom_{\mathsf{CAT}}(B, X)$ and the bundle $B \times G \times_G X \to B$ is simply the projection $B \times X \to B$ and so sections of this is easily seen to be $\hom_{\mathsf{CAT}}(B, X)$. It is not hard to check these are compatible with passage to G-orbits as indicated above.

Now consider the general case. Let $\{U_i\}_{i \in I}$ be an open cover of B such that $\xi_1^i := \xi_1 | U_i$ is trivial. There is a commutative diagram of equalizers

$$\begin{array}{ccccc}
\hom_{\mathsf{CAT}_G}(P_1, X) & \longrightarrow & \prod_{i\in I} \hom_{\mathsf{CAT}_G}(\xi_1^i, X)G & \rightrightarrows & \prod_{i,j\in I} \hom_{\mathsf{CAT}_G}(\xi_1 | (U_i \cap U_j), X) \\
\downarrow & & \downarrow \cong & & \cong \downarrow \\
\Gamma(\xi_1 \times_G X) & \longrightarrow & \prod_{i\in I} \Gamma(\xi_1^i \times_G X)G & \rightrightarrows & \prod_{i,j\in I} \Gamma(\xi_1 | (U_i \cap U_j) \times_G X)
\end{array}$$

with the isomorphisms following from the trivial principal G-bundle case. It follows that the induced map is an isomorphism. Since the solid vertical arrows are induced by passing to G-orbits as above, one can check that the dashed map is induced in the same way.

(c) This follows since G-equivariant maps between the total spaces of principal G-bundles are precisely the bundle maps. ■

We can now prove the second classification theorem.

*Proof (**Theorem 6.2.2**).* ($\Rightarrow$) Let $E \to B$ be a universal bundle and let $EG \to BG$ be the Milnor construction. Let $f\colon B \to BG$ classify $E \to B$ and let $g\colon BG \to B$ classify $EG \to BG$. Then by pasting pullbacks, $g \circ f\colon B \to B$ necessarily classifies $E \to B$ and therefore $E \to B$ is isomorphic to the bundle $(g \circ f)^* E$. Since $E \to B$ is universal and the identity map also classifies this bundle, $g \circ f \simeq \mathrm{id}_B$. The same reasoning shows that $f \circ g \simeq \mathrm{id}_{BG}$. Let $h\colon B \times I \to B$ be a homotopy from $g \circ f$ to id_B. By the ***homotopy invariance theorem***, we know that the corresponding pullback bundle

$$\begin{array}{ccc}
P & \longrightarrow & E \\
\downarrow & & \downarrow \\
B \times I & \xrightarrow[h]{} & B
\end{array}$$

is $P \cong (P | B \times \{0\}) \times I \cong (P | B \times \{1\}) \times I$ and so is isomorphic to $E \times I$. Similarly for $EG \times I \to EG$. Hence, the two composites of $E \to EG$ and $EG \to E$ are G-equivariantly homotopic to the identity and thus constitute a G-homotopy equivalence. Hence, since $EG \simeq *$, so too do we have $E \simeq *$.

($\Leftarrow$) Let $p\colon E \to B$ be a numerable principal G-bundle with $E \simeq *$ contractible and let $q\colon P \to X$ be another numerable principal G-bundle. By the associated bundle construction, since $E \curvearrowleft G$ is free and faithful (i.e., free and effective), the opposite action $G \curvearrowright E$ given by $g \cdot v = v \cdot g^{-1}$ is free and faithful and so we may "replace" the fiber of the principal G-bundle $q\colon P \to X$ by E itself—in particular, the construction is $P \times_G E \to X$. It is easy to see this bundle remains numerable by the construction of the associated bundle. By the preceding lemma, a section $X \to P \times_G E$ of this bundle is equivalent to providing a morphism $P \to E$. We must show that a section exists.

Since $\widetilde{p}\colon P \times_G E \to X$ is numerable, let $\{\rho_i\}_{i \in \mathbf{N}}$ be a countable locally finite partition of unity such that $W_i = \rho_i^{-1}((0, 1])$ is a trivializing open set for the bundle $P \times_G E \to X$ with trivialization ψ_i. We can further suppose this set is minimal in that by removing any W_i from the cover results in a collection of open sets that does not cover X.

Let $W_i \to \psi_i(\widetilde{p}^{-1}(W_i))$ be any section, then the poset $\mathscr{S}$ of sections with domains $\bigcup_{j \in J \subset \mathbf{N}} W_j$ is non-empty. Taking a union of a chain $\left\{ s_k\colon \bigcup_{j \in J_k} W_j \to P \times_G E \right\}_{k \in K}$ where K is totally ordered, we define $W = \bigcup_{k \in K} \bigcup_{j \in J_k} W_j$ and we define $s\colon W \to P \times_G E$ by $s(w) = s_k(w)$ where $w \in \bigcup_{j \in J_k} W_j$. This is well-defined by the chain condition, W is certainly of the form $\bigcup_{j \in J \subset \mathbf{N}} W_j$ and continuity follows from the pasting lemma.

Hence, by Zorn's lemma, a maximal element of $\mathscr{S}$ exists. Call it (s, W) and suppose $W = \bigcup_{j \in J \subset \mathbf{N}} W_j$. If $W \neq X$, then by our assumption on the partition of unity, $J \neq \mathbf{N}$ and so there is an index $n \in \mathbf{N}$ such that $n \notin J$. Let $f\colon W \cup W_n \to [0,1]$ be the function

$$f(x) = \begin{cases} 1 & \rho_n(x) \leq \sum_{j \in J} \rho_j(x) \\ \dfrac{1}{\rho_n(x)} \sum_{j \in J} \rho_j(x) & \rho_n(x) \geq \sum_{j \in J} \rho_j(x) \end{cases}$$

Note that this is well-defined since $\rho_n(x) \geq \sum_{j \in J} \rho_j(x) \geq 0$ for $x \in W \cup W_n$ means that either $x \in W$ so that $\rho_n(x) > 0$ or $x \in W_n$, so that $\rho_n(x) > 0$. It is continuous and $f(x) > 0$ if and only if $\sum_{j \in J} \rho_j(x) > 0$ so that $W \subset f^{-1}((0,1])$. If we can extend $s|\, f^{-1}(1)$ to a section over $\rho_n^{-1}((0,1])$, call it s', then

$$S(x) = \begin{cases} s(x) & \rho_n(x) \leq \sum_{j \in J} \rho_j(x) \\ s'(x) & \rho_n(x) \geq \sum_{j \in J} \rho_j(x) \end{cases}$$

is continuous by the pasting lemma applied to the closed subsets $(\rho_n - \sum_{j \in J} \rho_j)^{-1}((-\infty, 0])$ and $(\rho_n - \sum_{j \in J} \rho_j)^{-1}([0, \infty))$. This also extends the original section s since for $x \in W$, if $S(x) \neq s(x)$, then $f(x) < 1$ and so $\rho_n(x) > \sum_{j \in J} \rho_j(x) \geq 0$. This will be the contradiction of maximality we seek.

To see that such a section s' exists, notice that $W_n = \rho_n((0,1])$ is a trivializing open set and so it suffices to provide a section of $W_n \times E_1 \to W_n$ extending one defined on a relatively closed subset $f^{-1}(1) \cap W_n$. We already have an extension to the open set $W_0 = f^{-1}((0,1]) \cap W_n$ contained in W_n by the given section s. Let $* \in E$ be a point and $H\colon E \times I \to E$ the contraction to $*$. Define an extension of s by

$$S(x) = \begin{cases} (x, *) & x \in f^{-1}([0, 1/2]) \\ H(s(x), 1 - 2f(x)) & x \in f^{-1}([1/2, 1]). \end{cases}$$

This is continuous by the pasting lemma to the two evident closed sets and noting that when $f(x) = 1/2$ the two pieces agree. ■

Lecture 7 (Bonus)

7.1 Cohomology Computations for Classifying Spaces

7.1.1 Maximal Tori and the Action of the Weyl Group

Let us record the following useful observation.

Lemma 7.1.1. *Let G be one of the compact classical Lie groups. That is, we exclude $\mathrm{GL}_n(\Bbbk)$ and $\mathrm{Sp}_{2n}(\Bbbk)$.*

(a) *G admits a faithful unitary representation $\rho\colon G \hookrightarrow U(m)$ for some large enough m and moreover this is a closed map and embedding.*
(b) *G admits a locally compact Hausdorff classifying space.*
(c) *Let $H \leq G$ be a closed abelian subgroup and $N \trianglelefteq G$ be the normalizer of H in G. Then N is closed in G and thus a Lie group.*
(d) *Let $H \leq G$ be a closed abelian subgroup and $N \trianglelefteq G$ be the normalizer of H in G. There is a model of BH equipped with a continuous right action by $N/H = W(H)$, the* ***Weyl group*** *of H in G.*
(e) *$W(H)$ acts on the right by conjugation in a natural way on H—namely $h \cdot nH = n^{-1}hn$. For this conjugation action, the induced map $BH \to BH$ corresponding to the action of nH is the action given above.*
(f) *With H, N and G as above, let $\pi\colon BH \to BG$ be the usual fiber bundle. Then for all coefficient rings (or modules) R, $\pi^*\colon H^*(BG;R) \to H^*(BH;R)^{N/H}$ where $H^*(BH;R)^{N/H}$ indicates the fixed points under the action of the Weyl group.*

Remark. With some hard work, one consequence of the Peter-Weyl theorem in the representation theory of Lie groups is that *every* compact Lie group satisfies **(a)** and thus all parts of this lemma are true for any compact Lie group.

It is worth pointing out that we may drop the assumption that $H \leq G$ is closed if we give N the discrete topology. In this case each action of an element of n induces a map on classifying spaces with no difficulty and by discreteness of N the map $BH \times N \to BH$ is continuous.

Proof. **(a)** This is basically immediate from how we defined the classical groups along with the observation that there is a homomorphism and embedding $\rho\colon O(n) \hookrightarrow U(n)$ given in the evident way. Note that any group-homomorphism of Lie groups is smooth and an immersion—this latter thing can be seen by observing that such a map always has constant rank. By assumption our map is also injective so this is at least an injective immersion. Since G was assumed compact, the map is also *proper* and this is sufficient for the map to be an embedding of manifolds. The image of G is compact and thus closed by continuity; since all closed subsets of G are compact (and conversely) and since compactness transcends subspace inclusion, $\rho\colon G \to U(m)$ is a closed map.

(b) We have constructed $EU(m)$ as a locally compact Hausdorff space for which $EU(m)/U(m) \cong BU(m)$ is locally compact Hausdorff. We have seen that for $G \leq U(m)$ a subgroup, $EU(m)/G$ is a model for BG in **Lemma 6.2.9** and by this same lemma, $EU(m)/G \to EU(m)/U(m)$ is a fiber bundle with structure group $U(m)$ and fiber $U(m)/G$. Since $G \leq U(m)$ is a closed subgroup by **(a)**, we know that $U(m)/G$ is a smooth manifold and since $U(m)$ is compact it is itself compact. It follows that $EU(m)/G$ is locally of the form $V \times (U(m)/G)$ where $V \subset EU(m)/U(m)$ is open. Since $EU(m)/U(m)$ is Hausdorff, it follows easily that $EU(m)/G$ is Hausdorff.

Local compactness is a more tedious point-set argument. Since every space in sight is Hausdorff, we have access to results making our lives easier. Let $p \in EU(m)/G$ be in the fiber over $q \in EU(m)/U(m)$. Let V be a trivializing open nbhd for q. Since $BU(m) = EU(m)/U(m)$ is locally compact Hausdorff, all open subspaces are locally compact Hausdorff in their subspace topology. Thus, $p \in V \times (U(m)/G)$ a product of locally compact Hausdorff spaces which is therefore locally compact Hausdorff. This means that p has an open nbhd U in $V \times (U(m)/G)$ for which $p \in U \subset K$ with K compact in $V \times (U(m)/G)$. Since $V \times (U(m)/G)$ is (up to isomorphism) an open subspace of $EU(m)/G$, U is open in $EU(m)/G$ and it is a basic fact of topology that compact subspaces transcend subset inclusion; hence, we have exhibited a compact nbhd[1] of p in $EU(m)/G$.

[1] This just means that there is a compact set K and an open set U such that $p \in U \subset K$.

(c) Equivalently, $H \leq G$ is a compact abelian subgroup. We claim that its normalizer is a closed subgroup of G. By Cartan's theorem, this implies that N is in fact a Lie subgroup. We are in a situation where being sequentially closed is equivalent to being closed. Thus, let $p_i \in N$ be a sequence of points in N and suppose $p_i \to p$; we show $p \in N$. Now, since H is closed, it is sequentially closed, so for every $h \in H$, the sequence of points $(p_i^{-1} h p_i)_i$ lie in H and therefore converge to a point of H. On the other hand, $p_i^{-1} h p_i \to p^{-1} h p$, and thus for all $h \in H$, $p^{-1} h p \in H$ so that p is in the normalizer N of H as desired.

(d) We use the model of **(b)**. We have a map $EU(m) \times N \to EU(m)$. Thus, we have a composite

$$\begin{array}{ccc} EU(m) \times N & \longrightarrow & EU(m) \\ & \searrow & \big\downarrow \\ & & EU(m)/H \end{array}$$

and for each $n \in N$, this sends the H-orbit of $e \in EU(m)$ to $\{[ehn] : h \in H\} \subset EU(m)/H$ but when $n = h \in H \subset N$ this sends the H-orbit of $e \in EU(m)$ to $[e]$ a single point. Thus, since $EU(m)$ is locally compact Hausdorff, the map $EU(m) \times N \to EU(m)/H$ descends to $EU(m) \times N/H \to EU(m)/H$.

Since N is the normalizer of H, $Hn = nH$ for all $n \in N$ and so $eHn = enH$ for all $n \in N$. Hence, the H-orbit of e is mapped to $eHn = enH$ and the H-orbit class of enH is the single point en. Since H is closed in G and N is closed in G, $H \leq N$ is closed; hence, we know that the quotient N/H is a smooth manifold and so is locally compact Hausdorff. Hence, the map $EU(m) \times N/H \to EU(m)/H$ descends to the quotient $EU(m)/H \times N/H \to EU(m)/H$ as desired. This is then the assignment $eH \mapsto eHn = enH$ with the equality following from the fact that n is an element of the normalizer of H in G.

(e) In order to make this match up with what we want later, we must give a more contravariant definition of this action. Since N normalizes H and H is abelian, this will not end up mattering.

Define the right action $H \times W(H) \to H$ of $W(H)$ on H by $h \cdot nH = n^{-1} h n$. For any other representative of nH, say ntH with $t \in H$,

$$h \cdot ntT = (nt)^{-1} h (nt)$$

but since N is the normalizer of H and $t \in H$, $nT = Tn$ and so there is some $t_0 \in H$ for which $nt = t_0 n$. Hence,

$$h \cdot ntT = (nt)^{-1} h (nt) = (t_0 n)^{-1} h (t_0 n) = n^{-1} t_0^{-1} h t_0 n = n^{-1} h n$$

since H is abelian so that $t_0^{-1} h t_0 = h$. This shows the association is well-defined. One easily sees this also satisfies the unit and associativity axioms for a right group action. Then for each $nH \in W(H)$, we obtain a map (by abuse of notation)

$$nH \colon H \to H, \qquad h \mapsto n^{-1} h n.$$

This is a group-homomorphism since $nH(hh') = n^{-1}(hh')n = n^{-1} h n n^{-1} h' n = nH(h) nH(h')$. Hence, we can deloop this map to get a map $B(nH) \colon BH \to BH$. Our claim is that $B(nH)$ is homotopic to map $nH \colon BH \to BH$ afforded by the action of nH on the right of BH as described in **(d)**. For this, it suffices by **Lemma 6.2.8(b)** to check that the map nH is induced by passage to quotients by a map $\widetilde{f} \colon EH \to EH$ satisfying $\widetilde{f}(x \cdot h) = \widetilde{f}(x) \cdot nH(h)$ where $nH(h) = n^{-1} h n$. Indeed, in this case we may take the total spaces to be the same—namely $EU(m)$. Define $\widetilde{f}(e) = e \cdot n$. Then $\widetilde{f}(e \cdot n) = e \cdot hn$ and $\widetilde{f}(e) \cdot n^{-1} h n = e \cdot n \cdot n^{-1} h n = e \cdot hn$ and these two expressions are equal. Hence,

$$\widetilde{f}(e \cdot h) = \widetilde{f}(e) \cdot nH(h)$$

as required. Continuity of this map follows from the pasting lemma by working in local trivializing coordinates. Now, the only thing we need to check is that upon passage to quotients $\widetilde{f}$ induces $nH \colon EU(m)/H \to EU(m)/H$. For this, we simply observe that $\widetilde{f}(e \cdot H) = (e \cdot H)n = e \cdot Hn = e \cdot nH$. The equality here follows because $nH = Hn$ for all $n \in N$ by definition of the normalizer. Hence, upon passage to the quotient by H, $\widetilde{f}$ induces the map $eH \mapsto enH$ as desired.

(f) Using the model of $BH = EU(m)/H$ and $BG = EU(m)/G$ from above; the quotient map $\pi \colon EU(m)/H \to EU(m)/G$ is represents the homotopy class of $Bi \colon BH \to BG$ for $i \colon H \to G$ the inclusion. One can check this by observing that we can use the identity map on total spaces $EH = EU(m) = EG$ in

$$\begin{array}{ccc} EU(m) & \xrightarrow{\text{id}} & EU(m) \\ \big\downarrow & & \big\downarrow \\ EU(m)/H & \xrightarrow[\pi]{} & EU(m)/G \end{array}$$

which satisfies $\text{id}(e \cdot h) = \text{id}(e) \cdot i(h) = \text{id}(e) \cdot h$. Upon passage to the quotients, the map thereby induced is precisely π and so by **Lemma 6.2.8(b)** it is the correct model.

We show π^* lands in the fixed point set. Given a class $nH \in N/H$, let $nH\colon BH \to BH$ be the evident automorphism. Then $nH^* \circ \pi^* = (\pi \circ nH)^*$ but by construction $(\pi \circ nH)(eH) = \pi(eHn) = \pi(enH) = enG = eG$ because for any $g \in G$, $gG = G$. On the other hand, $\pi(eH) = eG$. Hence, for all $nH \in N/H$, $nH^* \circ \pi^* = \pi^*$. This means that π^* lands in the fixed-point set of the action of nH on $H^*(BH)$. ■

We can also compute the action of certain Weyl groups on cohomology.

Definition. For G any one of $U(n)$, $SU(n+1)$, $SO(n)$, $SO(2n+1)$ or $\mathrm{Sp}(n)$, the ***standard maximal torus*** T_G of G has dimension n and is defined as follows.

(a) When $G = U(n)$, $T_{U(n)} = U(1)^n \leq U(n)$ identified with the diagonal matrices.
(b) When $G = SU(n+1)$, $T_{SU(n+1)} \leq T_{U(n+1)}$ is the subgroup of matrices with determinant 1.
(c) When $G = SO(2n)$, $T_{SO(2n)}$ is the set of 2×2 block diagonal $2n \times 2n$ matrices with each such matrix being an element of $SO(2)$ (and thus a rotation matrix in any basis).
(d) When $G = SO(2n+1))$, $T_{SO(2n+1)}$ is the image of $T_{SO(2n)} \leq SO(2n+1)$ under the map $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
(e) When $G = \mathrm{Sp}(n)$, $T_{\mathrm{Sp}(n)} = T_{U(n)}$ under the extension of scalars inclusion $U(n) \to \mathrm{Sp}(n)$ corresponding to the natural inclusion $\mathbf{C} \to \mathbb{H}$ into the (real) quaternions.

Exercise 62. *Verify these are maximal tori.* [Hint: For $\mathrm{Sp}(n)$, a bigger torus containing this one cannot be abelian since multiplication in $\mathbb{H}$ is not commutative.]

Definition. The Lie group $O(n)$ is not connected, so it does not make sense to refer to its maximal tori. Nevertheless, we shall abuse notation and denote

$$T_{O(n)} = O(1)^{\times n} \subset O(n)$$

identified with the diagonal matrices of $O(n)$ and call this the "standard maximal torus of $O(n)$."

Warning. This definition of $T_{O(n)}$ as a "standard maximal torus" is non-standard and not even technically correct in light of the definition of maximal tori.

Lemma 7.1.2. *Let W denote the Weyl group of a maximal torus T_G for $G = U(n)$, $SU(n+1)$, $SO(2n)$, $SO(2n+1)$, or $\mathrm{Sp}(n)$. These all have dimension n. For $G = O(n)$, let $T_{O(n)} = O(1)^{\times n}$ identified with the diagonal matrices—this is a discrete Lie subgroup of $O(n)$.*

Then W acts effectively on $T_G \cong T_n = S^{1\times n}$ and admits the following descriptions.

(a) *W acts on $T_{U(n)}$ by permuting coordinates and, in particular, $W \cong \Sigma_n$.*
(b) *W acts on $T_{SU(n+1)})$ by permuting coordinates and, in particular, $W \cong \Sigma_n$.*
(c) *W acts on $T_{SO(2n)}$ by permuting coordinates and performing an even number of sign changes to the coordinates.*
(d) *W acts on $T_{SO(2n+1)}$ by permuting coordinates and changing signs of the coordinates.*
(e) *W acts on $T_{\mathrm{Sp}(n)}$ by permuting coordinates and changing signs of the coordinates.*
(f) *W acts on $T_{O(n)}$ by permuting coordinates and, in particular, $W \cong \Sigma_n$.*

Remark. Effectiveness of the action allows us to identify $W(T_{SO(2n)})$ with the subgroup of $\mathrm{Aut}(T_{SO(2n)})$ generated under composition by the coordinate permutations and an even number of sign changes. Similarly, $W(T_{SO(2n+1)})$ and $W(T_{\mathrm{Sp}(n)})$ can be identified with the respective subgroups of $\mathrm{Aut}(T_G)$ generated by the coordinate permutations and sign changes.

We omit the proof but sketch how it goes. The upshot is that this is mostly a tedious exercise in algebra. Complete details can be found as **Theorem 3.13** of *Topology of Lie Groups, I and II* by Mimura and Toda.

Proof (Sketch). For $G = U(n)$ or $SU(n+1)$, one simply computes the normalizer by hand and show that all coordinate permutations are achieved. For the case of $SU(n+1)$, this may involve changing the sign of the entry of a permutation matrix so that it is, up to that sign, an element of $SU(n+1)$ and so can be treated as a representative of an element in the corresponding Weyl group.

For $G = SO(2n)$ or $SO(2n+1)$, the argument similar but involving a slightly different computation of the normalizer. Namely, one computes that the matrices in the normalizer must have that their 2×2 diagonal blocks A_i (excluding the last entry on the diagonal in the case of $SO(2n+1)$) satisfy for each i one of the following:

- There is a 2×2 special orthogonal (i.e., rotation) matrix B such that $A_i B = B A_i$ then either $A_i = r B_i$ for some $r \in \mathbf{R}$ and $B_i \in SO(2)$ or $A_i = \pm I$.
- If $A_i \neq 0$ and there are 2×2 special orthogonal (i.e., rotation) matrices B_1 and B_2 such that $A_i B_1 = B_2 A_i$, then either $B_1 = B_2$ or $B_1 = B_2^{-1}$.

The case of $SO(2n+1)$ is similar to this computation but involves sticking a 1 onto the end of things.

For $G = \mathrm{Sp}(n)$, one first computes that for $B_1, B_2 \in \mathrm{Sp}(1)$ and $r \in \mathbb{H}$:

- $rB_1 = B_2 r$ with $r \neq 0$ implies $B_1 = B_2$ or $r = \overline{r}$.
- $rB_1 = B_1 r$ implies $r \in \mathbf{C} \subset \mathbb{H}$ or $r = \pm 1$.

As for the last item, we can show this directly. The normalizer of $T_n = T_{O(n)} = O(1)^{\times n}$ in $O(n)$ can easily be computed to be the set of matrices of the form PD where $D \in T_n$ and P is some permutation matrix. This immediately implies the description sought. With some persistence, one can read off from this description that the conjugation action of W on T_n is permutations. //

With this in hand, we can compute the action of the Weyl group $W(T_G)$ on $H^*(T_G)$. The preceding implies that it suffices to determine this action in degree 1.

Corollary 7.1.3. *Let W denote the Weyl group of the standard maximal torus of G where G is any of $U(n)$, $SU(n+1)$, $SO(2n)$, $SO(2n+1)$, $\mathrm{Sp}(n)$ or $O(n)$.*

(a) *$H^*(T_G; \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_1, \ldots, X_n]$ where these generators are degree 1 with X_i the unique element mapping to the preferred generator of $H^1(T^1)$ such that $i_j^*(X_i) = \delta_{ij} X_i$ (by abuse of notation) where $i_j \colon T^1 \to T_G$ is the inclusion of $T^1 = S^1$ onto the j-th factor as a subgroup and δ_{ij} is the Kronecker delta. Furthermore, $H^*(T_G; \mathbf{Z})$ is generated multiplicatively by $H^1(T_G; \mathbf{Z})$.*
(b) *$H^*(T_G; \mathbf{Z}/2) \cong \mathbf{Z}/2[X_1, \ldots, X_n]$ with the X_i as in **(a)**. $H^*(T_G; \mathbf{Z}/2)$ is generated multiplicatively by $H^1(T_G; \mathbf{Z}/2)$.*
(c) *The action of W on $H^1(T_{U(n)}; \mathbf{Z})$ is permutations of the generators. The action is effective.*
(d) *The action of W on $H^1(T_{SU(n+1)}; \mathbf{Z})$ is permutations of the generators. The action is effective.*
(e) *The action of W on $H^1(T_{\mathrm{Sp}(n)}; \mathbf{Z})$ is permutations of the generators and changing the signs of the generators. The action is effective.*
(f) *The action of W on $H^1(T_{SO(2n+1)}; \mathbf{Z})$ is permutations of the generators and changing the signs of the generators. This action is effective.*
(g) *The action of W on $H^1(T_{SO(2n)}; \mathbf{Z})$ is permutations of the generators and changing an even number of signs of the generators. This action is effective.*
(h) *The action of W on $H^1(T_{O(n)}; \mathbf{Z}/2)$ (note the coefficient ring) is permutations of the generators. This action is effective.*
(i) *If $G \neq O(n)$, then $H^*(BT_G; \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_1, \ldots, X_n]$ where $|X_i| = 2$.*
(j) *If $G = O(n)$, then $H^*(BT_{O(n)}; \mathbf{Z}/2) \cong \mathbf{Z}/2[X_1, \ldots, X_n]$ where $|X_i| = 1$.*

Proof. Parts **(c)**–**(h)** follow immediately from the descriptions of the actions given in the preceding lemma and **(a)** and **(b)**.

For **(a)**, the description of the cohomology ring is found via the Künneth isomorphism. As for uniqueness of the element, this follows by inspecting the map i_j^* on the level of the Künneth isomorphism, say by taking $S^1 \cong * \times \cdots \times * \times S^1 \times * \times \cdots \times *$ with S^1 in the j-th spot so that i_j is the inclusion of the relevant subgroup. In this case, we see immediately by naturality properties of cross product map defining the Künneth isomorphism that $i_j^*(X_i) = \delta_{ij} X_i$ so that the element so defined is unique (see **Proposition D.1.6**). It is easy to see that we can pin down the element from here. **(b)** is similarly an application of the Künneth isomorphism, noting that $-1 = 1$ in $\mathbf{Z}/2$. Generation by the cohomology group is immediate from the description of the cohomology rings.

For **(i)**, one simply observes that in this case

$$BT_G \simeq BS^{1 \times n} \simeq (BS^1)^{\times n} \simeq \mathbf{C}P^{\infty \times n}.$$

The cohomology $H^*(\mathbf{C}P^\infty; \mathbf{Z}) = \mathbf{Z}[X]$ with $|X| = 2$, so the Künneth isomorphism applies for $\mathbf{C}P^{\infty \times n}$ and one computes $H^*(BT_G; \mathbf{Z}) \cong \Lambda_{\mathbf{Z}}[X_1, \ldots, X_n]$. Similarly, for **(j)**,

$$BT_{O(n)} \simeq BO(1)^{\times n} = B((S^0)^{\times n}) \simeq (BS^0)^{\times n} \simeq \mathbf{R}P^{\infty \times n}.$$

The cohomology $H^*(\mathbf{R}P^\infty; \mathbf{Z}/2) = \mathbf{Z}/2[X]$ with $|X| = 1$, so the Künneth isomorphism applies for $\mathbf{R}P^{\infty \times n}$ and one computes $H^*(BT_{O(n)}; \mathbf{Z}/2) \cong \mathbf{Z}/2[X_1, \ldots, X_n]$. ■

We can similarly describe the action of $W(T_G)$ on $H^*(BT_G)$. The above implies that it suffices to determine this in degree 2 for $G \neq O(n)$ and in degree 1 for $G = O(n)$.

Corollary 7.1.4. *Let G be any one of $U(n)$, $SU(n+1)$, $SO(2n)$, $SO(2n+1)$, $\mathrm{Sp}(n)$ or $O(n)$ and let T_G be the standard maximal torus for G and let $W = W(T_G)$.*

(a) *The action of W on $H^1(BT_{O(n)}; \mathbf{Z}/2)$ is permutations of the generators. This action is effective.*
(b) *The action of W on $H^2(BT_{U(n)}; \mathbf{Z})$ is permutations of the generators. The action is effective.*
(c) *The action of W on $H^2(BT_{SU(n+1)}; \mathbf{Z})$ is permutations of the generators. The action is effective.*

(d) *The action of W on $H^2(BT_{\mathrm{Sp}(n)};\mathbf{Z})$ is permutations of the generators and changing the signs of the generators. The action is effective.*
(e) *The action of W on $H^2(BT_{SO(2n+1)};\mathbf{Z})$ is permutations of the generators and changing the signs of the generators. This action is effective.*
(f) *The action of W on $H^2(BT_{SO(2n)};\mathbf{Z})$ is permutations of the generators and changing an even number of signs of the generators. This action is effective.*

Proof. **(a)** Write $T = T_{O(n)}$. Model BT as $\mathbf{R}P^{\infty\times n}$ and ET as $S^{\infty\times n}$. Fix a permutation $P \in \Sigma_n \cong W(T)$ and let $\sigma\colon T \to T$ be the action it induces on $T = T_{O(n)}$ by conjugation. The automorphism $\widetilde{f}$ of ET which permutes the factors according to σ satisfies $\widetilde{f}(x) = x\cdot P$ and so $\widetilde{f}(x\cdot A) = x\cdot AP = x\cdot PP^{-1}AP = \widetilde{f}(x)P^{-1}AP = \widetilde{f}\sigma(A)$ where $P^{-1}AP = \sigma(A)$ is the conjugation action of the Weyl group. It follows by **Lemma 6.2.8(b)** that the map induced on quotients is $B\sigma$; we see immediately that $B\sigma$ simply permutes the base space according to P from which the conclusion follows.

The arguments for **(b)**–**(f)** are identical so fix G any of these groups. Write $T = T_G$. Model BT as $\mathbf{C}P^{\infty\times n}$ and ET as $S^{\infty\times n}$. Fix a permutation $P \in \Sigma_n \cong W(T)$ and let $\sigma\colon T \to T$ be the action it induces on $T = T_{O(n)}$ by conjugation. The automorphism $\widetilde{f}$ of ET which permutes the factors according to σ satisfies $\widetilde{f}(x) = x\cdot P$ and so $\widetilde{f}(x\cdot A) = x\cdot AP = x\cdot PP^{-1}AP = \widetilde{f}(x)P^{-1}AP = \widetilde{f}\sigma(A)$ where $P^{-1}AP = \sigma(A)$ is the conjugation action of the Weyl group. It follows by **Lemma 6.2.8(b)** that the map induced on quotients is $B\sigma$; we see immediately that $B\sigma$ simply permutes the base space according to P from which the conclusion follows. ∎

Remark. We now begin the procedure of picking out distinguished generators of $H^*(BO(n);\mathbf{Z}/2)$. To do this, we need to understand the cohomology of $H^*(T_{O(n)};\mathbf{Z}/2)$ and $H^*(T_{O(n)};\mathbf{Z}/2)^{W(T_{O(n)})}$. The computations in this case are much more subtle than for $H^*(BU(n);\mathbf{Z})$, $H^*(SU(n);\mathbf{Z})$ and $H^*(\mathrm{Sp}(n);\mathbf{Z})$ reflecting the more complicated nature of the spectral sequences involved. Morally, this explains the length of the next theorem. Indeed, in the theorem below, the crucial point is that the latter part of **(d)**, part **(e)** and **(f)** all have analogues in the case of $U(n)$ and $\mathrm{Sp}(n)$. All other theorems likewise will have appropriate analogues that are easier to prove.

Definition. The ***i-th elementary symmetric polynomial in n variables*** is

$$\sigma_i(X_1,\dots,X_n) = \sum_{1\le k_1<k_2<\cdots<k_i\le n} X_{k_1}\cdots X_{k_i}.$$

These polynomials are invariant under the evident right Σ_n (the n-th symmetric group) action permuting the coordinates and they generate all polynomials invariant under the Σ_n action. It is sometimes convenient (e.g., when we are dealing with graded things) to declare

$$\sigma_0 = 1.$$

Lemma 7.1.5. *Fix a commutative ring R. All symmetric polynomials in the indeterminants $X_1,\dots,X_n$ are uniquely expressible as a polynomial over R in the elementary symmetric polynomials. In other words, the map $R[\sigma_1,\dots,\sigma_n] \to \mathrm{SymPoly}[X_1,\dots,X_n]$ mapping $\sigma_i \mapsto \sigma_i(X_1,\dots,X_n)$ is an isomorphism of rings.*

Proof. This is a simple algebra exercise. ∎

Warning. Note that we use the same notation for the quotient by a subgroup and the quotient by a subspace. In the arguments below, one must interpret which one is meant correctly. Whenever there is a quotient by a subgroup, we mean the orbit space and note the quotient by the subspace.

Theorem 7.1.6. *Let $T_n = O(1)^{\times n} \le O(n)$ identified with the diagonal matrices, let $ST_n = T_n \cap SO(n)$ and let $j\colon T_n \to O(n)$ and $Sj\colon ST_n \to SO(n)$ be the evident inclusions. Let $i\colon ST_n \to T_n$ be the also evident inclusion. All cohomology is taken with $\mathbf{Z}/2$ coefficients.*

(a) *There are isomorphisms of Lie groups $ST_n \cong T_{n-1}$ and $SO(n)/ST_n \cong O(n)/T_n$.*
(b) *$H^*(BT_n) \cong \mathbf{Z}/2[X_1,\dots,X_n]$ where $|X_i| = 1$. The maps $i^*\colon H^*(T_n) \to H^*(ST_n)$ and $Bi^*\colon H^*(BT_n) \to H^*(BST_n)$ have kernel generated by $X_1+\cdots+X_n$. Furthermore, $O(n)/T_n$ is path-connected.*
(c) *Let $n \ge 2$. If $i\colon SO(n-1) \to SO(n)$ is the inclusion $A \mapsto \begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix}$, then $Bi^*\colon \mathbf{Z}/2[w_2,\dots,w_n] \to \mathbf{Z}/2[w_2,\dots,w_{n-1}]$ is surjective and for $n \ge 3$ it maps $w_n \mapsto 0$.*
(d) *$H^*(O(n)/T_n)$ is generated multiplicatively by $H^1(O(n)/T_n)$, $\dim_{\mathbf{Z}/2} H^1(O(n)/T_n) = n-1$ and the natural map $O(n)/T_n \to O(n+1)/T_{n+1}$ induces a surjection on cohomology.*
(e) *From the fiber inclusion of the fiber bundle $O(n)/T_n \to BT_n \to BO(n)$, the map $H^*(BT_n) \to H^*(O(n)/T_n)$ is surjective.*

(f) *There is an isomorphism $H^*(BT_n)^{W(T_n)} \cong \mathbf{Z}/2[\sigma_1, \dots, \sigma_n]$ where σ_i is the i-th elementary symmetric polynomial on the $X_1, \dots, X_n$ with $|\sigma_i| = i$. Moreover, $Bj^*\colon H^*(BO(n)) \to H^*(BT_n)$ is injective and, in particular, induces an isomorphism $H^*(BO(n)) \cong H^*(BT_n)^{W(T_n)}$.*

(g) *There are unique classes $w_i \in H^i(BO(n))$ for each $i \geq 0$ satisfying $Bj^* \sum w_i = \prod(1 + X_i) = \sum \sigma_i$ and such that $H^*(BO(n)) \cong \mathbf{Z}/2[w_1, \dots, w_n]$. In particular, w_i is the unique element mapping to $\sigma_i(X_1, \dots, X_n)$ where, by convention, we take $\sigma_0 = 1$.*

Proof. **(a)** The first follows since an element of ST_n is completely determined by the first $(n-1)$ elements on the diagonal as the determinant must be $+1$. This is for the homeomorphism at least. As for the diffeomorphism, observe that $\#(ST_n) = 2^{n-1}$ and every element of ST_n has order 2. The classification of finitely generated abelian groups immediately implies that, as a discrete group, $ST_n \cong (\mathbf{Z}/2)^{n-1}$. Since ST_n is a discrete *Lie group*, this immediately implies the group-isomorphism is a smooth isomorphism. Alternatively, every continuous group-homomorphism of Lie groups is smooth.

As for the latter, let $SO(n)$ act on $O(n)$ by left multiplication and consider the composite map $SO(n) \times O(n) \to O(n) \to O(n)/T_n$. If $T^1, T^2 \in T_n$ and $X \in O(n)$, then the action of $A \in SO(n)$ is AXT^1 and AXT^2 and these remain in the same orbit. Since everything in sight is locally compact Hausdorff, the action descends to an action $SO(n) \times O(n)/T_n \to O(n)/T_n$; this action is now transitive as well since modding out by T_n collapses the two components of $O(n)$. Since everything in sight is compact and smooth and since the stabilizer of this action at any point is isomorphic to SZ_n and the action is transitive, the quotient manifold theorem applies and we have an isomorphism $SO(n)/ST_n \cong O(n)/T_n$ by sending a class $A \cdot sZ_n \mapsto A \cdot T_n$, where we take this to be the stabilizer at the element $[T_n] \in O(n)/T_n$. This is an isomorphism of Lie groups because $T_n \leq O(n)$ and $ST_n \leq SO(n)$ are closed normal subgroups, being contained in the centers of each of these Lie groups.

(b) **(You will be asked to fill in the missing details and argument as an exercise.)** The first part is a direct application of the Künneth theorem with the observation that $B(O(1)^{\times n}) \simeq (BO(1))^{\times n} \simeq (\mathbf{R}P^\infty)^{\times n}$. As for the latter, note that by **(a)** $H^*(BST_n) \cong \mathbf{Z}/2[Y_1, \dots, Y_{n-1}]$. Up to isomorphism, we may consider i as a smooth embedding and homomorphism $i\colon ST_n \cong T_{n-1} \cong O(1)^{\times n-1} \to O(1)^{\times n} \cong T_n$. Such a map is determined by its n projections $i_k\colon O(1)^{\times n-1} \to O(1)$. The inclusion of this torus is easily seen to be the subset $\left\{(t_1, \dots, t_n) \in T_{O(n)} = T_n : \prod t_i = 1\right\}$. Thus, $ST_n \cong T_{n-1} \subset T_n$ by allowing the last entry t_n of ST_n to be such that $\prod_{i=1}^{n-1} t_i = t_n$, which is a group-homomorphism. Using **(b)** of **Corollary 7.1.3**, with sufficient determination using the identification of ST_n as $T_{n-1} \subset U(n-1)$ but appended with last entry t_n such that the determinant is 1, one can show that, on the level of cohomology, $X_n \mapsto \sum_{j=1}^{n-1} -X_i$ whereas for $1 \leq i \leq n$, $X_i \mapsto X_i$, whence the description of the kernel. The same analysis furnishes the description of Bi^* with a little extra work.

There is still another way to see this for Bi. The map Bi is easily seen to be the map $(\mathbf{R}P^\infty)^{\times n-1} \to (\mathbf{R}P^\infty)^{\times n}$ for which $(\mathbf{R}P^\infty)^{\times n-1} \to \mathbf{R}P^\infty$ is the unique map, up to homotopy, which on π_1 looks like i_k—indeed, it is a basic fact that for any abelian group A, the homotopy classes of maps $[K(A,n), K(A,n)] \cong \mathrm{End}_{\mathsf{Ab}}(A, A)$ for any $n \geq 0$. In particular, since $(\mathbf{R}P)^{\infty \times n}$ is a $K(\mathbf{Z}/2^n, 1)$, we see that this is well-defined up to homotopy (so long as we restrict to CW-complex models which is not a problem in this setting.) The result now follows from naturality of the Hurewicz theorem and the universal coefficient theorem, since we may identify by natural isomorphism

$$H^1((\mathbf{R}P^\infty)^{\times n}) \cong \mathrm{Hom}_{\mathbf{Z}/2}(H_1((\mathbf{R}P^\infty)^{\times n}), \mathbf{Z}/2) \cong \mathrm{Hom}_{\mathbf{Z}/2}(\pi_1((\mathbf{R}P^\infty)^{\times n}), \mathbf{Z}/2)$$

so that the map on cohomology is given in degree 1 by (written suggestively)

$$\pi_1(Bi)^*\colon \mathrm{Hom}_{\mathbf{Z}/2}(\pi_1((\mathbf{R}P^\infty)^{\times n}), \mathbf{Z}/2) \to \mathrm{Hom}_{\mathbf{Z}/2}(\pi_1((\mathbf{R}P^\infty)^{\times n-1}), \mathbf{Z}/2)$$

and it only remains to check that the map $\pi_1((\mathbf{R}P^\infty)^{\times n-1}) \to \mathbf{Z}/2$ sending $(1, \dots, 1) \mapsto 1$ pre-composed with $\pi_1(Bi)$ is 0 on homotopy groups and this follows by careful inspection of the map Bi.

(c) **(This is effectively Theorem 6.5.2 but we reproduce the argument for completeness.)** case of $n = 2$ corresponds to $* \to BSO(2)$ and so on cohomology is trivial so we may assume $n \geq 3$. In this case we have a spherical fibration $SO(n)/SO(n-1) \cong S^{n-1} \to BSO(n-1) \to BSO(n)$. The local coefficient system for this fibration is trivial, either because $BSO(n)$ is 1-connected or because we are considering $\mathbf{Z}/2$ coefficients and there is only one automorphism of $H^*(S^n)$ with such coefficients. Hence, we have access to the Gysin sequence

$$\cdots \to H^i(BSO(n-1)) \xrightarrow{\delta} H^{i-(n-1)}(BSO(n)) \xrightarrow{\gamma \smile -} H^{i+1}(BSO(n)) \xrightarrow{Bi^*} H^{i+1}(BSO(n-1)) \to \cdots$$

When $i \leq n-2$, $H^i(BSO(n-1)) \xrightarrow{\delta} H^{i-(n-1)}(BSO(n))$ is zero because the target is 0 and we see that $Bi^*\colon H^i(BSO(n)) \to H^i(BSO(n-1))$ is an isomorphism in this range. Now consider the case of $i = n-1$. We at least have the following exact sequence

$$H^{n-2-(n-1)}(BSO(n)) = 0 \to H^{n-1}(BSO(n)) \xrightarrow{Bi^*} H^{n-1}(BSO(n-1)) \xrightarrow{\delta} \mathbf{Z} \xrightarrow{\gamma \smile -} H^n(BSO(n)) \xrightarrow{Bi^*} H^n(BSO(n-1))$$

so that Bi^* is injective in degree n with image the kernel of δ. From our knowledge of the cohomology ring of $BSO(n)$ with $\mathbf{Z}/2$ coefficients, we know that cupping with any degree 0 element (i.e., the $\mathbf{Z}$) is injective. Hence, $\delta = 0$ and so Bi^* is surjective as well and is thus an isomorphism in degree $n-1$. From this, it follows that we have a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\gamma\smile -} H^n(BSO(n)) \to H^n(BSO(n-1)) \to 0$$

Here, $H^n(BSO(n)) \to H^n(BSO(n-1))$ surjects because $Bi^*\colon H^*(BSO(n)) \to H^*(BSO(n-1))$ has already been shown to hit all generators of the cohomology of the codomain. Since Bi^* is a ring map, it must be that the generator $w_2 \in H^*(BSO(n))$ maps to the generator $w_2 \in H^*(BSO(n-1))$, up to a sign. Hence, $\gamma \in H^n(BSO(n))$ is not a polynomial in the $w_1, \ldots, w_{n-1}$ and therefore $\gamma = w_n$ by degree considerations. This shows everything we wanted to in this part.

(d) Our claim is that $H^*(O(n)/T_n)$ is generated under cup product by the elements of $H^1(O(n)/T_n)$ and $\dim_{\mathbf{Z}/2} H^1(O(n)/T_n) = n-1$. Since $O(1)/T_1 = *$, the case of $n=1$ is trivial. The case of $n=2$ is likewise trivial since $O(2)/T_2 = O(2)/(O(1)\times O(1)) \cong \mathbf{R}P^1 \cong S^1$ and the assertion is certainly true for $H^1(S^1)$. **Thus, WLOG we shall suppose $n \geq 3$ from now on.**

The idea is to proceed by induction and find a fiber sequence relating $O(n)/T_n$ and $O(n-1)/T_{n-1}$. By **Corollary 6.2.10**, for each $n \geq 2$, we have with $K = T_n$, $H = O(n-1)\times \mathbf{Z}/2$ and $G = O(n)$ a fiber bundle with structure group $O(n)\times \mathbf{Z}/2$ given by

$$H/K \cong O(n-1)/T_{n-1} \xrightarrow{i} O(n)/T_n \to O(n)/(O(n-1)\times \mathbf{Z}/2) \cong \mathbf{R}P^{n-1}.$$

When $n=2$, this bundle is simply the identity map on $\mathbf{R}P^1 \cong S^1$ and so it is only interesting for $n \geq 3$ anyways.

To run the Serre spectral sequence for this fiber sequence, we should hope that its local coefficient system is trivial. The indicated bundle satisfies the hypotheses for **Theorem F.1.5(e)** except possibly the condition that i^* is surjective, but we have good reason to expect i^* to surject on cohomology since, among other things, by induction hypothesis $H^*(O(n-1)/T_{n-1})$ is generated in degree 1 and so it suffices to that i^* is surjective in degree 1.

To show i^* is surjective, we will consider auxiliary fiber sequences. Recall that we have computed $H^*(BSO(n))$ and from **(a)** and **(b)** we know $H^*(ST_n)$. We also know that $\pi_1 BSO(n) = \pi_0 BSO(n) = *$. The following morphism of fiber sequences which we label $(*)$ will be useful towards this end.

$$\begin{array}{ccccc} SO(n)/ST_n & \longrightarrow & BST_n & \longrightarrow & BSO(n) \\ \psi\uparrow & & \uparrow & & \uparrow \\ SO(n-1)/ST_{n-1} & \longrightarrow & BST_{n-1} & \longrightarrow & BSO(n-1) \end{array} \qquad (*)$$

The right-most square is B of the underlying commutative diagram of Lie groups with vertical maps the evident compatible inclusions—namely $ST_{n-1} \to ST_n$ and $SO(n-1) \to SO(n)$ pad the matrices by a 1 in the last diagonal spot and 0s elsewhere. In particular, we shall use the following explicit model furnishing a fiberwise map $BST_n \to BT_n$. Interpreting $ST_{n-1} \leq ST_n$ and $SO(n-1) \leq SO(n)$ compatibly and in the evident way as the inclusion

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

we may appeal to **Theorem 6.2.9** to write the right-hand square as

$$\begin{array}{ccc} ESO(n)/ST_n & \longrightarrow & ESO(n)/SO(n) \\ \uparrow & & \uparrow \\ ESO(n)/ST_{n-1} & \longrightarrow & ESO(n)/SO(n-1) \end{array}$$

where the quotients mean the quotient by the right-action and the displayed maps are all induced by the relevant subgroup inclusions.

Claim 18. The natural isomorphisms $SO(n)/ST_n \to O(n)/T_n$ of **(b)** make the following diagram commute

$$\begin{array}{ccc} SO(n)/ST_n & \xrightarrow{\cong} & O(n)/T_n \\ \psi\uparrow & & \uparrow i \\ SO(n-1)/ST_{n-1} & \xrightarrow[\cong]{} & O(n-1)/T_{n-1} \end{array}$$

where ψ is as in $(*)$. Thus, to show i^* surjects, it suffices to show ψ^* surjects.

To see this, observe that ψ, in our explicit model for $(*)$ is induced fiberwise by restricting the evident map $SO(n)/ST_{n-1} \to SO(n)/ST_n$ (which is itself induced by including $ST_{n-1} \hookrightarrow ST_n$) to the subspace $SO(n-1)/ST_{n-1}$. The effect is that ψ is the map $SO(n-1)/ST_{n-1} \to SO(n)/ST_n$ induced by the compatible subgroup identifications $SO(n-1) \hookrightarrow SO(n)$ and $ST_{n-1} \hookrightarrow ST_n$. Since all other maps in the above diagram are induced in the same way, it is then clear the diagram commutes and the latter part of the claim follows easily. //

Now, since the base spaces of the bundles in $(*)$ are simply connected, the local coefficient systems are trivial. It is easy to see that the other vertical maps in $(*)$ are all surjective on cohomology. To show ψ^* is surjective, we will utilize this along with naturality of the Serre spectral sequence in $(*)$.

Running the Serre spectral sequence for these bundles,

$$O(m)/T_m \cong SO(m)/ST_m \longrightarrow BST_n \longrightarrow BSO(n)$$

observe that we can compute part of the E_2 page (using the UCT) in a generic case as

3	$H^3(SO(m)/ST_m)$	0	$H^3(SO(m)/ST_m)$	$H^3(SO(m)/ST_m)$		
2	$H^2(SO(m)/ST_m)$	0	$H^2(SO(m)/ST_m)$	$H^2(SO(m)/ST_m)$		
1	$H^1(SO(m)/ST_m)$	0	$H^1(SO(m)/ST_m)$	$H^1(SO(m)/ST_m)$	$\bigoplus_m \mathbf{Z}/2w_2^2 \oplus \mathbf{Z}/2w_4$	
0	$\mathbf{Z}/2$	0	$\mathbf{Z}/2w_2$	$\mathbf{Z}/2w_3$	$\mathbf{Z}/2w_2^2 \oplus \mathbf{Z}/2w_4$	
	0	1	2	3	4	5

The displayed differential is the transgression $\tau = d_2$ and is either 0 or surjective. Since $SO(m)/ST_m \cong O(m)/T_m$, we have the following general deduction.

For $n \geq 2$, $\dim_{\mathbf{Z}/2} H^1(O(n)/T_n) = n-1$ (resp. n) if the displayed differential is 0 (resp. surjective).

Thus, from our induction hypothesis, we know that the differential $H^1(SO(n-1)/ST_{n-1}) \to \mathbf{Z}/2w_2$ is 0. By naturality of the Serre spectral sequence applied to $(*)$ and the isomorphism in second cohomology of the base spaces involved, this forces the displayed differential $\tau = d_2$ to be 0 as well because TFDC with the right vertical arrow an isomorphism:

$$\begin{array}{ccc} H^1(SO(n)/ST_n) & \xrightarrow{d_2} & \mathbf{Z}/2w_2 \\ \downarrow \psi^* & & \downarrow \cong \\ H^1(SO(n-1)/ST_{n-1}) & \xrightarrow[0]{} & \mathbf{Z}/2w_2 \end{array}$$

The natural map $BSO(n-1) \to BSO(n)$ induces a surjection on cohomology in degree 2 when $n \geq 2$ and thus by dimensional considerations an isomorphism—this is essentially part of **Theorem 6.5.2**. If $d_2 \neq 0$ in this diagram, then the only way it commutes is either if $\tau = 0$ or $H^1(O(n)/T_n) = 0$. The latter is precluded by the Serre exact sequence for this bundle which has initial segment reading as

$$0 \to \underbrace{H^1(BSO(n))}_{=0} \to H^1(BST_n) \to H^1(O(n)/T_n)$$

which shows that $H^1(O(n)/T_n)$ at least contains a copy of $H^1(BST_n)$ which is non-zero for all $n \geq 2$. Hence, we have shown $\dim_{\mathbf{Z}/2} H^1(O(n)/T_n) = n-1$ for all n and it only remains to show that $H^1(O(n)/T_n)$ generates $H^*(O(n)/T_n)$ multiplicatively. For this, we must still show that ψ^* (hence, i^*) is surjective.

For the two spectral sequences arising from $(*)$, we now have that the transgression on the E_2 page is zero. This means that $E_2^{0,1} = E_\infty^{0,1}$ and $E_2^{1,0} = E_\infty^{1,0}$ in both spectral sequences and, hence, that the inclusion of the fiber $O(m)/T_m \to BST_m$ induces an isomorphism $H^1(O(m)/T_m) \cong H^1(BST_m)$ for $m = n-1$ and $m = n$. In particular, by consideration of convergence, it suffices to show that $H^1(BST_n) \to H^1(BST_{n-1})$ surjects as this will imply that $\psi^* \colon H^1(O(n)/T_n) \to H^1(O(n-1)/T_{n-1})$ surjects (and thus i^* surjects by the claim).

Claim 19. For $n \geq 3$, the inclusion $ST_{n-1} \to ST_n$ deloops to a map which is surjective on first cohomology $H^1(BST_n) \to H^1(BST_{n-1})$.

The map $ST_{n-1} \to ST_n$, under the isomorphism $ST_m \cong T_{m-1}$ may be identified with the inclusion $T_{n-2} \to T_{n-1}$ identifying T_{n-2} with the first $n-2$ factors of $T_{n-1} = O(1)^{\times n-1}$. This map deloops to same inclusion of factors $(\mathbf{R}P^\infty)^{\times n-2} \to (\mathbf{R}P^\infty)^{\times n-1}$. By writing $(\mathbf{R}P^\infty)^{\times n-2} \times * \to (\mathbf{R}P^\infty)^{\times n-1}$, it follows by naturality of the Künneth isomorphism (i.e., naturality in the Künneth theorem) that this map, upon passage to cohomology, is surjective.

This shows that ψ^* and thus i^* are surjective. By **Theorem F.1.5(e)**, this means the local coefficient system for the bundle $O(n-1)/T_{n-1} \xrightarrow{i} O(n)/T_n \to O(n)/(O(n-1) \times \mathbf{Z}/2) \cong \mathbf{R}P^{n-1}$ is trivial and so we have access to the Serre spectral sequence in its usual form. Recall that $H^*(\mathbf{R}P^{n-1}) \cong \mathbf{Z}/2[X]/(X^n)$ with $|X| = 1$. Letting $Q = O(n-1)/T_{n-1}$, this takes the form

3	$H^3(Q)$	$H^3(Q)$	$H^3(Q)$	$H^3(Q)$	$H^3(Q)$	
2	$H^2(Q)$	$H^2(Q)$	$H^2(Q)$	$H^2(Q)$	$H^2(Q)$	
1	$H^1(Q)$	$H^1(Q)$	$H^1(Q)$	$H^1(Q)$	$H^1(Q)$	
0	$\mathbf{Z}/2$	$\mathbf{Z}/2X$	$\mathbf{Z}/2X^2$	$\mathbf{Z}/2X^3$	$\mathbf{Z}/2X^4$	
	0	1	2	3	4	5

By **Theorem F.1.5(e)** this spectral sequence collapses so $E_2 = E_\infty$. By **Theorem F.1.4(iii)**, the cup product pairing on this page is identified with the pairing on the filtration quotients of $H^*(O(n)/T_n)$. But more is in fact true.

To see what more is true, note that by induction hypothesis, $H^*(O(n-1)/T_{n-1})$ is generated multiplicatively by $H^1(O(n-1)/T_{n-1})$ so on the E_2 page it suffices to compute the displayed differential. But this differential must be 0 by convergence considerations! Indeed, we have seen that $H^1(O(n-1)/T_{n-1}) \cong \mathbf{Z}/2^{\oplus n-1}$ and that $H^1(O(n)/T_n) \cong \mathbf{Z}/2^{\oplus n}$ so if this differential were non-zero, we would be forced to conclude that $\mathbf{Z}/2 \oplus \mathbf{Z}/2^{\oplus n-3} \cong H^1(O(n)/T_n)$ which is impossible by dimension considerations. This implies that all differentials starting at the first column $E_2^{0,q} \to E_2^{2,q-1}$ are zero. But by **Theorem F.1.5(c)** the cup product structure on the E_2 page of this spectral sequence induces an isomorphism

$$E_2^{p,0} \otimes_{\mathbf{Z}/2} E_2^{0,q} \cong H^p(\mathbf{R}P^{n-1}) \otimes_{\mathbf{Z}/2} H^q(O(n-1)/T_{n-1}) \xrightarrow{\cong} E_2^{p,q} \cong H^q(O(n-1)/T_{n-1})$$

and, of course, for $0 \le p \le n-1$, $H^p(\mathbf{R}P^{n-1}) \cong \mathbf{Z}/2$. The Leibniz rule for this cup product structure states for $\psi \in E_2^{p,0}$ and $\varphi \in E_2^{0,q}$ that

$$d_2(\psi \cdot \varphi) = d_2(\psi) \cdot \varphi + (-1)^{0+q} d_2(\varphi).$$

Since we are working with $\mathbf{Z}/2$ coefficients, the power of (-1) is irrelevant. Since $d_2(\psi) = 0$ for all p and by what we have just shown $d_2(\varphi) = 0$ for all q, this implies that $d_2(\psi \cdot \varphi) = 0$ for all $\psi \in E_2^{p,0}$ and for all $\varphi \in E_2^{0,q}$. This provides another proof that the spectral sequence collapses and, in particular, gives an isomorphism of bigraded R-algebras

$$E_\infty \cong \bigoplus_{p,q} H^p(\mathbf{R}P^{n-1}) \otimes_{\mathbf{Z}/2} H^q(O(n-1)/T_{n-1})$$

by **Theorem F.1.4(ii)** and **Theorem F.1.4(iv)**.

Putting this all together and unpacking definitions, some thought shows that $H^1(O(n)/T_n)$ generates $H^*(O(n)/T_n)$ multiplicatively. To understand the notation of the following explanation, refer to **Theorem F.1.4**.

Under the identification $E_\infty \cong \bigoplus_{p,q} H^p(\mathbf{R}P^{n-1}) \otimes_{\mathbf{Z}/2} H^q(O(n-1)/T_{n-1})$, the first filtration piece of $H^k(O(n)/T_n)$, namely $F_k^k \cong H^k(O(n-1)/T_{n-1})$, is generated by $F_1^1 = H^1(O(n-1)/T_{n-1})$ under the product structure. Now we claim that the second filtration pieces F_{k-1}^k are contained in the subring generated by $H^1(O(n)/T_n)$. Consider, for example, $F_1^2 \cong H^1(O(n-1)/T_{n-1})$. Up to addition with an element of F_2^2 (which is itself generated by $F_1^1 \subset H^1(O(n)/T_n)$ multiplicatively), the isomorphism

$$E_\infty^{0,1} \otimes_{\mathbf{Z}/2} E_\infty^{1,0} = F_1^1 \otimes_{\mathbf{Z}/2} F_0^1/F_1^1 \cong H^1(\mathbf{R}P^{n-1}) \otimes_{\mathbf{Z}/2} H^1(O(n-1)/T_{n-1}) \xrightarrow{\cong} H^2(O(n-1)/T_{n-1}) \cong F_1^2/F_2^2 = E_\infty^{1,1}$$

coming from the cup product structure says that, up to addition by elements of F_2^2 the product $F_1^1 \otimes F_1^2 \to F_1^2$ surjects, but F_2^2 is multiplicatively generated by F_1^1 as we have just seen. Putting this together, we deduce that $F_1^2 \subset H^2(O(n)/T_n)$ is contained in the subring multiplicatively generated by $H^1(O(n)/T_n)$. Since $H^1(\mathbf{R}P^{n-1})$ generates $H^*(\mathbf{R}P^{n-1})$ multiplicatively, the same reasoning allows us to deduce that the same conclusion is true for $F_0^2 = H^2(O(n)/T_n)$. This pattern of reasoning repeats along each filtration sequence of $H^k(O(n)/T_n)$ and so we may conclude.

(e) This constitutes a refinement of the above. Note that the cases of $n = 1, 2$ are trivial. We have a bundle $O(n)/T_n \xrightarrow{i} BT_n \xrightarrow{Bj} BO(n)$ with structure group $O(n)$ which is not connected. However, the local coefficient system with $\mathbf{Z}/2$ coefficients is still trivial because $O(n)/T_n$ is path-connected and, as we shall show, i^* is surjective. The bundle $O(n)/T_n \xrightarrow{\kappa} BST_n \to BSO(n)$ has connected structure group $SO(n)$ with path-connected base and fiber and hence has a trivial local coefficient system. Hence, we may use the piece of the Serre exact sequence

$$0 \to \underbrace{H^1(BSO(n))}_{=0} \to H^1(BST_n) \to H^1(O(n)/T_n)$$

showing that $H^1(BST_n) \to H^1(O(n)/T_n)$ is injective. We also have a map of fiber sequences

$$\begin{array}{ccccc} O(n)/T_n & \xrightarrow{\kappa} & BST_n & \longrightarrow & BSO(n) \\ {\scriptstyle\phi}\downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow \\ O(n)/T_n & \xrightarrow[i]{} & BT_n & \longrightarrow & BO(n) \end{array}$$

where the map ϕ is an isomorphism because it is in fact the natural map $SO(n)/ST_n \to O(n)/T_n$ which we have seen is isomorphism—one can make this precise by setting for $H \leq O(n)$ a subgroup $BH = EO(n)/H$ in the diagram above to build the right-hand square of fiber bundles.

Now, we have seen that $H^*(BT_n) \to H^*(BST_n)$ surjects in **(b)**. We thus have the following commutative diagram involving the Serre exact sequence of the bundle $O(n)/T_n \to BST_n \to BSO(n)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(SO(n)) = 0 & \longrightarrow & H^1(BST_n) & \rightarrowtail & H^1(O(n)/T_n) \\ & & & & \uparrow\!\uparrow & \nearrow{\scriptstyle \phi^* i^*} & \\ & & & & H^1(BT_n) & & \end{array}$$

and dimensional considerations force the injective map $H^1(BST_n) \to H^1(O(n)/T_n$ to be an isomorphism (recall we are working with field coefficients $\mathbf{Z}/2$). Hence, the map $\phi^* i^*$ is surjective. But ϕ^* is an isomorphism, so i^* is surjective.

(f) From the preceding lemma, we know that $Bj^* \colon H^*(BO(n)) \to H^*(BT_n)$ lands in the subring $H^*(BT_n)^{W_{O(n)}(T_n)}$ where $W_{O(n)}(T_n) = N_{O(n)}(T_n)/T_n$ is the Weyl group. From what we just saw above, $H^*(BT_n) \to H^*(O(n)/T_n)$ is surjective, so the spectral sequence collapses and thus by **Theorem F.1.5(a)**, Bj^* is injective. We must show it hits everything. Having computed $H^*(BO(n))$, this is a simple dimension argument. Note that since the Weyl group acts by permutation of the generators on $H^*(BT_n)$, $H^*(BT_n)^{W_{O(n)}(T_n)} \cong \mathbf{Z}/2[\sigma_1, \dots, \sigma_n]$ with σ_i the i-th elementary symmetric polynomials in the X_j.

(g) One can show by induction $\prod(1 + X_i) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n$. We let w_0 be the unique element mapping to 1 (which since this is a ring map is just the unit of the domain) and w_i be the unique element mapping to σ_i for $i \geq 1$. ■

Corollary 7.1.7. *An analogous result goes through with $O(n)$ and T_n replaced by $SO(n)$ and ST_n respectively.*

The interested reader can formulate and prove this for themselves. Most of the tools required have been shown above. The analogue of part **(e)** can be shown using the morphism of fiber sequences

$$\begin{array}{ccccc} O(n)/T_n & \longrightarrow & BST_n & \longrightarrow & BSO(n) \\ \| & & \downarrow & & \downarrow \\ O(n)/T_n & \longrightarrow & BT_n & \longrightarrow & BO(n) \end{array}$$

Remark. Essentially the same arguments apply to specify Chern classes. The argument above picked out Stiefel-Whitney classes. Another modification works for symplectic Pontryagin classes. A variation on this analysis is needed for the usual Pontryagin classes. We defer these analyses to the exercises since they are simpler.

Convention. In light of **Theorem 7.1.6(f)**, let us henceforth agree to write $H^*(BO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_1, \dots, w_n]$ where the w_i are such that $w_i \mapsto \sigma_i$ under the isomorphism of **Theorem 7.1.6(f)**.

Corollary 7.1.8. *All cohomology is taken with $\mathbf{Z}/2$ coefficients and we suppress the coefficient ring. Let $B\rho_{m,n} \colon BO(m) \times BO(n) \to BO(m+n)$ be the delooping of the block sum map $\rho_{m,n} \colon O(m) \times O(n) \to O(m+n)$. Then $\rho_{m,n}^*(w_k) = \sum_{i+j=k} w_i \times w_j$ where $\times$ denotes the cohomology cross product $H^i(BT_i) \otimes H^j(BT_j) \to H^k(BT_i \times BT_j)$.*

Proof. From our description of $T_n = O(1)^{\times n}$, we know that there are models of BT_i for which $BT_m \times BT_n \cong BT_{m+n} \cong BT_1^{\times m+n}$—namely taking $\mathbf{R}P^\infty = BT_1$ and in general $BT_i = (\mathbf{R}P^\infty)^{\times i}$. From **(f)** of the theorem above, Bj_{m+n}^* is an isomorphism mapping $w_k \mapsto \sigma_k$. It is easy to check that, under this map, $Bj_{m+n}^*(w_i \times w_j) = \sigma_i(X_1, \ldots, X_m)\sigma_j(X_{m+1}, \ldots, X_{m+n})$. Set $\sigma_0 = 1$. Then

$$\sum_{\ell=0}^{m+n} \sigma_\ell = \prod_{\ell=1}^{m+n}(1 + X_\ell) = \prod_{\ell=1}^{m}(1 + X_\ell) \prod_{\ell=m+1}^{m+n}(1 + X_\ell) = \sum_{\ell=0}^{m} \sigma_\ell(X_1, \ldots, X_m) \sum_{\ell=0}^{m} \sigma_\ell(X_{m+1}, \ldots, X_{m+n}).$$

The LHS of this expression is $Bj_{m+n}^*(1 + w_1 + \cdots + w_{m+n})$. The RHS of this expression is easily checked to expand to

$$\sum_{k=0}^{m+n} \sum_{i+j=k} \sigma_i(X_1, \ldots, X_m)\sigma_j(X_{m+1}, \ldots, X_{m+n}) = \sum_{k=0}^{m+n} \sum_{i+j=k} Bj_{m+n}^*(w_i \times w_j) = Bj_{m+n}^*(\sum_{k=0}^{m+n} \sum_{i+j=k} w_i \times w_j)$$

so since Bj_{m+n}^* is an isomorphism, the desired equality holds. ■

Remark. This is essentially the Whitney sum formula for the Stiefel-Whitney classes.

7.1.2 Exercises

Exercise 63. *Fill in the details to* ***Theorem 7.1.6(b)****.* [Hint: You may find **Proposition D.1.6** useful.]

Exercise 64. *Verify that* $Bj_{m+n}^*(w_i \times w_j) = \sigma_i(X_1, \ldots, X_m)\sigma_j(X_{m+1}, \ldots, X_{m+n})$ *as in* ***Corollary 7.1.8****.*

Exercise 65. *Suppose* $\eta = \xi_1 \oplus \xi_2$ *is a numerable vector bundle over a space* X *with* $\operatorname{rank} \xi_1 = m$ *and* $\operatorname{rank} \xi_2 = n$*.*

(a) *Let* $f \colon X \to BO(m+n)$ *classify* η *and let* $B\rho_{m+n} \colon BO(m) \times BO(n) \to BO(m+n)$ *be as in* ***Corollary 7.1.8****. Show that* f *factors through* $B\rho_{m+n}$ *up to homotopy.* [Hint: There are maps $X \to BO(m)$ and $X \to BO(n)$ classifying ξ_1 and ξ_2 respectively.]

(b) *Let* $(g_1, g_2) \colon X \to BO(n) \times BO(m)$ *be the map furnished by part* ***(a)****. Let* $w_i(\eta) = f^* w_i$ *and* $w_i(\xi_j) = g_j^* w_i$ *with the* w_i *chosen as in* ***Theorem 7.1.6(g)****. Show that*

$$w_k(\eta) = \sum_{i+j=k} w_i(\xi_1) w_j(\xi_2).$$

Exercise 66. *Fix* $n \leq m$ *and let* $i_{m,n} \colon O(n) \to O(m)$ *be the standard inclusion*

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{(m-n)\times(m-n)} \end{pmatrix}$$

with $I_{(m-n)\times(m-n)}$ *the identity matrix. Show that* $Bi_{m,n} \colon BO(n) \to BO(m)$ *on* $\mathbf{Z}/2$ *cohomology maps*

$$w_i \mapsto \begin{cases} w_i & 1 \leq i \leq n \\ 0 & i > n. \end{cases}$$

[Hint: Build a diagram which commutes (perhaps up to homotopy)

$$\begin{array}{ccc} BT_n & \longrightarrow & BT_m \\ \downarrow & & \downarrow \\ BO(n) & \xrightarrow[Bi_{m,n}]{} & BO(m) \end{array}$$

and compute $BT_n \to BT_m$ on cohomology. For this, you may find **Proposition D.1.6** useful.]

7.2 Axioms for Stiefel-Whitney Classes

Definition. Define a vector bundle γ_n^1 over $\mathbf{R}P^n$ called the ***tautological line bundle*** or the ***canonical line bundle*** as follows.

(1) The total space
$$E(\gamma_n^1) = \{(v,[w]) \in \mathbf{R}^{n+1} \times \mathbf{R}P^n : v \| w \text{ that is, } v \text{ and } w \text{ are parallel}\}$$
is a subbundle of the the trivial bundle $\pi\colon \mathbf{R}^{n+1} \times \mathbf{R}P^n \to \mathbf{R}P^n$ with projection $E(\gamma_n^1) \to \mathbf{R}P^n$ inherited from this bundle.

(2) For $U \subset S^n$ any open subset not containing a pair of antipodal points. Its image $U_1 \subset \mathbf{R}P^n$ is open and the bundle γ_n^1 is trivializable over U_1 with trivialization $\varphi^{-1}\colon U_1 \times \mathbf{R} \to \pi^{-1}(U_1)$ defined by $\varphi^{-1}([w], r) = ([w], rw)$ for each $(w,r) \in U$.

Proposition 7.2.1. *The bundle γ_n^1 is not trivial for any $n \geq 1$.*

Proof. Let $s\colon \mathbf{R}P^n \to E(\gamma_n^1)$ be any section and consider the composite $S^n \xrightarrow{\times 2} \mathbf{R}P^n \to E(\gamma_n^1)$ which sends $v \in S^n$ to $([v], t(v)v)$ for some $t\colon S^n \to \mathbf{R}$ which satisfies $t(-v) = -t(v)$. Since S^n is connected, the intermediate value theorem implies that some $t(v) = 0$. Hence, $E(\gamma_n^1)$ has no global section and thefore $E(\gamma_n^1)$ could not possibly be the trivial bundle, which concludes the proof. ∎

Lemma 7.2.2. *The bundle γ_1^1 is the open Möbius bundle over S^1.*

Proof. Each point of $E(\gamma_1^1)$ can be written as $([(\cos\theta, \sin\theta)], t(\cos\theta, \sin\theta))$ where $0 \leq \theta \leq \pi$ and $t \in \mathbf{R}$. This is a unique continuous assignment except at $\theta = 0, \pi$ where $([(\cos 0, \sin 0)], t(\cos 0, \sin 0)) = ([(\cos\pi, \sin\pi)], -t(\cos\pi, \sin\pi))$. It is also a continuous map $[0,\theta] \times \mathbf{R} \to E(\gamma_1^1)$. One can easily verify that it is in fact a quotient map and therefore it follows that $E(\gamma_1^1) \cong 0, \theta] \times \mathbf{R}/\sim$ where $\sim$ identifies $\{\pi\} \times \mathbf{R}$ with $\{0\} \times \mathbf{R}$ by $(0,t) \sim (\pi, -t)$. ∎

Theorem 7.2.3. *The* ***Stiefel-Whitney classes*** *for vector bundles $\xi \in \mathsf{Vect}_{\mathbf{R}}$ of finite rank are cohomology classes $w_i(\xi) \in H^i(B(\xi); \mathbf{Z}/2)$ satisfying the following properties.*

(1) *$w_i(\xi) \in H^i(B(\xi); \mathbf{Z}/2)$ for $i \geq 1$, $w_0(\xi) = 1$ the unit of the cohomology and ring, and for $m \geq \dim \xi$, $w_m(\xi) = 0$.*
(2) *Given a fiberwise-isomorphism map of rank n-vector bundles $(\widetilde{f}, f)\colon \xi \to \xi'$, $w_i(\xi) = f^* w_i(\xi')$.*
(3) *$w_k(\xi \oplus \eta) = \sum_{i=0}^k w_i \xi \smile w_{k-i}\eta$; this is called the* ***Whitney sum formula****.*
(4) *$w_1(\gamma_1^1) \neq 0$.*

Moreover, these conditions uniquely characterize these classes.

Remark. Bj in this proof is as in **Theorem 7.1.6**. We will implicitly understand all cohomology to be with $\mathbf{Z}/2$ coefficients in the proof.

Proof. Recall that $\mathbf{R}P^1 \cong S^1$. We study $BO(n)$. Define $w_i \in H^*(BO(m))$ as in **Theorem 7.1.6(g)** for each m and each i and define $w_i(\xi) = f^* w_i$ where $f\colon B \to BO(n)$ is the map classifying the relevant bundle. The Stiefel-Whitney classes for the universal vector bundle over $BO(n)$ are then simply the w_i themselves, and we call them the ***universal Stiefel-Whitney classes***. We show existence and uniqueness for this choice.

(3) This is **Exercise 61** which itself follows quickly from **Corollary 7.1.8**.

(1) This follows immediately by pulling back the universal Stiefel-Whitney classes.

(2) A morphism of bundles is induced, up to isomorphism, by a morphism between the classifying maps from which this follows immediately.

(4) $w_1(\gamma_1^1) \in H^1(S^1) \cong \mathbf{Z}/2$. We saw in the course of the proof of **Theorem 6.5.2** that $\mathsf{P}(\gamma_1^1)$ is the connected double cover $O(1) \to S^1 \to S^1$ because we could exclude the possibility it was the disconnected covering space. We know that γ_1^1 is classified by a map $S^1 \to BO(1) \simeq \mathbf{R}P^\infty$. By the classification theorem, $S^\infty \simeq EO(1)$ since S^∞ is contractible and $S^\infty \xrightarrow{\times 2} \mathbf{R}P^\infty$ is the universal cover and hence a principal $O(1)$-bundle. View $\mathbf{R}P^\infty$ as a cell complex with a cell in every dimension attached by a degree 2 map and let $f\colon S^1 \to \mathbf{R}P^\infty$ be such that it is not null-homotopic. We may suppose f is cellular by standard topology arguments (cellular approximation). We claim that $\mathsf{P}(\gamma_1^1) \cong f^* S^\infty$. Since $\mathbf{R}P^\infty$ is a simple space (also called an abelian space), the homotopy classes of maps $S^1 \to \mathbf{R}P^\infty$ are in bijection with $\pi_1(\mathbf{R}P^\infty) \cong \mathbf{Z}/2$. The attaching map of the 2-cell is not null-homotopic and thus represents this homotopy class. It follows that $f^* S^\infty$ is not trivial and thus since γ_1^1 is not trivial as we have just seen, we must have that $\mathsf{P}(\gamma_1^1) \cong f^* S^\infty$ (there are only two possible such bundles by the classification theorem).

A short computation in cellular cohomology now shows that f^* must be an isomorphism on degree 1 cohomology and therefore $w_1(\gamma_1^1) = f^* w_1 \neq 0$. An alternative argument goes through using naturality of the Hurewicz map and the universal coefficient theorem.

This establishes existence of the Stiefel-Whitney classes for rank n-bundles. For uniqueness, observe that every rank n-bundle $E \to B$ fits into a pullback

$$\begin{array}{ccc} E & \longrightarrow & EO(n) \times_{O(n)} \mathbf{R}^n \\ \downarrow & & \downarrow \\ B & \xrightarrow[f]{} & BO(n) \end{array}$$

where f is the map classifying the vector bundle $E \to B$. Thus, by naturality of the Stiefel-Whitney classes, $w_i(E \to B) = f^* w_i$. This is uniqueness.

Conversely, given choices $w_i(\xi)$ satisfying the above criteria and denote $W_i \in H^i(BO(n))$ the choices for the universal bundle. It suffices to show that $W_i = w_i$ since $w_i(\xi) = f^* W_i$ by the naturality axiom where f classifies ξ. Since $w_1(\gamma_1^1) \neq 0$, the map $f\colon S^1 \to \mathbf{R}P^\infty$ classifying γ_1^1 then pulls back the Stiefel-Whitney classes of the universal line bundle γ^1 (i.e., the universal bundle $EO(1) \to BO(1)$) and thus we deduce that $W_1 = X$ for $W_1 \in H^*(BO(1))$ and hence $w_1 = W_1$. Keeping γ^1 the universal bundle over $BO(1)$, with $\xi = (\gamma^1)^{\times n} \cong \mathrm{pr}_1^* \gamma^1 \oplus \cdots \oplus \mathrm{pr}_n^* \gamma^1$, one can easily show using the axioms that

$$\sum w_i(\xi) = \prod_{i=1}^n (1 + X_i) = \prod_{i=1}^n (1 + W_1) = \prod_{i=1}^n (1 + w_1)$$

where the last two equalities must be interpreted suitably. The map classifying this bundle is Bj and we have seen that this map induces an injection on cohomology into $H^*(BT_n)$ in **Theorem 7.1.6(f)**. Hence, for this equality to hold, it must have been that $w_n = W_n$ for all n, as desired. ■

Remark. In **Exercise 67**, you will be asked to verify that Bj really does classify $(\gamma^1)^{\times n} \cong \mathrm{pr}_1^* \gamma^1 \oplus \cdots \oplus \mathrm{pr}_n^* \gamma^1$.

7.3 Exercises

Remark. Keep Bj as in **Theorem 7.1.6**.

Exercise 67. *Show that all characteristic classes of numerable vector bundles are polynomials in the Stiefel-Whitney classes.* [Hint: This is really nothing more than the definition and the computation of $H^*(BO(n); \mathbf{Z}/2) \cong \mathbf{Z}/2[w_1, \ldots, w_n]$.]

The following exercise is called the splitting principle. It is adapted from Peter May's generalized splitting principle.

Exercise 68 (Splitting Principle). *Let G be a compact connected Lie group and $T = T_G$ a maximal torus of G. It is known*[2] *that $H^*(G/T_G)$ is concentrated in even degrees, finitely generated and free abelian.*

(a) *Show that $H^*(G/T_G; R)$ is free and finitely generated in each degree.*
(b) *Show that $H^*(BT; R) \cong H^*(BG; R) \otimes H^*(G/T; R)$.* [Hint: The Serre spectral sequence for $j\colon BT \to BG$ collapses.]
(c) *Conclude that $Bj\colon BT \to BG$ induces a monomorphism in cohomology with R coefficients.*
(d) *Let $f\colon X \to BG$ classify a bundle ξ over X. For convenience, we may suppose WLOG that X is connected. Consider the following pullback diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & BT \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Bj} \\ X & \xrightarrow[f]{} & BG \end{array}$$

(i) *Show that $q\colon Y \to X$ is a fiber bundle with fiber G/T and that the action of $\pi_1(X)$ on $H^*(G/T; R)$ is trivial.* [Hint: Show that the action of $\pi_1(X)$ on $H^*(G/T; R)$ is constructed by pulling it back from the action of $\pi_1(BG) = 0$ on G/T and is therefore trivial.]
(ii) *Show that $H^*(Y) \cong H^*(X; R) \otimes_R H^*(G/T; R)$.* [Hint: Show that all classes of $H^*(G/T)$ are permanent cycles in the Serre spectral sequence of Bj (i.e., it collapses). Run the Serre spectral sequence for q show that it collapses by using naturality of the Serre spectral sequence for the map $(f, g)\colon (Y, X) \to (BT, BG)$ and thus all elements of $H^*(G/T; R)$ are once again permanent cycles (i.e., it collapses).]
(iii) *Show that q^* is the canonical map $H^*(X; R) \to H^*(X; R) \otimes_R H^*(G/T; R)$ sending $\varphi \mapsto \varphi \otimes 1$.* [Hint: Use the identification of the edge homomorphism as q^*.]

(e) *Show that $q^*\xi$ is a principal G-bundle with a reduction of structure group to a principal T-bundle.*
(f) *Show that the same statements are true when $G = O(n)$ and $T = O(1)^{\times n}$ when $R = \mathbf{Z}/2$.* [Hint: By **Theorem 7.1.6(d)** $H^*(O(n)/T; \mathbf{Z}/2)$ is generated multiplicatively by $H^*(O(n)/T; \mathbf{Z}/2) \cong (\mathbf{Z}/2)^{\oplus n-1}$. Use this and the description of the pairing on the E_2 page of the Serre spectral sequence to deduce that the Serre spectral sequence for Bj still collapses.]

To appreciate this reduction of structure group, consider the following exercise.

Exercise 69. *Show that a real vector bundle with structure group $O(n)$ and a reduction of structure group to $O(1)^{\times n}$ splits a sum of line bundles. Show that a complex vector bundle with structure group $U(n)$ and a reduction of structure group to $SU(n)$ splits as the sum of n complex line bundles $\zeta_1, \ldots, \zeta_n$ such that $\zeta_1 \otimes \cdots \otimes \zeta_n$ is the trivial bundle.*

[2] For a proof, see Mimura and Toda's *Topology of Lie Groups, I & II*; in particular, **Theorem 4.21**.

The following is also sometimes called the splitting principle but we will consider it a consequence of the preceding exercise.

Exercise 70. *Let γ^n be the universal vector bundle over $BO(n)$ (the associated vector bundle to the universal principal $O(n)$-bundle over $BO(n)$).*

(a) *Show that $(\gamma^1)^{\times n} \cong \mathrm{pr}_1^* \gamma^1 \oplus \cdots \oplus \mathrm{pr}_n^* \gamma^1$ is classified by Bj. Conclude that $w_i(\gamma^1 \times \cdots \times \gamma^1) = \sigma_i$ the i-th symmetric polynomial in the generators $X_1, \ldots, X_n$ of the $\mathbf{Z}/2$-cohomology of $(\mathbf{R}P^\infty)^{\times n}$.*

(b) *Let $\mathcal{O}$ be an m-ary operation on vector bundles induced by a fiberwise construction as in **Exercise 28**. Let $f_i \colon B \to BO(n_i)$ classify a bundle ξ_i. Show that $\mathcal{O}(\xi_1, \ldots, \xi_m) \cong (f_1, \ldots, f_m)^* \mathcal{O}(\mathrm{pr}_1^* \gamma^{n_1}, \ldots, \mathrm{pr}_m^* \gamma^{n_m})$.* [Hint: This boils down to showing that $(f_1, \ldots, f_m)^* \mathcal{O}(\mathrm{pr}_1^* \gamma^{n_1}, \ldots, \mathrm{pr}_m^* \gamma^{n_m}) \cong \mathcal{O}((f_1, \ldots, f_m)^* \mathrm{pr}_1^* \gamma^{n_1}, \ldots, (f_1, \ldots, f_m)^* \mathrm{pr}_m^* \gamma^{n_m})$.]

(c) *Let $\mathcal{O}$ be as above. Using the splitting principle, show that to verify an equation relating the Stiefel-Whitney classes of $\mathcal{O}(\xi_1, \ldots, \xi_m)$ in terms of those of $\xi_1, \ldots, \xi_m$, it suffices to verify the equation for $\mathcal{O}(\gamma^1, \ldots, \gamma^1)$.* [Hint: Part **(b)** implies it is enough to verify it for all possible combinations of the universal vector bundles. Use the splitting lemma to reduce this considering as inputs only the universal line bundle $BO(1)$ alone.]

Remark. The next two exercises offer two approaches to computing the Stiefel-Whitney classes of a tensor product of bundles.

Exercise 71. *Given ξ and η numerable vector bundles over a space B of rank m and n respectively. Show that*

$$\sum w_i(\xi \otimes \eta) = p_{m,n}(w_1(\eta), \ldots, w_m(\eta), w_1(\eta), \ldots, w_n(\eta))$$

where

$$p_{m,n}(X_1, \ldots, X_m, Y_1, \ldots, Y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 + X_i + Y_j).$$

[Hint: Use the splitting principle. Show that for two line bundles, ξ and η, $w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta)$ by showing it for $\gamma^1 \otimes \gamma^1$ and then arguing that this is sufficient.]

Exercise 72. *Let $\otimes_{mn} \colon G(m) \times G(n) \to G(mn)$ be the smooth map sending an $(m \times m)$-matrix $A = (a_{ij})$ and an $(n \times n)$-matrix $B = (b_{ij})$ to their **Kronecker product** written in block form as*

$$A \otimes_{mn} B = (a_{ij} B)_{i,j}.$$

(a) *Verify that $\otimes_{mn}$ is indeed smooth by showing it is a continuous homomorphism of Lie groups.*

(b) *Fixing bases, show that $\otimes_{mn}$ is the tensor product $A \otimes B$ of the two linear maps A and B.*

(c) *Show that $B\otimes_{mn}$ classifies the principal $G(mn)$-bundle $\otimes_{mn*}(\gamma^{mn})$ over $BG(n) \times BG(m)$, where γ^{mn} is the universal $G(mn)$ bundles.* [Hint: Appeal to a previous result in the notes.]

(d) *Given $f \colon X \to BG(m) \times BG(n)$, show that, on the level of vector bundles, the bundle $f^* \otimes_{mn*} (\gamma^{mn})$ is the tensor product of the vector bundles classified by $\mathrm{pr}_{BG(m)} f$ and $\mathrm{pr}_{BG(n)} f$.*

(e) *Use this to determine a formula for the Stiefel-Whitney classes $w_i(\xi \otimes \eta)$ of the tensor product of two vector spaces.* [Hint: It suffices to compute this for $B\otimes_{mn}^*$ on cohomology (why?) and for this one has access to the Künneth isomorphism and from naturality of the Künneth isomorphism it suffices to understand the inclusion $BG(m) \to BG(mn)$ induced by $G(m) \times e \xrightarrow{\otimes_{mn}} G(mn)$ and similarly for the case of n. Use a commutative square involving $BT_m \to BT_{mn}$.]

Appendix

Technicalities and Manifolds with Corners

A.1 General Notions of Smoothness in Local Coordinates

A.1.1 Important Notation and Definitions

Notation. We make the following notational conventions.

$$\mathbf{R}^n_+ \overset{\text{def}}{=} [0,\infty)^n$$
$$\mathbf{H}^n \overset{\text{def}}{=} \mathbf{R}^{n-1} \times \mathbf{R}_+$$
$$\mathbf{R}^n_k \overset{\text{def}}{=} \mathbf{R}^{n-k} \times \mathbf{R}^k_+$$

We will denote

$$i_{n,k} \colon \mathbf{R}^n_k = \mathbf{R}^{n-k} \times \mathbf{R}^k_+ \to \mathbf{R}^n$$

the canonical embedding given by the evident subset inclusion for each $0 \le k \le n$.

Definitions.

(a) For $A \subset \mathbf{R}^k$, a function $f \colon A \to \mathbf{R}^n$ is ***smooth*** if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k$ and a smooth function $\overline{f} \colon U \to N$ such that $\overline{f}\big| A = f$.

(b) Similarly, for $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$, a function $f \colon A \to \mathbf{R}^n$ is ***smooth*** if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k \supset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$ and a smooth function $\overline{f} \colon U \to \mathbf{R}^n$ such that $\overline{f}\big| A = f$.

(c) For a subset $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$ and a function $f \colon A \to \mathbf{R}^m_n$, we will say that f is ***smooth*** if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k \supset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$ and a smooth function $\overline{f} \colon U \to \mathbf{R}^m$ such that $\overline{f}\big| A = f$. In other words, for the purposes of smoothness, we consider a function into $\mathbf{R}^m_n$ to be smooth **iff** it is smooth considered as a function into $\mathbf{R}^n$—in other words, f is said to be ***smooth*** if $i_{m,n} \circ f$ is smooth in the sense given above.

(d) We will define ***manifolds with corners*** in the section below. Given two such manifolds M^m and N^n, we will say a function $f \colon M \to N$ is ***smooth*** if for each $p \in M$, there are charts (x, U) about p and (y, V) about $f(p)$ such that the map

$$y \circ f \circ x^{-1} \colon \underbrace{x(U \cap f^{-1}y^{-1}(V))}_{\subset \mathbf{R}^m_k} \to \underbrace{y(V)}_{\subset \mathbf{R}^n_\ell}$$

is smooth in the sense just described.

A.1.2 Basic Results

Theorem A.1.1. *Let $A \subset \mathbf{R}^k$ be a set and $f \colon A \to N$ be a function. Then f is smooth **iff** there is an open set $U \subset \mathbf{R}^k$ with $A \subset U \subset \mathbf{R}^k$ and a smooth function $\overline{f} \colon U \to N$ is smooth and $\overline{f}\big| A = f$.*

Proof. This is a partition of unity argument. ■

Corollary A.1.2. *The same is true if $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$.*

Proof. ∂A in $\mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$ is ∂A in $\mathbf{R}^k$. Indeed, $\mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+ \subset \mathbf{R}^k$ is closed, and so contains all of its limit points and hence the limit points of A in $\mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_+$ is the same as the limit points of A in $\mathbf{R}^k$. ■

A.2 Manifolds With Corners

A.2.1 Basic Definitions and Facts

Definition (Model Spaces). Consider $\mathbf{R}^n_k \subset \mathbf{R}^n$. We give this the following standard smooth structure where a smooth chart of $\mathbf{R}^n_k$ is a smooth homeomorphism onto an open subset of some $\mathbf{R}^n_\ell$ where smoothness is defined as before for subsets of Euclidean spaces. Smooth compatibility of these charts boils down to a simple exercise in point-set topology. These will be our ***model spaces*** after which we pattern manifolds with corners.

Definitions (Manifold with Corners). A smooth ***manifold with corners*** of dimension n is a second countable, Hausdorff space that is locally patterned after the spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ (k is not fixed, $k \geq 0$) with a maximal smooth atlas $\mathscr{A}$ comprised of such charts that are smoothly compatible—smooth compatibility of these charts is defined in the way given above. The definition of a smooth function between two manifolds with corners is then patterned after the notion of smoothness for functions $\mathbf{R}^m_k \to \mathbf{R}^n_\ell$ introduced above. See, specifically, **(d)** of the definitions given in the preceding section.

We shall say that a chart (x, U) for an n-manifold-with-corner M is a ***boundary chart*** if it is a homeomorphism from U onto an open subset of $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ such that $x(U) \cap \mathbf{R}^{n-k} \times \mathbf{R}^{k-\ell}_+ \times \mathbf{0} \neq \emptyset$ for some $1 \leq \ell \leq k$. We shall say that a chart (x, U) for an n-manifold-with-corner M is a ***corner chart*** if it is a homeomorphism from U onto an open subset of $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ with $k \geq 2$ such that $x(U) \cap \mathbf{R}^{n-k} \times \mathbf{R}^{k-\ell}_+ \times \mathbf{0} \neq \emptyset$ for some $2 \leq \ell \leq k$.

Definition (Boundary and Corners). By abuse of notation, we shall refer to the ***boundary*** ∂M of a smooth manifold with corners M to be the set of all points that are mapped by some chart to the boundary of one of model spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ ($k \geq 1$) and we shall call the set of points which are mapped by some chart to the boundary of one of the model spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ with $k \geq 2$ the ***corner set*** of M and denote it by $\angle M$.

Definition (Corner Depth). Let M be a manifold-with-corners of dimension n. For each $1 \leq k \leq n$, let $\angle_k M$ be the set of points $p \in M$ for which there is a chart (x, U), $x \colon U \to \mathbf{R}^{n-k} \times \mathbf{R}^k_+$ such that $x(p) \in \mathbf{R}^{n-k} \times \mathbf{0} \subset \mathbf{R}^{n-k} \times \mathbf{R}^k_+$. We call the set $\angle_k M$ the set of ***k-th order corners*** or ***corners of depth*** k. We denote by $\mathrm{depth}_M(p)$ or simply $\mathrm{depth}(p)$ the smallest integer k for which there exists a chart (x, U) about p where $x \colon U \to \mathbf{R}^{n-k} \times \mathbf{R}^k_+$. We call this the ***depth*** of p.

Remark. The upshot of the remainder of this chapter is that *what you expect to be true is indeed true.*

The following theorem is a standard result in algebraic topology.

Theorem A.2.1 (Topological Invariance of the Boundary). *Given a topological n-dimensional manifold with boundary M, if there is a chart (x, U) for which $x(p) \in \partial \mathbf{R}^n_+$, then the same is true for all other charts of M.*

Proof. This is a local homology argument. By shrinking U if necessary and shifting, we may suppose $x(U)$ is an open half ball of some fixed radius $\varepsilon > 0$ centered at $x(p) = \mathbf{0} \in \mathbf{R}^n_+$. By excision, $H_n(M, M \setminus \{p\}) \cong H_n(U, U \setminus \{p\}) \cong H_n(x(U), x(U) \setminus \{\mathbf{0}\})$. By the LES of the pair and contractibility of $x(U)$, $H_n(x(U), x(U) \setminus \{\mathbf{0}\}) \cong H_{n-1}(x(U) \setminus \{\mathbf{0}\})$ and $x(U) \setminus \{\mathbf{0}\} \simeq S^{n-1}$ by the radial contraction onto the boundary. Hence, the local homology of p is non-trivial and evidently concentrated in degree $n-1$ with a factor of $\mathbf{Z}$. Since local homology is a homeomorphism invariant, this shows that any other chart must send p to a point with non-trivial local homology and some thought shows that the only such points lie on the boundary of $\mathbf{R}^n_+$ as desired. ■

Theorem A.2.2 (Smooth Invariance of Corner Points). *Let M be a manifold-with-corner.*

- **(a)** *If $p \in \angle M$, then p is topologically a boundary point in the sense that there is a homeomorphism $\mathbf{R}^n_k \cong \mathbf{R}^n_+$ for $k \geq 1$.*
- **(b)** *If $p \in \partial M$, then the defining condition is true for every chart about p in the smooth and topological case.*
- **(c)** *If $p \in \angle M$, then the defining condition is true for every chart about p in the smooth case. In particular, there is no diffeomorphism $\mathbf{R}^n_+ \not\cong \mathbf{R}^{n-k} \times \mathbf{R}^k_+$ for any $k \geq 2$.*
- **(d)** *If $i \neq j$ and $p \in \angle_i M$, then $p \notin \angle_j M$.*
- **(e)** *Any diffeomorphism $\mathbf{R}^{n-k} \times \mathbf{R}^k_+ \to \mathbf{R}^{n-k} \times \mathbf{R}^k_+$ preserves $\angle_k(\mathbf{R}^{n-k} \times \mathbf{R}^k_+)$ for each $1 \leq k \leq n$. This lifts to manifolds with corners in the obvious way.*

Proof (Sketch). The idea is that you can successively flatten the walls of $\mathbf{R}^n_k$ to get a homeomorphism $\mathbf{R}^n_k \cong \mathbf{R}^n_+$, but it cannot be smooth because things go "too quickly" around the origin. This can be made precise by contradiction, supposing there is a diffeomorphism $f \colon \mathbf{R}^n_k \to \mathbf{R}^n_+$, taking a smooth curve γ in $\partial \mathbf{R}^n_+$ passing through $f(\mathbf{0})$ at time $t = 0$ *with non-zero derivative* and then observing that $f^{-1}(\gamma)$ has a kink at time $t = 0$ and does not slow to speed 0, so could not possibly be smooth.

(c) and **(d)** are proved in essentially the same manner. The gist of it is that $\partial \mathbf{R}^n_k \setminus \angle \mathbf{R}^n_k$ is disconnected with components consisting of the boundary points of $\mathbf{R}^n_k$ for which exactly one of the coordinates $x^{n-k+1}, \ldots, x^n$ are equal to 0. //

A.2.2 Constant Rank Theorem

Theorem A.2.3. *Suppose M^m and N^n are smooth manifolds (without boundary) and that $f\colon M \to N$ is smooth.*

(a) *If f has rank k at $p \in M$, then is some coordinate system (x,U) about p and some coordinate system (y,V) about $f(p)$ with $y \circ f \circ x^{-1}$ in the form*

$$(y \circ f \circ x^{-1})(a^1, \dots, a^m) = (a^1, \dots, a^k, \psi^{k+1}(a), \dots, \psi^n(a)).$$

Moreover, given any coordinate system y, the appropriate coordinate system on N can be obtained by permuting the component functions of y.

(b) *If f has rank k in a nbhd of p, then there are coordinate systems (x,U) about p and (y,V) about $f(p)$ with $y \circ f \circ x^{-1}$ in the form*

$$(y \circ f \circ x^{-1})(a^1, \dots, a^m) = (a^1, \dots, a^k, 0, \dots, 0).$$

(c) *If $n \leq m$ and f has rank n at p, then for any coordinate system (y,V) about $f(p)$, there is some coordinate system (x,U) about p with*

$$y \circ f \circ x^{-1}(a^1, \dots, a^m) = (a^1, \dots, a^n).$$

(d) *If $m \leq n$ and f has rank m at p, then for any coordinate system (x,U) about p, there is a coordinate system (y,V) about $f(p)$ with*

$$y \circ f \circ x^{-1}(a^1, \dots, a^m) = (a^1, \dots, a^m, 0, \dots, 0).$$

(e) *(**Equivariant Rank Theorem**) Let G be a Lie group acting on M and N and suppose the action on M is transitive. Let $f\colon M \to N$ be G-equivariant and smooth. Then f has constant rank.*

Remark. Note that $\operatorname{rank} f \leq \min\{m,n\}$. Hence, in **(c)** and **(d)**, f has full rank at p and therefore f has full rank in a nbhd of p since the condition of being full rank is an open condition.

Proof. **(a)** Fix a coordinate system (y,V) about $f(p)$ and choose some coordinate system u about p. Since $\operatorname{rank}(df_p) = k$, there is some $k \times k$ submatrix of df_p (in coordinates) whose determinant is nonzero. Thus, by performing some diffeomorphisms (i.e., permuting the coordinate functions u^i and y^i and thereby performing row/column operations) and relabeling, we can bring this $k \times k$-submatrix into the upper left-hand corner of $D(y \circ f \circ u^{-1})$:

$$\det\left(\frac{\partial(y^\alpha \circ f)}{\partial u^\beta}(p)\right) \neq 0 \qquad \alpha, \beta = 1, \dots, k.$$

Now, define

$$\begin{aligned} x^\alpha &= y^\alpha \circ f & \qquad \alpha &= 1, \dots, k \\ x^r &= u^r & \qquad r &= k+1, \dots, m. \end{aligned}$$

Then, recalling that $\dfrac{\partial(y^\alpha \circ f)}{\partial u^\beta} \stackrel{\text{def}}{=} D_\beta(y^\alpha \circ f \circ u^{-1})(u(p))$, we see that the determinant $m \times m$ matrix $\left(\dfrac{\partial x^i}{\partial u^j}(p)\right)$ is in fact

$$\det\left(\begin{array}{c|ccc} \left(\dfrac{\partial(y^\alpha \circ f)}{\partial u^\beta}\right)_{\alpha,\beta=1,\dots,k} & \begin{matrix} D_{k+1}(y^1 \circ f \circ u^{-1})(u(p)) & \cdots & D_m(y^1 \circ f \circ u^{-1})(u(p)) \\ \vdots & \cdots & \vdots \\ D_{k+1}(y^n \circ f \circ u^{-1})(u(p)) & \cdots & D_m(y^n \circ f \circ u^{-1})(u(p)) \end{matrix} \\ \hline 0_{k\times k} & \mathbf{1}_{(m-k)\times(m-k)} \end{array}\right) \neq 0$$

because the columns are clearly linearly independent. Unraveling what this matrix is (namely, $D_k(x^\alpha \circ u^{-1})$), it follows by the Inverse Function Theorem that $x \circ u^{-1}$ is a diffeomorphism in a nbhd of $u(p)$. Hence, $x = (x \circ u^{-1}) \circ u$ is a coordinate system in some nbhd of p in M: it will be a homeomorphism and if (z,W) were any other coordinate system about p in M, then the transition map will likewise clearly be smooth. The cases of $\partial z/\partial x$ are taken care of by noting that the Inverse Function Theorem (really the chain rule, I think) gives us a description of $\partial z/\partial x$ as $(\partial x/\partial z)^{-1}$.

Now, if $q = x^{-1}(a^1, \dots, a^m)$, then $x(q) = (a^1, \dots, a^m)$ and therefore $x^i(q) = a^i$ and hence,

$$\begin{cases} y^\alpha \circ f(q) = a^\alpha & \alpha = 1, \dots, k, \\ u^r(q) = a^r & r = k+1, \dots, m, \end{cases}$$

so

$$\begin{aligned} y \circ f \circ x^{-1}(a^1, \dots, a^m) &= y \circ f(q) \qquad \text{for } q = x^{-1}(a^1, \dots, a^n) \\ &= (a^1, \dots, a^k, \underline{\quad}). \end{aligned}$$

This is **(a)**.

(b) As above, choose coordinate systems x and v so that $v \circ f \circ x^{-1}$ has the form obtained in **(a)**. Since $\operatorname{rank}(df_p) = k$ in a nbhd of p, the lower rectangle in the $n \times m$ matrix $\left(\dfrac{\partial(v^i \circ f)}{\partial x^j}\right)$ must vanish in a nbhd of p. That is, the lower (right) rectangle of

$$\left(\begin{array}{c|ccc} \mathbf{1}_{k\times k} & & 0_{k\times(m-k)} & \\ \hline & D_{k+1}\psi^{k+1} & \cdots & D_m\psi^{k+1} \\ \times & \vdots & \cdots & \vdots \\ & D_{k+1}\psi^n & \cdots & D_m\psi^n \end{array}\right)$$

Hence, $\psi^{k+1}, \dots, \psi^n$ are independent of $a^{k+1}, \dots, a^m$ on said nbhd. Since the ψ^{k+i} are smooth, this means that we can write

$$\psi^r(a) = \overline{\psi}^r(a^1, \dots, a^k) \qquad r = k+1, \dots, n.$$

To see this, "walk along coordinate lines," use the MVT and possibly regroup—we can always walk in an open, path-connected subset of $\mathbf{R}^n$ from one point to another along coordinate lines by using compactness and a metric d to put an ε-tube around a curve connecting the two points (I think... see for instance HW 5).

Define

$$\begin{aligned} y^\alpha &= v^\alpha & \alpha &= 1, \dots, k \\ y^r &= v^r - \overline{\psi}^r \circ (v^1, \dots, v^k) & r &= k+1, \dots, n. \end{aligned}$$

Since

$$\begin{aligned} y \circ v^{-1}(b^1, \dots, b^n) &= y(q) \qquad \text{for } v(q) = (b^1, \dots, b^n) \\ = (b^1, \dots, b^k, b^{k+1} &- \overline{\psi}^{k+1}(b^1, \dots, b^k), \dots, bb^m - \overline{\psi}^n(b^1, \dots, b^k)), \end{aligned}$$

the $n \times n$ Jacobian matrix

$$\left(\frac{\partial y^i}{\partial v^j}\right) = \begin{pmatrix} \mathbf{1}_{k\times k} & 0_{k\times(n-k)} \\ \times & \mathbf{1}_{(n-k)\times(n-k)} \end{pmatrix}$$

has nonzero determinant, clearly, as the columns are linearly independent. Therefore y is a coordinate system in a nbhd of $f(p)$ by the same reasoning as in **(a)** (i.e., diffeomorphism, etc.). Moreover, from the previous centered equation,

$$\begin{aligned} y \circ f \circ x^{-1}(a^1, \dots, a^m) &= y \circ v^{-1} \circ v \circ f \circ x^{-1}(a^1, \dots, a^m) \\ &= y \circ v^{-1}(a^1, \dots, a^k, \psi^{k+1}(a), \dots, \psi^n(a)) \\ &= (a^1, \dots, a^k, \psi^{k+1}(a) - \overline{\psi}^{k+1}(a^1, \dots, a^k), \dots, \psi^n(a) - \overline{\psi}^n(a^1, \dots, a^k)) \\ &= (a^1, \dots, a^k, 0, \dots, 0), \end{aligned}$$

as desired.

(c) This is basically a special case of **(a)**. Except, when $k = m$, it is unnecessary to permute the y^i (i.e., the column space), only the u^i (i.e., the rows) need to be permuted in order that

$$\det\left(\frac{\partial(y^\alpha \circ f)}{\partial u^\beta}(p)\right) \neq 0 \qquad \alpha, \beta = 1, \dots, k.$$

(d) Since the rank of f at any point must be $\leq m$, the rank of f equals m in some nbhd of p (i.e., full rank at a point implies full rank in a nbhd). It is easier to think of the case that $M = \mathbf{R}^m$ and $N = \mathbf{R}^n$ and find the coordinate system y when we are given $x = \mathrm{id}_{\mathbf{R}^m}$—since this result is local, we don't really lose anything. Then **(b)** yields coordinate systems φ on $\mathbf{R}^m$ and ψ for $\mathbf{R}^n$ such that

$$\psi \circ f \circ \varphi^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^m, 0, \ldots, 0).$$

Even without φ^{-1}, $\psi \circ f$ takes $\mathbf{R}^m$ into $\mathbf{R}^m \times \{0\} \subseteq \mathbf{R}^n$ except—as Spivak puts it—the points of $\mathbf{R}^m$ just get moved to the wrong place in $\mathbf{R}^m \times \{0\}$. This is corrected by defining a diffeomorphism $\lambda\colon \mathbf{R}^n \to \mathbf{R}^n$. In particular,

$$\lambda(b^1, \ldots, b^n) = (\varphi^{-1}(b^1, \ldots, b^m), b^{m+1}, \ldots, b^n).$$

Then, if $\varphi^{-1}(b^1, \ldots, b^m) = (a^1, \ldots, a^m)$, we have

$$\begin{aligned} \lambda \circ \psi \circ f(a^1, \ldots, a^n) &= \lambda \circ \psi \circ f \circ \varphi^{-1}(b^1, \ldots, b^n) \\ &= \lambda(b^1, \ldots, b^m, 0 \ldots, 0) \\ &= (\varphi^{-1}(b^1, \ldots, b^m), 0, \ldots, 0) \\ &= (a^1, \ldots, a^m, 0, \ldots, 0), \end{aligned}$$

which shows that $\lambda \circ \psi$ is the coordinate system y we sought (of course, since these are smooth manifolds, the diffeomorphism λ being compatible with the maximal atlas will obviously be a chart). If we are given a coordinate system x on $\mathbf{R}^m$ other than the identity, we define

$$\lambda(b^1, \ldots, b^n) = (x(\varphi^{-1}(b^1, \ldots, b^m), b^{m+1}, \ldots, b^n),$$

and is not hard to check that $y = \lambda \circ \psi$ is the coordinate system we sought.

(e) Choose $g \in G$ such that $gp = q$ in M for any two points $p, q \in M$. By transitivity, this g exists. Since $g \cdot f = f(g \cdot -)$ (equivariance) TFDC:

$$\begin{array}{ccc} T_pM & \xrightarrow{f_{*p}} & T_{f(p)}N \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ T_qM & \xrightarrow[f_{*q}]{} & T_{f(q)}N \end{array}$$

with the linear maps isomorphisms. Hence, f must have constant rank. ■

Corollary A.2.4. *Suppose $f\colon M^m \to N^n$ has full rank at $p \in M$ and suppose that M and N have corners.*

(a) *Suppose $n \le m$. For any coordinate system (y, V) about $f(p)$ (say a k-corner chart) and any coordinate system (x, U) about p and any smooth extension of $i_{n,k} \circ y \circ f \circ x^{-1}$ to a smooth function defined on an open subset of $\mathbf{R}^m$, there is a coordinate system (z, W) of $\mathbf{R}^m$ about $x(p)$ with*

$$i_{n,k} \circ y \circ f \circ x^{-1} \circ z^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^n).$$

(b) *Suppose $m \le n$. For any coordinate system (x, U) about p, any coordinate system (y, V) about $f(p)$ (say a k-corner chart) and any smooth extension of $i_{n,k} \circ y \circ f \circ x^{-1}$ there is a coordinate system (z, W) about $(i_{n,k} \circ y \circ f)(p)$ with*

$$z \circ i_{n,k} \circ y \circ f \circ x^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^m, 0, \ldots, 0).$$

Remark. In practice, it is convenient to drop the standard embeddings $i_{k,\ell}$ from these expressions.

Proof. Since the condition of full rank is an open condition (since the rank function is a ***lower semicontinuous*** function), any smooth extension of $y \circ f \circ x^{-1}$ to a function from an open subset of $\mathbf{R}^m$ into $\mathbf{R}^n$ has full rank in a sufficiently small nbhd of the original domain. We will use this in the short argument below.

(a) For *any* charts y and x, by definition of smoothness, we may suppose $y \circ f \circ x^{-1}$ is defined on an open nbhd $U \subset \mathbf{R}^m$ into $\mathbf{R}^n$ and, furthermore, since max rank is an open condition, we may suppose that f has max rank on this extension and then apply **(c)** of the constant rank theorem.

(b) This argument is entirely analogous. ■

A.2.3 Submanifolds

Warning. The following definition is wordy and seemingly difficult to parse but the basic idea is completely tractable and that is how one should remember it. We will give the idea immediately after the definition.

Definition (Submanifold). Let M be an m-dimensional manifold with corner or boundary. A subset $N \subset M$ is a ***submanifold*** of dimension n or an ***n-dimensional submanifold*** of M if the following holds.

For each point $q \in N$ there is a chart $x\colon U \to \mathbf{R}^{m-k} \times \mathbf{R}^k_+$ of M about q (note that necessarily $k \geq \operatorname{depth}_M(q)$ by smooth invariance of corner points) such that for each $p \in i_{m,k}(x(U \cap N))$, there is a chart (φ_p, V_p) of $\mathbf{R}^m$ about p such that for some $0 \leq \ell \leq n$,

$$V_p \cap (i_{m,k} \circ x)(U \cap N) = \varphi_p^{-1}(\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+)$$

or, equivalently,

$$\varphi_p(V_p \cap i_{m,k}(x(U \cap N))) = \varphi_p(V_p) \cap (\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+).$$

In other words, φ_p sends $V_p \cap i_{m,k}(x(U \cap N))$ homeomorphically onto its image in $\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+ \cong \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+$.

Lemma A.2.5. *In the above definition, one can replace the set $\mathbf{0}_{m-n} \times \mathbf{R}^n_\ell$ by any permutation of the factors of the product.*

Proof. Permute the components functions of φ in the definition—this permutation is a diffeomorphism. ∎

Remark. The idea this definition captures is relatively simple. A submanifold should be a subset that sits nicely in charts of the original manifold. This condition is too restrictive when we do not map into a full Euclidean space since we haven't allowed ourselves the room to massage a subset into a locally nice form.

Thus, the idea here is that a submanifold of a manifold with corners is a subset which can be "straightened out" locally *after* embedding the model space $\mathbf{R}^n_k$ in $\mathbf{R}^n$. Thus, in some sense, this condition is no different from the one that is encountered for manifold without boundary.

In the following definition, ∂ means the generalized boundary, as usual.

Definition. A ***neat submanifold*** of a manifold-with-corners M^n is a submanifold N^m of M, in the sense of being immersed and topologically embedded, such that

(a) $(\partial M) \cap N = \partial N$;
(b) $(\partial M) \cap \overline{N} = (\partial M) \cap N$;
(c) For every point $p \in \partial N$, $\operatorname{depth}_N(p) = \operatorname{depth}_M(p)$ and there is a (corner) chart (x, U) of M about p such that $x^{-1}(\mathbf{0} \times \mathbf{R}^{m-\operatorname{depth}_N(p)} \times \mathbf{R}_+^{\operatorname{depth}_N(p)}) = U \cap N$.

(b) is an item of convenience in the sense that it's possible only items **(b)** and **(c)** may matter some application. For tubular neighborhoods, however, **(b)** is essential, as we remark below.

Remarks.

(a) In the case of a manifold with boundary but no corners, the idea is that a neat submanifold is a submanifold that meats the boundary transversely.
(b) Observe that when $\partial N = \emptyset$, this recovers the definition of submanifold we used previously when we only discussed manifolds without boundary. The only difference is that we previously asked that it sit nicely in the first m-coordinates—we have to modify this to make notation easier.
(c) One essential difference between a neat submanifold and an ordinary submanifold is that we require the submanifold be able to be straightened out *natively* in the ambient manifold M, as opposed to straightening it out in the codomain $\mathbf{R}^n$ of some chart for M.
(d) Sometimes people require a neat submanifold to be in addition a closed submanifold (i.e., a closed subset as well) instead of the somewhat weak condition that $\partial M \cap \overline{N} = \partial M \cap N$. The reason why is that we may want to throw away pathological examples like $M = \mathbf{H}^2 = \{(x, y) \in \mathbf{R}^2 : y \geq 0\}$ and $N = \{(0, y) \in \mathbf{R}^2 : y > 0\}$ because these subspaces *will not* admit tubular neighborhoods.
(e) The condition that $\operatorname{depth}_N(p) = \operatorname{depth}_M(p)$ is superfluous if we restrict ourselves only to manifolds with or without boundary. Otherwise, this guarantees that we avoid something like $N = \{(t, t, t) : t \geq 0\} \subset \mathbf{R}^3_+$, where N meets ∂M at a depth 3 corner point.
(f) If we restrict to $\mathbf{Man}_\partial$, a neat submanifold $N \subset M$ is exactly a submanifold satisfying $\partial M \cap N = \partial M \cap \overline{N} = \partial N$ and $T_pN \nsubseteq T_p\partial M$ for all $p \in \partial M$. For manifolds with corners, this extra stipulation doesn't make sense since ∂M isn't a manifold with corners (it's not even smooth in a sensible way), but it does still serve to guide intuition. We will prove this later after we have collars.

Observation. For manifolds without boundary, this definition recovers the usual one since the composite of two diffeomorphisms is a diffeomorphism and so the two charts at play compose to give a single chart for the smooth structure.

Example 6 (Kissing the Disk). Let $M \cong D^2$ be the unit disk with boundary in $\mathbf{R}^2$ centered at $(x, y) = (0, 1)$ and let N be the image of $(-1/2, 1/2)$ of the curve $t \mapsto (t, t^2)$. For the moment, let us forget that $N \subset \mathbf{R}^2$ and $M \subset \mathbf{R}^2$.

One can check that $N \subset M$ and that N meets ∂M tangentially at the single point $(0,0)$. Then there is no chart (x,U) of M about $(0,0)$ such that $x(U \cap N) = x(U) \cap \mathbf{R} \times \{a\}$ in $\mathbf{R}^2_+$ for any $a \geq 0$. This is because, by smooth invariance of the boundary, boundary points must be sent to boundary points, so any such chart of M sends $(0,0) \mapsto \partial\mathbf{R}^2_+$ and similarly every other point of N in this chart must be mapped to an interior point. Moreover, since N meets the boundary of M tangentially, we are precluded from straightening N out as $\{a\} \times \mathbf{R}_+$.

Now let us embed this picture in $\mathbf{R}^2$ by remembering that $N \subset \mathbf{R}^2$ and $M \subset \mathbf{R}^2$. We can now imagine a chart of $\mathbf{R}^2$ that "unfurls" the boundary of the disk locally near $(0,0)$ and so sends N near $(0,0)$ onto $\mathbf{R} \times \{0\}$. Here are some words about this. The desired chart of $\mathbf{R}^2$ can be produced by sending $(x,y) \mapsto (x, y - x^2)$. This is certainly smooth and it is bijective since $(x, y-x^2) = (x_0, y_0 - x_0^2)$ if and only if $x = x_0$ and hence $y = y_0$ (from the equation $y - x^2 = y_0 - x^2$). This is invertible because the Jacobian of $(x,y) \mapsto (x, y-x^2)$ is $\begin{pmatrix} 1 & 0 \\ -2x & 1 \end{pmatrix}$ with determinant $1 \neq 0$ and so this is a bijective, smooth, locally invertible function and so it is a diffeomorphism. This sends x^2 to the line $y = 0$.

Definition (Submanifold Chart). Let M be an m-dimensional manifold with corner or boundary and let $N \subset M$ be a subset which is an n-dimensional submanifold. Given a chart $x \colon U \to \mathbf{R}^{m-k} \times \mathbf{R}^k_+$ such that $U \cap N \neq \emptyset$ and for which there exists a chart (φ, V) of $\mathbf{R}^m$ satisfying $V \cap (i_{m,k} \circ x)(U \cap N) = \varphi^{-1}(\mathbf{0}_{m-n} \times \mathbf{R}^n_\ell)$ as above, we say that $\varphi \circ i_{m,k} \circ x$ is a ***submanifold chart*** for N. As usual, we will think of the submanifold chart $\varphi \circ i_{m,k} \circ x$ as a smooth function onto an open subset of some $\mathbf{R}^n_\ell$.

Remark. This is guaranteed to exist when N is a submanifold by restricting the chart x to the open set $U \cap x^{-1}(V_p)$.

Theorem A.2.6. *Suppose $N \subset M$ is an n-dimensional submanifold with corners, where $\dim M = m$. Then N can be given the structure of a smooth manifold with corners determined by collection of submanifold charts and this makes $N \hookrightarrow M$ a smooth embedding. In particular, the corner points of N are well-defined.*

Conversely, any smooth embedding $i \colon N \hookrightarrow M$ has submanifold charts in this way with the smooth structure on N determined by them and, hence, the smooth structure on N is the unique one for which the topological embedding $N \hookrightarrow M$ is an immersion. In other words, $i(N)$ is a submanifold of M and $N \to i(N)$ is a diffeomorphism.

Proof. It is probably easier to understand some of the arguments below if we reduce to working with model spaces.

($\Rightarrow$) Before proceeding, we should point out that the property of being Hausdorff and second-countable are all inherited by subspaces.

The smooth structure on N is obtained by giving it the atlas (extended to a maximal atlas as usual) consisting of submanifold charts for N. $(\varphi i_{m,k} x, (i_{m,k} x)^{-1}(V) \cap U \cap N)$. To see smoothness of transitions, let us write

$$(\varphi' i_{m,k'} y)(\varphi i_{m,k} x)^{-1}) = \varphi' i_{m,k'} y x^{-1} i_{m,k}^{-1} \varphi^{-1}$$

where we are now required to show that smoothness of φ^{-1} and $i_{m,k}^{-1}$ makes sense in this context. Let us consider their composite. Smoothness of $i_{m,k}^{-1}\varphi^{-1}$ means that there is a smooth extension to a function into $\mathbf{R}^m$, by definition. Recalling that φ is a chart of $\mathbf{R}^m$, it is clear that the smooth extension of this composite is simply φ^{-1} on its full domain. This shows, additionally, that the corners of N are well-defined. To see this, let φ and ψ be two of the charts as above. Then smoothness of $\varphi \circ \psi^{-1}$ means that $\operatorname{depth}(\psi(p)) = \operatorname{depth}(\varphi(p))$ by smooth invariance of corner points.

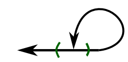
We should like, additionally, for N to be paracompact in the subspace topology. This follows since manifolds are ***hereditarily paracompact***. We argue this in a remark below the end of this proof. We could also appeal to the fact that every manifold is metrizable and every metric space is paracompact—since subspaces of metric spaces are metric spaces this is enough.

($\Leftarrow$) Now suppose N is a manifold with corners and $i \colon N \to M$ is a smooth embedding. Let $q \in N$ and pick a coordinate system (x,U) about q and a coordinate system (y,V) about $i(q)$ and consider the composite $y \circ i \circ x^{-1}$, which is smooth. By shrinking U and shifting things as necessary, we may suppose this is a map $x(U) \to y(i(U)) \subset \mathbf{R}^{m-k} \times \mathbf{R}^k_+$ and where $x(U) \subset \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+$. In other words, WLOG we henceforth suppose $x(U) \subset V$.

Since the composite $i_{m,k} \circ y \circ i \circ x^{-1}$ is smooth, we know it extends to a function on an open subspace of $\mathbf{R}^n$ and since it is full rank, which is an open condition, we may suppose that the function has full rank on this open subspace. By **(d)** of the constant rank theorem, it follows that there is a chart (z, V_p) about each $p \in i_{m,k}(y(i(U)))$ in $\mathbf{R}^m$ such that $z_p \circ i_{m,k} \circ y \circ i \circ x^{-1}(a^1, \ldots, a^n) = (0, \ldots, 0, a^1, \ldots, a^n) \in \mathbf{R}^m$ (recall that we're being fiddly with the way coordinates go so WLOG we make them go this way). This means that $z_p \circ i_{m,k} \circ y$ *almost* constitutes a submanifold chart for $i(N)$ (after intersecting the domain with $i(N)$). It remains to show that when $z_p \circ i_{m,k} \circ y$, when restricted to $V \cap N$, or perhaps some $V' \cap N$ where $V' \subset V$ is open, has the desired form. This is where it is important that i be a topological embedding. Since i is an embedding, $i(U)$ is an open subspace of $V \cap N$, and so by definition of the subspace topology there is some W such that $W \cap N = i(U)$—we may suppose $W \subset V$ by the obvious modification and thus for the chart (W, y) we have what we want—$z_p \circ i_{m,k} \circ y$ has the right form and is a submanifold chart.

Now we consider uniqueness of the smooth structure. Let $i \colon N \to i(N)$ be a smooth embedding.
Recall that the collection of all submanifold charts determines a subbase for the subspace topology

on $i(N)$ and likewise determine the submanifold smooth structure on $i(N)$. We've just shown in one direction that these charts are smoothly compatible with N—namely, we just showed that $i\colon N \to i(N)$ is smooth with the charts. Now let us consider the other way around $i^{-1}\colon i(N) \to N$, which certainly exists since i is a topological embedding and so homeomorphism onto its image.This will be smooth if we can show that $x \circ i^{-1} \circ (z_p \circ i_{m,k} \circ y)^{-1}$ is smooth. This is the part where i being a topological embedding is important—we need to throw away the possibility of the immersed line $j\colon \mathbf{R} \to \mathbf{R}^2$ at right, where the map j^{-1} back to $\mathbf{R}$ from the interval indicated will necessarily discontinuous in the subspace topology. Just as before, there is some $W \subset V$ such that $W \cap N = i(U) \subset V$. Hence, for the shrunken chart (W, y), we know that $x \circ i^{-1} \circ (z_p \circ i_{m,k} \circ y)^{-1}$ has the form $(0, \dots, 0, a^1, \dots, a^n) \mapsto (a^1, \dots, a^n)$ which is obviously smooth—hence, i^{-1} is smooth and therefore $i\colon N \to i(N)$ is a diffeomorphism. ∎



Remark. Thus, i being a topological embedding lets us exclude the possibility that some disparate piece of N intersects every open nbhd in M of $V \cap N$.

Remark (Hereditarily Paracompact). All manifolds are hereditarily paracompact. According to the Wikipedia article for paracompactness, this is equivalent to having all open subspaces being paracompact. In fact, by the Whitney embedding theorem, it suffices to show that all subspaces of $\mathbf{R}^n$ are paracompact, so let $U \subset \mathbf{R}^n$ be open. In any case, there's a shortcut to this result. Any locally compact second-countable Hausdorff space is paracompact, say by **Theorem 2.6** here. The property of being second-countable and Hausdorff is hereditary. Clearly any open subspace of $\mathbf{R}^n$ is locally compact since ε-balls are precompact. Similarly for any model space $\mathbf{R}^n_k$.

Corollary A.2.7. *Fix $N \subset M$ a submanifold. A submanifold chart $y = \varphi \circ i_{m,k} \circ x$ considered as a smooth function defined on an open nbhd U of M is a diffeomorphism onto its image—in particular, $y(U)$ is a submanifold of $\mathbf{R}^n$.*

Proof. The map is a smooth embedding and so by the above theorem determines a smooth structure on its image. The inverse map restricted to its image is certainly a homeomorphism and it is smooth as the map $x \circ x^{-1} \circ i^{-1}_{m,k} \circ \varphi^{-1} = i^{-1}_{m,k} \circ \varphi^{-1}$ defined on a subset of Euclidean space has smooth extension given simply by φ^{-1}. ∎

Theorem A.2.8 (Universal Property of Submanifolds). *Let $S \subset N$ be a submanifold and let $i\colon S \to N$ be the inclusion. A map $f\colon M \to S$ is smooth **iff** $i \circ f$ is smooth. Say* $\dim M = m$, $\dim N = n$ *and* $\dim S = s$.

Proof. ($\Rightarrow$) Easy since $i\colon S \to N$ is smooth. ($\Leftarrow$) Suppose $i \circ f$ is smooth. By definition of a submanifold, about each point in S, there is a nbhd V and a diffeomorphism onto its image $y\colon V \to y(V) \subset \mathbf{R}^n$, such that $y(V \cap N) = y(V) \cap (0 \times \mathbf{R}^{s-\ell} \times \mathbf{R}^{\ell}_{+})$—that is, a submanifold chart. We have concluded y is a diffeomorphism onto its image by the above corollary. Thus, in coordinates, $y \circ i \circ f \circ x^{-1}$ looks like a map onto these last s coordinates and is assumed smooth. But this has the same form as $y|\, V \cap N \circ f \circ x^{-1}$ using the submanifold chart constructed as above and, hence, $y|\, V \cap N \circ f \circ x^{-1}$ is smooth. Hence, f is smooth. ∎

A.3 Whitney Theorems

Remark. All of the following material is adapted from Lee's *Introduction to Smooth Manifolds*.

Lemma A.3.1 (Lee, 2.26). *Let M be a manifold with corners, $A \subset M$ closed, and $f\colon A \to \mathbf{R}^k$ smooth.*[1] *For any open nbhd U of A, there is a smooth function $\widetilde{f}\colon M \to \mathbf{R}^k$ such that $\widetilde{f}\Big|\, A = f$ and* $\operatorname{supp} \widetilde{f} \subset U$.

Proof. This is a partition of unity argument. ∎

Warning. If A is not closed, then we have no control over the boundary behavior and this will therefore fail in general. For example, consider $1/x$ defined on the set $(0, 1] \subset \mathbf{R}$—we cannot extend this at 0.

Theorem A.3.2 (Whitney Approximation Theorem for Functions). *Let M be a manifold with corners and $F\colon M \to \mathbf{R}^k$ continuous. Given any positive continuous function $\delta\colon M \to \mathbf{R}$, there is a smooth function $\widetilde{F}\colon M \to \mathbf{R}^k$ that is **δ-close** to F—that is $\Big|F(x) - \widetilde{F}(x)\Big| < \delta(x)$ for all $x \in M$. If F is smooth on a closed subset $A \subset M$, then $\widetilde{F}$ can be chosen such that $\widetilde{F}\Big|\, A = F|\, A$.*

Proof. Partition of unity argument along with the lemma above. ∎

[1] Recall that this means that there is a smooth extension of f in an open nbhd of each point $p \in A$.

Corollary A.3.3. *If M is a manifold with corners and $\delta\colon M \to \mathbf{R}$ a continuous function, then there is a smooth positive function $\varepsilon\colon M \to \mathbf{R}$ with $0 < \varepsilon(x) < \delta(x)$ for all $x \in M$.*

Proof. Apply Whitney approximation to construct a smooth $e\colon M \to \mathbf{R}$ such that $\left|e(x) - \frac{1}{2}\delta(x)\right| < \frac{1}{2}\delta(x)$. ■

Remark. This gives an easy way to construct the smooth function used in the proof of the collar nbhd theorem for smooth manifolds.

Theorem A.3.4 (Whitney Approximation Theorem). *Let N be a manifold with corners, M is a manifold* without *boundary and let $F\colon N \to M$ be continuous. Then F is homotopic to a smooth map $\widetilde{F}\colon N \to M$. If F is already smooth on a closed subset $A \subset N$, then the homotopy can be taken relative to A (this means that the homotopy is fixed on A).*

Remark. It will turn out that dropping the relative homotopy assumption makes this go through for manifolds M with boundary, but perhaps not necessarily with corners.

Corollary A.3.5. *Suppose M has no boundary and we are given a homotopy $H\colon N \times I \to M$ between smooth maps $f, g\colon N \to M$. Then there is a smooth homotopy $\widetilde{H}\colon N \times I \to M$ between f and g such that H and $\widetilde{H}$ are themselves homotopic rel $N \times \partial I$.*

Proof. Let $A = N \times \partial I$ be a closed subset and note that H is already smooth on it. The Whitney approximation theorem tells us that there exists a smooth homotopy $\widetilde{H}$ satisfying the properties we want. ■

Remark. In particular, this shows that for a manifold M with empty boundary, the homotopy groups of M may defined in the *smooth* category by taking $A = * \times I$ where $* \in S^n$ is a chosen basepoint.

Corollary A.3.6. *If N is a manifold with corners, M has no boundary, $A \subset N$ is closed and $f\colon A \to M$ is smooth, then f has a smooth extension to N **iff** it has a continuous extension to N.*

Proof. Whitney approximation! ■

Here's an example of what goes wrong when M has boundary and we insist the homotopy be fixed on a closed subset.

Example 7 (6-7). Let $F\colon \mathbf{R} \to \mathbf{H}^2$ by $t \mapsto (t, |t|)$, $A = [0, \infty)$. Then no such homotopy fixed on A exists.

To get this to work for manifolds with boundary, but without corners, we need to construct a smooth "flowing in" map $R\colon M \to \operatorname{Int} M \overbrace{\subset}^{\iota} M$ and a smooth homotopy $H\colon M \times I \to M$ satisfying the following properties: H is a smooth homotopy from $\iota \circ R$ to id_M and the restriction of H to $\operatorname{Int} M \times I$ gives a smooth homotopy from $R|\operatorname{Int} M$ to $\mathrm{id}_{\operatorname{Int} M}$. Let us show this exists.

Warning. See the errata for the following. It's not totally clear to me the Lee needs the properness assumption so I have not used it.

Construction 1 (Lee 9.26). Let $C\colon [0,1) \times \partial M \to M$ be an open collar nbhd. Observe that $M \setminus \operatorname{Im}(C|\,[0, \frac{1}{3}) \times \partial M)$ is closed because the collar is an embedding of an open submanifold.

Let $\psi\colon [0,1) \to [\frac{1}{3}, 1)$ be an increasing diffeomorphism which is the identity on $[\frac{2}{3}, 1)$, and define an embedding $R\colon M \to \operatorname{Int} M$ by flowing in along the collar as

$$R(p) = \begin{cases} p, & p \in M \setminus \operatorname{Im}(C|\,[0, \frac{2}{3}) \times \partial M) \\ (\psi(s), x), & p = C(s, x). \end{cases}$$

The two pieces agree on their overlap by definition of ψ and so ψ is smooth since each piece is smooth. R is a diffeomorphism onto the closed subset $M \setminus \operatorname{Im}(C|\,[0, \frac{1}{3}) \times \partial M)$ and hence it is a smooth embedding of M into $\operatorname{Int} M$, where, recall, proper means the preimage of compact sets are compact. Since $\operatorname{Int} M \subset M$ is a submanifold, the same things should be true of R viewed a map into M.

Let $\iota\colon \operatorname{Int} M \to M$. There is a smooth homotopy $H\colon M \times I \to M$ by "flowing back," defined by

$$H(p, t) = \begin{cases} p, & p \in M \setminus \operatorname{Im}(C|\,[0, \frac{2}{3}) \times \partial M) \\ (ts + (1-t)\psi(s), x), & p = C(s, x). \end{cases}$$

H also gives a smooth homotopy from $\iota \circ R$ to id_M and the restriction of H to $\operatorname{Int} M \times I$ gives a smooth homotopy from $R|\operatorname{Int} M$ to $\mathrm{id}_{\operatorname{Int} M}$. There is a way to make this, moreover, a *proper* map and thus an embedding.

Remark. An injective immersion that is proper is an embedding. This is a consequence of a theorem in the chapter **Point-Set Results**.

Theorem A.3.7 (Whitney Approximation Theorem). *Let N be a manifold with corners, M a manifold* with *boundary but no corners and let $F\colon N \to M$ be continuous. Then F is homotopic to a smooth map $\widetilde{F}\colon N \to M$.*

Proof. With this in hand, we see that $R \circ F\colon N \to \operatorname{Int} M$ is smoothly homotopic to a map G by the standard Whitney approximation theorem. Let $\iota\colon \operatorname{Int} M \to M$ be the inclusion. Then the flow back homotopy gives a homotopy $\iota \circ G \simeq \iota \circ R \circ F \simeq F$, so $\iota \circ G\colon N \to M$ is a smooth map homotopic to F. ∎

Theorem A.3.8. *Let N be a manifold with corners and M a manifold with boundary. If $F, G\colon N \to M$ are homotopic, then they are smoothly homotopic.*

Proof. Let R be the flow-in constructed above. Then $R \circ G$ and $R \circ F$ are homotopic smooth maps from N into $\operatorname{Int} M$, so they are smoothly homotopic. Thus we have smooth homotopies $F \simeq \iota RF \simeq \iota RG \simeq G$ as desired. Obviously smooth homotopy is an equivalence relation so we're good. ∎

A.4 Collars and Boundaries

Lemma A.4.1. *Let M be a manifold-with-boundary. Then TM is a manifold-with-boundary and, in particular, $\partial TM = T\partial M$.*

Proof. This is essentially the vector bundle construction lemma, Lee 10.6, and is not hard to see directly. The bundle charts are the same, they are still $(x^1, \ldots, x^n, \partial_1, \ldots, \partial_n)$ and so we see we only run into issues when the chart x in question is a boundary chart. ∎

Lemma A.4.2. *Let M be a manifold-with-boundary. Then, in coordinates, for every $p \in \partial M$, $T_p\partial M \subset T_pM$ consists of the vectors with last coordinate* 0.

Proof. This is easiest to see with curves. ∎

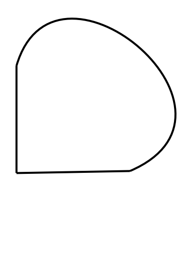
Definition. Let M be a manifold-with-boundary and $p \in \partial M$. It is easy to see that one may still take T_pM to be the vector space of derivations of germs of smooth functions. Moreover, T_pM has a distinguished class of ***inward pointing*** vectors, defined as those vectors with a strictly positive last coordinate. This definition is invariant under choice of coordinates. One similarly defines ***outward pointing*** vectors.

Remark. We might be tempted to define T_pM in terms of smooth curves, but this seems to require annoying modifications—we must allow ourselves to consider *smooth* curves with domain $(-\varepsilon, 0]$ and $[0, \varepsilon)$ (really just one by symmetry) to make sense of this. There is a geometric interpretation of inwards pointing vectors in terms of smooth curves.

Exercise 73. *The above definitions are invariant under choice of coordinates and can be detected using curves (in the appropriate sense) and derivations.*

Definition (Collar). A ***collar*** of a manifold-with-boundary M is an embedding $i\colon \partial M \times [0,1) \to M$ such that $i|_{\partial M \times \{0\}}$ is the canonical inclusion of $\partial M \subset M$. In particular, a collar is a ***neat submanifold*** (see above for the definition). Say a ***closed collar*** is an embedding (in the loose sense) $i\colon \partial M \times [0,1] \to M$. A closed collar always contains a collar.

Warning. While it might be tempting to try and define collars for manifolds with corners, we run into a serious issue with smoothness. Namely, consider the (filled) teardrop. This is a smooth manifold with corners of dimension 2. But its boundary could not possibly be a manifold with corners with its subspace topology, because it has a singularity! This is basically because, as remarked before, the boundary of a manifold with corners *does not* have a smooth structure unless there is no corner set. However, if we were content to work *outside* some category of smooth manifolds, then we strongly suspect that collars will exist in some modified sense and the same argument will work.



Proposition A.4.3. *A collar $i\colon [0,1) \times \partial M \hookrightarrow M$, if it exists, is an open submanifold of M. A closed collar is a closed submanifold. In particular, they are open (resp. closed) maps.*

Proof. The first part suffices since the latter will be the closure of the restriction to $[0,1)$.

The invariance of domain implies that any embedding between manifolds with empty boundary of the same dimension is an open map, since it amounts to giving an injective map from an open subspace of $\mathbf{R}^n$ into itself sends the subspace to another open subspace, and being an open map is a local property when the map in question is injective. Hence, on the interior of the collar $(0,1)\times\partial M$, at least, the map C an open map. We can cheat for points on the boundary. Fix a coordinate nbhd for the boundary of $[0,1)\times\partial M$. In coordinates, we might as well assume the map looks like an embedding $\mathbf{H}^n\supset U\to\mathbf{H}^n$. We can then extend this to a smooth map $\mathbf{R}^n\supset\widehat{U}\to\mathbf{R}^n$. Since collar map is an embedding of full rank, this is an open condition and so we may assume the extended map has full rank. This means that in a nbhd of p the map is a local embedding and therefore by invariance of domain an open map. But this means that its restriction to U is open by inspecting what the subspace topology does. ■

Neatness is essentially automatic since the only points to worry about from the definition are the boundary points and we gave ourselves the entire boundary!

Remark. We will prove these always exist. First we need a few lemmas. We will go about this in the most natural way to prove it, at least I think. Another way to prove it is to use tubular neighborhoods by embedding the manifold in $\mathbf{R}^N$ for big enough N (here we simply mean an immersion and topological embedding). Kupers takes this approach in his differential topology lecture notes but it seems somewhat incorrect in the sense that he is not using his own stipulated definition of a submanifold!

Say a vector field on a manifold-with-boundary or corners M is an ***inward pointing vector field*** if for all $p\in\partial M$, X_p points inward.

Lemma A.4.4. *Let M be a manifold-with-boundary or corners of dimension n. Then there exists an inward pointing vector field X on M.*

Proof. This is a partition of unity argument where we stipulate that on a non-boundary coordinate patch U_α, $X_\alpha=\dfrac{\partial}{\partial x^n}$, and on a coordinate patch for a corner with order k, we set $X_\alpha=\sum_{i=n-k}^{n}\dfrac{\partial}{\partial x^i}$. Then we set $X=\sum_\alpha\rho_\alpha X_\alpha$. It is easy to see that X_p is inward pointing since only boundary charts intersect the boundary. ■

The idea is to flow in along this vector field.

Remark. It is important to point out that the flow for an inward pointing vector field exists and is smooth. The proof is a variation upon the usual argument which we sketch below.

Theorem A.4.5 (Collar Neighborhood Theorem). *Let M be a manifold-with-boundary of dimension n.*

(a) *M has a (closed) collar. In addition, for a collar $C\colon[0,1)\times\partial M\to M$, the complement of $C(a)=\operatorname{Im}(C|\,[0,a)\times\partial M)$ is closed. In particular, a collar is an open submanifold and the collar map is an open map.*

(b) *Suppose $N\subset M$ is a neat submanifold. Then we can find a collar for M that restricts to a collar for N.*

Remark. We give two proofs. The first will be for **(a)** and the second for **(b)**, which implies **(a)**. For **(b)**, the idea is roughly that we can find a fat enough covering of N by neat submanifold charts and then cover M by charts that never meet ∂N. It is worth pointing out that *we do not* need to assume $\overline{N}\cap\partial M=\partial N$ and *we do not* need to assume N is closed for this argument to work.

*Proof (**(a)**).* Let X be an inward pointing vector field on M and consider the ODE on M given by $\dot\gamma=X(\gamma)$ with initial condition $\gamma(o)=p\in M$. In coordinates, this locally has the form $y'=f(t,y(t))$ where $f(t,y(t))=y(t)$ and this is Lipschitz continuous in the dummy variable $y(t)$ so that the Picard-Lindelöf theorem applies (and one can easily check that transitions preserve solutions). Kosinski I.6.3 shows that the flow exists and, because of the time tube argument for flows extending to a global flow, we know that in general the valid times for the flow must taper off to 0 unless the manifold is compact. So let A be the maximal flow domain about $M\times\{0\}$ in $M\times\mathbf{R}$, and let the flow be Φ. Let $\mathcal{U}=A\cap(\partial M\times\mathbf{R})$ and note that this is open in $\partial M\times\mathbf{R}_{\geq 0}$. Then for $(q,0)\in\mathcal{U}$, $\Phi_{*,(q,0)}(\partial_i,r\cdot d/dt)=\partial_i+rX^n(q)$ and so clearly is an isomorphism between tangent spaces $T_{(q,0)}\mathcal{U}\to T_qM$, since we have arranged that $X^n\neq 0$ for any $q\in\partial M$. We used the fact that $X(q)$ only has component in the inwards direction from the construction above and hence by the inverse function theorem $\Phi|\mathcal{U}$ is a local diffeomorphism—one might worry that the inverse function theorem does not apply because of the boundary, but we just extend everything where we need to and use properties of the subspace topology to see that $\Phi|\mathcal{U}$, which is certainly an immersion, is additionally a local topological embedding and hence a local diffeomorphism.

Observation. We can glue the local inverses together once we know that it is injective on an open subspace of the union of the nbhds upon which Φ is invertible.

This follows from the tubular neighborhood trick.

Thus, we may also suppose WLOG that Φ is an embedding on $\mathcal{U}$, perhaps by shrinking it first—note that $\mathcal{U}$ will always contain $\partial M \times \{0\}$. (It is clearly an embedding.) Suppose we have a smooth function $\varepsilon\colon \partial M \to (0,\infty)$ such that $(q, \varepsilon(q)t) \in \mathcal{U}$ for all $t \in [0,1]$ for the moment. Then $c\colon \partial M \times [0,1] \to M$ by $(p,t) \mapsto \Phi(p, t\varepsilon(p))$ is an embedding that is neat on $[0,1)$. It is certainly smooth because everything in sight is smooth and to show it is an embedding it suffices to show that $(p,t) \mapsto (p, t\varepsilon(p))$ is an embedding into $\mathcal{U} \subset \partial M \times \mathbf{R}_{\geq 0}$ since Φ is an embedding on $\mathcal{U}$ by hypothesis now. This function is also certainly smooth and injective. It has differential $(\mathrm{id}, ?)$ into $\partial M \times \mathbf{R}_{\geq 0}$ so it suffices to determine the differential of $(p,t) \mapsto t \cdot \varepsilon(q)$. In coordinates, the matrix for this will be $1 \times (n+1)$ or a row vector of length $n+1$ and it is clear that this will be (using the identity chart on the time part) $(t\partial_1 \varepsilon \ \cdots \ t\partial_n \varepsilon \ \varepsilon(q))$. Since $\varepsilon(q) > 0$ for all q, this will always have full rank. Hence, the differential is componentwise $(\mathrm{id}, \text{full rank})$ and so is clearly an isomorphism. It therefore remains to construct ε.

The construction of ε is a partition of unity argument in ∂M by noting that every $q \in \partial M$ has a coordinate nbhd U such that $U \times [0, \varepsilon(q)) \in \mathcal{U}$ where $\varepsilon(q) > 0$. Pick an open cover of ∂M be charts $\{U_\alpha\}_{\alpha \in J}$ such that for each $\alpha \in J$, there exists $u_\alpha > 0$ such that $\{q\} \times [0, u_\alpha] \subset \mathcal{U}$ for all $q \in U_\alpha$. To see this exists, simply shrink everything as needed. WLOG we may suppose by paracompactness that $\{U_\alpha\}$ is locally finite.

Let $I_{\alpha,2}$ be the (finite) set of all $\gamma \in J$ for which there exists $\beta \in J$ such that $U_\alpha \cap U_\beta \neq \varnothing$ and $U_\beta \cap U_\gamma \neq \varnothing$. Let I_α be the set of $\beta \in J$ such that $U_\alpha \cap U_\beta \neq \varnothing$ and let $N_\alpha = \max\{\#(I_\beta) : \beta \in I_{\alpha,2}\}$.

Observation. Notice that for each $\beta \in I_\alpha$, $N_\beta \geq \#(I_\alpha)$ since $\alpha \in I_{\beta,2}$ and, in particular, I_β.

Set $t_\alpha = \min\left\{u_\beta / \max\left\{N_\alpha^2, N_\beta^2\right\} : \beta \in I_{\alpha,2}\right\}$. Running the partition of unity subordinate to $\{U_\alpha\}$, we put $\varepsilon = \sum \rho_\alpha t_\alpha$. For $q \in U_\alpha$, we now wish to show that $\varepsilon(q) \leq u_\beta$ for each $\beta \in I_\alpha$. Suppose we set $\rho_\beta \equiv 1$ for $\beta \in I_\alpha$. Fix $\gamma \in I_\alpha$ and pick $t_\beta \leq u_\gamma / \max\left\{N_\beta^2, N_\gamma^2\right\}$ for each $\beta \in I_\alpha$. Then

$$\varepsilon(q) = \sum_{\beta \in I_\alpha} t_\beta \leq \sum_{\beta \in I_\alpha} u_\gamma / \max\left\{N_\beta^2, N_\gamma^2\right\} = \#(I_\alpha) u_\gamma \sum_{\beta \in I_\alpha} 1/\max\left\{N_\beta^2, N_\gamma^2\right\} \leq \#(I_\alpha) u_\gamma / \#(I_\alpha) = u_\gamma$$

hence, $\varepsilon(q) \leq u_\beta$ for all $\beta \in I_\alpha$ so we've achieved our goal, ε is smooth into where this is an embedding.

The last part follows from a preceding lemma. ■

Here's a slightly different and more terse proof for **(b)**.

*Proof ((**b**)).* Cover $\partial N \subset \partial M$ by neat submanifold charts in M with image coordinate balls of radius 2, say $\{(z_i, V_i)\}_{i \in I}$. WLOG we may assume this collection is locally finite by paracompactness since manifolds are hereditarily paracompact. Let U be the union of the restriction of each neat submanifold chart (z_i, V_i) to the coordinate balls of radius 1—call the resulting chart (z_i, U_i)—and let F be the union of the closed balls of radius 3/4s for each such chart. Note that since the collection in question is locally finite, F is closed.

In the coordinates of the neat submanifold charts, the last coordinate points inward for *both* N and M. We must be prudent about how we extend this covering. For each $p \in \partial M \setminus U \cap \partial M = \partial M \cap (U \cap \partial M)^c$, there is an open nbhd in ∂M of p disjoint from $U \cap \partial M$. Indeed, we need to find nbhds separating p and F and this amounts to saying that a manifold is a regular space. Thus, we may find a sufficiently small boundary chart (x, V) about p such that $V \cap (U \cap \partial M) = \varnothing$.

Cover the rest of ∂M by such charts and then observe that $M \setminus \partial M$ is open and we cover it by charts. Now we construct a partition of unity subordinate to this open cover where we use the radius 1 charts constructed in the first paragraph. Let $X = \sum \rho_\alpha X_\alpha$ where X_α is, in coordinates, $\dfrac{\partial}{\partial x^m}$ the last coordinate. Then for any $p \in \partial N$, X_p is inward pointing, being a sum of inward pointing vectors and similarly for any $p \in \partial M$. This is a consequence of the above construction.

Let $W_1 \subset M \times \mathbf{R}_+$ be the open subset on which the flow of X is defined, call the flow η, and let $W \subset \partial M \times \mathbf{R}_+$ be $W_1 \cap \partial M \times \mathbf{R}_+$. Then since W_1 is open, W is open in $\partial M \times \mathbf{R}_+$. We must shrink W to yet another open subset to make things work out. Begin by noting that for $q \in \partial M$ and working in one of our neat submanifold charts about this point, $\Phi_{*(q,0)}(\partial_i + r \cdot d/dt)$ can be computed as

$$\begin{aligned}
(\partial_i + r\frac{d}{dt})(x^j \circ \Phi) = (\partial_i + r\frac{d}{dt})(x^j \circ \Phi) = (\partial_i + r\frac{d}{dt})\Phi^j \\
= \partial_i \Phi^j + r\frac{d}{dt}\Phi^j = \partial_i \Phi^j + r\left.\frac{d}{dt}\gamma_q^j\right|_{t_0} = \partial_i \Phi^j + r\dot{\gamma}_q^j(0) \\
= \partial_i \Phi^j + r\dot{\gamma}^j_{\Phi(q,t_0)}(0) = \partial_i \Phi^j + rX_q^j = \partial_i + X_q^j
\end{aligned}$$

where we have used the group law to deduce this for the X term and since $\Phi(-,0) = \mathrm{id}$, so the directional derivative ∂_i of id at $(q,0)$ is still ∂_i. It follows easily that $\Phi_{*(q,0)}$ has full rank. Hence, even though we have boundary from $\mathbf{R}_+$, the inverse function theorem implies that this is a local diffeomorphism and thus we may shrink W to an open subset where $\Phi_{*(q,t)}$ has full rank.

As above, we can construction an embedding $\partial M \times [0,1) \hookrightarrow W$ and now the desired collar map is

$$\partial M \times [0,1) \hookrightarrow W \xrightarrow{\eta} M$$

since everything in sight here has full rank. The open part follows as before.

We now want to show that we can restrict this to a collar for N. At this point, we might worry that η may shoot W out of N, despite pointing into N, so we need to shrink W yet again. To fix this, let U be the union of the boundary charts in our open cover and let $W' = W \cap \eta^{-1}(U)$. Redoing the above construction with W' in place of W gives us a collar that restricts as a consequence of the delicate construction of our given open cover. Essentially, restricting to $W \cap \eta^{-1}(U)$ makes us shoot into points of only U—by working in the nice submanifold coordinates, for points $p \in \partial N$, we see that we are simply flowing vertically inward for both N and M in U.

Openness of the restricted collar is the same argument as usual. ■

Let us call such a function ε as above a smooth ***shrinking function***.

Lemma A.4.6. *Shrinking functions exist.*

This lemma should be interpreted appropriately.

Corollary A.4.7. *Every open nbhd of ∂M contains a collar.*

Proof. An open nbhd U of ∂M is an open submanifold and, in particular, it is neat submanifold-with-boundary, so the same argument applies to show a collar exists. ■

Although we didn't need the collar nbhd theorem to show the following, it makes it particularly straightforward and easy to see.

Corollary A.4.8. *Suppose M is orientable. Then $TM|\partial M \cong T\partial M \oplus \underline{\mathbf{R}}$ where as usual $\underline{\mathbf{R}}$ is the trivial bundle over ∂M with fiber $\mathbf{R}$. In particular, the normal bundle of ∂M in M is trivial.*

Proof. Let $i\colon \partial M \to M$ be the inclusion and let $j\colon \partial M \times [0,1) \to M$ be a collar nbhd so that $j|\partial M \times \{0\} = i$. First note that $TM|\partial M \cong i^*TM$. The collar neighborhood is an open submanifold of M and has tangent bundle diffeomorphic to $T\partial M \times \underline{\mathbf{R}}$ over $\partial M \times [0,1)$ and, as before, this is diffeomorphic to j^*TM. The collar has a submanifold (and note that the condition of being a neat submanifold is transitive) $\partial M \times \{0\}$. By pasting pullbacks we get the following rectangle with every rectangle a pullback

$$\begin{array}{ccccc} i^*TM & \longrightarrow & j^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow & & \downarrow \\ \partial M & \xrightarrow{i_0} & \partial M \times [0,1) & \xrightarrow{j} & M \end{array} \qquad (i = j \circ i_0\colon \partial M \to M)$$

where $j^*TM \cong T\partial M \times \underline{\mathbf{R}}$ as we said above. Hence, we must compute $i_0^* j^*TM$. Of course, one sees immediately that this is what we described. ■

Remark. To identify the normal bundle $\nu_{\partial M}$ with $\underline{\mathbf{R}}$, one can simply use a partition of unity argument and a collar to produce a Riemannian metric on M which is a product metric in a nbhd of ∂M. Say we make it the product metric at least on $[0,1/4)$ by covering M with open sets that only intersect the collar at $[1/4,1) \times \partial M$. This can be done using coordinate balls whose closure in M is compact.

For this next corollary, it helps to know that M is orientable **iff** TM is orientable as a vector bundle over M. First, we make a definition.

Definition (Induced Orientation). Let M be an orientable manifold with boundary (but not corners) of dimension n. Then ∂M inherits an ***induced orientation*** from M. The natural way of specifying this for which Stokes' theorem has a nice form is the ***outward pointing first convention***. Namely, for each $p \in \partial M$, we define an orientation class for $T_p\partial M$ by declaring a tuple of vectors $(v_1, \ldots, v_{n-1}) \in T_p\partial M$ to be in this orientation class iff for each outward pointing vector (hence, any outward pointing vector) $w \in T_pM$, $(w, v_1, \ldots, v_{n-1})$ defines a positively oriented basis in T_pM. One could similarly make a definition by using the ***inward pointing first convention*** but we do not need this.

Of course, we must check that these actually define an orientation.

Corollary A.4.9. *Let M be an orientable manifold with boundary. Then ∂M inherits a natural orientation by the* ***outward pointing first convention****. Namely, for each $p \in \partial M$, we define an orientation class for $T_p\partial M$ by declaring a tuple of vectors $(v_1, \dots, v_n) \in T_p\partial M$ to be in this orientation class iff for each outward pointing vector (hence, any outward pointing vector) $w \in T_pM$, $(w, v_1, \dots, v_n)$ defines a positively oriented basis in T_pM.*

Proof. This is straightforward using the definitions. ■

Transversality and Regular Value Theorems

Here is the basic concept of transversality.

Definition. Let K, L and N be manifolds with corners and let $f\colon K \to N$ and $g\colon L \to N$ be smooth maps. Then we say that f ***is transversal to*** g, denoted by $f \pitchfork g$, if whenever we have $f(k) = g(l) = p$, we have $f_* T_k K + g_* T_l L = T_p N$. We can also say that f ***and*** g ***are transverse***.

Remark. If $f(K) \cap g(L) = \emptyset$, then transversality holds vacuously. Basically, the idea is that the two maps intersect as generically as possible.

Remark. The only way, it seems, to get good results for transversality, at least with little effort, is to assume neatness in some places. Essentially, the issue is that the regular value theorem, as we know how to prove it, is insensitive to the corners or boundaries. Basically, the argument one wants to use relies upon not having extra structure floating around on M. It is possible to compensate for this by imposing additional constraints on the map f to get an analogous result for manifolds with boundary. A neat submanifold is assumed to only have corner points of depth k match up with the corner points of depth k in the ambient manifold, and this assumption eliminates the extra data needed to make certain arguments go. Another issue is that the regular value theorem only makes sense in the category DIFF and if M has corners then ∂M is not smooth.

Before we begin with the regular value theorem, let us introduce an auxiliary lemma and use it to prove a proposition.

Lemma B.0.1. *Let M be a smooth manifold without boundary and let $g\colon M \to \mathbf{R}$ be smooth. Suppose g has regular value* 0 *and $g^{-1}(0) \neq \emptyset$. Then $g^{-1}([0,\infty))$ is a submanifold with boundary $g^{-1}(0)$ and dimension equal to that of M. In particular, the submanifold charts for $g^{-1}(\mathbf{R}_+)$ can be chosen in such a way that $g^{-1}(\mathbf{R}_+)$ sits as $\mathbf{H}^m \subset \mathbf{R}^m$ without further straightening—these submanifold charts would exhibit $g^{-1}(\mathbf{R}_+)$ as a neat submanifold in a different context.*

Proof. Since 0 is a regular value of g, $g^{-1}(0)$ is a codimension one submanifold of M by the usual constant rank theorem. We have that $g^{-1}((0,\infty))$ is an open submanifold being open in M. We only need to check that there is a smooth structure on this and that we have submanifold charts. Really the only issue is with the boundary. Each $p \in g^{-1}(0)$ admits a submanifold chart for $g^{-1}(0)$ and we must show we can make this a submanifold chart for $g^{-1}(\mathbf{R}_+)$.

A submanifold chart exists for each $p \in g^{-1}(0)$, say (x, U), such that $U \cap g^{-1}(0) = x^{-1}(\mathbf{R}^{m-1} \times \{0\})$. We want to show that $g^{-1}((0,\varepsilon)) \cap U$ sits in this chart as $\mathbf{H}^m$. With respect to the given chart, since $g|\, g^{-1}(0)$ is constant and $g^{-1}(0) \subset \mathbf{R}^{m-1} \times \{0\}$, g has trivial derivatives in the directions lying in the $\mathbf{R}^{m-1} \times \{0\}$ subspace. Hence, in these coordinates, for each $p \in g^{-1}(0)$, $g_{*p} = (0, \ldots, 0, v)$ for some $v \neq 0$, $v \in \mathbf{R}$—and so in any chart, $v > 0$ or $v < 0$ since by the IVT it will otherwise be 0 somewhere—since 0 is a regular value, forcing $v \neq 0$. Therefore suppose in our chart $v > 0$. Then our coordinates, each $q \in x(U)$, $q = (q^1, \ldots, q^m)$, with $q^m > 0$ has $g(q) > 0$. Hence, $U \cap g^{-1}(\mathbf{R}_+) \subset x^{-1}(\mathbf{H}^m)$ as desired. This is a submanifold chart because the boundary of $g^{-1}(0)$ already sits neatly in the chart and we do not need to do any more straightening. ∎

Lemma B.0.2. *Let M have boundary but no corners and let $f\colon M \to N$ be smooth,* $\dim M = m$, $\dim N = n$. *No point $q \in \partial N$ can be a regular value for* both *f and $f|\,\partial M$ unless $f^{-1}(q) = \emptyset$. In particular, q can only be a regular value for $f|\,\partial M$.*

Proof. We have seen in the first subsection of the annoying part of the appendix that $f^{-1}(q) \subset \partial M$ is forced if q is a regular value for f. Now, if $p \in f^{-1}(q) \subset \partial M$, then since in coordinates about p, after extending f to an open nbhd where it remains maximal rank, f looks like a projection and $f|\, f^{-1}(q) \equiv p$, $\operatorname{Ker} f_{*p} \subset T_p f^{-1}(q)$, where we are identifying $T_p f^{-1}(q)$ with its image $i_{*p} T_p f^{-1}(q)$. But then by the usual regular value theorem, $W = (f|\,\partial M)^{-1}(q) = f^{-1}(q)$ is a submanifold of dimension $m-1-n$, and each $p \in W$, $\operatorname{Ker} f|\,\partial M_{*p} = (T_{\partial M})_p f^{-1}(q)$, where this notation means the image of $T_p W$ in $T_p \partial M$.

The usual regular value theorem for manifolds M' with $\partial M' = \emptyset$ is proved by using **(c)** of the constant rank theorem and observing these yield submanifold charts. Just because N has boundary points or corner points does not mean the usual argument fails. Indeed, after performing a diffeomorphism of the domain chart we get a submanifold chart for W and we see that W has no boundary. We have to throw away the possibility that $f\colon M' \to N$ can have regular value $q \in \partial N$ with $f^{-1}(q) \neq \emptyset$, but this was argued in the appendix—roughly, when $\partial M' = \emptyset$, if f has max rank locally at a point $p \in M'$, then **(c)** of the constant rank theorem carries through and careful analysis shows that we must have $f^{-1}(q) = \emptyset$.

Now, on the other hand, f_{*p} surjects $T_pM \to T_qN$ and so has kernel dimension $m-n$, so there is a vector $v \in T_pM$ for which $f_{*p}(v) = 0$ but $v \notin \operatorname{Ker} f|\,\partial M_{*p}$. It cannot be that $v \in T_p\partial M \subset T_pM$ since then $v \in \operatorname{Ker} f|\,\partial M_{*p}$ and therefore v is an outward or inward pointing vector. Working in coordinates (x, U) and (y, V), after extending, there is a coordinate system for $x(U)$ by **(c)** of the constant rank theorem such that f looks like a projection $\mathbf{R}^m \to \mathbf{R}^n$, say projecting onto the first $m-n$ coordinates. We casually identify vectors for these Euclidean spaces with vectors in the naive sense. Let w be the image of the vector v in this coordinate system. In these coordinates, f_* is the block diagonal matrix that is $I_{m-n\times m-n}$ in the upper-left corner and 0 everywhere else. Hence, for f_* to have vanishing derivative in the direction of w, w must be a linear combination of the last n coordinates of $\mathbf{R}^m$ and therefore, in particular, $f(rw) = \mathbf{0}$ for all sufficiently small $r \in \mathbf{R}$.

Claim 20. It is not hard to see that for small enough r with one of either $r \geq 0$ or $r \leq 0$, rw remains in the image of $x(U)$ under the diffeomorphism taking us to the coordinates in which f is a projection.

One can verify the claim by noting that in the original coordinate (x, U), $x(U) \subset \widetilde{x(U)}$ where $\widetilde{x(U)}$ is the domain of the extension, a vector pointing into or out of $x(U)$ viewed as a subset of $\mathbf{R}^m$ will still do so after we perform a diffeomorphism of $\widetilde{x(U)}$—the diffeomorphism must take half-balls inside (resp. outside) the boundary of $x(U)$ to half-balls inside (resp. outside) its image. By outside, we mean its complement.

Hence, $f^{-1}(0)$ must contain points not lying in ∂M and this is a contradiction. ■

Remark. We can get a feel for what's going on here by the following corollary, which essentially states that what goes wrong is dimensional when $q \in \partial N$ is a critical point of f but not the restriction $f|\,\partial M$.

Corollary B.0.3. *If $q \in \partial N$ is a regular value for f, then for each $p \in f^{-1}(q)$,* $\operatorname{Ker} f_{*p} \subset T_p\partial M$*.*

Proof. Suppose $\operatorname{Ker} f_{*p} \nsubseteq T_p\partial M$ and let $V = \operatorname{Ker} f_{*p} \cap T_p\partial M$. Since q is a regular value for f, f_{*p} has rank $n = \dim N$ and $\dim \operatorname{Ker} f_{*p} = m-n$ and $\dim V \leq m-n-1$. Working in a boundary chart, one deduces $V = \operatorname{Ker}(f|\,\partial_M)_{*p} \subset T_p\partial M \subset T_pM$. By the rank-nullity theorem, $\dim V + \operatorname{rank}(f|\,\partial_M)_{*p} = m-1$ and therefore

$$\operatorname{rank}(f|\,\partial_M)_{*p} = m - 1 - \dim V \geq m - 1 - m + n + 1 = n$$

but also $\operatorname{rank}(f|\,\partial_M)_{*p} \leq n$ since $\dim T_qN = n$ so in fact

$$\operatorname{rank}(f|\,\partial_M)_{*p} = n$$

so q is a regular value for $f|\,\partial M$. This contradicts the above lemma. ■

Theorem B.0.4 (Regular Value Theorem). *Let M and N be smooth manifolds with boundary but no corners of dimension m and n, respectively and let $f\colon M \to N$ be smooth. If $q \in N$ is a regular value of both f and $f|\,\partial M$, then $f^{-1}(q)$ is a neat submanifold of M of codimension n (i.e., $\dim f^{-1}(q) = m-n$).*

Remark. For $q \in N$ to be a regular value of f means that for all $p \in f^{-1}(q)$, $\operatorname{rank}(df_p) = \dim N$, and this forces $\dim N \leq \dim M$. We must throw out the vacuous case in this theorem which is why we additionally stipulated that $f^{-1}(q) \neq \emptyset$.

For our assumptions, it will turn out that for $q \notin \partial N$, $\dim N \leq \dim M - 1$ if $\partial N \cap f^{-1}(q) \neq \emptyset$ and $\dim N \leq \dim M$ if $\partial N \cap f^{-1}(q) = \emptyset$. For $q \in \partial N$ it will turn out we only need $\dim N \leq \dim M$ because $f^{-1}(q) \subset \partial M$ in this case and it is furthermore not possible for q to be a regular value of both f and $f|\,\partial M$. This follows from the preceding lemma.

Proof. We have seen in the first subsection of the annoying part of the appendix that $f^{-1}(q) \subset \partial M$ is forced whenever $q \in \partial N$, so we first suppose that $q \in \partial N$ and suppose it is a regular value of $f|\,\partial M$ and thus not f. Then $(f|\,\partial M)^{-1}(q) = f^{-1}(q)$ is a submanifold of ∂M and hence M by the usual regular value theorem, the proof of which only relies on the domain manifold not having boundary (see a similar comment in a lemma above).

Now suppose $q \in \partial N$ is a critical point of f and thus not $f|\,\partial M$. Fixing any coordinate system (y, V) about q and (x, U) about p where, say, WLOG $x(p) = 0$ and $y(q) = 0$. After extending from the domain $x(U) \subset \mathbf{H}^m$ to $\widetilde{U} \subset \mathbf{R}^m$ while keeping f maximal rank, and performing a diffeomorphism of the domain $\widetilde{U}$, call it say $\widetilde{x}$—we may assume it is a diffeomorphism of the entire domain by shrinking things where necessary—f looks like a projection $\mathbf{R}^m \to \mathbf{R}^n$ onto the *last* $m-n$ coordinates. Strictly speaking, the extension $\widetilde{f}$ is an extension of yfx^{-1} to have domain $\widetilde{U}$, and then the final function in question is $\widetilde{f}\widetilde{x}^{-1}$. WLOG assume that $p = 0$ in these coordinates so that $\widetilde{f}(p) = \widetilde{f}(0) = 0$. Then $\widetilde{f}^{-1}(0)$ is a submanifold of codimension n by the usual regular value theorem.

Claim 21. $\widetilde{x}\widetilde{f}^{-1}(0) \cap \widetilde{x}x(U) = \widetilde{x}xf^{-1}y^{-1}(0) = \widetilde{x}xf^{-1}(q)$ and $\widetilde{x}x$ is an honest chart that gives a submanifold chart $f^{-1}(q)$ about p.

Now, $\widetilde{x}\widetilde{f}^{-1}(0) = \widetilde{x}(\widetilde{U}) \cap (\mathbf{0} \times \mathbf{R}^{m-n})$ since $\widetilde{f}\widetilde{x}^{-1}$ is a projection, so $\widetilde{x}\widetilde{f}^{-1}(0) \cap \widetilde{x}x(U) = \widetilde{x}x(U) \cap (\mathbf{0} \times \mathbf{R}^{m-n})$ and the RHS is just the preimage of 0 after restricting to $\widetilde{x}x(U)$ so these are equal. Hence, $\widetilde{x}x(U \cap f^{-1}(q)) \subset \mathbf{0} \times \mathbf{R}^{m-n}$ and in particular $\widetilde{x}x(U \cap f^{-1}(q)) = \widetilde{x}x(U) \cap \mathbf{0} \times \mathbf{R}^{m-n}$ so that $U \cap f^{-1}(q) = x^{-1}\widetilde{x}^{-1}(\mathbf{0} \times \mathbf{R}^{m-n})$ which shows that, if it is a chart, then it is a submanifold chart. For this last part, observe that points with last coordinate positive are sent to points with last coordinate positive, so U still gets mapped to a half space and so by restriction we then get a chart.

Now suppose $q \notin \partial N$. We begin by supposing $p \in f^{-1}(q)$ is not in ∂M for this hypothesis. Then $f^{-1}(q)$ is a submanifold in a nbhd of p. This is because, in coordinates, we may write this locally as a projection from an open subset of $\mathbf{R}^m$ onto $\mathbf{R}^n$, say killing off the first $m-n$ coordinates, with no other words needed. Hence, if $p = (a^1, \dots, a^m)$ in this coordinate system, then this is clearly a submanifold chart for $f^{-1}(q)$ about p since all points of the form $(x^1, \dots, x^{m-n}, a^{m-n+1}, \dots, a^m)$ are sent to the image of p under f in these coordinates. This takes care of the points not in the boundary of M. Next, we must consider points in the boundary of M and verify neatness as well.

Now consider the case $p \in \partial M \cap f^{-1}(q)$. Pick charts (x, U_0) and (y, V_0) such that $x(p) = 0$ and $y(q) = 0$ and set

$$x(U_0) = U \qquad y(V_0) = V.$$

We have a smooth map $U \to \mathbf{R}^m$ with U open in $\mathbf{H}^m$ the upper half-space which we may extend to an open subset $\widetilde{U} \subset \mathbf{R}^m$ and get $\widetilde{f} \colon \widetilde{U} \to \mathbf{R}^n$. Since max rank is an open condition, we may suppose this extension has max rank. It follows that $\widetilde{f}^{-1}(0)$ is a submanifold of $\mathbf{R}^m$ of codimension n (i.e., of dimension $m-n$). WLOG suppose U is an open unit coordinate ball in $\mathbf{H}^n$ and $\widetilde{U}$ is the completion of it to a full open unit coordinate ball in $\mathbf{R}^m$—we can arrange for this by shrinking; the point is that we want $\widetilde{f}$ to agree with f on the $\partial \mathbf{H}^m \subset \mathbf{R}^m$.

Let $\pi \colon \widetilde{f}^{-1}(0) \to \mathbf{R}$ be the projection onto the m-th coordinate and recall that this coordinate for any boundary chart is the outward/inward pointing direction. This has regular value 0—i.e., $\widetilde{f}^{-1}(0)$ has non-trivial tangent vectors in the x^m-direction. Suppose this was not the case. Then the tangent space to $\widetilde{f}^{-1}(0)$ at 0 (i.e., $x(p)$) would lie completely in some collection of n of the direction $\frac{\partial}{\partial x^i}$ where $i \neq m$ and so $\widetilde{f}^{-1}(0)$ lies in a subset of the first $m-1$ coordinates. But for these coordinates, one easily verifies that $(f|\partial M)^{-1}(0) = (\widetilde{f}\big|\partial \mathbf{H}^m)^{-1}(0)$ and so as a consequence of how we constructed $\widetilde{U}$ and U (see above) we have that (working in coordinates) $(f|\partial \mathbf{H}^m)^{-1}(0) = \widetilde{f}^{-1}(0) \cap \partial \mathbf{H}^m$ (i.e., those points with $x^m = 0$). Since q is a regular value for $f|\partial M$, this submanifold must have dimension $m-n-1$, but if 0 is not a regular value of the m-th coordinate projection map, then in fact $T_0(f|\partial M)^{-1}(0) \subset T_0\partial \mathbf{H}^m$ and therefore is a submanifold of dimension $m-n$, which is a contradiction.

Now, $f^{-1}(0) = \pi^{-1}(\mathbf{R}_+)$ and by the **Lemma**, $\pi^{-1}(\mathbf{R}_+)$ is a submanifold of $\widetilde{f}^{-1}(0) \subset \widetilde{U}$ contained in U with boundary $\pi^{-1}(0)$—that is, $\widetilde{f}^{-1}(0) \cap U = f^{-1}(0)$. Thus, $f^{-1}(0)$ admits reasonable submanifold charts in $\widetilde{f}^{-1}(0)$ and has codimension 0 therein. We also know that $f^{-1}(0)$ is a submanifold of U since U is a submanifold of $\widetilde{U}$ for the obvious reasons (consider how we constructed U and $\widetilde{U}$). It remains to show that it is *in addition* neat.

The only trouble arises for points in $\pi^{-1}(0)$, so let (α, U_α) be a submanifold chart for $\pi^{-1}(0)$ in $\widetilde{f}^{-1}(0)$. Then (after rearranging) $U_\alpha \cap \pi^{-1}(0) = \alpha^{-1}(\mathbf{0} \times \mathbf{R}^{m-n-1} \times \{0\})$. Since $i \colon \widetilde{f}^{-1}(0) \to \widetilde{U}$ is an embedding between manifolds without boundary, **(d)** of the constant rank theorem guarantees that there is a chart (β, V_β) such that (after rearranging) $\beta i \alpha^{-1}(a^1, \dots, a^{m-n}) = (0, \dots, 0, a^1, \dots, a^{m-n})$. The reasoning of the preceding **Lemma** shows us that $\pi^{-1}(\mathbf{R}_+)$ must sit as the collection of points in the image having the form $(0, \dots, 0, a^1, \dots, a^{m-n-1}, v)$ where either $v \geq 0$ for all such a^i or $v \leq 0$ for all such a^i. ■

Theorem B.0.5. *Let M^m and N^n be smooth manifolds with boundary of dimension m and n, respectively. Let $A \subset N$ be a k-dimensional submanifold without boundary. If $f \colon M \to N$ is smooth and $f \pitchfork A$ and $f|\partial M \pitchfork A$, then $f^{-1}(A)$ is a neat submanifold of codimension $n-k$ (i.e., dimension $m-n+k$) with $\partial f^{-1}(A) = f^{-1}(\partial A)$.*

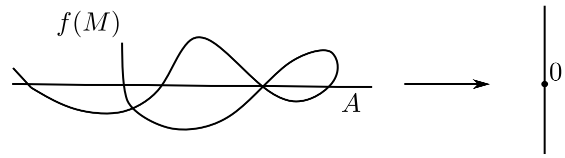
Remark. If A has no boundary, A is not automatically neat because of the example of the parabola kissing the disk.

Proof. Either $\partial A = \emptyset$ or $A \cap \partial N = \partial A$. First consider the interior points of A. These are points which, by definition, also lie in the interior of N. In particular, $A \setminus \partial A$ is a smooth boundary-less manifold and $N \setminus \partial N$ is too. Since $A \cap \partial N = \partial A$, we may choose our submanifold chart about for each $q \in A \setminus \partial A \cap \operatorname{Im}(f)$ to be an *interior chart* of N and, perhaps by shrinking, we may suppose our submanifold chart (y, W) about q has image a product nbhd $y(W) = U \times V \subset \mathbf{R}^k \times \mathbf{R}^{n-k}$ such that $y(A \cap W) = U \times \mathbf{0}$. Pick coordinates, (x, Z) about $p \in f^{-1}(q)$ in M with Z so small that $f(Z) \subset W$, so we don't have to worry about intersecting things. To avoid breaking into cases, suppose $x(Z) \subset \mathbf{H}^m$ is open but we do not specify whether (x, Z) is a boundary chart or not. Transversality of f to A then becomes transversality of $y \circ f \circ x^{-1}$ to $U \times \mathbf{0}$ and transversality of $f|\partial M$ to A then similarly becomes transversality of $y \circ f \circ x^{-1}\big|(Z \cap \partial M)$ to $U \times \mathbf{0}$. The first of these is equivalent to the assertion that the composite

$$g\colon \mathbf{H}^m \supset x(Z) \xrightarrow{f\circ x^{-1}} W \xrightarrow{y} U \times V \xrightarrow{\mathrm{pr}} V \subset \mathbf{R}^{n-k}$$

has regular value $\mathbf{0}$ and the latter that $g|\,\partial\mathbf{H}^m$ has regular value $\mathbf{0}$. Transversality of $f|\,\partial M$ to A then becomes transversality of $y \circ f \circ x^{-1}\big|\,(Z \cap \partial M)$ to $U \times \mathbf{0}$.

This shows that $g^{-1}(\mathbf{0})$ is a submanifold of W having codimension $n-k$ (i.e., dimension $m-n+k$) as a consequence of the regular value theorem proved above. In other words, $x(f^{-1}(y^{-1}(U \times \{0\}))) = x(f^{-1}(A \cap W))$ is a submanifold of Z. But x is a diffeomorphism from Z onto $x(Z)$, so $f^{-1}(A \cap W)$ must be a submanifold of M. Now suppose $q \in \partial A$ and so by neatness of A, $q \in A \cap \partial N$. Since A is neat, we may replace our target chart (y, W) by a neat submanifold chart for $q \in A \cap \partial N$. Then the same argument above works, replacing V by an open $V \subset \mathbf{H}^{n-k}$ intersecting the boundary. ■



Corollary B.0.6. *Let M^m have boundary and no corners and let $K, L \subset M$ be neat submanifolds of dimensions k and ℓ, respectively. If $K \pitchfork L$ and $\partial K \pitchfork L$, then $K \cap L$ is a neat submanifold of M of dimension $k + \ell - m$. In fact, in this setup, $\partial K \pitchfork L$ is equivalent to $K \pitchfork \partial L$ and $K \cap L$ is a neat submanifold of both K and L of dimension $k + \ell - m$.*

Remark. Since K and L are submanifolds, $\dim K, \dim L \le m$ ($k, \ell \le m$) and since they are transverse, $\dim K + \dim L \ge \dim M$ ($k + \ell \ge m$) because for all $p \in K \cap L$, $T_pK + T_pL = T_pM$. When K and L *both* have boundary, then this inequality tightens to $k + \ell - 1 \ge m$ because we assumed $\partial K \pitchfork L$.

Proof. The last statement follows from showing it for one submanifold by symmetry. Let $f\colon K \to M$ be the neat embedding of K into M. Since $f \pitchfork L$ and $f|\,\partial K \pitchfork L$, it follows by the the preceding that $f^{-1}(L)$ is a neat submanifold of K of dimension $k - m + \ell = k + \ell - m$ (i.e., of codimension $m - \ell$). We want to show that the neat embedding f restricts to a neat embedding $f\colon K \cap L \to M$.

The result now follows from the following claim, whose proof is exemplary of the utility of thinking locally.

Claim 22. If $A \subset B \subset C$ and B is neat in C and A is neat in B, then A is neat in C (neatness forces $\dim A \ge 1$).

Say $\dim A = i$, $\dim B = j$ and $\dim C = k$. We make some reductions. Pick a neat submanifold chart for $a \in \partial B \cap \partial a$ in C, call it (y, V). Using this chart, we may reduce to the Euclidean case where we suppose, in particular, that $C = \mathbf{H}^k$, $B = \mathbf{0} \times \mathbf{H}^j$ and $A \subset B$ is neat—we may make this assumption by shrinking to a subset diffeomorphic to the open unit half-ball in $\mathbf{H}^k$ via our chart and then using the evident radial diffeomorphism. We have this reduced to the case that $A \subset \mathbf{0}_{k-j} \times \mathbf{H}^j \subset \mathbf{H}^k$ with A neat in $\mathbf{0} \times \mathbf{H}^j$.

Suppose WLOG $0 \in A$ is our new a. Pick a neat submanifold chart (x, U) for A about 0 in $\mathbf{0} \times \mathbf{H}^j$ and suppose U is the open unit half-ball in $\mathbf{H}^j$. Then $x\colon U \to \mathbf{H}^j$ is a diffeomorphism for which $x(U \cap A) = x(U) \cap \mathbf{0}_{j-i} \times \mathbf{H}^i \subset \mathbf{H}^j$. We can now extend this to a chart for $\mathbf{H}^k$ having domain the open unit half-ball B_1 as follows. For $a = (a^1, \ldots, a^{k-j}, a^{k-j+1}, \ldots, a^k) \in B_1$, we define a chart (y, B_1) by $a \mapsto (a^1, \ldots, a^{k-j}, x(a^{k-j+1}, \ldots, a^k))$. Since $y = (\mathrm{pr}, x)$ on its domain, where pr is the projection onto the first $k - j$ coordinates, it is clearly a diffeomorphism. The inverse is $y^{-1} = (\mathrm{pr}, x^{-1})$ which is likewise smooth. Thus, this is a chart and moreover $y(V \cap A) = y(U \cap A) = x(U \cap A) = x(U) \cap \mathbf{0}_{k-i} \times \mathbf{H}^i \subset \mathbf{H}^k$ as desired. ■

Lemma B.0.7. *If $p\colon E \to B$ is an orientable vector bundle of rank $n \ge 1$ and $i\colon X \to B$ is an embedding, then the induced bundle $i^*p\colon i^*E \to X$ formed by the pullback is orientable.*

Proof. Since i is an embedding, one easily verifies that there is bundle isomorphism $i^*E \cong p^{-1}(X) = E|\,X$. This is verified topologically by universal properties and one then checks that the homeomorphism given is in fact a bundle isomorphism by recalling how the vector space structure is defined on the fibers of i^*E.

We therefore give each fiber $p^{-1}(x)$ the orientation μ_x is had originally. Fix a trivializing open nbhd U in B of a point $x \in X$. Then $U \cap X$ is a trivializing open nbhd in X. Moreover, one quickly verifies that $p^{-1}(X) \supset p^{-1}(U \cap X) \hookrightarrow p^{-1}(U) \cong U \times \mathbf{R}^n$ is therefore orientation preserving or orientation reversing everywhere, and so $i^*E \cong p^{-1}(X)$ is orientable in the obvious way. ■

It once again helps to know the definition of orientability of a vector bundle over M.

Theorem B.0.8. *Fix $n \ge 1$. Let $N \subset M$ is be a submanifold of an orientable manifold with corners M and suppose $\dim N = \dim M - 1$ (i.e., a hypersurface). Then N is orientable **iff** the normal bundle of N is trivial.*

Remark. M being orientable is surely needed since the Möbius band M is not orientable and $\partial(M \times [0,1)) \cong M$ is not orientable, where $\dim \partial M = \dim(M \times [0,1)) - 1$.

Proof. ($\Leftarrow$) Suppose the normal bundle of N is trivializable. It follows that $TM|\,N \cong TN \oplus \underline{\mathbf{R}}$. Since M is orientable, $TM|\,N = TN \oplus \underline{\mathbf{R}}$ is orientable, we claim, and this follows from the preceding lemma. The other lemma now shows that TN must be orientable and hence N is orientable. ($\Rightarrow$) Is N is orientable, then TN is orientable. Hence, $0 \to TN \to TN \oplus \nu_N \to \nu_N \to 0$ is a SES of vector bundles and the middle one is orientable once again because M is orientable and we have an isomorphism $TM|\,N \cong TN \oplus \underline{\mathbf{R}}$. Hence, ν_N must be orientable. But the only orientable line bundle is trivial, so we conclude. ■

Bundles, Normal Bundles, Tubular Neighborhoods

C.1 Bundle Potpourri

Proposition C.1.1. *Let B be a paracompact Hausdorff space and $p\colon E \to B$ be a vector bundle. Then E admits a metric (i.e., inner product).*

Proof. Define $E^* \otimes E^*$ as before and define S^2E^* as before. Construct local sections $\omega\colon U_\alpha \to S^2E^*\big|_{U_\alpha}$. $\omega(x) = \sum_{ij} \omega_{ij}(x)(\ell_i(x) \otimes \ell_j(x))$ (in general). Set $\omega(x) = \sum_i \ell_i(x) \otimes \ell_i(x)$. Then ω is positive definite. Partition of unity $\{\lambda_i\}$. Convex linear combination (adds to 1, not negative) $\sum \lambda_i \omega_i$ for positive definite ω_i. Since this is a convex linear combination of positive definite forms, the resulting function is positive definite. ■

Remark. Paracompact Hausdorff is equivalent to the statement that every open cover admits a subordinate partition of unity.

Lemma C.1.2. *Let $p\colon E \to B$ be a vector bundle of (as we always implicitly assume) finite rank. Then the dual bundle $E^\vee$ exists and there is a natural isomorphism of bundles $E^{\vee\vee} \cong E$. Moreover, $E^\vee \cong \operatorname{Hom}(E, \underline{\mathbf{R}})$.*

Proof. $E^\vee$ is constructed as in the vector/fiber bundle construction lemma. To show that $E^{\vee\vee} \cong E$ naturally, we simply let $E_p^{\vee\vee} \cong E_p$ be the natural double duality isomorphism for FDVSs. On trivializations, this is basically just $U \times \mathbf{R}^{\vee\vee} \to U \times \mathbf{R}$.

For the next part, pick a trivialization U for E. Then $\operatorname{Hom}(E, \underline{\mathbf{R}})$ on U has trivialization given essentially by doing φ^{-1*}—that is, on fibers it is $\operatorname{Hom}(E_p, \mathbf{R}) \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$. ■

Theorem C.1.3. *Let $f\colon E' \to E$ be a morphism of smooth vector bundles over M. The function $p \mapsto \dim \operatorname{Ker} f_p$ is locally constant **iff** there is a covering of M by open sets U_i such that $E'|\,U_i$ admits a trivializing frame containing a subset whose specialization in each fiber over each point $p \in U_i$ is a basis of $\operatorname{Ker} f_p$ (i.e., a subset of the collection of specified local sections on U_i are at each point a basis for the kernel).*

Proof. ($\Leftarrow$) This is obvious. ($\Rightarrow$) WLOG we may assume the U_i are path-connected. Admitting a trivializing frame is the same as saying the U_i are trivializing, we remark. Since we have assumed local constant-ness, we may assume that for all $p \in U_i$, $\dim \operatorname{Ker} f_p = d$. Let $\{s_i'\}$ and $\{s_j\}$ be trivializing frames with $1 \le i \le n'$ and $1 \le j \le n$ so that $r = n' - d$ is the common rank of the maps f_p on U_i. We can write

$$f(s_j') = \sum_i a_{ij} s_i$$

since the s_i are a local frame where $a_{ij}\colon U_i \to \mathbf{R}$ are smooth functions. For each $p \in U_i$, since f_p has rank r (i.e., for all $p \in U_i$, $\operatorname{rank} f_p = r$). It follows from standard linear algebra that an $r \times r$ submatrix of $(a_{ij}(p))$ has full rank (i.e., is invertible), call it $A(p)$ where A is the function which is this *particular* submatrix at all points. Since rank is a lower semi-continuous function, the set of points $q \in U_i$ for which $\operatorname{rank} A(q) > r - 1$ is open. Hence, we can cover U_i by open sets for which some submatrix satisfies this property—say we cover U_i by U_α for which a submatrix A_α is invertible and let I_α and J_α be the sets of indices picking out A_α in (a_{ij}).

Fix α and restrict attention to U_α. WLOG suppose that the upper left $r \times r$ matrix of (a_{ij}) is A_α, perhaps by rearranging indices. Since (a_{ij}) has rank r on U_α, it is easy to see that the first r columns of (a_{ij}) span the image of (a_{ij}) at each point—basically this is because a linear dependency among the full column vector would imply a linear dependency for A_α which is impossible because A_α is invertible. Hence, for each $j > r$ and $p \in U_\alpha$, there is a unique linear combination in E_p'

$$f(s_j')(p) = \sum_{k=1}^{r} c_{kj}\, f(s_k')(p) = \sum_{k=1}^{r} \sum_{i=1}^{n'} c_{kj} a_{ik}(p) s_i(p).$$

Of course, also, by linear independence of the s_i everywhere, we must have that

$$a_{ij}(p)s_i(p) = \sum_{k=1}^{r} c_{kj}a_{ik}(p)s_i(p)$$

or in other words

$$a_{ij}(p) = \sum_{k=1}^{r} a_{ik}(p)c_{kj}.$$

This gives a system of n' equations for fixed j by varying i. Since $(a_{ij})_{1\le i,j\le r}$ is invertible everywhere, ***Cramer's Rule*** allows us to solve for each c_{kj} uniquely such that all of these n' equations are satisfied. In particular, Cramer's rule tells us that each c_{kj} is a rational function of the $a'_{ij}s$ with denominator the determinant polynomial which is non-vanishing by assumption. So these are all smooth.

Hence, we get d sections

$$v_j = s'_{j+r} - \sum_{k=1}^{r} c_{k,j+r}s'_k$$

with $1 \le j \le d$ such that $v_j(p) \in \operatorname{Ker}(f|_p)$ for all $p \in U_\alpha$. One sees this since we just showed for $j > r$ that $f(s'_j) = \sum_{k=1}^{r} c_{kj}f(s'_k)$ and f is linear on each fiber so this means that $f(s'_j) - f(\sum_{k=1}^{r} c_{kj}s'_k) = 0$ and so $s'_j - \sum_{k=1}^{r} c_{kj}s'_k$ is in the kernel of f at each point but $s'_j - \sum_{k=1}^{r} c_{kj}s'_k \neq 0$ by linear independence of the s'_i.

By inspection, the d vectors v_j are linearly independent essentially because if $j \neq j'$ then v_j has a factor of s'_{j+r} whereas $v_{j'}$ has a factor of $s_{j'+r}$. Hence, dimension considerations force $v_1, \ldots, v_d$ to span $\operatorname{Ker} f|_p$ at each point $p \in U_\alpha$.

Finally, consider the n' sections $s'_1, \ldots, s'_r, v_1, \ldots, v_d$. By construction, for each $p \in U_\alpha$, $f(s'_1(p)), \ldots, f(s'_r(p))$ are a basis for the image of $f|_p$ whereas $v_1(p), \ldots, v_d(p)$ are a basis for its kernel. Hence, together they form a basis for E'_p by dimension considerations and the Rank-Nullity theorem. ■

Corollary C.1.4. *Let $f\colon E \to E'$ be a bundle surjection over B, then $p \mapsto \operatorname{Ker} f_p$ is locally constant* ***iff*** $\operatorname{Ker} f$ *is a subbundle of E.*

Proof. (⇐) Trivial. (⇒) We have local trivializing frames by the preceding theorem. ■

Corollary C.1.5. *If $f\colon E \to E'$ is a bundle surjection then* $\operatorname{Ker} f$ *is a subbundle of E.*

Theorem C.1.6. *Let $f\colon E \to E'$ be a smooth bundle map over B. Then f is a bundle isomorphism* ***iff*** *it is a fiberwise (linear) isomorphism.*

Proof. (⇒) This is obvious.

(⇐) Choose local coordinates about a point $b \in B$—say with the same trivializing open set U WLOG—with trivializations g and h for E and E', respectively, and consider the composite $U \times \mathbf{R}^n \xrightarrow{h^{-1}fg} U \times \mathbf{R}^n$—we must show this is a diffeomorphism.

Since f is smooth, $h^{-1}fg$ is smooth. Working in the evident local frame, we see that this map is therefore given by mapping $(b, x) \mapsto (b, y)$ with

$$y^i = \sum_j a_{ij}(b)x^j$$

and, furthermore, that the non-singular matrix $(a_{ij}(b))_{i,j}$ varies smoothly with b. The formula for the inverse matrix involves dividing by the determinant and the cofactors of the given matrix—these are all polynomial in the entries of (a_{ij}) and thus is smooth on the same domain. Denote the inverse matrix by $(A_{ij}(b))$. Then

$$g^{-1}f^{-1}h(b, y) = (b, x) \qquad x^i = \sum_j A_{ij}(b)y^j$$

and this is smooth since A_{ij} depends smoothly on b as we have just argued. ■

Reminder. Recall that a subbundle of a vector bundle $p\colon E \to B$ is a subspace $E' \subset E$ such that for all $p \in B$

(a) $E'_p \subset E_p$ is equipped with the natural vector subspace structure coming from E_p;
(b) $E'_p \subset E_p$ has rank constant k (at least, say, on each connected component of E if we really want to include that possibility).

We also demand that $p\colon E' \to B$ has the structure over a vector bundle over B. If we forget to say smooth before subbundle, we will probably mean a smooth subbundle, which is a subbundle that is also a submanifold of E.

Lemma C.1.7. *Let $p'\colon E' \to B$ and $p\colon E \to B$ be smooth vector bundles over B of rank n' and n respectively and let $i\colon E' \to E$ be a smooth bundle morphism which is injective on fibers (a bundle monomorphism).*

(a) *Then $i(E')$ is a smooth subbundle of E. In particular, i is a closed embedding and immersion (hence, a submanifold inclusion) and i locally looks like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} = U \times \mathbf{R}^n$.*

(b) *If $f\colon E'_1 \to E$ is a bundle map over B with $f(E'_1) \subset i(E')$, then there is a unique smooth bundle map $\phi\colon E'_1 \to E'$ over B such that $i\phi = f$. If f is a fiberwise isomorphism, then ϕ is a smooth bundle isomorphism. In particular, the subset $i(E') \subset E$ uniquely determines the pair (E', i) up to a unique smooth bundle isomorphism.*

We shall do this by showing that there are local trivializations determined by frames such that n' of the local sections lie entirely in E' entirely and constitute a frame for E'—we then extend this to a local frame for E.

Proof. **(a)** i is obviously injective. We will first show that i is a closed immersion. Let U be a common trivialization of E' and E perhaps by shrinking things enough. We may also suppose U is path-connected. Restricting to U, we may suppose that the bundles in question are both trivial. Henceforth we assume the bundles over B are trivial.

Pick trivializing frames $\{s'_k\}$ and $\{s_j\}$. There is an $n \times n'$ matrix (a_{jk}) such that $i_p s'_k(p) = \sum_j a_{jk}(p)s_j(p)$ where $a_{jk}\colon B \to \mathbf{R}$ are smooth. This has rank n' at all points i is injective on all fibers. It is a standard linear algebra fact that at each point $p \in B$, an $n' \times n'$ submatrix of $(a_{jk}(p))$ is invertible. Since rank is lower semi-continuous, this is an open condition. Hence, we can once again pass to smaller (connected) neighborhood, say $V \subset U$ on which the *same* $n' \times n'$ submatrix of (a_{jk}) is invertible at all points. Hence, we might as well assume that the bundles are trivial and, furthermore, that upper left $n' \times n'$ submatrix of (a_{jk}) is everywhere invertible on B (perhaps after rearranging indices).

Notation. Denote is'_k the function $i(s'_k)$ for each $1 \le k \le n'$.

Denote this submatrix by $A(p)$ at each point $p \in B$.

Observation. The $n \times n$ matrix (call is M) of smooth functions representing $\Sigma = (is'_1, \dots, is'_{n'}, s_{n'+1}, \dots, s_n)$ in the basis of the s_j's has upper left $n' \times n'$ submatrix A. Furthermore the upper right $n' \times (n-n')$ submatrix is 0, the lower right $(n-n') \times (n-n')$ submatrix is the identity matrix.

These observations imply that the matrix M is invertible at all points $p \in B$—for instance, expanding the determinant along the last column each time will reduce us to computing $\det A$ so that $\det M = \pm \det A$. It follows that $M(p)$ is a basis for the vector space over p for each $p \in B$. In particular, Σ comprises a trivializing frame.

The bundle morphism i in the bundle charts determined by $\{s'_k\}$ and Σ is then

$$(p, (v_1, \dots, v_{n'})) \mapsto (p, (v_1, \dots, v_{n'}, 0, \dots, 0)).$$

It is easy to see from this description that i is an immersion and an embedding. To see that $\mathrm{Im}(i)$ is closed, let $v \in E \setminus \mathrm{Im}(i)$ and say it lies over the fiber over $p \in B$. In coordinates, this looks like $v \in V \times \mathbf{R}^n \setminus \mathbf{R}^{n'} \times \mathbf{0}$ and from this description it is clear that the complement is open so that $\mathrm{Im}(i)$ is closed.

Since i is a closed injective immersion, it is a proper injective immersion. Proper injective immersions are exactly the submanifold inclusions with closed image. Since the given map is in fact closed, this is already satisfied.

(b) Once we build ϕ uniquely as a bundle map, then when f is a fiberwise isomorphism we will have that ϕ is a bundle map that is bijective on fibers and hence a bundle isomorphism. Uniqueness of ϕ follows immediately since i is a fiberwise injection. As for existence of ϕ, it is certainly a set map that is fiberwise linear. To check smoothness, it is enough to work locally. By **(a)**, we may assume i is locally the standard inclusion. Then we are reduced to showing that the smooth map $U \times \mathbf{R}^{n'_1} \to U \times \mathbf{R}^n$ which lands in the submanifold $U \times \mathbf{R}^{n'} \subset U \times \mathbf{R}^n$ is smooth, and this is clear even without using the universal property of submanifolds because of the niceness of the standard smooth structure on Euclidean space. ■

Lemma C.1.8. *Let $E \to B$ be a vector bundle of rank n and let $E' \subset E$ be a fiberwise subset having constant dimension n'. Then E' is a subbundle of rank n' over B **iff** there is a covering $\{U_i\}_{i \in I}$ of B by trivializing open sets such that over each U_i there exists a vector bundle E''_i and bundle isomorphisms $\varphi_i\colon\ E'|U_i \oplus E''_i \cong E|U_i$ satisfying that the composite $E'|U_i \to E'|U_i \oplus E''_i \cong E|U_i$ is the inclusion map over U_i.*

Remark. The idea is take local frames for E' and E and apply linear algebra to see that at a point p there is a basis for the fiber E_p that contains the frame for E' evaluated at p. Then we use calculus to show this holds in fact holds locally.

Proof. ($\Rightarrow$) We can construct frames for both bundles $\{s'_i\}$ and $\{s_j\}$ over a small enough trivializing nbhd U. Fix $p \in U$. Then some subcollection of the s_j's append to $\{s'_i\}$ to construct a linearly independent set at p, WLOG say $s_{n'+1}, \dots, s_n$. The $n \times n'$ matrix (a_{jk}) of smooth functions satisfying $s'_k = \sum a_{jk}s_j$ has rank n' everywhere and therefore has an $n' \times n'$ invertible submatrix at p, which we may suppose after rearranging indices is the block $(a_{jk})_{1 \le j,k \le n'}$. This is an open condition so let $p \in V \subset U$ be open where this block is invertible. On V it follows that the matrix of coefficients for $\{s'_1, \dots, s'_{n'}, s_{n'+1}, \dots, s_n\}$ in terms of the $\{s_j\}$ has upper left $n' \times n'$ block $(a_{jk})_{1 \le j,k \le n'}$ (perhaps after rearranging), upper right $n' \times (n-n')$ block 0 and lower right $(n-n') \times (n-n')$ block the identity matrix. Hence, this matrix is invertible and so is invertible locally on $p \in V' \subset V \subset U$ and so furnishes a frame.

This construction gives us a trivialization for which $E'|\,V' \cong V' \times \mathbf{R}^{n'} \times \mathbf{0} \subset V' \times \mathbf{R}^n \cong E|\,V'$. Let $E'' = V' \times \mathbf{0} \times \mathbf{R}^{n-n'}$. That $E'|\,V' \oplus E'' \cong E|\,V'$ in the desired manner follows by

$$E'|\,V' \oplus E''|\,V' \cong (X \times \mathbf{R}^{n'}) \oplus (X \times \mathbf{R}^{n-n'}) \cong X \times (\mathbf{R}^n \oplus \mathbf{R}^{n-n'}) \cong X \times \mathbf{R}^n \cong E|\,V'$$

where in the first isomorphism we used the local frame $\{s'_1, \ldots, s'_{n'}\}$ on E' over V' to construct the isomorphism, noting that $E''|\,V' = X \times \mathbf{R}^{n-n'}$, and in the last isomorphism we used the inverse of the trivialization afforded by $\{s'_1, \ldots, s'_{n'}, s_{n'+1}, \ldots, s_n\}$. This obviously respects the inclusion in the sense that the composite $E'|\,V' \to E'|\,V \oplus E''|\,V' \cong E|\,V'$ is the inclusion.

($\Leftarrow$) The conditions here imply that E' has the structure of a smooth vector bundle since smoothness is local and it is clearly subbundle from the condition here as well. ■

Corollary C.1.9. *If $E' \subset E$ is a subbundle of $p\colon E \to B$ where E' has rank n' and E has rank n, then there are bundle charts of E covering B such that $\varphi_i\colon (p^{-1}(U_i), p^{-1}(U_i) \cap E') \cong (U_i \times \mathbf{R}^n, U_i \times \mathbf{R}^{n'} \times \mathbf{0})$.*

Proof. We constructed these charts above. ■

Corollary C.1.10. *Let $E' \subset E$ be a subbundle of rank n' of the vector bundle $p\colon E \to B$ of rank n. Then the **quotient bundle** $E/E' \to B$ exists.*

Proof. Using the charts above, we may fix and consistently use the obvious isomorphism $\mathbf{R}^n/\mathbf{R}^{n'} \times \mathbf{0} \cong \mathbf{R}^{n-n'}$ sending a vector to the element defined by its last $n - n'$ coordinates. Define E/E' to be fiberwise the quotient E_b/E'_b. Pick bundle charts U_i for E such that $p^{-1}(U_i) \cap E'$ maps under the trivialization to $U_i \times \mathbf{R}^{n'} \times \mathbf{0}$ and let $q\colon E/E' \to B$ be the obvious projection. We topologize $q^{-1}(U_i)$ by declaring the isomorphism of sets $q^{-1}(U_i) \cong U_i \times \mathbf{R}^{n-n'}$ induced by $p^{-1}(U_i) \cong U_i \times \mathbf{R}^n \to U_i \times \mathbf{R}^n/\mathbf{R}^{n'} \cong U_i \times \mathbf{R}^{n-n'}$ by the universal property of the quotient to be a *homeomorphism*. By giving U_i the inherited smooth structure, we can pull back the smooths structure on $U_i \times \mathbf{R}^{n-n'}$ to give $q^{-1}(U_i)$ a smooth structure. We generate topologies/take maximal atlases everywhere. ■

Corollary C.1.11. *Every subbundle of rank k of a real bundle $p\colon E \to B$ of rank n over a paracompact Hausdorff space B admits a complement. In particular, if $E_1 \subset E$ is a subbundle, then $E/E_1 \cong E_1^\perp$ (non-canonically, I think) for any choice of metric on E. In particular, $E \cong E_1 \oplus E_1^\perp$ and $E/E_1 \cong E_1^\perp$.*

Proof. Let $E_1 \subset E$ be a subbundle over B. Fix a metric g and let $E_1^\perp$ be its fiberwise orthogonal complement. One can check that $E_1^\perp$ is a subbundle and that $E \cong E_1 \oplus E_1^\perp$. Denote $q\colon E/E_1 \to B$ and $q_1\colon E_1^\perp \to B$ the bundle projections (the latter being the restriction of p to $E_1^\perp$ and the former being defined in essentially the same manner). Note further that we can give the bundle $q_1\colon E_1^\perp \to B$ the *same* trivializations as q and as p. For E/E_1, the trivializations are defined as above.

There is a fiberwise isomorphism $E_1 \oplus E_1^\perp \to E$ by sending vectors to their sum. Note that this sends the obvious subbundle $E_1 \oplus 0$ to the subbundle E_1 diffeomorphically, clearly. To see that this is smooth, note that in coordinates this looks like $U \times \mathbf{R}^k \times \mathbf{R}^{n-k} \to U \times \mathbf{R}^n$ sending $(p, v, w) \mapsto (p, v + w)$ and this is certainly smooth. To get this description, we just have to observe that local frames for E_1 and $E_1^\perp$ yield a local frame for their direct sum as well as for E. Since this is smooth and bijective, it is a diffeomorphism.

The last thing to check is that $E_1 \oplus E_1^\perp/E_1 \cong E_1^\perp$, since it surely must be that $E_1 \oplus E_1^\perp/E_1 \cong E/E_1$ because the isomorphism given above preserves the copies of E_1. Define $E_1 \oplus E_1^\perp \to E_1^\perp$ by sending $(p, v, w) \mapsto (p, w)$. This descends to the desired fiberwise quotient as a function. The description of the quotient given above immediately shows that it is smooth with little effort. ■

Before this next theorem, we need an easy auxiliary result.

Proposition C.1.12. *If $f\colon M \to N$ is smooth and $q \in \partial N$ is a regular value for f, then $f^{-1}(q) \subset \partial M$. More generally, for smooth $f\colon M \to N$ with $\dim N \leq \dim M$ where f has maximal rank at $p \in M$, if $f(p) \in \partial N$ then $p \in \partial M$.*

Proof. For f to even have a regular point p, we must have that $\dim N \leq \dim M$. We will prove parts in one go, since nothing we do below will depend on q being a regular value of f, only that $f(p) = q$ and f has maximal rank $\dim N \leq \dim M$ at p.

Let $q \in \partial N$ be a regular value. If $p \in f^{-1}(q) \cap \operatorname{Int}(M)$, then we can take a small enough coordinate nbhd about p such that the coordinate chart is strictly Euclidean. Working in coordinates, our map, then has the form

$$\mathbf{R}^m \supset U_0 \xrightarrow{f} \mathbf{R}^{n-k} \times \mathbf{R}^k_+ \subset \mathbf{R}^n.$$

By **(c)** of the constant rank theorem, there is a chart for $\mathbf{R}^m$ about $p \in U_0$, call it (x, U), perhaps by shrinking U and by abuse of notation, such that the new coordinate form of $(*)$ looks like a projection $U \to \mathbf{R}^n$ killing the last $m - n$ coordinates. Shrinking U_0 to be contained in U, we now wish to analyze what the image of U_0 is under this projection.

Observation. Note that we only change the domain chart, so the image remains unchanged.

Since $U_0 \subset \mathbf{R}^m$ is open and the projection $\mathbf{R}^m \to \mathbf{R}^n$ is an open map, the restriction to U_0 is an open map onto its image which is an open subset of $\mathbf{R}^n$—by the observation, regardless of the domain chart, f takes U_0 into $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$. But the open subsets of $\mathbf{R}^n$ contained in $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$ are precisely the ones that miss $\partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_+)$, since there is no open subset V of $\mathbf{R}^n$ for which $W = V \cap \partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_+) \neq \emptyset$ is open, and this is because $W \cap \partial W \neq \emptyset$.Thus, $f(p) = q$ could not possibly be in ∂N since, in coordinates, it misses $\partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_+)$. This is the first part. As mentioned at the beginning, this has secretly shown the second part. ■

Warning. I do not think this analysis can be extended further because we would have to extend from an open subset of $\mathbf{R}^{m-\ell} \times \mathbf{R}^\ell_+$ to an open subset of $\mathbf{R}^m$ and we can no longer guarantee that the extension stays in $\mathbf{R}^{n-k} \times \mathbf{R}^k_+$.

Remark. The contrapositive of the above is that for smooth f with maximal rank $\dim N$ at p, $p \in \operatorname{Int}(M)$ implies that $f(p) \in \operatorname{Int}(N)$. So for a submersion, it appears to be possible that boundary points in M are send to interior points in N.

Corollary C.1.13. *There is no surjective smooth submersion $f\colon M \to N$ for $\partial M = \emptyset$ and $\partial N \neq \emptyset$.*

Proof. Easy consequence of the above. ■

Theorem C.1.14. *Let $\pi\colon E \to B$ be a smooth vector bundle and let $i\colon Z \hookrightarrow E$ be a closed injective immersion such that $Z \cap E_b$ is a linear subspace for all $b \in B$ whose dimension is locally constant as a function of $b \in B$ (in particular, $Z \cap E_b \neq \emptyset$ for any $b \in B$). Suppose in addition the following properties hold.*

(i) *Locally, i can be made to look like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^n$.*
(ii) *For every $z \in Z$, there is a smooth local section about $\pi(z) \in B$ of π such that $\pi(z) \mapsto z$.*

Then Z admits a unique structure as a smooth vector bundle over B for which i is a subbundle inclusion. If Z has no boundary, then B has no boundary and the local assumptions are automatically satisfied.

Proof. WLOG B is connected. Then Z inherits a linear structure on its fibers and we must only check there are compatible local trivializations for this structure. By the universal property of submanifolds (a closed injective immersion is an embedding, after all), the zero section $B \to E$ which lands in Z is smooth into Z. Thus, $B \xrightarrow{0} Z \xrightarrow{\pi} B$ is smooth and the identity and so by the chain rule $Z \to B$ is a submersion. It is surjective from our assumptions.

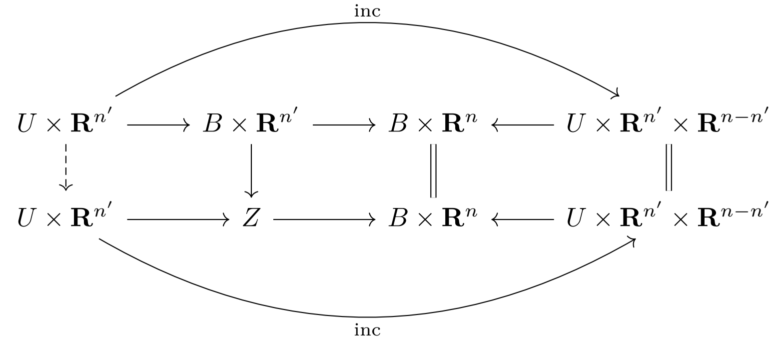
When Z has no boundary, there can be no smooth surjective submersion onto B unless $\partial B = \emptyset$ as well by the above. Hence, by the constant rank theorem, the desired local sections will clearly exist. Thus, for each $b \in B$, if $X_1, \ldots, X_{n'}$ is a basis of Z_b, then we can find, locally, smooth sections $s_1, \ldots, s_{n'}$ such that $s_i(b) = X_i$ for each i.

Having shown the above, we return to the general case. By shrinking, we may suppose the problem is local and thus we may suppose the bundle E is trivial over B, say

$$E = B \times \mathbf{R}^n.$$

Let the $s_1, \ldots, s_{n'}$ be as in the paragraph above for some $b \in B$. At b, these smooth sections $s_1, \ldots, s_{n'}$ form an $n \times n'$ matrix of rank n' and therefore there is an $n' \times n'$ submatrix which is invertible and thus invertible on a nbhd of b. Shrinking again we may suppose that this is globally invertible and thereby suppose the sections $s_1, \ldots, s_{n'}$ are fiberwise linearly independent *for all* $b \in B$.

Now consider the map $S\colon B \times \mathbf{R}^{n'} \to B \times \mathbf{R}^n$ defined by $(b, r_1, \ldots, r_{n'}) \mapsto (b, r_1 s_1(b), \ldots, r_{n'} s_{n'}(b))$. This is clearly smooth and fiberwise injective with $\operatorname{Im}(S) = Z$ and defines a subbundle essentially because we have constructed the sections s_i. Thus, S is a closed immersion and thus also an embedding. Since $i\colon Z \to E$ is another map with the same properties and same image. Hence, there are unique continuous maps $Z \to B \times \mathbf{R}^{n'}$ and $B \times \mathbf{R}^{n'}$ which factor i and S through each other. To show that this is smooth, we simply use the fact that, locally, each map $Z \to B \times \mathbf{R}^n$ and $B \times \mathbf{R}^{n'} \to B \times \mathbf{R}^n$ look like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} \cong U \times \mathbf{R}^n$. Essentially, TFDC:



which, by projecting, $U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} \to U \times \mathbf{R}^{n'}$ shows that the dashed arrow is simply the identity. Similarly in the other direction for $Z \to B \times \mathbf{R}^{n'}$. This shows that the two maps $Z \to B \times \mathbf{R}^{n'}$ and $B \times \mathbf{R}^{n'}$ are in fact smooth and fiberwise linear. They necessarily inverse to each other, so this establishes the desired fiberwise linear isomorphism. What we have actually shown (un-reducing all of our assumptions) is that Z has local trivializations, as desired. ∎

Theorem C.1.15. *Over paracompact Hausdorff spaces, all short exact sequences of bundles split, but as usual the splitting is not natural. In particular, in the smooth category, the splitting is additionally smooth.*

Proof. We just need access to partitions of unity. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of bundles (i.e., fiberwise short exact). We construct a section $s\colon B \to A$ of $i\colon A \to B$. Pick a local trivialization of A and extend this to a local trivialization of B in such a way that the trivialization has A sit as $\mathbf{R}^k \times \mathbf{0}$ in $\mathbf{R}^n$ —this exists as we have seen. The section is obvious then. Doing this locally everywhere by a partition of unity argument, we must show that the resulting thing is a global left inverse. One can do this with careful analysis.

Now we must show that this implies that B splits. This follows by showing that $A \oplus B/A \cong B$, which can be done. ∎

Remark. Alternatively, equip the bundle B with a Riemannian metric by a partition of unity argument and take the orthogonal complement of A in B. The argument fails in the holomorphic category because we need not have a holomorphic partition of unity.

Warning. Kernels are only guaranteed to exist in the category of vector bundles when we take the kernel of an epimorphism. See Hirsch's book on page 93.

Definition. A ***(linear) sphere bundle*** (resp. ***(linear) disk bundle***) is a fiber bundle in which every fiber is (homeomorphic to) the standard metric (i.e., unit) sphere (resp. metric disk) in Euclidean space having structure group the orthogonal group.

Reminder. This means that there is a covering with homeomorphisms $p^{-1}(U) \cong U \times F$.

Lemma C.1.16. *A (smooth) vector bundle (of rank n) $E \to B$ is the same thing as a fiber bundle $F \to E \to B$ with structure group $GL_n(\mathbf{R})$ and a (smooth) $GL_n(\mathbf{R})$-equivariant isomorphism $F \cong \mathbf{R}^n$ for all $p \in B$.*

If B is paracompact Hausdorff, then a (smooth) vector bundle (of rank n) $E \to B$ is additionally the same thing as a (smooth) vector bundle with structure group $O(n)$ which is the same thing as a fiber bundle $F \to E \to B$ with structure group $O_n(\mathbf{R})$ and a (smooth) $O_n(\mathbf{R})$-equivariant isomorphism $F \cong \mathbf{R}^n$.

Proof. For the first part, the inclusion $\subset$ is clear from the trivializations. For $\supset$, make F into a vector space by pulling back the vector space structure on $\mathbf{R}^n$. We can then define new trivializations by composing with the isomorphism $F \cong \mathbf{R}^n$: $\psi_j : q^{-1}(U_i) \cong U_i \times F \cong U_i \times \mathbf{R}^n$. Define a vector space structure on E_p by fixing a trivialization about p and pulling back the vector space structure from any trivialization. The choice of trivialization does not matter up to isomorphism of vector spaces. To see this, begin by letting $p \in U_i \cap U_j$. Then the transition functions relate the homeomorphisms/diffeomorphisms $\psi_j\colon E_p \cong \mathbf{R}^n$ and $\psi_i\colon E_p \cong \mathbf{R}^n$ by a linear isomorphism, since $F \cong \mathbf{R}^n$ is $GL_n(\mathbf{R})$-equivariant. The claim, then, is that the two induced structures on E_p are isomorphic, and this is clear because pulling back this structure means that the two structures will themselves be related by an element of $GL_n(\mathbf{R})$. Thus, for the trivialization $\psi_i\colon q^{-1}(U_i) \cong U_i \times \mathbf{R}^n$, we have for $p \in U_i \cap U_j$ and E_p the structure coming from the index j that $\psi_i| E_p$ is still linear since it becomes linear after post-composition with $t_{ji}(p) = t_{ij}(p)^{-1} \in GL_n(\mathbf{R})$, which is a linear isomorphism and so forces $\psi_i| E_p$ to be.

For second part, give the vector bundle a (smooth) metric and on each trivialization let $e_1^i, \dots, e_n^i\colon U_i \to U_i \times \mathbf{R}^n$ be a (smooth) orthonormal frame for the metric. Let the transition functions now be defined by letting $t'_{ij}(p)$ be the change of basis matrix taking $(e_1^j(p), \dots, e_n^j(p)) \mapsto (e_1^i(p), \dots, e_n^i(p))$. This is clearly smooth and the resulting vector is still isomorphic to the one with the old t_{ij} via the identity map. The last part is analogous to the above. ∎

Proposition C.1.17. *Over a paracompact Hausdorff base space, a real vector bundle of rank n having structure group $O(n)$ determines and is determined by linear sphere bundles and linear disk bundles. That is, these notions are "the same."*

Proof. Strictly speaking, this follows by the equivalence of categories $\mathsf{Bun}_{O(n)}^{\mathbf{R}^n} \simeq \mathsf{Prin}_{O(n)} \simeq \mathsf{Bun}_{O(n)}^{S^{n-1}}$ and similarly for linear disk bundles. ∎

Lemma C.1.18. *Let V and W be vector bundles over X. Then* $\mathrm{Hom}(V, W) \cong V^* \otimes W$ *and if V and W have common rank n, then the subset* $\mathrm{Iso}(V, W)$ *is a fiber bundle over X with typical fiber* $\mathrm{GL}_n(\mathbf{R})$ *and* $\Gamma(\mathrm{Iso}(V, W)) \cong \{$*bundle isos* $V \cong W\}$.

Proof. A section $X \to \mathrm{Iso}(V, W)$ is a choice of isomorphism $V_p \to W_p$ for all $p \in X$. We must show that this determines an isomorphism of bundles. In a nbhd of $U \subset X$, this is a section $U \to U \times \mathrm{GL}_n(\mathbf{R}))$ and is therefore determined by $f_U\colon U \to \mathrm{GL}_n(\mathbf{R})$. Such a map determines at each $p \in U$ a map $\mathbf{R}^n \to \mathbf{R}^n$ and so an assignment $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$ given by $(p, v) \mapsto (p, f_U v)$ which is therefore as continuous or smooth as f_U is. We worked locally and these all glue. ∎

Lemma C.1.19. *Let V and W have the same rank. Then* $\mathrm{Iso}(V, W)$ *is an open subset of* $\mathrm{Hom}(V, W)$

Proof. In the trivializations, this looks something like $U \times \mathbf{R}^{n^2}$ and the isos are the matrices of full rank which is an open condition. ∎

C.2 Some Further Recollections on Bundles

Lemma C.2.1. *Let $f, g\colon M \to \mathbf{R}$ be functions from a manifold into $\mathbf{R}$ and let $0 \le k \le \infty$. If $f_1 + \cdots + f_n = h$ is C^k and $f_1, \ldots, f_{n-1}$ are C^k, then f_n is C^k.*

Proof. $f_n = h - (f_1 + \cdots + f_{n-1})$ and so must be C^k since h and the sum $f_1 + \cdots + f_{n-1}$ are. ■

Proposition C.2.2. *Let $p\colon E \to B$ and $p'\colon E' \to B$ be vector bundles of rank n and let $f\colon E \to E'$ be a smooth map that is a linear isomorphism on each fiber. f is then a bundle isomorphism—that is, it is a diffeomorphism over B.*

Proof. In bundle coordinates, f looks like a map $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$ by $(p, v) \mapsto (p, f_p(\mathbf{v}))$ for f_p the bundle coordinate version of the relevant linear isomorphism. Define f^{-1} by $(p, \mathbf{v}) \mapsto (p, f_p^{-1}(\mathbf{v}))$. Let $A\colon U \to \mathrm{GL}_n(\mathbf{R})$ be such that $A(p)\mathbf{v} = f_p(\mathbf{v})$ so that f is $(p, \mathbf{v}) \mapsto (p, A(p)\mathbf{v})$.

Claim 23. The action $(p, \mathbf{v}) \mapsto A(p)\mathbf{v}$, $U \times \mathbf{R}^n \to \mathbf{R}^n$, is smooth. Therefore the adjoint of A is smooth into $\mathrm{GL}_n(\mathbf{R})$, which is equivalent to saying that A is smooth into $\mathbf{R}^{n^2}$ and hence equivalent to saying that the component functions of A are smooth.

For convenience, we will write A for $A(p)$. Since $\mathrm{GL}_n(\mathbf{R})$ is an open subset of $\mathbf{R}^{n^2}$, being the preimage under det of $\mathbf{R} \setminus \{0\}$, smoothness into $\mathrm{GL}_n(\mathbf{R})$ is equivalent to smoothness into $\mathbf{R}^{n^2}$. Recall that we are in bundle coordinates $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$—WLOG suppose U is the domain of a chart on B perhaps by shrinking if necessary. Observe that smoothness of $(p, \mathbf{v}) \mapsto (p, A(p)\mathbf{v})$ means that the assignment $(p, \mathbf{v}) \mapsto A(p)\mathbf{v}$, $U \times \mathbf{R}^n \to \mathbf{R}^n$, is smooth. This is because finite products exist in the category of manifolds. In particular, fix $\mathbf{v}_0 = (\delta^i_j)$. Then $U \times \{\mathbf{v}_0\} \to \mathbf{R}^n$ is smooth since $U \times \{\mathbf{v}_0\}$ is a submanifold of $U \times \mathbf{R}^n$. This map is then $(p, \mathbf{v}_0) \mapsto (A_{1i}, \ldots, A_{ni})$ and so for this to be smooth in $\mathbf{R}^n$, each component must be smooth. Now the map $A\colon U \to \mathrm{GL}_n(\mathbf{R})$ is simply the map $p \mapsto (A_{ij}(p))$ and by the above observation that $\mathrm{GL}_n(\mathbf{R})$ is open in $\mathbf{R}^{n^2}$, this is smooth because each component is smooth.

Claim 24. The inversion $(-)^{-1}\colon \mathrm{GL}_n(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ is smooth.

The inverse of matrix has entries rational functions which in the (i, j) spot has numerator a polynomial in the various relevant entries for the relevant minor and has numerator the determinant of the matrix. Since $\det\colon \mathrm{GL}_n(\mathbf{R}) \to \mathbf{R}$ is smooth and non-vanishing, the denominator is a smooth and non-vanishing function, so everything checks out.

Putting this together, the function defined in bundle coordinates as $(p, \mathbf{v}) \mapsto (p, A^{-1}(p)\mathbf{v})$ is smooth, it is well-defined since we have defined it in bundle coordinates locally, and it is clearly inverse to the given map. ■

Lemma C.2.3. *Let $p\colon E \to B$ be a smooth rank n vector bundle. Let $\underline{\mathbf{R}}$ be the trivial rank 1 bundle over B. Then the bundle maps $m\colon \underline{\mathbf{R}} \oplus E \to E$ and $+\colon E \oplus E \to E$ are smooth, where this is the Whitney sum.*

Proof. These are the Whitney sums of the bundles. Let U be a trivializing nbhd for E, which we can assume exists by shrinking if necessary any trivializing nbhd. The resulting trivialization of $\underline{\mathbf{R}} \times E$ is then simply the one sending $(p, r, v) \mapsto (p, r, \Phi_p(v))$ where $\Phi\colon p^{-1}(U) \to U \times \mathbf{R}^n$ is the trivializing diffeomorphism. The first map in coordinates is given by $U \times \mathbf{R} \times \mathbf{R}^n \to U \times \mathbf{R}^n$ sending $(p, r, v) \mapsto (p, rv)$. This is basically a diagram chase since for $p \in B$ $m_p(r, v_p) = rv_p \in E_p$ since the trivializations respect vector space operations. This map is further in coordinates $\mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$ by $(p, r, v) \mapsto (p, rv)$. The multiplication map $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is clearly smooth. For the second map, one argues as before and notes that addition $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is clearly smooth. ■

Lemma C.2.4 (Lee, 10.19). *Let $p\colon E \to B$ be a smooth vector bundle of rank n and let $U \subset B$ be an open neighborhood. Denote $\widetilde{e}_i\colon U \to U \times \mathbf{R}^n$ the i-th standard section $p \mapsto (p, \mathbf{e}_i)$. For any smooth local frame $\{s_1, \ldots, s_n\}$ on U, there exists a diffeomorphism—in fact trivialization—$\Psi\colon p^{-1}(U) \to U \times \mathbf{R}^n$ such that $\Psi^{-1} \circ \widetilde{e}_i = s_i$. Hence, smooth sections over an open set U determine a smooth bundle trivialization and conversely.*

Proof. We will define Ψ^{-1} and show it is fiberwise linear and a diffeomorphism, justifying the inverse notation. Define $\Psi^{-1}(p, (v_1, \ldots, v_n)) = \sum_i v_i s_i(p)$ and note that this is certainly fiberwise linear! To show this is smooth, we only need to check that the operation of summing is smooth on $p^{-1}(U)$. This is true since for any $V \subset U$ a trivializing open nbhd with Φ the trivialization, Φ is a diffeomorphism linear on each fiber and so commutes with the sum and hence $\sum_i v^i s_i(p) = \Phi^{-1}\Phi(\sum_i v^i s_i(p)) = \Phi^{-1}(\sum_i \Phi(v^i s_i(p)))$ and the fiberwise sum on $U \times \mathbf{R}^n$ is smooth as part of the definition of a smooth vector bundle from the above. Thus, if Ψ^{-1} is smooth, then Ψ is a smooth local trivialization and clearly we have $\Psi^{-1} \circ \widetilde{e}_i = s_i$.

It is clear that Ψ^{-1} is a bijection since the s_i form a frame, so to show it is a diffeomorphism, it suffices to show it is a local diffeomorphism. Let $V \subset U$ be a trivializing open nbhd as above. If we can show that $\Phi \circ \Psi^{-1}\big|\, V \times \mathbf{R}^n$ is a diffeomorphism of $V \times \mathbf{R}^n$ with itself, then since Φ is a diffeomorphism, we will have that Ψ^{-1} is a diffeomorphism $V \times \mathbf{R}^n \to p^{-1}(V)$.

Now, $\Phi \circ s_i$ is smooth as a composite of smooth functions. Hence, in coordinates $\Phi(s_i(p)) = (p, (\sigma_1^i(p), \ldots, \sigma_n^i(p)))$ and the σ_i must be smooth in p for this function to be smooth. Thus,

$$\Phi \circ \Psi^{-1}(p, (v_1, \ldots, v_n)) = \Phi(\sum_i v_i s_i(p)) = (p, (\sum_i v_i \sigma_1^i(p), \ldots, \sum_i v_i \sigma_n^i(p))) = \sum_i \Phi(v_i s_i(p))$$

which is smooth as the sum operation is smooth as soon as we know that the sum operation is smooth and we do know this (essentially the last equality). What's happening here is that the smooth matrix $(\sigma_j^i)_{i,j}$ is at each point p the matrix $(\sigma_j^i(p))_{i,j}$ which transforms something in the ordered basis $(s_1(p), \ldots, s_n(p))$ for E_p to something in the standard basis for $\mathbf{R}^n$. In other words, this is a change of basis matrix and it is therefore invertible. Thus, $\Phi \circ \Psi^{-1}(p, (v_1, \ldots, v_n)) = (\sigma_j^i(p))(v_1, \ldots, v_n)^t$ the matrix multiplication—this is smooth because the matrix multiplication just gives polynomials in smooth functions. It follows that the inverse is given by $(\Phi \circ \Psi^{-1})^{-1}(p, (w_1, \ldots, w_n)) = (\sigma_j^i(p))^{-1}(w_1, \ldots, w_n)^t$ and since (σ_j^i) is everywhere invertible, its determinant is always non-zero and smooth, so the inverse matrix is a smooth function being a rational function of smooth functions where the denominator never vanishes. ∎

Remark. Nothing we used above relied on using $\mathbf{R}^n$ for the typical fiber. We could just as well have consider complex vector bundles with typical fiber $\mathbf{C}^n$.

Corollary C.2.5. *If an open nbhd $U \subset B$ admits a smooth local frame for $p\colon E \to B$ a smooth vector bundle of rank n, then U is a trivializing open nbhd.*

Corollary C.2.6. *A smooth local trivialization is equivalent to a smooth local frame by sending $v \in E_p$ to $(v_1, \ldots, v_n)$ where $\sum_i v_i s_i(p)$.*

Proof. This just deconstructs what the construction above did. ∎

Corollary C.2.7. *Let $\pi\colon E \to B$ and $\pi'\colon E' \to B$ be smooth vector bundles of rank n and n' respectively with say $\dim B = m$. Let $f\colon E \to E'$ any fiberwise linear function (not assumed to be continuous or anything). Then f is smooth **iff** each point $p \in B$ is contained in the domain of a smooth local frame $\mathscr{F}$ such that f sends each section in $\mathscr{F}$ to a smooth function.*

Proof. The direction (⇒) is trivial since f is fiberwise linear, so let $s_1, \ldots, s_n$ be smooth sections of the first in a nbhd of a point that form a frame and let $\sigma_i = f \circ s_i$ and suppose the σ_i are smooth. Then in the trivialization constructed from the smooth local frame $\mathscr{F}$, we know this is $U \times \mathbf{R}^n \to (\pi')^{-1}(U)$ by

$$(p, (v_1, \ldots, v_n)) \mapsto \sum_i v_i s_i(p) \mapsto \sum_i v_i \sigma_i(p).$$

Note that we have used the fact that f is fiberwise linear to pull the coefficients out at the last step—this is evidently an indispensable assumption.

Let s_i' be local frame for E' on this same nbhd (perhaps by shrinking). Since the σ_i are smooth, $\sigma_i = \sum_{k=1}^{n'} c_{ik} s_k'$ where the c_{ik} are smooth real-valued functions. Thus, this can be written

$$(p, (v_1, \ldots, v_n)) \mapsto \sum_i \sum_{k=1}^{n'} v_i c_{ik}(p) s_k'(p) = \sum_{k=1}^{n'} \left(\sum_{i=1}^{n} v_i c_{ik}(p) \right) s_k'(p)$$

Hence, in the local trivializations afforded to us by these frames as we constructed above, the assignment is

$$(p, (v_1, \ldots, v_n)) \mapsto (p, (\sum_{i=1}^{n} v_i c_{i1}(p), \ldots, \sum_{i=1}^{n} v_i c_{in'}(p))).$$

This is smooth because each of the components are smooth. Indeed, using a chart for U, this is basically just

$$((x_1, \ldots, x_m), (v_1, \ldots, v_n)) \mapsto ((x_1, \ldots, x_m), (\sum_{i=1}^{n} v_i c_{i1}(x_1, \ldots, x_n), \ldots, \sum_{i=1}^{n} v_i c_{in'}(x_1, \ldots, x_n)))$$

All mixed partial derivatives with respect to each coordinate $x_1, \ldots, x_n, v_1, \ldots, v_n$ clearly exist and are always smooth, clearly. ∎

Corollary C.2.8. *Let $\pi\colon E \to B$ be a smooth vector bundle over B and $f\colon E \to \mathbf{R}$ a map that is linear on each fiber. Then f is smooth **iff** f sends some smooth local frame in a neighborhood of every point to smooth functions $B \to \mathbf{R}$.*

Proof. f is the composite $E \to \mathbf{R} \times B \to \mathbf{R}$ where the last map is the projection and is therefore smooth and the first map sends $v \in E_p$ to $(f(v), \pi(p))$ which is smooth precisely if f is smooth (since π is assumed to be smooth). This reduces us to the case above for the map (f, π) where it suffices to show that (f, π) satisfies the conclusions of the preceding corollary and surely it does. ■

C.3 Normal Bundles & Tubular Neighborhoods

C.3.1 Normal Bundles

Reminder. Recall that we have seen that $E'/E \cong E^\perp$.

Definition. Let $f\colon M \to N$ be an immersion. Denote $\nu_f = (f^*TN)/TM$ the ***normal bundle of the immersion*** f. Here, the quotient by TM occurs via the identification of TM with its image in TN. When f is an embedding of M into N, we denote this by ν_M.

Remark. Recall that $f^*TN = \{(p,v) \in M \times TN : f(p) = \pi_N(v)\}$.

Lemma C.3.1. *If N is a Riemannian manifold, then ν_f may be taken to be the subbundle of*

$$f^*TN = \{(p,v) \in M \times TN : f(p) = \pi_N(v)\}$$

consisting of all pairs (p, v) where $v \in T_pM^\perp$ (identifying T_pM with its image).

Proof. Should be similar to the proof that $E'/E \cong E^\perp$. ■

Theorem C.3.2. *Let $f\colon M \to N$ be an immersion. Then $f^*TN \cong TM \oplus \nu_f$.*

Proof. Use a metric. Define $TM \oplus \nu_f \to f^*TN$ by sending $(p, v, w) \mapsto (p, v+w)$. This is smooth and a fiberwise isomorphism so it is a diffeomorphism. ■

Remark. Everything above ought to hold for manifolds with boundary.

C.3.2 Exponential Map and Shrinking

Taken from Riemannian Geometry class notes. All manifolds are without boundary.

Reminder. Recall that for a Riemannian manifold M with $\dim M = n$, we call $\gamma_{p,v}$ the geodesic having $\dot\gamma(0) = v$ and $\gamma(0) = p$. In coordinates, the geodesic equation is $\ddot\gamma^\ell(t) + \Gamma^\ell_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0$ for $1 \le \ell \le n$, where $\Gamma^\ell_{ij} = \dfrac{1}{2} g^{\ell k}(g_{ik,j} + g_{jk,i} - g_{ij,k})$. More concisely, this is $D_t\dot\gamma(t) = 0$, where D_t is the covariant derivative along γ.

Proposition C.3.3 (Naturality of geodesics). *Let M and $\widetilde{M}$ be two Riemannian manifolds and $\varphi : M \to \widetilde{M}$ a Riemannian isometry. If $p \in M$ and γ is a geodesic on M such that $\gamma(0) = p$ and $\dot\gamma(0) = v \in T_pM$, then $\widetilde\gamma := \varphi \circ \gamma$ is a geodesic on $\widetilde{M}$ such that $\widetilde\gamma(0) = \varphi(p)$ and $\dot{\widetilde\gamma}(0) = \varphi_*(v)$.*

Remark. Note that the geodesic equation $D_t\dot\gamma(t) = 0$ is a *non-linear* differential equation.

Lemma C.3.4. *There exists a unique vector field G on TM whose integral curves are of the form $t \mapsto (\gamma(t), \dot\gamma(t))$ where γ is a geodesic. The flow of G is called the **geodesic flow**.*

Proof. The geodesic equations are in local coordinates $\ddot x^\ell + \Gamma^\ell_{ij}\dot x^i \dot x^j = 0$. We reduce this to a first order equation by introducing the variable $y^k = \dot x^k$. Then in bundle coordinates for TU, a solution to the geoedesic equation $t \mapsto (x^1(t), \dots, x^n(t), \dot x^1(t), \dots, \dot x^n(t))$ satisfies the system of first order equations

$$\begin{cases} \dot x^k = y^k & 1 \le k \le n, \\ \dot y^k = -\Gamma^k_{ij} y_i y_j & 1 \le k \le n. \end{cases}$$

where, here, this is in terms of the coordinates afforded by the trivializing frame $(x^1, \ldots, x^n, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. By standard results, there is a flow for this (the centered equations just above) pinned down by the usual specification. We recall that the flow is obtained by piecing together the integral curves, and it is unique by uniqueness of integral curves as usual—in particular, the integral curves are geodesics where the geodesic through (p, v) is precisely $\gamma_{p,v}$. ■

Corollary C.3.5 (Local Existence and Uniqueness). *Let $p_0 \in M$ and $u_0 \in T_{p_0}M$. Then there exists $\varepsilon_0 > 0$ and an open neighborhood $U_0 \subset TM$ of (p_0, u_0) with the following properties:*

1. *For any $(p, u) \in U_0$, there exists a unique geodesic $\gamma_{p,u} : (-\varepsilon, \varepsilon) \to M$ such that $\gamma_{p,u}(0) = p$ and $\dot\gamma_{p,u}(0) = u$.*
2. *The map $\gamma_{\cdot,\cdot}(\cdot) : U_0 \times (-\varepsilon_0, \varepsilon_0) \to M$ defined by $((p, u), t) \mapsto \gamma_{p,u}(t)$, is smooth.*

Proof. This follows by consideration of the properties that flows have. ■

Corollary C.3.6. *Fix $s \in \mathbf{R}$. If $\gamma_{p,sv}(1)$ exists, then $\gamma_{p,v}(s)$ exists and $\gamma_{p,tv}(1) = \gamma_{p,v}(s)$. In particular, $\gamma_{p,sv} = \gamma_{p,v}(s \cdot -)$.*

Proof. If $s = 0$, then we can check by hand that this is true. So suppose $s \neq 0$. In local coordinates, one checks that $\gamma_{p,v}(s \cdot -)$ is the solution to the IVP for

$$\ddot\gamma^\ell(t) + \Gamma^\ell_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0 \qquad 1 \le \ell \le n$$

subject to the initial conditions $\dot\gamma(0) = sv$ and $\gamma(0) = p$. This is because we can divide through by the common factor of s^2. Hence, uniqueness forces our hand. ■

Set

$$O_p \stackrel{\text{def}}{=} \{v \in T_pM : \gamma_{p,v}(t) \text{ is defined for all } t \in [0, 1]\} \subset T_pM.$$

Notice that by the preceding, there exists $\delta > 0$ such that $B^{T_pM}_\delta(0_p) \subset O_p$ (an open ball). It will turn out that O_p is open and that $O = \bigcup_{p \in M} O_p$ are both open.

Definition. For $p \in M$, define the ***exponential map*** at p as $\exp_p\colon O_p \to M$ by $v \mapsto \gamma_{p,v}(1)$.

Remarks.

1. For p fixed, the map $\exp_p$ is C^∞.
2. For $t \in \mathbf{R}$ and $v \in T_pM$ such that $tv \in O_p$, we have $\exp_p(tv) = \gamma_{p,tv}(1) = \gamma_{p,v}(t)$.

Proposition C.3.7. *Let $\dim M = n$. The differential map $d\exp_p(0_p)$ is the identity where we understand $T_0T_pM \cong \mathbf{R}^n$ and $T_pM \cong \mathbf{R}^n$.*

Proof. Pick $v \in T_pM$. Since $\gamma_{p,tv}(1) = \gamma_{p,v}(t)$, we have

$$d\exp_p(0_p)(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_{p,v}(t) = \dot\gamma_{p,v}(t)\Big|_{t=0} = v.$$

Corollary C.3.8. *On a neighborhood of $0_p \in T_pM$, the exponential map $\exp_p$ is a diffeomorphism onto its image in M.*

Proof. This follows from the inverse function theorem since $d\exp_p(0_p)\colon T_{0_p}O_p \to TM$ is an isomorphism. ■

Lemma C.3.9. $\exp$ *is smooth on an open subset of O. In particular, O is open in TM, O_p is open in T_pM, and* $\exp$ *is smooth on O.*

Proof. Suppose $\dim M = n$. Let G denote the geodesic flow, which we assume is maximal, as always—let A denote the maximal flow domain, which we know is an open subset of $\mathbf{R} \times TM$. By **Corollary 38**, $TM_1 = \{(p, v) \in TM : (1, p, v) \in A\}$ is open in TM. In particular, if $(p, v) \in TM_1$, then $(p, v) \in TM_t$ for all $t \in [0, 1]$ since one constructs the maximal flow domain as the union of the maximal integral curves (see way above for this). We can write therefore write the exponential function on its maximal domain of definition as the composite $TM_1 \xrightarrow{(1,\mathrm{id})} A \xrightarrow{G} TM \xrightarrow{\pi} M$. All functions in sight are smooth and TM_1 is open in TM. Now observe that $TM_1 = O$ and hence that $TM_{1,p} = TM_1 \cap T_pM = \{(p, v) \in T_pM : (1, p, v) \in A\} = O_p$ is open in T_pM in the subspace topology—the subspace topology on T_pM is equivalent to the topology it inherits from being diffeomorphic with $\mathbf{R}^n$. One observes easily now that $O = TM_1$ and so is open as well. ■

Corollary C.3.10. *Consider the map $E\colon O \to M \times M$ given by $(p, v) \mapsto (p, \exp_p(v))$. Then for each $p \in M$,*

$$dE((p, 0_p))\colon T_{(p,0_p)}TM \to T_{(p,p)}(M \times M)$$

is nonsingular.

Proof. Let (x, U) be chart about p in M. Note that any basis $\left.\frac{\partial}{\partial dx^i}\right|_{(p,0_p)}$ has for $1 \leq i \leq m$ $\left.\frac{\partial}{\partial dx^i}\right|_{(p,0_p)} = \left.\frac{\partial}{\partial x^i}\right|_p$ essentially by definition. Equipping the codomain with the basis induced by the chart $x \times x$, we see that the matrix of $dE((p, 0_p))$ must have the form $\begin{pmatrix} \mathrm{id}_{m\times m} & 0_{m\times m} \\ X & Y \end{pmatrix}$ as the projection is independent of ∂_j for $j \geq m+1$. On the other hand, for $m+1 \leq i \leq 2m$, we already know that $d\exp_p(0_p)$ is the identity by the above. Hence, $Y = \mathrm{id}_{m\times m}$. Hence, in coordinates, we must have

$$dE((p, 0_p)) = \begin{pmatrix} \mathrm{id}_{m\times m} & 0_{m\times m} \\ X & \mathrm{id}_{m\times m} \end{pmatrix}$$

which is upper triangular and therefore invertible. Hence, for each $p \in M$, $dE((p, 0_p))$ is non-singular. ■

Theorem C.3.11 (Naturality exponential map). *Let M and $\widetilde{M}$ be two Riemannian manifolds, $\Phi : M \to \widetilde{M}$ be a Riemannian isometry and p a point in M. Denote by $\exp^M$ and $\exp^{\widetilde{M}}$ the exponential maps of M and $\widetilde{M}$, respectively. Then*

$$\exp^{\widetilde{M}}_{\Phi(p)} \circ \Phi_* = \Phi \circ \exp^M_p .$$

Theorem C.3.12. *Let M and $\widetilde{M}$ be two Riemannian manifolds, and $\Phi_1, \Phi_2 : M \to \widetilde{M}$ be two Riemannian isometries. If there exists $p \in M$ such that $\Phi_1(p) = \Phi_2(p)$ and $d\Phi_1(p) = d\Phi_2(p)$, then $\Phi_1 \equiv \Phi_2$.*

Proof. Exercise. (*Hint:* Prove that the set where the two isometries agree is both open and closed.)

Algebraic Topology

D.1 Products and Pairings in Homology and Cohomology

Warning. Milnor and Stasheff make at least two non-standard sign conventions.

(i) For $\psi \in H^n(X)$ and $\sigma \in H_{n+1}(X)$, their connecting homomorphism δ in the LES in cohomology is characterized by the stipulation that

$$\delta\psi(\sigma) = (-1)^{n+1}\psi(\partial\sigma).$$

In the usual account of algebraic topology, the connecting homomorphism following relation holds

$$\boxed{\delta\psi(\sigma) = \psi(\partial\sigma).}$$

(ii) Let $\ell \leq k$. Milnor and Stasheff define the cap product $C_k(X) \otimes C^\ell(X) \to C_{k-\ell}(X)$ by

$$\sigma \frown \psi = (-1)^{\ell(k-\ell)}\psi(\sigma|\,[v_{k-\ell}, \dots, v_k])\;\sigma|\,[v_0, \dots, v_{k-\ell}].$$

The more standard definition is

$$\boxed{\sigma \frown \psi = \psi(\sigma|\,[v_0, \dots, v_\ell])\;\sigma|\,[v_\ell, \dots, v_k].}$$

D.1.1 Cup and Cap Products

Definition (Excisive Triad). A ***triad*** is a triple $(X; A, B)$ where $A, B \subset X$ and $A \cup B = X$. Given a homology (resp. cohomology) theory E_* (resp. E^*), we say that a triad $(X; A, B)$ is ***excisive*** for E if the inclusion $(A, A \cap B) \to (X, B)$ induces an isomorphism on all homology (resp. cohomology) groups for E.

Theorem D.1.1. *$(X; A, B)$ is excisive iff $(X; B, A)$ is excisive.*

Proof. This is **7.13** in Switzer. ■

Remark. The excision theorem in algebraic topology says roughly that when $A, B \subset X$ such that $\mathrm{Int}(A) \cup \mathrm{Int}(B) = X$, then $(X; A, B)$ is excisive for all homology and cohomology theories. This is further refined for CW-complexes as follows. If X is a CW-complex and $A, B \subset X$ are subcomplexes such that $(X; A, B)$ is a triad, then this triad is excisive for all homology and cohomology theories.

Definition (Cup Product). Given $\varphi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$, define $\varphi \smile \psi \in C^{k+\ell}(X; R)$ by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|\,[v_0, \dots, v_k])\psi(\sigma|\,[v_k, \dots, v_{k+\ell}]).$$

This is a bilinear pairing that descends to a bilinear pairing

$$H^k(X; R) \otimes_R H^\ell(X; R) \xrightarrow{\smile} H^{k+\ell}(X; R).$$

The same formula yields relative versions

$$H^k(X; R) \otimes_R H^\ell(X, A; R) \xrightarrow{\smile} H^{k+\ell}(X, A; R)$$
$$H^k(X, A; R) \otimes_R H^\ell(X; R) \xrightarrow{\smile} H^{k+\ell}(X, A; R)$$
$$H^k(X, A; R) \otimes_R H^\ell(X, A; R) \xrightarrow{\smile} H^{k+\ell}(X, A; R)$$

This is called the ***cup product***.

Proposition D.1.2. *When $A, B \subset X$ are open subsets or when $A, B \subset X$ are subcomplexes of the CW-complex X, there is a a cup product*

$$H^k(X, A; R) \otimes_R H^\ell(X, B; R) \xrightarrow{\smile} H^{k+\ell}(X, A \cup B; R).$$

Proof. This goes by showing that the inclusion of $C^*(X, A \cup B; R)$ into the subcomplex of $C^*(X; R)$ consisting of cochains that vanish on sums of chains in A and B is a cochain homotopy equivalence. ■

Theorem D.1.3. *Fix any ring R.*

(a) *For the differential δ of $C^*(X, A; R)$ and for $\varphi \in C^k(X, A; R)$ and $\psi \in C^\ell(X, A; R)$,*

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

(b) *The cup product turns $H^*(X, A; R) = \bigoplus_i H^i(X, A; R)$ into an associative, graded commutative, unital, R-algebra. If $|\alpha| = k$ and $|\beta| = \ell$, then $\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$. This is called the **cohomology ring**.*

(c) *Given $f: (X, A) \to (Y, B)$, the induced maps on relative cohomology f^* satisfies*

$$f^*(\alpha \smile \beta) = f^*\alpha \smile f^*\beta.$$

That is, f^* is a ring-homomorphism.

Definition (Cap Product). Fix $\ell \leq k$, spaces $A \subset X$ and a ring R. Define a bilinear pairing

$$C_k(X; R) \otimes_R C^\ell(X; R) \xrightarrow{\frown} C_{k-\ell}(X; R)$$

by

$$\sigma \frown \varphi = \varphi(\sigma|\,[v_0, \ldots, v_\ell])\; \sigma|\,[v_\ell, \ldots, v_k].$$

This descends to a bilinear pairing on cohomology

$$\frown : H_k(X; R) \otimes_R H^\ell(X; R) \to H_{k-\ell}(X; R).$$

The same formula yields relative versions

$$H_k(X, A; R) \otimes_R H^\ell(X; R) \xrightarrow{\frown} H_{k-\ell}(X, A; R)$$
$$H_k(X, A; R) \otimes_R H^\ell(X, A; R) \xrightarrow{\frown} H_{k-\ell}(X, A; R)$$

Theorem D.1.4. *Fix any ring R.*

(a) *Given $\sigma \in C_k(X, A; R)$ and $\varphi \in C^\ell(X, A; R)$ with $\ell \leq k$,*

$$\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi).$$

(Using Milnor and Stasheff's conventions, this has a somewhat nicer form.)

(b) *Given $f: (X, A) \to (Y, B)$,*

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))).$$

Theorem D.1.5.

(a) *Given $\alpha \in C_{k+\ell}(X, A; R)$, $\varphi \in C^k(X, A; R)$ and $\psi \in C^\ell(X, A; R)$,*

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha).$$

This holds on the level of cohomology as well.

(b) *For $\sigma \in C_k(X; R)$, $\varphi \in C^\ell(X; R)$ and $\psi \in C^n(X; R)$,*

$$\sigma \frown (\varphi \smile \psi) = (\sigma \frown \varphi) \frown \psi.$$

This holds on the level of cohomology as well and makes $H_*(X; R)$ a right $H^*(X; R)$-module.

D.1.2 Cohomology and Homology Cross Products

Definition (Cohomology Cross Product). Define a bilinear pairing

$$H^n(X;R) \otimes_R H^n(Y;R) \xrightarrow{\times} H^{n+m}(X \times Y;R)$$

called the ***cross product*** by

$$\varphi \otimes \psi \mapsto \varphi \times \psi \stackrel{\text{def}}{=} \text{pr}_X^*(\varphi) \smile \text{pr}_Y^*(\psi).$$

If $A \subset X$ is open (or a subcomplex) and $B \subset Y$ is open (or a subcomplex), there is a more general cross product

$$H^n(X,A;R) \otimes_R H^n(Y,B;R) \xrightarrow{\times} H^{n+m}(X \times Y, A \times Y \cup X \times B;R),$$

defined by the same formula, where $\text{pr}_X^*(\varphi) \in H^n(X \times Y, A \times Y;R)$ and $\text{pr}_Y^*(\psi) \in H^m(X \times Y, X \times B;R)$.

Remark. The cohomology cross product is manifestly natural with respect to maps of such pairs of spaces (X, A).

Proposition D.1.6. *If R is a commutative ring, then*

$$H^*(X) \otimes_R H^*(Y) \stackrel{def}{=} \bigoplus_n \bigotimes_{i+j=n} H^i(X) \otimes_R H^j(Y)$$

acquires the structure of a graded R-algebra where multiplication is defined on decomposable tensors by (the cup product is being suppressed)

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd.$$

In this case, the cross product

$$H^n(X;R) \otimes_R H^m(Y;R) \xrightarrow{\times} H^{n+m}(X \times Y;R)$$

is a natural homomorphism of graded R-algebras (natural in X and Y) with the following properties as a consequence of the Künneth theorem.

(i) *It is an isomorphism of graded R-algebras when R is a commutative ring and $H^*(Y)$ is a finitely generated free R-module.*
(ii) *When R is a field, the cross product is a natural isomorphism of graded R-algebras.*

Remark. By induction, this has the evident generalization for products of n spaces where $n \geq 3$ is finite.

D.1.3 Duality and Orientability

Notation. For $A \subset M$, let $H_*(M \mid A;R) = H_*(M, M \setminus A;R)$. When $A = \{p\}$, we denote this by $H_*(M \mid p;R)$.

Convention. We will understand "manifold" to mean a manifold without boundary here. All n-manifolds will have $n \geq 1$.

Lemma D.1.7. *For an n-manifold M and each $p \in M$, $H_*(M \mid p;R)$ is concentrated in degree with n with $H_n(M \mid p;R) \cong R$.*

Proof. Choose a coordinate system (x, U) about p with $x \colon U \to \mathbf{R}^n$ a homeomorphism. WLOG $x(p) = \mathbf{0}$ perhaps by shifting.

Since $M \setminus \{p\}$ and U are open and $M \setminus \{p\} \cup U = M$, we have an excision isomorphism

$$H(U \mid \{p\};R) = H_*(U, U \setminus \{p\};R) = H_*(U, M \setminus \{p\} \cap U;R) \xrightarrow{\cong} H_*(M, M \setminus \{p\};R) = H_*(M \mid p;R)$$

induced by the evident inclusion of pairs $(U, M \setminus \{p\} \cap U) \to (M, M \setminus \{p\})$. There is also a homeomorphism of pairs $(U, U \setminus \{p\}) \cong (\mathbf{R}^n, \mathbf{R}^n \setminus \{\mathbf{0}\})$. By the LES in homology for the pair $(\mathbf{R}^n, \mathbf{R}^n \setminus \{\mathbf{0}\})$, we see that,

$$H_*(\mathbf{R}^n \mid \mathbf{0};R) \cong H_{*-1}(\mathbf{R}^n \setminus \{\mathbf{0}\};R) \qquad \text{for } * \geq 1.$$

There is a natural homotopy equivalence $\mathbf{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ given by $v \mapsto v/\|v\|$. Hence,

$$H_*(\mathbf{R}^n \mid \mathbf{0};R) \cong H_{*-1}(\mathbf{R}^n \setminus \{\mathbf{0}\};R) \cong H_{*-1}(S^{n-1};R) \qquad \text{for } * \geq 1.$$

When $* = 0$, $H_0(\mathbf{R}^n \mid \mathbf{0})$ can be computed by hand to be 0 and so $H_0(M \mid p;R) = 0$. ■

Lemma D.1.8. *Let M be an n-manifold. If $p \in M$ and $x \colon U \cong \mathbf{R}^n$ is a coordinate nbhd of p in M, then*

$$H_*(M \mid B; R) \cong H_*(M \mid p; R)$$

where $B \subset U$ is any open subset contained in U mapping under x to an open ball of finite radius. Furthermore, this isomorphism is induced by the map of pairs $(M, M \setminus B) \to (M, M \setminus \{p\})$. In particular, $H_n(M \mid B; R) \cong R$.

Proof. Perhaps by shifting, we may suppose WLOG that $x(p) = \mathbf{0}$. Note that $M \setminus \{p\}$ deformation retracts onto $M \setminus B$. Indeed, it suffices to show that $\mathbf{R}^n \setminus \{\mathbf{0}\}$ deformation retracts onto $\mathbf{R}^n \setminus B(\mathbf{0}, r)$, where B is the open ball centered at $\mathbf{0}$ of some finite radius r. This map is obtained by

$$H(x,t) = \begin{cases} x & \|x\| \geq r, \\ (1-t)x - \dfrac{rx}{\|x\|}t & \|x\| \leq r. \end{cases}$$

This is clearly well-defined and it is continuous by the pasting lemma. In particular, this shows that $(M, M \setminus B) \to (M, M \setminus \{p\})$ is a homotopy equivalence of pairs. Another way to see this is that what we have just shown is $H_*(M \setminus B) \to H_*(M \setminus \{p\})$ is an isomorphism so it follows by naturality of the LES in homology and the five lemma applied to the inclusion $(M, M \setminus B) \to (M, M \setminus \{p\})$ that $H_*(M \mid B) \to H_*(M \mid p)$ is an isomorphism. ■

Definition. Let M be an n-manifold and R be a ring. We suppress coefficients.

(a) An R-***fundamental class*** of M at a subspace X is an element $\mu \in H_n(M \mid X)$ such that for each $x \in X$, the map

$$H_n(M \mid X) \to H_n(M \mid x)$$

sends μ to a generator of $H_n(M \mid x)$.

(b) An R-***orientation*** μ of M is an assignment $p \mapsto \mu_p \in H_n(M \mid p)$ which we require to satisfy the following condition. For each $p \in M$, there is a coordinate nbhd (x, U) of p with $x(U) \cong \mathbf{R}^n$ and an open subset $B \subset U$ such that $x(B)$ is an open ball of finite radius under x and a choice of generator $\mu_B \in H_n(M \mid B)$ such that for every $q \in B$, the natural map $H_n(M \mid B) \to H_n(M \mid q)$ maps $\mu_B \mapsto \mu_q$.

(c) M is R-***orientable*** if such an assignment as above exists.

(d) When $R = \mathbf{Z}$, we say M is ***orientable*** if such an assignment as above exists.

Theorem D.1.9. *Let M be a closed n-manifold and R be a ring. We suppress R coefficients.*

(a) *If M is R-orientable, then the map $H_n(M) \to H_n(M \mid p) \cong R$ is an isomorphism for $p \in M$.*

(b) *If M is not R-orientable, then the map $H_n(M) \to H_n(M \mid p) \cong R$ is not an isomorphism but it is injective and has image $\{r \in R : 2r = 0\}$.*

(c) *$H_i(M) = 0$ for $i > n$.*

(d) *The torsion subgroup of $H_{n-1}(M; \mathbf{Z})$ is trivial if M is orientable and is $\mathbf{Z}/2$ if M is non-orientable.*

(e) *If M' is any non-compact manifold of dimension n, the $H_n(M; R) = 0$.*

Reminder. Recall that the ***characteristic*** of a ring R, denoted $\operatorname{char}(R)$, is defined to be minimum integer $n \geq 1$ such that

$$\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$$

if it exists, and if such an integer does not exist, it is defined to be 1.

Corollary D.1.10. *Fix a ring R with $\operatorname{char}(R) > 2$ or $\operatorname{char}(R) = 0$. If M is a closed manifold with $H_n(M; R) \cong R$, then M is orientable.*

Proof. If M is not orientable, then $H_n(M) \to H_n(M \mid p) \cong R$ is injective with image the set of $r \in R$ with $2r = 0$. Our assumptions preclude this, however, since $R \neq 0$ and $2r = 0$ if and only if $r = 0$. ■

Corollary D.1.11. *Every closed n-manifold is $\mathbf{Z}/2$-orientable.*

Proof. Since $2 \cdot 1 = 1 + 1 = 0$ and $2 \cdot 0 = 0 + 0 = 0$ in $\mathbf{Z}/2$, if M is not $\mathbf{Z}/2$-orientable, then the map $H_n(M; \mathbf{Z}/2) \to H_n(M \mid p; \mathbf{Z}/2) \cong \mathbf{Z}/2$ is injective and has image all of $\mathbf{Z}/2$ and so is an isomorphism, but this contradicts the assumption that M is not $\mathbf{Z}/2$-orientable. ■

Theorem D.1.12 (Brown). *All topological manifolds with boundary have collars—that is, all topological manifolds with boundary admit an open embedding $\partial M \times [0,1) \to M$ which restricts to the inclusion of ∂M into M at $t = 0$.*

Corollary D.1.13. *If M is a topological manifold with boundary, then $M \setminus \partial M = \operatorname{Int}(M) \to M$ is a homotopy equivalence and $(M, \partial M)$ is a* ***good pair*** *in the sense of Hatcher.*

Definition. Let M be a *compact* n-manifold with boundary and R be a ring. We suppress coefficients.

(a) An R-***orientation*** μ of M is defined to be an orientation of $\operatorname{Int} M$.
(b) M is R-***orientable*** if $\operatorname{Int} M$ is R-orientable.
(c) When $R = \mathbf{Z}$, we say M is ***orientable*** if $\operatorname{Int} M$ is orientable.

Proposition D.1.14. *If M is a manifold with boundary, then an R-orientation of M determines an R-orientation of ∂M.*

Proof. We drop coefficients throughout the proof.

Pick a coordinate nbhd U of a point p on the boundary of M and suppose U is contained in a collar nbhd of the boundary. Suppose moreover that U maps to an open half ball in $\mathbf{H}^n$ and that U is contained in the domain of larger chart. Let $\partial U = U \cap \partial M$. Let $N = \operatorname{Int} M$, let $V = U \setminus \partial U$ and let $q \in V$.

Note that $H_*(N \mid q) \cong H_*(M \mid q)$ from the LES in homology and the five lemma since $N \to M$ and $N \setminus \{q\} \to M \setminus \{q\}$ are homotopy equivalences as a consequence of the collar theorem. Similarly $(M \setminus V) \setminus \{p\} \to M \setminus U$ is a homotopy equivalence.

We then have the following chain of isomorphisms

$$\begin{aligned} H_n(N \mid q) &\cong H_n(M \mid q) \cong H_n(M, M \setminus V) \\ &\xrightarrow[\cong]{\partial} H_{n-1}(M \setminus V, (M \setminus V) \setminus \{p\}) \cong H_{n-1}(\partial M \mid p) \\ &\cong H_{n-1}(\partial M \mid \partial U) \end{aligned}$$

The penultimate isomorphism is excision of $N \setminus V$. The connecting homomorphism ∂ arises from the LES of the triple $(M, M \setminus V, M \setminus U)$ and is an isomorphism since $H_*(M, M \setminus U) \cong H_*(M, M) = 0$ since $M \setminus U \to M$ is a homotopy equivalence—indeed, one can simply construct this by showing that the complement of an open hall ball in $\mathbf{H}^n$ is homotopy equivalent to $\mathbf{H}^n$.

This map sends a local orientation to a local orientation and all isomorphisms in sight are natural so they descend to restrictions between local generators. ∎

Corollary D.1.15. *If M is a compact n-manifold with boundary and R-orientable, then there is a unique class $\mu_M \in H_n(M, \partial M)$ mapping to the fundamental class of $H_{n-1}(\partial M)$.*

Proof. Since $\operatorname{Int}(M)$ is not compact and homotopy equivalent to M, the LES of the pair $(M, \partial M)$ satisfies that $H_n(M, \partial M) \to H_{n-1}(\partial M)$ is injective. Let C_0 be a collar of the boundary of the form $\partial M \times [0, 1)$ and let C be the image of $\partial M \times [0, 1/2)$ in M. Setting $N = M \setminus C$, N is compact, closed and a deformation retract of $\operatorname{Int} M$. Hence,

$$H_*(\operatorname{Int} M \mid N) \cong H_*(M \mid \operatorname{Int} M) = H_n(M, \partial M).$$

The R-orientation of $\operatorname{Int} M$ maps the fundamental class $\mu_{\operatorname{Int} M}$ for $\operatorname{Int} M$ to an element of $H_n(\operatorname{Int} M \mid N)$. This is itself a fundamental class for the subspace N. Indeed, this follows from the following commutative diagram of inclusions of pairs

$$\begin{array}{ccc} (\operatorname{Int} M, \operatorname{Int} M \setminus M) & \longrightarrow & (\operatorname{Int} M, \operatorname{Int} M \setminus N) \\ \downarrow & & \downarrow \\ (\operatorname{Int} M, \operatorname{Int} M \setminus \{x\}) & \longrightarrow & (\operatorname{Int} M, \operatorname{Int} M \setminus \{x\}) \end{array}$$

Let μ_M be the image of $\mu_{\operatorname{Int} M}$ int $H_n(M, \partial M)$. Naturality of the chain of isomorphisms above now implies that $\partial \mu_M$ gives a generator of $H_{n-1}(\partial M \mid x)$ for all $x \in \partial M$ furnishing an orientation. ∎

Proposition D.1.16. *Let R be a field or* $\mathbf{Z}$ *and M is a closed connected R-orientable n-manifold.*

(a) *The cup product pairing $H^k(M; R) \times H^{n-k}(M; R) \to H^n(M; R) \xrightarrow{\mu_M \frown -} R$ with $\mu_M \frown -$ an isomorphism by duality is a non-degenerate bilinear form. In particular, the adjunction maps sends a class φ to the map $\psi \mapsto (\psi \smile \varphi)(\mu_M)$.*
(b) *A class $\varphi \in H^k(M; R)$ generates an R-summand of $H^k(M; R)$* ***iff*** *there exists $\psi \in H^{n-k}(M; R)$ such that $\varphi \smile \psi$ is a generator of $H^n(M; R)$.*

Proof. **(a)** A bilinear pairing is non-degenerate **iff** both adjoint maps are isomorphisms. Hence, by adjunction, it suffices to check that $H^k(M; R) \to \operatorname{Hom}_R(H^n(M; R), R)$ and $H^{n-k}(M; R) \to \operatorname{Hom}_R(H^k(M; R), R)$ are isomorphisms. The adjunction map sends a class φ to the map $\psi \mapsto \varphi(\mu_M \frown \psi) = (\psi \smile \varphi)(\mu_M)$ and this factors as

$$H^*(M; R) \to \operatorname{Hom}_R(H_*(M; R), R) \xrightarrow{(\mu_M \frown -)^*} \operatorname{Hom}_R(H^{n-*}(M; R), R)$$

with the first map the natural comparison map arising in the universal coefficient theorem. Indeed, the adjunction map is $\varphi \mapsto (\psi \mapsto (\varphi \smile \psi)(\mu_M))$. For $R = \mathbf{Z}$ or R a field, the first map is an isomorphism since the Ext term vanishes and the second map is an isomorphism by duality.

(b) When R is a field, this is essentially automatically but we will give the proof generally.

($\Rightarrow$) We have a projection map $p\colon H^k(M) \to R$ sending $\varphi \mapsto \pm 1$. By **(a)**, the cup product pairing $H^k(M) \times H^{n-k}(M) \to R$ is non-degenerate and, hence, the adjoint map $H^{n-k}(M) \to \mathrm{Hom}_R(H^k(M), R)$ is an isomorphism which means that every $H^k(M) \to R$ is given by a cup product with some unique element of $H^{n-k}(M)$ so that there exists $\psi \in H^{n-k}(M)$ for which $p(\varphi) = (\varphi \smile \psi)(\mu_M)$ by the above so $\varphi \smile \psi$ is a generator. ($\Leftarrow$) If there exists $\psi \in H^{n-k}(M; R)$ such that $\varphi \smile \psi$ is a generator of $H^n(M; R)$, then the map sending $\alpha \in H^k(M)$ to $(\alpha \smile \psi)(\mu_M)$ is surjective with $(\varphi \smile \psi)(\mu_M) = \pm 1$. But such a map corresponds by non-degeneracy to the image of φ under the natural isomorphism $H^k(M) \to \mathrm{Hom}_R(H^{n-k}(M), R)$. Hence, $r\varphi \neq 0$ for any $r \in R$ since this would imply that $r(\varphi \smile \psi) = (r\varphi) \smile \psi = 0$ which is impossible as this necessarily is equal to R. ■

D.2 The Gysin Sequene

Theorem D.2.1. *Let $S^n \to E \xrightarrow{\pi} B$ be a fibration with trivial local coefficients for R-coefficients in singular cohomology where R is a commutative ring. Then there is an exact sequence (suppressing the coefficient group)*

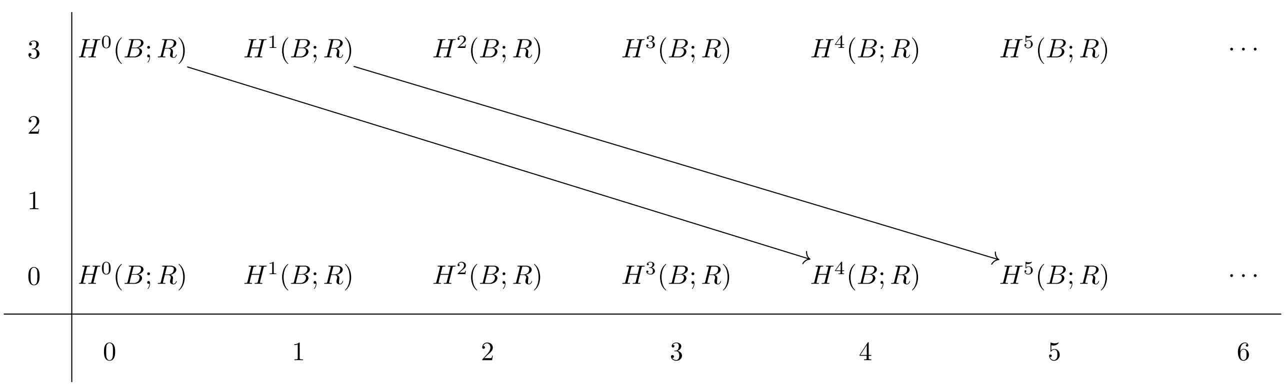
$$\cdots \to H^i(E) \xrightarrow{\delta} H^{i-n}B \xrightarrow{\gamma \cdot -} H^{i+1}B \xrightarrow{\pi^*} H^{i+1}E \to \cdots$$

where $\gamma \in h^{n+1}B$ is $d_{n+1}(\iota)$ where $\iota \in \iota \in E_2^{0,n} = E_{n+1}^{0,n} \cong H^0(B; R)$ corresponds to the multiplicative unit in $H^0(B; R)$. Moreover, this is natural with respect to morphisms of such sequences and sends γ of the first sequence to γ of the second sequence.

Proof. The Serre spectral sequence runs as usual with signature $E_2^{p,q} = H^p(B, H^q(S^n; R)) \to H^{p+q}(E; R)$. This has E_2 pages

$$E_2^{p,q} \cong \begin{cases} H^p(B; R) & q = 0, n \\ 0 & \text{else.} \end{cases}$$

The only non-trivial differential is d_{n+1}. For example, if $n = 3$, we have (not displaying all differentials)



Let $\iota \in E_2^{0,n} \cong H^0(B; R)$ correspond to the multiplicative unit of $H^*(B; R)$ and let $\gamma = d_{n+1}(\iota) \in E_{n+1}^{n+1,0}$. The product structure on this page is, up to a sign, the cup product of the terms followed by the cup product on coefficients, since it agrees with the E_2 page. In particular, from this description, we deduce that every element of $E_{n+1}^{-,n}$ is of the form $\iota \cdot \varphi$ for $\varphi \in E_{n+1}^{-,0}$ where $\cdot$ has the structure just described from the E_2 page. This means that for $\varphi \in E_{n+1}^{p,0}$, we can compute

$$\begin{aligned} (-1)^p d_{n+1}\varphi &= d_{n+1}((-1)^p \iota \smile \varphi) = d_{n+1}(\iota\varphi) = d_{n+1}(\iota) \cdot \varphi + (-1)^n \iota \cdot d_{n+1}(\varphi) \\ &= d_{n+1}(\iota) \cdot \varphi = (-1)^p d_{n+1}(\iota) \smile \varphi. \end{aligned}$$

Hence, $d_{n+1}(\iota) \smile \varphi = d_{n+1}\varphi$ and this identification respects the identification of the $E_2 = E_{n+1}$ page so we have SESs

$$0 \to E_\infty^{p,n} = \operatorname{Ker} d_{n+1} \to H^p(B; R) \xrightarrow{d_{n+1}\iota \smile -} H^{p+n+1}(B; R) \to \operatorname{Coker} d_{n+1} = E_\infty^{p+n+1,0} \to 0$$

By convergence, we also have SESs

$$0 \to E_\infty^{p,0} \to H^p(E; R)) \to E_\infty^{p-n,n} \to 0.$$

Piecing these together affords the long exact sequence. In the Serre spectral sequence, the map $\pi^*\colon H^i(B;R) \to H^i(E;R)$ factors through the edge homomorphism and the inclusion $E_\infty^{i,0} \subset H^iE$ as $H^iB = E_2^{i,0} \to E_\infty^{i,0} \subset H^iE$. This allows us to identify the map $H^i(B;R) \to H^i(E;R)$ in this long exact sequence as π^*.

As for naturality, this now follows from naturality of the Serre spectral sequence. In particular, we will obtain a diagram

$$\begin{array}{ccc} H^0(B') & \xrightarrow{\gamma' \smile -} & H^{n+1}(B') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^0(B) & \xrightarrow{\gamma \smile -} & H^{n+1}(B) \end{array}$$

and by chasing 1 this shows

$$\begin{array}{ccc} 1 & \longmapsto & \gamma' \\ \downarrow & & \downarrow \\ 1 & \longmapsto & \gamma \end{array}$$

which is what we sought to show. ■

D.3 The Steenrod Algebra, Graded Modules and Graded Rings

D.3.1 Definitions and Properties

Convention. We fix a prime p throughout, but indicate the special cases of $p = 2$.

Definitions. For a monoid M, an ***M-graded ring*** is a ring R with $R \cong \bigoplus_{m\in M} R_m$ as abelian groups and $R_m \cdot R_{m'} \subset R_{m\cdot m'}$. When we say a ***graded ring***, we mean $\mathbf{Z}$-graded, and we henceforth restrict to these and we will understand certain $\mathbf{Z}$-graded modules and rings to be trivial in negative degrees.

A ***graded left module*** over a graded ring R is a left R-module M such that as abelian groups $M = \bigoplus_{n\in\mathbf{Z}} M_n$ and $R_i \cdot M_j \subset M_{i+j}$. A ***graded right module*** is defined similarly with $M_i \cdot R_j \subset M_{i+j}$. A ***homomorphism of graded R-modules*** M and N is an R-linear map $f\colon M \to N$ that respects the grading in the sense that $f = \bigoplus_i f_i$ where $f_i\colon M_i \to N_i$.

If M is a graded right R-module and N is a graded left R-module, we define $M \otimes_R N$ to be the graded abelian group with underlying abelian group $M \otimes_R N$ the usual tensor product and with grading defined by letting $(M \otimes_R N)_k$ be the *subgroup* (not submodule, of course) generated by elements $m \otimes n$ with $\dim m + \dim n = k$.

If R is a graded ring and $n \in \mathbf{Z}$, let $R(n)$ be graded the R-module which in degree k is given by $R(n)_k = R_{n+k}$. A ***free graded module*** over a graded ring is any direct sum of the form

$$\bigoplus_{i\in I} R(n_i).$$

Suppose R is a ***commutative graded ring***. This simply means R is a graded ring and additionally the multiplication is commutative. A ***graded R-algebra*** over a commutative graded ring R is an R-algebra A—that is, a ring A that has an R-module structure for which the ring multiplication is R-bilinear—where we additionally require the multiplication map $A \otimes_R A \to A$ to be a morphism of graded R-modules. We always assume algebras are associative. We say A is a ***commutative graded R-algebra*** if for $x \in A_i$, $y \in A_j$, $xy = (-1)^{ij}yx$. In fact, the map $T\colon A \otimes A \to A \otimes A$ generated by sending $a_i \otimes a_j \mapsto (-1)^{ij} a_j \otimes a_i$ for $a_i \in A_i$ and $a_j \in A_j$ is R-bilinear so descends to a map $A \otimes_R A \to A \otimes_R A$.

Continuing to suppose R is a commutative graded ring, if A and B are graded R-algebras, then $A \otimes_R B$ acquires a graded R-algebra structure by defining $\mu_{A,B}\colon (A \otimes_{\mathbf{Z}} B) \otimes_{\mathbf{Z}} (A \otimes_{\mathbf{Z}} B) \to A \otimes_{\mathbf{Z}} B$ by $(\mu_A \otimes \mu_B) \circ (1 \otimes T \otimes 1)$, so that $(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\dim a_2 \cdot \dim b_1}(a_1a_2 \otimes b_1b_2)$.

Continuing to suppose R is a commutative graded ring, let M be an ordinary R-module (not graded). Then we may define its ***graded tensor algebra*** $\Gamma(M)$ to be $\Gamma(M) = \bigoplus_{n\geq 0} M^{\otimes_R n}$ with $M^{\otimes_R 0} = R$, and where "juxtaposition" defines the product structure and where addition is obvious. The R-algebra structure follows since the tensor product "bilinearizes" the r-action.

Definition. Suppose first that $p > 2$. The ***Steenrod algebra*** $\mathcal{A} = \mathcal{A}_p$ is $\mathbf{Z}/p$-algebra

$$\mathcal{A}_p = (\mathbf{Z}/p)[\beta, P^0, P^1, P^2, \ldots]/I$$

where I is the ideal closed under $\mathbf{Z}/p$-multiplication generated by $P^0 - 1$, β^2, and the ***Adem relations***

$$P^a P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j} P^j \qquad a < pb$$
$$P^a \beta P^b = \sum_j (-1)^{a+j} \binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j} P^j \qquad a \leq pb$$

the binomial coefficients necessarily being taken mod p.

Here β is the ***Bockstein homomorphism*** and the P^a terms are call the ***Steenrod reduced p-th powers***. The relation $P^0 = 1$ is conceptually useful because it's nice to sum from 0.

When $p = 2$, the Steenrod algebra has a somewhat more tractable description as the $\mathbf{Z}/2$-algebra

$$\mathcal{A}_2 = (\mathbf{Z}/2)[Sq^0, Sq^1, Sq^2, Sq^3, \ldots]/I$$

where I is the ideal closed under $\mathbf{Z}/2$-multiplication generated by $Sq^0 - 1$ and the ***Adem relations***

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \qquad a < 2b$$

the binomial coefficients necessarily being taken mod 2. The terms Sq^a are called the ***Steenrod squares***.

The Steenrod algebra $\mathcal{A}_p$ is naturally graded. For $p = 2$, $\deg Sq^i = 2i$. For $p > 2$, $\deg \beta = 1$ and $\deg P^i = 2i(p-1)$.

Theorem D.3.1. *Let H^* denote the cohomology functor $H^*(-,-;\mathbf{Z}/p)\colon \mathsf{Top}^{(2)} \to \mathsf{Ab}_{\mathbf{N}}$ on pairs of spaces to graded abelian groups. Then, in fact, $H^*\colon \mathsf{Top}^{(2)} \to \mathsf{gMod}_{\mathcal{A}_p}$ lands in graded modules over the Steenrod Algebra.*

Theorem D.3.2. *Let $p = 2$ and let $H^* = H^*(-,-;\mathbf{Z}_p)$. Then the Steenrod squares Sq^i satisfy the following list of properties on cohomology.*

(1) *$Sq^i\colon H^*(-,-;\mathbf{Z}_2) \to H^{*+i}(-,-;\mathbf{Z}_2)$ is a natural transformation of cohomology theories, meaning $Sq^i\colon H^n \to H^{n+i}$ is natural and Sq^i commutes with the connecting homomorphism in the LES in cohomology. This implies, for instance, that $Sq^i(\alpha+\beta) = Sq^i(\alpha) + Sq^i(\beta)$.*
(2) *If $i > j$, then $Sq^i(x) = 0$ for all $x \in H^j(K, L; \mathbf{Z}_2)$.*
(3) *$Sq^i(x) = x^2$ for all $x \in H^i(K, L; \mathbf{Z}_2)$.*
(4) *$Sq^0 = \mathrm{id}$.*
(5) *Sq^1 is the Bockstein homomorphism arising from the connecting homomorphism induced by hitting the SES*

$$0 \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to 0$$

with the cochain complex function $C^*(K, L; -)$ and then noting that a SES of cochain complexes gives rise to a long exact sequence in cohomology.
(6) ***Cartan formula**: $Sq^i(xy) = \sum_j (Sq^j x)(Sq^{i-j} y)$.*
(7) ***Adem relations**: For $a < 2b$, $Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$, the binomial coefficient is taken mod 2.*

Fix $p > 2$ and let $H^* = H^*(-,-;\mathbf{Z}_p)$. Then β and the Steenrod reduced p-th powers satisfy the following list of properties on cohomology.

(1) *$P^i\colon H^*(-,-;\mathbf{Z}_p) \to H^{*+i}(-,-;\mathbf{Z}_p)$ is a natural transformation of cohomology theories, meaning $P^i\colon H^n \to H^{n+i}$ is natural and P^i commutes with the connecting homomorphism in the LES in cohomology. This implies, for instance, that $P^i(\alpha+\beta) = P^i(\alpha) + P^i(\beta)$.*
(2) *If $2i > j$, then $P^i(x) = 0$ for all $x \in H^j(K, L; \mathbf{Z}_p)$.*
(3) *$P^i(x) = x^p$ for all $x \in H^{2i}(K, L; \mathbf{Z}_p)$.*
(4) *$P^0 = \mathrm{id}$.*
(5) *β is the Bockstein homomorphism arising from the connecting homomorphism induced by the SES*

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$$

with the cochain complex function $C^*(K, L; -)$ and then noting that a SES of cochain complexes gives rise to a long exact sequence in cohomology.
(6) ***Cartan formula**: $P^i(xy) = \sum_j (P^j x)(P^{i-j} y)$.*
(7) ***Adem relations:***

$$P^aP^b = \sum_j (-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j \qquad a < pb$$
$$P^a\beta P^b = \sum_j (-1)^{a+j}\binom{(p-1)(b-j)}{a-pj} \beta P^{a+b-j}P^j \qquad a \leq pb.$$

the binomial coefficients being taken mod p.

Remark. Since the P^is are coomology natural transformations, it will follow that they commute with the suspension isomorphisms for reduced suspension on well-pointed spaces. This is essentially because of the way reduced and unreduced co/homology theories are related.

Definition. Fix $p > 2$. The ***admissible monomials*** in the Steenrod algebra $\mathcal{A}_p$ are the monomials of the form

$$\beta^{\varepsilon_1}P^{i_1}\beta^{\varepsilon_2}P^{i_2}\cdots\beta^{\varepsilon_n}P^{i_n}$$

for $n \in \mathbf{N}$, where $\varepsilon_i = 0$ or 1 and where $i_j \geq \varepsilon_{j+1} + pi_{j+1}$ for all j. In other words, the ***admissible*** monomials are the ones to which we *cannot* apply the Adem relations. In general, any monomial is of this form, and we may give lexicographic ordering to such monomials by associating to them the $\mathbf{N}$-length tuples of integers $(\varepsilon_1 + pi_1, \varepsilon_2 + pi_2, \ldots)$. We define the ***excess*** of an admissible monomial by $\sum_j(2i_j - 2pi_{j+1} - \varepsilon_{j+1})$ (the reason why is buried in Hatcher's book where he defines this).

Fix $p = 2$. The ***admissible monomials*** in the Steenrod algebra $\mathcal{A}_2$ are the monomials $Sq^{i_1}Sq^{i_2}\cdots Sq^{i_{n-1}}Sq^{i_n}$ for $n \in \mathbf{N}$, where $i_j \geq 2i_{j+1}$ for all j. In other words, the ***admissible*** monomials are the ones to which we *cannot* apply the Adem relations. It is convenient to write such monomials as Sq^I where $I = (i_1, \ldots, i_n)$. In general, any monomial is of this form, and we may give lexicographic ordering to such monomials by associating to them the $\mathbf{N}$-length tuples of integers $(i_1, i_2, i_3, \ldots)$. We define the ***excess*** of an admissible monomial by $\sum_j(i_j - 2i_{j+1})$. For the Steenrod squares, we can write this as $e(I)$.

Theorem D.3.3. *All monomials can be written as a sum of admissible monomials.*

Proof. Use the lexicographic ordering and apply the Adem relations. ■

Proposition D.3.4. *Let $Sq = 1 + Sq^1 + Sq^2 + Sq^3 + \cdots$ (here $1 = Sq^0$ is the identity) the* ***total Steenrod square*** *and view this as a map $Sq\colon H^*(K, L; \mathbf{Z}/2) \to H^*(K, L; \mathbf{Z}/2)$.*

(a) *This is well-defined.*
(b) *The total Steenrod square is a ring-homomorphism. In particular, $Sq(xy) = Sq(x)Sq(y)$.*

Proof. **(a)** By axiom **(2)**, the action of Sq on each piece of the graded cohomology ring $H^*(K, L; \mathbf{Z}/2)$ with only finitely many non-trivial terms. This shows it is indeed well-defined.

(b) By additivity, $Sq(x + y) = Sq(x) + Sq(y)$. By axiom **(2)**, the unit $1 \in H^0(K, L; \mathbf{Z}/2)$ satisfies $Sq(1) = 1$. Now we simply observe that the total Steenrod square gives a slick way to repackage the Cartan formula as the Cartan formula implies that

$$Sq(xy) = Sq(x)Sq(y),$$

as desired. ■

Proposition D.3.5. *$Sq(x \times y) = Sq(x) \times Sq(y)$ for all $x \in H^*(X; \mathbf{Z}/2)$ and $y \in H^*(Y; \mathbf{Z}/2)$. In particular, degree considerations imply that $Sq^k(x \times y) = \sum_j Sq^j(x) \times Sq^{k-j}(y)$.*

Proof. If we take as our definition of $\times$ that it is $x \times y = \mathrm{pr}_X^* x \smile \mathrm{pr}_Y^* y$, then we know that

$$Sq(x \times y) = Sq(\mathrm{pr}_X^* x)Sq(\mathrm{pr}_Y^* y)$$

and by axiom **(1)** (naturality of the Steenrod squares) we know that this is

$$Sq(\mathrm{pr}_X^* x)Sq(\mathrm{pr}_Y^* y) = \mathrm{pr}_X^* Sq(x) \smile \mathrm{pr}_Y^* Sq(y)$$

and this expression is the precisely $Sq(x) \times Sq(y)$. ■

Corollary D.3.6. *Let $H^*(X_i; \mathbf{Z}/2)$ be finitely generated in each degree for $i = 2, \ldots, n$. Then the cohomology cross product furnishes a Künneth isomorphism $\bigotimes_{\mathbf{Z}/2,\ i=1}^n H^*(X_i; \mathbf{Z}/2) \to H^*\left(\prod X_i; \mathbf{Z}/2\right)$ and this is furthermore an isomorphism of modules over the Steenrod algebra where the Steenrod squares act componentwise on decomponsable tensors and by shifting degree (explained in the proof).*

Proof. The first part is an easy induction—using as a base case the usual Künneth isomorphism statement—breaking off X_n. The final map thereby induced is exactly the iterated cohomology cross product. Fix a decomposable tensor $x_1 \otimes \cdots \otimes x_m$ in the left-hand side with $|x_i| = n_i$ and $\sum_i n_i = m$. Define

$$Sq^k(x_1 \otimes \cdots \otimes x_m) = \sum_{i_1+\cdots+i_m=k} Sq^{i_1}x_1 \otimes \cdots \otimes Sq^{i_m}x_m$$

and extend this linearly (it is certainly $\mathbf{Z}/2$-bilinear). One can easily check that this respects the identifications for the Steenrod algebra and defines a module structure over the Steenrod algebra. Finally, we note from the preceding proposition that

$$Sq^k(x_1 \times \cdots \times x_n) = \sum_{i_1+\cdots+i_m=k} Sq^{i_1}x_1 \times \cdots \times Sq^{i_m}x_m$$

so that this morphism commutes with the action of the Steenrod squares and thus constitutes a morphism of algebras over the Steenrod algebra. ■

Theorem D.3.7. *The mod p Steenrod algebra is equivalently the collection of all stable cohomology operations on H^* (or, equivalently, $H\mathbf{F}_p^*(H\mathbf{F}_p)$, if you know about spectra).*

Definition. Let $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} : \sum t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$ be the ***standard n-simplex***. We order the vertices of Δ^n by letting, for each $0 \leq i \leq n$, v_i be the point that is 0 everywhere except the i-th entry. There is a canonical linear homeomorphism from this n-simplex onto any other such ordered n-simplex $[v_0, \ldots, v_n]$ (i.e., the v_i are the vertices) preserving the order of the vertices by $(t_0, \ldots, t_n) \mapsto \sum t_i v_i$.

Let X be a a Δ-complex. That is, X is a space along with the additional structure of a collection of characteristic maps $\sigma_\alpha \colon \Delta^{n_\alpha} \to X$ satisfying

(i) $\sigma_\alpha|\operatorname{Int}(\Delta^{n_\alpha})$ is injective and each $x \in X$ lies in the image of exactly one of the maps $\sigma_\alpha|\operatorname{Int}(\Delta^{n_\alpha})$.
(ii) The restriction of each σ_α to a face is already a map $\sigma_\beta \colon \Delta^{n_\alpha-1} \to X$ in this collection upon using the canonical linear homeomorphism from the standard n-simplex to this n-simplex, where the relevant face of Δ^{n_α} inherits the evident ordering of vertices.
(iii) A set $U \subset X$ is open iff $\sigma_\alpha^{-1}(U)$ is open in each Δ^{n_α} for all α.

Remark. Unlike a simplicial complex, a Δ-complex already has an implicit ordering on each simplex. The compatibility condition of **(ii)** often forces your hand in ways when building explicit Δ-complexes for quotient spaces such as the Klein bottle.

Definition. Fix $i \geq 0$ and fix the coefficient ring $\mathbf{Z}/2$. Given an n-chain σ, define $\Delta_i \colon C_*(X; \mathbf{Z}/2) \to C_*(X; \mathbf{Z}/2)^{\otimes 2}$ by

$$\Delta_i(\sigma) \stackrel{\text{def}}{=} \sum_{\substack{U=\{v_1<\cdots<v_{n-i}\} \\ v_j \in \{0,\ldots,n\}}} d_{V^0}(\sigma) \otimes d_{V^1}(\sigma)$$

where $V^0 = \{v_j \in V : v_j \equiv j \mod 2\}$ and $V^1 = V \setminus V_j$ and where for a given ordered such set V,

$$d_V(\sigma) = d_{v_1} \cdots d_{v_r}(\sigma)$$

with $d_{v_i}(\sigma) = \sigma|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$ where the hat indicates omission.

Remark. Δ_0 is the Alexander-Whitney map. On an n-chain φ, $\Delta_n(\varphi) = \varphi \otimes \varphi$.

Theorem D.3.8 (Anibal M. Medina-Mardones). *The map Δ_i constructed above is a **cup-i coproduct**.*

This means that we can construct the Steenrod squares as follows.

Corollary D.3.9 (Anibal M. Medina-Mardones). *The Steenrod squares*

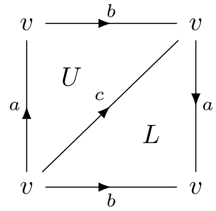
$$Sq^k \colon H^*(X; \mathbf{Z}/2) \to H^{*+k}(X; \mathbf{Z}/2)$$

are given by $Sq^k(\varphi) = (\varphi \otimes \psi)(\Delta_{n-k}(-))$

D.3.2 Example Applications

In simple cases, it is often possible to compute the module structure of the cohomology ring with $\mathbf{Z}/2$ coefficients as a module over the Steenrod algebra directly, using only the elementary properties of the Steenrod Squares.

Example 8. The Klein bottle K is a Δ-complex in the evident way (see for instance Hatcher's book). It has a single vertex, three edges and two faces. The idea is that the Klein bottle is formed by starting with a square and identifying one pair of opposite edges with the same orientation but then identifying the other pair of opposite edges with reverse orientation. Illustrated below are the simplices of this Δ-complex structure for K.



Using the simplicial cochain complex with $\mathbf{Z}/2$ coefficients, one can compute that $H^*(K;\mathbf{Z}/2) \cong \mathbf{Z}/2[X,Y]/(X^3, Y^2, X^2 - XY)$ where $|X| = |Y| = 1$. In the standard Δ-complex structure on K given in Hatcher's book, X is the element of $C^1(K;\mathbf{Z}/2)$ that returns $1 \in \mathbf{Z}/2$ upon evaluation on the 1-simplices a and b and Y returns 1 upon evaluation on the diagonal 1-simplex c and the 1-simplex b. One likewise computes its $\mathbf{Z}/2$-fundamental class is $U + L$.

To determine the action of the Steenrod squares, observe that Sq^k with $k \geq 3$ all act by 0 and Sq^2 acts by squaring things in degree 2 and killing everything else. $Sq^0 = 1$ is the identity and Sq^1 acts by squaring things in the middle degree, so we only need to compute the action of Sq^1 and Sq^2 in degree 2. One can check that X^2 is the generator of $H^2(K;\mathbf{Z}/2)$. But, of course, a degree argument shows that

$$Sq^1(X^2) = 0$$

since $\left|Sq^1(X^2)\right| = 3$ and there is no cohomology groups in degree 3 and similarly

$$Sq^2(X^2) = X^4 = 0$$

since there is no degree 4 cohomology groups. Alternatively, by naturality of the squaring operation,

$$Sq^1(X^2) = Sq^0(X)Sq^1(X) + Sq^1(X)Sq^0(X).$$

By graded commutativity,

$$Sq^0(X)Sq^1(X) = (-1)^{2+1}Sq^1(X)Sq^0(X) = -Sq^1(X)Sq^0(X) = Sq^1(X)Sq^0(X)$$

where the last equality follows since the cohomology ring is in fact an algebra over $\mathbf{Z}/2$ so that the additive inverse of an element is always itself. Thus,

$$Sq^1(X^2) = 2X^2 = 0$$

since for all modules over $\mathbf{Z}/2$, multiplication by 2 is multiplication by 0. Since Sq^k is linear, this suffices.

The total Wu class

$$v = 1 + v_1 + v_2$$

can now be computed. Let μ be the fundamental class of K and note that since K is a closed manifold there is a unique choice for this with $\mathbf{Z}/2$-coefficients. Now, there is no ambiguity for v_0 since $v_0 = 1$ is a definition. For v_2, either $v_2 = X^2$ or $v_2 = 0$ since $H^2(K;\mathbf{Z}/2) \cong \mathbf{Z}/2$ generated by X^2. To see that $v_2 = 0$, recall that for a closed n-manifold M, its k-th Wu class v_k is the unique mod 2 cohomology class of degree k satisfying for all $\varphi \in H^{n-k}(M;\mathbf{Z}/2)$ that

$$(v_k \smile \varphi)(\mu_M) = Sq^k(\varphi)(\mu_M)$$

or, equivalently, such that

$$v_k(\mu_M \frown \varphi) = Sq^1(\varphi)(\mu_M)$$

where μ_M is the $\mathbf{Z}/2$-fundamental class of M. Hence, $v_2(K) = 0$ if and only if for all $\varphi \in H^0(K;\mathbf{Z}/2)$, $Sq^2(\varphi)(\mu_M) = 0$. Of course, $2 > |\varphi| = 0$ so $Sq^k(\varphi) = 0$. In fact, for any closed n-manifold M, $v_k(M) = 0$ whenever $k > n/2$.

To compute v_1, we recall that v_1 is the unique element of $H^1(K;\mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ such that

$$(v_1 \smile \varphi)(\mu) = Sq^1(\varphi)(\mu)$$

for each $\varphi \in H^1(K;\mathbf{Z}/2)$. Note that the $\mathbf{Z}/2$-fundamental class of K is the generator of $H_2(K;\mathbf{Z}/2) \cong \mathbf{Z}/2$ and is thus Poincaré dual to the unique generator of $H^0(K;\mathbf{Z}/2)$. Now, in the middle dimension, $Sq^1(\varphi) = \varphi^2$. Since $Sq^1(X+Y) = (X+Y)^2 = X^2 + 2XY + Y^2 = X^2$, $Sq^1(X) = Sq^1(X+Y)$ and $Sq^1(Y) = Y^2 = 0$—since K is non-orientable, we know that $w_1(K) \neq 0$. Our life is relatively easy in this case: since $Y^2 = 0$, $YX = XY = X^2$ and thus also $Y(X+Y) = XY = X^2$, the only interesting cup products to check are with X and $X+Y$, but since $Sq^1(X) = Sq^1(X+Y) = X^2$, it suffices to check this identity with X alone, in which case the equality $(Y \smile X)(\mu_M) = Sq^1(X)(\mu_M)$ follows trivially since $Sq^1(X) = Y \smile X = X^2$.

Thus, the Wu formula lets us the total Steenrod square as

$$w(K) = Sq(v) = Sq(1 + Y + 0) = Sq(1) + Sq(Y).$$

Obviously $Sq(1) = 1$ and

$$Sq(Y) = Sq^0(Y) + Sq^1(Y) = Y + Y^2 = Y.$$

Altogether, then

$$w(K) = 1 + Y.$$

Hence, we have obtained the non-trivial computation that for the tangent bundle of K,

$$w_1(K) = Y \in H^1(K; \mathbf{Z}/2) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2.$$

Of course, w_1 of a manifold vanishes if and only if it is orientable, so we could already conclude that $w_1(K) \neq 0$, but now we know the element precisely. We have also computed $w_2(K) = 0$.

Recall that for any rank n vector bundle ξ over a CW-complex B, for $1 \leq k \leq n$, if $n-k+1$ even and less than n, then $w_{n-k+1}(\xi)$ is the obstruction to the existence of a k-frame over the $(n-k+1)$-skeleton of B and for all other n $w_{n-k+1}(\xi)$ is the mod 2 reduction of this obstruction class under the coefficient reduction map $H^1(K; \mathbf{Z}) \to H^1(K; \mathbf{Z}/2)$.

Remark. Of course, other times it is more advantageous to have an explicit formula on hand when the cohomology ring is more complicated; we lucked out in the above example for v_1 because of the low complexity of the cohomology ring. In fact, on any closed 2-manifold, the second Wu class vanishes.

Exercise 74. *Compute the cohomology ring $H^*(S^1 \times S^1 \times S^1; \mathbf{Z}/2)$ as an algebra over the Steenrod algebra.* [Hint: Use the version of the Künneth theorem which identifies this algebra over the Steenrod algebra with $H^*(S^1; \mathbf{Z}/2)^{\otimes_{\mathbf{Z}/2} 3}$ via the cohomology cross product.]

Exercise 75. *Let M be a closed 2-manifold.*

(a) *The Wu formula states that $w = Sq(v)$ for any smooth manifold. Show that this implies $v_2 = w_2 + w_1^2$. Conclude that when M is a surface, then $w_2 = w_1^2$. Deduce a generalization of this.*

(b) *Suppose N is an orientable m-manifold for which TN splits as a sum of line bundles. Show that the tangent bundle of N admits a reduction of structure group to $S(O(1)^{\times m})$ and, hence, the classifying map $N \to BSO(m)$ factor up to homotopy through the map $B(SO(1)^{\times m}) \to BSO(m)$ induced by the evident inclusion of subgroups.* [Hint: Construct the reduction by using local frames.]

(c) *Suppose M is a closed orientable 2-manifold. Show that TM splits as a sum of two line bundles **iff** its Euler characteristic $\chi(M)$ is 0. Conclude that for such orientable surfaces M, $M \cong T^2$ the torus.* [Hint: Use the following facts: if ξ is an oriented vector bundle, then the Euler class $e(\xi)$ is the obstruction to finding a non-vanishing section of ξ; $e(M)(\mu_M) = \chi(M)$, where $\chi(M)$ is the Euler characteristic; $e(M)$ is obtained from the classifying map $M \to BSO(2)$ and if TM splits, then the classifying map factors through $BS(O(1)^{\times 2}) \simeq \mathbf{R}P^\infty$; use the classification of surfaces for the converse.]

(d) *Suppose M is a closed 2-manifold. If TM splits as a sum of two line bundles, then $\chi(M) = 0$.* [Hint: Pass to the orientation double cover $p\colon \widetilde{M} \to M$ and use the fact $T\widetilde{M} \cong p^*TM$ and the fact that the Euler characteristic is multiplicative for finite covering spaces.]

(e) *Show that, more generally, if TM splits as a sum of a line bundle and a rank $n-1$ bundle, then $\chi(M) = 0$.* [Hint: Passing to the orientation double cover, assume M is orientable. Find a smooth double cover $p\colon X \to M$ such that takes the line bundle summand of M to $p^*(L)$ a trivial bundle. Note that the Euler characteristic is multiplicative for finite covering spaces.]

Point-Set Results

E.1 Miscellany

Theorem E.1.1 (May, Thm 7.4.1). *Let $p\colon E \to B$ be a map and $\mathscr{U}$ be a numerable open cover of B. Then p is a Hurewicz fibration **iff** $p\colon p^{-1}(U) \to U$ is a Hurewicz fibration for all $U \in \mathscr{U}$.*

Proof. Omitted. There are two typos in May's proof. u_j should be $u_j = \sum_{i=1}^{j} \gamma_{T_i}(\beta)/\sum_{i=1}^{q} \gamma_{T_i}(\beta)$ and $s(e,\beta)$ should be $s(e,\beta)(0) = e$.

Corollary E.1.2. *Every numerable fiber bundle is a Hurewicz fibration.*

Proof. For an element U of a numerable open cover by trivializing open sets, it suffices to show in the coordinates of the trivialization that $U \times F \to U$ is a Hurewicz fibration. Of course, the dashed lift in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & U \times F \\ {\scriptstyle i_0}\big\downarrow & \nearrow & \big\downarrow \\ X \times I & \xrightarrow[H]{} & U \end{array}$$

always exists and can be taken to be the map $(H, g \circ \mathrm{pr}_X)$. Hence, the previous theorem allows us to conclude. ∎

Theorem E.1.3. *Every fiber bundle $E \to B$ is a Serre fibration.*

Proof. Omitted.

Theorem E.1.4 (Lee, A.57). *A proper continuous map to a locally compact Hausdorff space is a closed map.*

Proof. We show that for $f\colon X \to Y$ continuous and proper and $C \subset X$ closed, $f(C)^c$ is open. Since Y is LCH, each $y \in f(C)^c$ has an open nbhd V containing y that is precompact (open set whose closure is compact). So $K = f^{-1}(\overline{V})$ is compact as f is proper and so $C \cap K$ is a closed subset of the compact space K and so is compact in K and, hence, also X. Hence, $f(C \cap K) = f(C) \cap \overline{V}$ is compact. Since Y is Hausdorff, it is also closed. Hence, $V \setminus (f(C) \cap \overline{V}) = V \setminus f(C)$ is an open nbhd of y not intersecting $f(C)$. ∎

E.2 Submanifolds are Locally Closed

Definition. Say a subspace $A \subset X$ is ***locally closed*** if it A is a closed subspace of an open subspace V of X.

Lemma E.2.1. *Let $A \subset X$. TFAE:*

(a) *A is locally closed.*
(b) *Each $p \in A$ has an open nbhd $U \subset X$ such that $A \cap U$ is closed in U.*
(c) *A is open in its closure $\overline{A}$.*

Proof. (a) ⇒ **(b)** $A \subset V \subset X$. The nbhd if V since $V \cap A = A$ is closed in V.

(b) ⇒ **(c)** Let U_p be a nbhd of $p \in A$ asserted to exist. Then $\mathrm{Cl}_{U_p}(U_p \cap A) = U \cap \mathrm{Cl}_X(A)$ since if $x \in \mathrm{Cl}_{U_p}(U_p \cap A)$, then every nbhd of x in U contains points of A and therefore since U_p is open $x \in \overline{A}$, which is the non-trivial inclusion. Since $U_p \cap A$ is closed in U_p, it follows that $U_p \cap A = U_p \cap \overline{A}$ and so $U_p \cap A$ is a nbhd of p in the subspace topology on $\overline{A}$. Since p was arbitrary, $A \subset \overline{A}$ is open in the subspace topology.

(c) ⇒ **(a)** Since $A \subset \overline{A}$ is open in the subspace topology, there is an open subspace U of X such that $U \cap \overline{A} = A$. ∎

Theorem E.2.2. *Submanifolds are locally closed.*

Proof. Let $N^n \subset M^m$ be a submanifold. By **(b)** above, this is a local problem, so fix $p \in N$. Then there is a chart (x, U) of M about p which, for convenience, we assume $x\colon U \to \mathbf{R}^m$ is a diffeomorphism onto an open subspace of some $\mathbf{R}^{m-k} \times \mathbf{R}^k_+ \subset \mathbf{R}^m$ and we assume $x(U)$ is an open ball, as well as a straightening diffeomorphism $\varphi\colon V \to \mathbf{R}^m$ where we may as well assume $x(U) \subset V$, where V is open in $\mathbf{R}^m$. Then $\varphi x(U \cap N) = \varphi x(U) \cap \mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+ \subset \mathbf{R}^m$. But this is closed in $\varphi x(U)$ since its complement is

$$\varphi x(U) \cap \varphi x(U \cap N)^c = \varphi x(U) \cap (\varphi x(U) \cap \mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+ \subset \mathbf{R}^m)^c = \varphi x(U) \cap (\mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+)^c$$

and $\mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_+$ is closed so its complement is open, and therefore the intersection is an open subset in $\varphi x(U)$. This shows that U is an open nbhd of $p \in N$ for which $N \cap U$ is closed in U. We conclude by **(b)**. ■

Remark. The preceding theorem allows us to throw away the closed hypothesis in many assertions in the literature. It can be useful to pair this with the corollary of the following theorem. Note that we phrase it differently from Kosinski, however, because it seems that his statement is not quite correct.

E.3 Tubular Neighborhood Trick

In order to prove the following theorem in the smooth case, we need the following auxiliary lemma.

Lemma E.3.1. *If $f\colon M \to N$ is a local diffeomorphism and $C \subset M$ is a submanifold for which $f|\,C$ is a homeomorphism onto its image, then $f(C)$ is a submanifold of N and hence $f|\,C$ is a diffeomorphism onto its image.*

Proof. This is an exercise in definitions. Since $f|\,C$ is a homeomorphism onto its image, it is a topological embedding. We therefore only need to verify that it is an immersion, and this follows because the property of being an immersion is local and f is locally a diffeomorphism. ■

The following theorem is taken from Daniel Tausk's notes, Lemma 8.12, where it is proved carefully.

Theorem E.3.2 (Tubular Neighborhood Trick). *If $f\colon X \to Y$ is a local homeomorphism where Y is hereditarily paracompact and Hausdorff and f is a homeomorphism on a subspace $C \subset X$, then f is a homeomorphism on a nbhd U of C.*

This can be upgraded to DIFF *as follows. If $f\colon X \to Y$ is a local diffeomorphism which is a homeomorphism on a submanifold $C \subset X$, then f is a diffeomorphism on a nbhd U of C.*

Since closed subspaces of a paracompact Hausdorff spaces are themselves paracompact, the proof admits minor modifications showing the following.

Corollary E.3.3. *If $f\colon X \to Y$ is a local homeomorphism where Y is paracompact Hausdorff and f is a homeomorphism on a subspace $C \subset X$ such that $f(C)$ is closed in Y, then f is a homeomorphism on a nbhd U of C.*

Remark. We have already shown that manifolds are hereditarily paracompact.

Proof (of Theorem). First, let us agree on some ad hoc terminology. For an open subset V of X, we will call the map $f|\,V$ a *chart for f* if $f|\,V$ is a homeomorphism onto its image. We will let $C' = \overline{f(C)}$. Now, the trickiest part of this is showing that a nbhd of $f(C)$ of the correct form exists. Lang, as usual, does not explain this well, or even really try to explain this.

Claim 25. For each point of $x \in C$ and nbhd U in X of x, there is a nbhd $V \subset U$ of x such that $f(V \cap C) = f(V) \cap f(C)$.

Since $U \cap C$ is open in C, $f(U \cap C)$ is open in $f(S)$. Hence, there is an open subset $A \subset Y$ such that $f(U \cap S) = A \cap f(S)$. Let $V = U \cap f^{-1}(A)$. Then V is an open nbhd of x contained in U and trivially we have $f(V' \cap C) \subset f(V') \cap f(C)$. On the other hand,

$$f(V) \cap f(C) \subset A \cap f(C) = f(U \cap C) = f(V \cap C).$$

For the last equality, observe that $V \subset U$ so $V \cap C \subset U \cap C$, while on the other hand, $U \cap C \subset f^{-1}(A)$ (basically just apply f^{-1} to $f(U \cap C) = A \cap f(C)$) so that by intersecting both sides of $U \cap C \subset f^{-1}(A)$ with U and C, we obtain $U \cap C \subset U \cap f^{-1}(A) \cap C = V \cap C$ and so $f(U \cap C) \subset f(V \cap C)$ and therefore have equality.

Note that a local homeomorphism that is injective is a homeomorphism. Therefore it suffices to find an open set $Z \subset X$ containing C such that $f|\,Z$ is injective. For each $x \in C$, let

$$f_x = f|\,U'_x\colon U'_x \to V'_x$$

be a local homeomorphism. By the claim, we may assume WLOG that $f(U'_x \cap C) = V'_x \cap C$. Let $Y_0 = \bigcup_{x \in C} V'_x$. Then this is open and paracompact Hausdorff since Y is hereditarily paracompact and Hausdorff. Therefore $\{V'_x\}$ admits a locally finite open refinement, say $\{V_i\}_{i \in I}$ (the family $\{V_i\}_{i \in I}$ is locally finite *in* Y_0).

For each index i, choose $x \in C \cap V_i$ such that $V_i \subset V'_x$ and set

$$U_i = f_x^{-1}(V_i) = (f|\, U'_x)^{-1}(V_i) \subset U'_x,$$

which is open since Y_0 is open and therefore its open subsets are open in Y. Then

$$f_i = f|\, U_i \colon U_i \to V_i$$

is a local homeomorphism and

$$f(U_i \cap C) = V_i \cap f(C).$$

This latter thing follows because f_x is a homeomorphism and therefore

$$f_x^{-1}(V_i \cap f(C)) = f_x^{-1}(V_i \cap f_x(C)) = f_x^{-1}(V_i) \cap f_x^{-1} f(C) = U_i \cap C.$$

Since paracompact Hausdorff spaces are normal, the shrinking lemma guarantees a locally finite open refinement of the V_i on the same index set, say $\{W_i\}$ with $W_i \subset V_i$ such that $\mathrm{Cl}_{Y_0}(W_i) \subset V_i \subset V'_x$. For each $i \in I$, let

$$Z_i = f_i^{-1}(W_i).$$

Then $Z_i \subset U_i \subset U'_x$ is open in X and, by abuse of notation, $f_i = f|\, Z_i \colon Z_i \to W_i$ is a homeomorphism. Once again, since f_x is a homeomorphism, we have that

$$f(Z_i \cap C) = W_i \cap f(C).$$

Now we claim that

$$C \subset \bigcup_{i \in I} Z_i.$$

Indeed, for $x \in C$, there exists $i \in I$ such that $f(x) \in W_i$ and therefore $f(x) \in W_i \cap f(C) = f(Z_i \cap C)$; it follows that there exists $y \in Z_i \cap C$ with $f(y) = f(x)$ but since $f|\, C$ is injective, $x = y$, proving the claim.

For each $x \in C$, let

$$I_x = \{i \in I : f(x) \in \mathrm{Cl}_{Y_0}(W_i)\}.$$

Since the closed cover $\{\mathrm{Cl}_{Y_0}(W_i)\}$ is locally finite as $\overline{W}_i \subset V_i$ and $\{V_i\}$ is locally finite in Y_0 so $\#(I_x) < \infty$. Moreover, $I_x \neq \varnothing$ from the above.

Keep $x \in C$. If $i \in I_x$, then from what we have shown,

$$f(x) \in \mathrm{Cl}_{Y_0}(W_i) \cap f(C) \subset V_i \cap f(C) = f(U_i \cap C),$$

and so since $f|\, C$ is injective, $x \in U_i$ and, in particular

$$x \in \bigcap_{i \in I_x} U_i,$$

and this holds for all $x \in C$.

Let us find an open nbhd G_x of $f(x)$ in Y_0 with the following properties:

(a) for each $i \in I$, $G_x \cap W_i \neq \varnothing$ **iff** $i \in I_x$;
(b) $G_x \subset f(\bigcap_{i \in I_x} U_i)$.

Such a set G_x can be defined by

$$G_x = \underbrace{(Y_0 \setminus \bigcup_{i \in I \setminus I_x} \mathrm{Cl}_{Y_0}(W_i))}_{\textbf{(a)}} \cap \underbrace{f(\bigcap_{i \in I_x} U_i)}_{\textbf{(b)}}.$$

We claim that G_x is open in Y_0 (and hence Y). Since f is an open map and $\#(I_x) < \infty$, $f(\bigcap_{i \in I_x} U_i)$ will be open in Y_0 and hence Y. Since $\{\mathrm{Cl}_{Y_0}(W_i)\}$ is locally finite and the union of any collection of locally finite sets is closed, $Y_0 \setminus \bigcup_{i \in I \setminus I_x} \mathrm{Cl}_{Y_0}(W_i)$ is open in Y_0 and hence Y—therefore G_x is open in Y_0 and hence Y. Note that for *any* locally finite collection of sets, the closure operator distributes over the union, which is where the penultimate assertion comes from.

Let $G = \bigcup_{x \in S} G_x$ and let $Z = f^{-1}(G) \cap \bigcup_{i \in I} Z_i$. Then G is open in Y_0 and hence Y and therefore Z is open in X. Moreover, $S \subset Z$ since $C \subset \bigcup_{i \in I} Z_i$ and clearly $f|\,Z \colon Z \to G$. Since Z is open and f is a local homeomorphism, $f|\,Z$ is a local homeomorphism. It therefore suffices to show it is injective to complete the proof.

Let $x, y \in Z$ with $f(x) = f(y)$. Pick indices $i, j \in I$ with $x \in Z_i$ and $y \in Z_j$. Now, $f(x) = f(y) \in G_z$ for some $z \in C$ so $f(x) \in G_z \cap W_i$ and $f(y) \in G_z \cap W_j$ and therefore $i, j \in I_z$ by property **(a)**. Property **(b)** implies $G_z \subset f(U_i \cap U_j)$ and therefore there exists $p \in U_i \cap U_j$ with $f(x) = f(p) = f(y)$. But since f is injective on U_i and on U_j individually, f is injective on $U_i \cap U_j$. Therefore $x = p = y$.

Observe that everything we did above made no explicit mention of whether we worked in TOP or DIFF. Indeed, because smoothness is a local property, everything still goes through in the smooth. ■

The Serre Spectral Sequence

F.1 Serre Spectral Sequence: Basic Terminology and Machinery

Convention. Throughout this section, we only consider cohomology and homology theories that are representable in the stable ∞-category Sp of spectra. In particular, this means we require all cohomology and homology theories send weak equivalences to isomorphisms and we require that all homology theories satisfy the ***wedge axiom*** which states that the natural map $\bigoplus_{i\in I}\widetilde{E}_*(X_i) \to \widetilde{E}_*(\bigvee_{i\in I} X_i)$ is an isomorphism for any collection of spaces. We now fix a representing spectrum E for our cohomology and homology theory. You may think of E as $H(-;R)$ where R is some commutative ring and $H(-;R)$ indicates either singular homology or cohomology, depending on where we indicate the grading goes.

Definition (Serre Fibration). A map $E \to B$ is a ***Serre fibration*** if for all $n \geq 0$, the dashed lift exists in any solid commutative diagram of the form

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ D^n \times I & \longrightarrow & B \end{array}$$

Remark. It turns out for any sufficiently "cellular" map $A \to B$, the dashed lift exists in any solid commutative diagram of the form

$$\begin{array}{ccc} A & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & B \end{array}$$

For example, if $A \to B$ is $A \times \{0\} \to A \times I$ and A is a CW-complex, this lift exists. The precise statement follows from model category theory. We will only use the case of A being a CW-complex and $A \times \{0\} \to A \times I$, although it is important to note that $A \to A \times I$ is cellular because I is a compact CW-complex—in general the product of cellular objects need not be cellular since, for example, the product of two non-compact CW-complexes need not be a CW-complex in general.

Proposition F.1.1. *Let $p\colon E \to B$ be a Serre fibration (e.g., a fiber bundle). Then any two fibers $p^{-1}(b)$ and $p^{-1}(b')$ with b and b' in the same path-component of B are connected by a zig-zag of weak equivalences.*

One can prove this by hand using lifting properties. Alternatively, one can appeal to model categorical machinery, which we will do to make this short.

Proof. The fiber $p^{-1}(b)$ is the pullback

$$\begin{array}{ccc} p^{-1}(b) & \longrightarrow & E \\ \downarrow & & \downarrow\!\!\downarrow \\ * & \xrightarrow[b]{} & B \end{array}$$

Since $E \to B$ is a fibration in the Quillen model structure on spaces and since this model structure is right proper, this pullback is a homotopy pullback. Let $\gamma\colon I \to B$ be a path connecting b and b' with $\gamma(0) = b$ and $\gamma(1) = b'$. Since the model category is right-proper, the weak equivalences of diagrams

$$\begin{array}{ccccc} * & \longrightarrow & B & \twoheadleftarrow & E \\ {\scriptstyle t=0}\downarrow\wr & & \| & & \| \\ I & \xrightarrow{\gamma} & B & \twoheadleftarrow & E \\ {\scriptstyle t=1}\uparrow\wr & & \| & & \| \\ * & \longrightarrow & B & \twoheadleftarrow & E \end{array}$$

pass upon taking pullbacks to weak equivalences of their respective pullbacks, yielding

$$p^{-1}(b) \xrightarrow{\sim} E \times_\gamma I \xleftarrow{\sim} p^{-1}(b')$$

which is the zig-zag of weak equivalences claimed. ■

Corollary F.1.2. *Let $p\colon E \to B$ be a Serre fibration and denote $F_b \overset{def}{=} p^{-1}(b)$. Let $\Pi_1(B)$ be the* ***fundamental groupoid***[1] *of B. There is a functor $\mathcal{E}^i(F)\colon \Pi_1(B) \to \mathsf{Mod}_{\mathbf{Z}}$ sending $b \mapsto E^i(F_b)$ and a path homotopy class $[\gamma]$ to $E^i(\gamma)$. This is what is called a* ***local coefficient system****. The same is true for homology. If $E^i(F)$ has the structure of a module over a commutative ring R, we may assume that $\mathcal{E}^i(F)\colon \Pi_1(B) \to \mathsf{Mod}_R$*

Proof. Let $b \in B$ and let $\gamma\colon b \to b'$ be any path. The above argument supplies a zig-zag of weak equivalences between fibers F_b and $F_{b'}$. On cohomology, this induces a map $E^*(F_b) \to E^*(F_{b'})$ and we claim that it is independent of the path-homotopy class of γ. If $\gamma \simeq \gamma'$ fixing the endpoints and we let $H\colon I \times I \to B$ be this homotopy—so $H(-,1) \equiv b'$, $H(-,0) \equiv b$ and $H(0,-) = \gamma$ and $H(1,-) = \gamma'$—then, denoting it by $H(s,t)$, this is

$$\begin{array}{ccccc}
I & \xrightarrow{\gamma} & B & \twoheadleftarrow & E \\
{\scriptstyle s=0}\downarrow\wr & & \| & & \| \\
I \times I & \xrightarrow{\gamma} & B & \twoheadleftarrow & E \\
{\scriptstyle s=1}\uparrow\wr & & \| & & \| \\
I & \xrightarrow[\gamma']{} & B & \twoheadleftarrow & E
\end{array}$$

Each of these diagrams further admit maps from the diagrams

$$* \xrightarrow{b} B \twoheadleftarrow E \qquad\qquad * \xrightarrow{b'} B \twoheadleftarrow E$$

picking out fibers in such a way that, upon taking pullbacks, the model categorical argument from before gives a commutative diagram of zig-zags of weak equivalences

$$\begin{array}{ccccc}
F_b & \xrightarrow{\sim} & E \times_\gamma I & \xleftarrow[\sim]{} & F_{b'} \\
\| & & \downarrow\wr & & \| \\
F_b & \xrightarrow{\sim} & E \times_H (I \times I) & \xleftarrow[\sim]{} & F_{b'} \\
\| & & \wr\uparrow & & \| \\
F_b & \xrightarrow{\sim} & E \times_\gamma I & \xleftarrow[\sim]{} & F_{b'}
\end{array}$$

Hitting this with cohomology and inverting suitable isomorphisms, this shows the independence upon the choice of path. From this, it follows easily that there is a functor $\mathcal{E}^i\colon \Pi_1(B) \to \mathsf{Ab}$ sending $b \mapsto E^i(F)$ and a path homotopy class $[\gamma]$ to $E^i(\gamma)$—the argument for composition is similar to the sorts already given. The case of modules goes through mutatis-mutandis. ■

Remark. We say a local coefficient system for a fibration over a path-connected base space is ***trivial*** if the functor $\Pi_1(B) \to \mathsf{Mod}_R$ sending $b \mapsto E^i(F_b)$ is naturally isomorphic to the constant local coefficient system sending $b \in B$ to $E^i(F_{b_0})$ and all morphisms to the identity map, where $b_0 \in B$ is any point—if this is true for one point, it is true for all points.

Example 9. If $E^i = H^i(-;\mathbf{Z}/2)$ and $F \simeq S^n$, then for each i, the local coefficient systems arising from a Serre fibration $S^n \to E \to B$ with $b \mapsto E^i(F_b)$ is trivial. This is because there is only one automorphism of $\mathbf{Z}/2$.

Definition. Let R be a ring and let $\mathcal{L}\colon \Pi_1(X) \to \mathsf{Mod}_R$ be a local coefficient system on X and define

$$C_n(X,\mathcal{L}) \subset \bigoplus_{x \in X} \mathcal{L}(x) \otimes_R C_n(X;R)$$

by

$$C_n(X,\mathcal{L}) = \left\{ \sum g_i \otimes \sigma_i \mid \sigma_i \in C_n(X;R) \text{ and } g_i \in \mathcal{L}(\sigma_i(v_0)) \right\}$$

where $v_0 = (1,0,\ldots,0)$ and define $\partial\colon C_n(X,\mathcal{L}) \to C_{n-1}(X,\mathcal{L})$ on decomposable tensors by

[1] This is the category whose objects are the points of B and whose morphisms are *homotopy classes* of paths between two points and composition is concatenation—the homotopy classes are taken with endpoints fixed. All morphisms in this category are therefore isomorphisms.

$$\partial(g \otimes \sigma) = \mathcal{L}(\lambda_\sigma)(g) \otimes d_0\sigma + \sum_{i=1}^{n}(-1)^i g \otimes d_i\sigma = \mathcal{L}(\lambda_\sigma)(g) \otimes (\sigma|[\widehat{v_0}, v_1, \ldots, v_n]) + \sum_{i=1}^{n}(-1)^i g \otimes (\sigma|[v_0, \ldots, \widehat{v_i}, \ldots, v_n]).$$

Here, $\lambda_\sigma \colon [0,1] \to X$ is the path from $\sigma(v_1)$ to $\sigma(v_0)$ defined by

$$\lambda_\sigma(t) = \sigma((1-t)v_1 + tv_0).$$

Define

$$C^n(X, \mathcal{L}) = \left\{ f \colon C_n(X;R) \to \bigoplus_{x \in X} \mathcal{L}(x) \mid f(\sigma) \in \mathcal{L}(\sigma(v_0)) \right\} \subset \mathrm{Hom}_R(C_n(X;R), \bigoplus_{x \in X} \mathcal{L}(x))$$

and define $\delta \colon C^n(X, \mathcal{L}) \to C^{n+1}(X, \mathcal{L})$ by

$$(\delta f)(\sigma) = \mathcal{L}(\lambda_\sigma)^{-1} f(d_0\sigma) + \sum_{i=1}^{n+1}(-1)^i f(d_i\sigma)$$

where $\mathcal{L}(\lambda_\sigma)^{-1} = \mathcal{L}(\lambda_\sigma^{-1})$ is the inverse of the path given above.

Then $\partial^2 = 0$ and $\delta^2 = 0$ and we define

$$H_n(X; \mathcal{L}) = H_n C_*(X; \mathcal{L})$$

the ***singular homology with local coefficients*** and

$$H^n(X; \mathcal{L}) = H^n C^*(X; \mathcal{L})$$

the ***singular cohomology with local coefficients***.

Exercise 76. *Show that $\partial^2 = 0$ and $\delta^2 = 0$ above.*

Exercise 77. *Let $F_b \to E \to B$ be a Serre fibration and fix a ring R and cohomology with R coefficients.*

(a) *If B is simply-connected, then every local coefficient system on B is trivial.* [Hint: Work from the definitions.]

(b) *If B is path-connected, show that the local coefficient system for the fibration is trivial **iff** for some $b' \in B$, $i_0 \colon b' \to B$ induces an isomorphism $H^0(B; \mathcal{H}^i(F)) \cong H^0(B; \mathcal{H}^i(F)(b')) = H^0(B; H^i(F_{b'}))$ for each i.* [Hint: Work from the definitions.]

(c) *If the Serre fibration given is a fiber bundle with connected structure group G and typical fiber F, then the local coefficient system is trivial.* [Hint: Given a path in B, say γ, the pullback bundle $\gamma^*E \cong I \times F_{\gamma(0)}$ is trivial and the map $i \colon I \times F_{\gamma(0)} \to E$ is a morphism of G-bundles so $i(1, -) \colon F_{\gamma(0)} \to F_{\gamma(1)} \subset E$ is in bundle coordinates given by the action of an element $g \in G$. Conclude that this map must be homotopic to the identity.]

(d) *Suppose B is path-connected. If the action of $\pi_1(B)$ on the (co)homology of the fiber is trivial, then the local coefficient system for the fibration is trivial.*

Lemma F.1.3. *Suppose B is path-connected and $\mathcal{L} \colon \Pi_1(B) \to \mathsf{Mod}_R$ is any local coefficient system. Then $H^0(B; \mathcal{L})$ is naturally a submodule of $H^0(b; \mathcal{L}(b))$.*

Proof. Identify $C_0(B;R) \cong R[B]$ the free abelianization on the underlying set of B and identify $\Delta^1 \cong [0,1]$. Thus, by adjunction, a homomorphism $f \colon R[B] \to \bigoplus_{b \in B} \mathcal{L}(b)$ is the same as a map $f \colon B \to \bigoplus_{b \in B} \mathcal{L}(b)$. In particular, suppose $f \colon B \to \bigoplus_{b \in B} \mathcal{L}(b)$ satisfies $f(b) \in \mathcal{L}(b)$ so that f represents an element of the relevant cochain complex. Observe that $(\delta f)(\sigma) = \mathcal{L}(\sigma)^{-1} f(\sigma(1)) - f(\sigma(0))$. It follows easily that the kernel of δ consists of those morphisms f such that for any $b, b' \in B$ and any path $\sigma \colon b \to b'$,

$$f(b) = \mathcal{L}(\sigma)^{-1} f(b')$$

and, of course, if $\lambda \colon b \to b'$ is a path in a different path-homotopy class, then it must be that

$$f(b) = \mathcal{L}(\sigma)^{-1} f(b') = \mathcal{L}(\lambda)^{-1} f(b')$$

and similarly for automorphisms of $\Pi_1(B)$ and so it follows that for each $b \in B$, the action of $\pi_1(B, b)$ on $f(b)$ is trivial (i.e., if $\gamma \in \pi_1(B, b)$ then $\mathcal{L}(\gamma) f(b) = f(b)$). Conversely, any such function defines an element of the kernel since for any two paths $\sigma, \lambda \colon b \to b'$ $\sigma^{-1} \cdot \lambda \in \pi_1(B, b)$ and therefore acts trivially on $f(b)$ so that, with some thought, one sees it must be that $\mathcal{L}(\lambda) f(b) = \mathcal{L}(\sigma) f(b)$ and therefore $\mathcal{L}(\sigma) f(b)$ is fixed under the action of $\pi_1(B, b')$. It is now easy to see that any element of $\mathrm{Ker}\,\delta$ is uniquely specified by the fixed point element $f(b) \in \mathcal{L}(b)$. It follows immediately that $H^0(B; \mathcal{L}) \cong \mathcal{L}(b)^{\pi_1(B,b)}$ (the fixed points), which is a submodule of $\mathcal{L}(b)$.

Let $i_0 \colon b \to B$ be the inclusion. Then on cohomology this induces a map $H^0(B; \mathcal{L}) \to H^0(b; \mathcal{L}(b))$ which is the inclusion of this submodule. ■

Theorem F.1.4 (Serre Spectral Sequence: Trivial Coefficients Case). *Let $F \to E \to B$ be a Serre fibration (e.g., a fiber bundle) and suppose the local coefficient system is trivial for singular cohomology with R-coefficients.*

(a) *To say there is a spectral sequence $E_2^{p,q} = H^p(B; H^q(F;R)) \implies H^{p+q}(E;R)$ means that for each n there is a filtration $0 = F_{n+1}^n \subset F_n^n \subset F_{n-1}^n \subset \cdots \subset F_1^n \subset F_0^n = H^n(E;R)$ such that $E_\infty^{p,q} = F_p^{p+q}/F_{p+1}^{p+q}$ with $E_2^{p,q}$ page naturally identified with $H^p(B; H^q(F;R))$ and with differentials on the page E_n having bidegree $(n, -(n-1))$.*

(b) *To say there is a spectral sequence $E^2_{p,q} = H_p(B; H_q(F;R)) \implies H_{p+q}(E;R)$ means that for each n there is a filtration $0 = F_n^{-1} \subset F_n^0 \subset F_n^1 \subset \cdots \subset F_n^n = H_n(E;R)$ such that $E^\infty_{p,q} = F^p_{p+q}/F^{p-1}_{p+q}$ with $E_2^{p,q}$ page naturally identified with $H_p(B; H_q(F;R))$ and with differentials on the page E^n having bidegree $(-n, n-1)$.*

*In fact, there is an analogous spectral sequence with non-trivial local coefficients where we replace $H^q(F;R)$ by $\mathcal{H}^q(F;R)$ the local coefficient system. This is the **Serre spectral sequence**, which is first quadrant.*

Furthermore, if R is a commutative ring and the local coefficient system for $F \to E \to B$ is trivial for singular cohomology with R-coefficients, then in cohomology there is also a product pairing $E_r^{p,q} \otimes_R E_r^{s,t} \to E_r^{p+s,q+t}$ with the following properties.

(i) *Each differential d_r is a derivation satisfying $d_r(xy) = (d_r x)y + (-1)^{p+q} x(d_r y)$ for $x \in E_r^{p,q}$ and $y \in E_r^{s,t}$ (i.e., a Leibniz rule). This implies that the product $E_r^{p,q} \otimes_R E_r^{s,t} \to E_r^{p+s,q+t}$ induces a product $E_{r+1}^{p,q} \otimes_R E_{r+1}^{s,t} \to E_{r+1}^{p+s,q+t}$, which is the product structure on the page E_{r+1}. The product in E_∞ is the one induced from the products in E_r for finite r.*

(ii) *The product $E_2^{p,q} \otimes_R E_2^{s,t} \to E_2^{p+s,q+t}$ is $(-1)^{qs}$ times the standard cup product*

$$H^p(B; H^q(F;R)) \otimes_R H^s(B; H^t(F;R)) \xrightarrow{\smile} H^{p+s}(B; H^q(F;R) \otimes_R H^t(F;R)) \xrightarrow{H^{p+s}(B;\smile)} H^{p+s}(B; H^{q+t}(F;R))$$

which sends a pair of cocycles (φ, ψ) to $\varphi \smile \psi$ where coefficients are multiplied via the cup product $H^q(F;R) \otimes_R H^t(F;R) \to H^{q+t}(F;R)$.

(iii) *The cup product in $H^*(E;R)$ restricts to maps $F_p^m \otimes_R F_s^n \to F_{p+s}^{m+n}$ on pieces of the filtration. These induce quotient maps $F_p^m/F_{p+1}^m \otimes_R F_s^n/F_{s+1}^n \to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$ that coincide with the products $E_\infty^{p,m-p} \otimes_R E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$ of **(i)**.*

(iv) *When R has characteristic 2, then all signs vanish. (This is more of an observation.)*

Definition. A spectral sequence is said to ***collapse*** at page n if all differentials $d_m = 0$ for $m \geq n$. If the spectral sequence collapses at the first interesting page (usually the second page), the spectral sequence is simply said to ***collapse***. Thus, we say the Serre spectral sequence collapses if it collapses at the second page.

Remark. Usually these are used imprecisely and we shall probably do so too.

Theorem F.1.5. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and suppose B is path-connected—suppose $F = F_b$ is taken as the fiber over some fixed $b \in B$. We fix a commutative ring R and understand all cohomology to be taken with R coefficients.*

(a) $E_r^{p,q} = E_\infty^{p,q}$ *for* $r > \max\{p, q+1\}$.

(b) *If the cohomological Serre spectral sequence collapses and the local coefficient system is trivial, then i^* is surjective and p^* is injective.*

(c) *Suppose the local coefficient system is trivial and suppose F is path-connected. If $H^*(F)$ is a finitely-generated free R-module for each $*$, then the product pairing $E_2^{p,0} \otimes_R E_2^{0,q} \to E_2^{p,q}$ induces an isomorphism $H^p(B;R) \otimes_R H^q(F;R) \cong E_2^{p,q}$. If R is a PID and $H^*(B)$ is a finitely-generated free R-module for each $*$, then the product pairing $E_2^{p,0} \otimes_R E_2^{0,q} \to E_2^{p,q}$ induces an isomorphism $H^p(B;R) \otimes_R H^q(F;R) \cong E_2^{p,q}$.*

(d) *Suppose F is path-connected. If $H^*(F)$ is a finitely-generated free R-module for each $*$ and i^* is surjective, then the local coefficient system is trivial and the Serre spectral sequence collapses.*

(e) *Suppose F is path-connected and R is a PID. If $H^*(B)$ is a finitely-generated free R-module for each $*$ and i^* is surjective, then the local coefficient system is trivial and the Serre spectral sequence collapses.*

(f) *Suppose F is path-conected and the local coefficient system is trivial. If $H^*(F)$ is a finitely generated free R-module in each degree. Let $\operatorname{Im} p^* \subset H^*(E)$ denote the obvious subring.*

  (1) *$H^*(E)$ is a free $\operatorname{Im} p^*$-module with a basis $\{\varphi_k\}$ such that $i^*\varphi_k$ is a homogeneous basis of $H^*(F)$;*

  (2) *$\operatorname{Im} p^*$ is a direct summand of $H^*(E)$ for which $H^*(B) \cong H^*(E)/\left(\operatorname{Im} p^* \cap \bigoplus_{i \geq 1} H^i(E)\right)$.*

Proof. **(a)** We will take this for granted. See Hatcher's book for a proof.

(b) This follows form the construction of the spectral sequence as an exact couple. In the exact couple procedure, one sees that the map $i^* : H^n(E) \to H^n(F)$, for path-connected B, factors as

$$H^n(E) \xrightarrow{\text{pr}} E_\infty^{0,n} \subset \cdots \subset E_2^{0,n} = H^0(B; \mathcal{H}^n(F)) \xrightarrow{j^*} H^0(b; H^n(F)) \cong H^n(F)$$

with the last isomorphism being natural. Here $j\colon b \to B$ is the inclusion. Since the local coefficient system is trivial, j^* is an isomorphism, so since the spectral sequence collapses this is clear as all the subsets are equalities. Similarly, for path-connected F, one sees that p^* factors as

$$H^n(B) \xrightarrow{j_*} H^n(B; \mathcal{H}^0(F))E_2^{n,0} \xrightarrow{\text{pr}} E_3^{n,0} \xrightarrow{\text{pr}} \cdots \xrightarrow{\text{pr}} E_{n+1}^{n,0} = E_\infty^{n,0} \subset H^n(E)$$

where j_* is a natural transformation of local coefficient systems $R \to \mathcal{H}^0(F)$ with R the trivial local coefficient system given by picking any $b \in B$ and mapping $r \in R$ to $r \smile 1 \in H^0(F_b)$. Since the local coefficient system is trivial, j_* is an isomorphism, so since the spectral sequence collapses, this is clear as all the projections are equalities.

(c) First suppose H^*F is a finitely-generated free R-module in each dimension. Since $H^p(B; H^q(F)) \cong H^p(B; R^{\oplus k})$ is computed as the p-th cohomology of the cochain complex $\operatorname{Hom}_R(C_*(X,R), R^{\oplus k})$, we have a string of isomorphisms

$$E_2^{p,q} \cong H^p \operatorname{Hom}_R(C_*(X,R), R^{\oplus k}) \cong H^p \oplus^k \operatorname{Hom}_R(C(X,R), R) \cong \oplus^n H^p(B) \cong H^p(B) \otimes_R R^{\oplus k} \cong H^p(B) \otimes_R H^q(F)$$

so we are good; the pairing on the E_2 page agrees, up to a sign, with the cup product of the cocycles of B followed by the cup product of the coefficients coming from the fiber—in our case, this amounts to multiplying a cocycle of B by a coefficient, up to a sign and it is not hard to see this is an isomorphism—in any case, the modules $H^p(B) \otimes_R H^q(F) \cong E_2^{p,q}$ are isomorphic.

Now suppose H^*B is a finitely-generated free R-module in each dimension with R a PID. Then by the universal coefficient theorem, $H^p(B; H^q(F)) \cong \operatorname{Hom}_R(H_p(B); H^q(F))$ and since $H^p(B) \cong \operatorname{Hom}_R(H_p(B), R)$ by the universal coefficient theorem is finitely generated and free, the natural map $\operatorname{Hom}_R(H_p(B), R) \otimes_R H^q(R) \to \operatorname{Hom}_R(H_p(B); H^q(F))$ is an isomorphism. The same reasoning as above now shows that the product structure induces an isomorphism $H^p(B) \otimes_R H^q(F) \cong E_2^{p,q}$.

(d) From our description above of i^*, if it is surjective, we have that $E_\infty^{0,n} = E_{n+1}^{0,n} = \cdots = E_2^{0,n}$. Since i^* is surjective, so too is the map $j^*\colon H^0(B; \mathcal{H}^n(F)) \to H^0(b; H^n(F_b)) \cong H^n(F_b)$ since i^* factors through this map and is assumed surjective, the description we gave above forces j^* to be surjective as well. On the other hand, i_0^* is injective by **Lemma F.1.3**. Hence, it is an isomorphism and the preceding exercise then implies that the coefficient system is trivial.

Since the pairing on the E_2 page induces an isomorphism $H^p(B) \otimes_R H^q(F) \cong E_2^{p,q}$, it follows that $d_n = 0$ for all $n \geq 2$ and $p \geq 1$ since every element of $E_2^{p,q}$ is of the form $b \cdot f$ and $d_2(b \cdot f) = d_2(b) \cdot f + (-1)^p b \cdot d_2(f) = (-1)^p b \cdot d_2(f)$. The equalities $E_\infty^{0,n} = E_{n+1}^{0,n} = \cdots = E_2^{0,n}$ imply that $d_2(f) = 0$. This pattern holds for all d_n with $n \geq 2$ and so the spectral sequence collapses with $d_n \equiv 0$ for all $n \geq 2$.

(e) This proceeds exactly as in **(d)**, mutatis-mutandis.

(f) The details are technical so we omit them but they may be found as **Theorem 4.2** of *Topology of Lie Groups, I and II* by Mimura and Toda.

Theorem F.1.6 (Serre Exact Sequence). *Let R be a ring and let $F_b \xrightarrow{i} E \xrightarrow{p} B$ be a fibration such that B and F are path-connected and the local coefficient system is trivial. Let $m, n \geq 1$.*

(a) *If $H_i(B; R) = 0$ for $1 \leq i < m$ and $H_j(F; R) = 0$ for $1 \leq j < n$ (the vacuous cases of $m = 1$ and $n = 1$ are allowed!), then there is a long exact sequence in homology (suppressing coefficients)*

$$H_{m+n-1}(F) \xrightarrow{i_*} H_{m+n-1}(E) \xrightarrow{p_*} H_{m+n-1}(B) \xrightarrow{\tau} H_{m+n-2}(F) \to \cdots \to H_2(B) \xrightarrow{\tau} H_1(F) \xrightarrow{i_*} H_1(E) \xrightarrow{p_*} H_1(B) \to 0.$$

(b) *If $H^i(B; R) = 0$ for $1 \leq i \leq m-1$ and $H^j(F; R) = 0$ for $1 \leq j \leq n-1$, then there is a long exact sequence in cohomology (suppressing coefficients)*

$$0 \to H^1(B) \xrightarrow{p^*} H^1(E) \xrightarrow{i^*} H^1(F) \xrightarrow{\tau} H^2(B) \to \cdots \to H^{m+n-2}(F) \xrightarrow{\tau} H^{m+n-1}(B) \xrightarrow{p^*} H^{m+n-1}(E) \xrightarrow{i^*} H^{m+n-1}(F)$$

where τ is the relevant transgression.

(c) *Moreover, these are both natural with respect to morphisms of fiber sequences. In particular, if $m_1 + n_1 - 1 < m_2 + n_2 - 1$, then the commuting ladder of exact sequences terminates at $m_1 + n_1 - 1$.*

Proof. **(a)&(b)** The two cases are dual so we prove the first.

The vanishing of the indicated homologies means that the only interesting things happening in the associated spectral sequence (ignoring $E_{p,q}^2 \xrightarrow{d^p} E_{0,p+q-1}^2$ for the moment) are for $k < m+n$ the transgressions $d_{k,0}^k\colon E_{k0}^2 = H_k(B) \to H_{k-1}(F) = E_{0,k-1}^2$. In this range, since the Serre spectral sequence is, in particular, a first quadrant spectral sequence, it must be that the filtration of $H_k(E) = \bigcup_{p+q=k} F_{p,q}$ has only two distinct terms—to see this, observe that on the line $p + q = k$, all terms off the axes are already at $E_{p,q}^\infty$ being zero and, in particular, since $E_{p,q}^\infty = F_{pq}/F_{p-1,q+1}$. Hence, the filtration of $H_k(E)$ has two terms $E_{k0}^\infty = F_{k0} = \operatorname{Ker} d^k$ and E_{0k}^∞ and observe that $E_{0k}^\infty = F_{0k}$ (because $F_{-1,k+1} = 0$) is a submodule of $H_k(E)$. Note well that E_{k0}^∞ is a quotient of F_{k0} which by exhaustiveness and *finiteness* of the filtration of

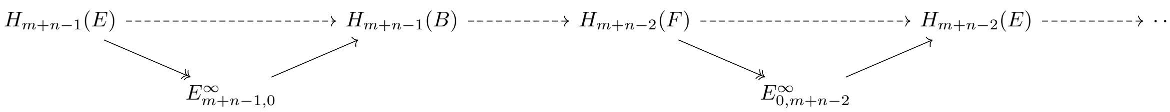
$H_n(E)$, must satisfy $F_{k0} = H_n(E)$ and hence that E^∞_{0k} is the quotient of $H_n(E)$ by $E^\infty_{k0} = F_{k0}$. We thus obtain a SES for each $k \leq m+n-1$.

$$0 \to E^\infty_{0k} \to H_k(E) \to E^\infty_{k0} \to 0.$$

But also $F_{k0} = E^\infty_{k0} = \operatorname{Ker} d^k$ and, in particular, for $k < m+n-1$ it is clear that $\operatorname{Coker} d^k = E^\infty_{0,k-1}$ by dimension considerations. Thus, we also obtain SESs for each $k < m+n-1$ arising from the transgression in the spectral sequence

$$0 \to E^\infty_{k0} \to H_k(B) \xrightarrow{\tau} H_{k-1}(F) \to E^\infty_{0,k-1} \to 0.$$

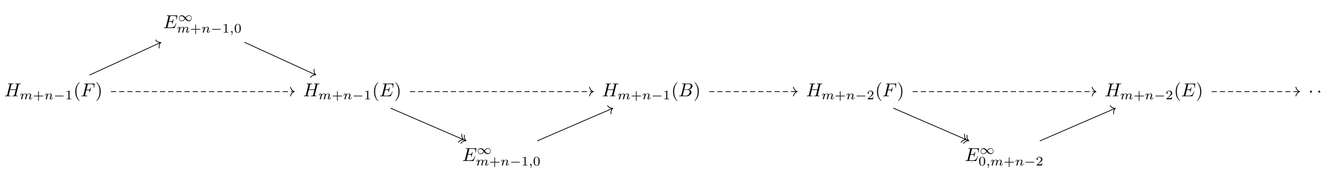
By induction, it follows easily that the following sequence is exact



It remains to show that we can stick $H_{m+n-1}(F)$ onto the end of this. However, in dimension $k = m+n-1$, there is a differential $d^p\colon E^2_{p,q} \to E^2_{0,m+n-1}$. Thus, $E^{p+1}_{0,m+n-1} = \operatorname{Coker} d^p$. Thus, the best we can say is that the following sequence is exact

$$0 \to E^\infty_{m+n,0} \to H_{m+n}(B) \to E^{p+1}_{0,m+n-1} \to E^\infty_{0,m+n-1} \to 0.$$

However, since the SSS is a first quadrant spectral sequence, the differentials out of $E^r_{0,m+n-1}$ are always trivial—hence, we have a surjection $H_{m+n-1}(F) \to E^\infty_{0,m+n-1}$ by way of the quotient map. Finally, it is easily observed that the following sequence is exact because of this surjection and as $E^\infty_{m+n-1,0} \to H_{m+n-1}(E) \to E^\infty_{0,m+n-1}$ is exact



Notice that $H_1(B) = E^\infty_{1,0}$ and, thus, by exactness, $H_1(B) \to H_0(F)$ is the zero map. Thus, we may truncate the LES as displayed in the statement.

(c) Given a morphism of fiber sequences

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \longrightarrow & B_1 \\ \downarrow & & \downarrow & & \downarrow \\ F_2 & \longrightarrow & E_2 & \longrightarrow & B_2 \end{array}$$

the only thing to check to get the conclusion is that the transgression τ respects this morphism. Since τ is a differential of the spectral sequence, naturality of the Serre spectral sequence implies that τ commutes with the morphisms of the map of fiber sequences. All other squares in the ladder commute by naturality of cohomology and homology. ■

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