MORSE THEORY AND (HAMILTONIAN) FLOER HOMOLOGY MINI-COURSE

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Contents

1 Morse Theory: Introduction and Preliminary Motivation

Warning. None of the exercises here are particularly important. The most important part of this section are the ideas. The interested reader is encouraged to work out the details spelled out below. The author has followed closely the textbook account afforded in Jet Nestruev's *Smooth Manifolds and Observables*.

We begin with some motivation. To what extent does the ring (in fact, $\mathbf{R}$-algebra) $C^\infty(M, \mathbf{R}) = C^\infty(M)$ of smooth real-valued functions on a smooth manifold M determine the geometry of the smooth manifold and the relationships it has with other smooth manifolds? If this is a good, rich sort of algebraic invariant of a smooth manifold, then we might expect that we can study the geometry and topology of smooth manifold M using smooth functions $M \to \mathbf{R}$.

Aggregating all results below, what we shall end up showing is the following.

Theorem

There is an equivalence of categories

$$C^\infty(-) : \mathsf{Man}^{\mathrm{op}} \leftrightarrow \mathsf{SmAlg}_{\mathbf{R}} : |-|$$

between the opposite category of smooth manifolds with corners and the full subcategory of $\mathbf{R}$-algebras[a] consisting of those $\mathbf{R}$-algebras that are commutative, geometric, complete and smooth.

In particular, the essential image of $M \mapsto C^\infty(M)$ in commutative $\mathbf{R}$-algebras is precisely characterized as those that are commutative, geometric, complete and smooth.

[a] We always assume $\mathbf{R}$-algebras are associative and unital.

This says that a manifold is totally determined by its $\mathbf{R}$-algebra of smooth functions.

We start with some reminders about category theory.

1.1 Categorical Reminders

Let us give a somewhat imprecise definition of a category first, ignoring set-theoretic difficulties. We shall implicitly assume all categories are *locall small*, meaning that the collections of morphisms between two objects is always a set, and not a proper class.

If this looks somewhat mysterious, see the list of examples following this definition.

Definition 1

A *category* is a class objects $\mathrm{Ob}(\mathsf{C})$ and arrows (also called morphisms) $\mathrm{Ar}(\mathsf{C})$ (which should be thought of as maps between objects in the category) such that

(1) There are source and target functions $s, t \colon \mathrm{Ar}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{C})$ picking out the source (domain) object and target (codomain) object for each morphism.

(2) There is an operation of composition of morphisms when one morphism has codomain the domain of the other: $f \circ g \colon s(g) \xrightarrow{g} t(g) = s(f) \xrightarrow{f} t(f)$. We demand that composition is associative.

(3) There is an *identity map* id_c for each $c \in \mathrm{Ob}\,(()\,\mathsf{C})$ such that for any $f \colon c' \to c$ and $g \colon c \to c''$, $g \circ \mathrm{id}_c = g$ and $\mathrm{id}_c \circ f$.

As with functions, we often denote a morphism f with source c and target c' by $f \colon c \to c'$ or $c \xrightarrow{f} c'$. We also denote for any two objects $c, c' \in \mathsf{C}$ the set of arrows between them by $\hom_{\mathsf{C}}(c, c') = \hom(c, c')$ (where we drop the subscript indicating the category where it is clear).

Definition 2

A morphism $f \colon c \to c'$ in C is said to be an *isomorphism* if there is a morphism $g \colon c' \to c$ such that $f \circ g = \mathrm{id}_{c'}$ and $g \circ f = \mathrm{id}_c$.

Example 1

The following are all categories.

(1) The category Set of sets whose objects are sets and morphisms are functions between them.

(2) The category Grp (resp. Ab) of groups (resp. abelian groups) whose objects are (resp. abelian) groups and morphisms are homomorphisms.

(3) The category Mod_R of modules over a ring R whose morphism are R-linear maps.

(4) The category Vect_k of vector spaces over a field whose morphisms are k-linear maps.

(5) The category Top of spaces and continuous maps between them.

(6) The category $\mathsf{Man} = \mathsf{DIFF}$ of smooth manifolds with corners and smooth maps between them.

(7) The category $\mathsf{Alg}_{\mathbf{R}}$ of $\mathbf{R}$-algebras and $\mathbf{R}$-algebra homomorphisms between them. (We will define this precisely in the next subsection.)

(8) Just as well, the category $\mathsf{GAlg}_{\mathbf{R}}$ of geometric commutative $\mathbf{R}$-algebras with $\mathbf{R}$-algebra homomorphisms between them.

Definition 3

Given any category C, its ***opposite*** category C^{op} has the same underlying objects and arrows as C except all arrows are reversed. To clarify that an arrow f actually belongs to the opposite category of some category C, we sometimes decorate it as f^{op}.

More precisely, if $f \in \mathrm{Ar}(\mathsf{C})$ with $f\colon c \to c'$, then $f^{\mathrm{op}}\colon c' \to c$ (i.e., $s(f^{\mathrm{op}}) = t(f)$ and $t(f^{\mathrm{op}}) = s(f)$). Composition $\circ_{\mathsf{C}^{\mathrm{op}}}$ in C^{op} is defined by the formula

$$g^{\mathrm{op}} \circ_{\mathsf{C}^{\mathrm{op}}} f^{\mathrm{op}} \overset{\mathrm{def}}{=} (f \circ_{\mathsf{C}} g)^{\mathrm{op}}$$

Some thought shows that this is well-defined and yields a category.

Thus, for any category C, C^{op} is another category.

Definition 4

A ***functor*** between two categories $F\colon \mathsf{C} \to \mathsf{D}$ is a rule that assigns to each object $c \in \mathsf{C}$ an object $F(c) \in \mathsf{D}$ and assigns to each morphism $f\colon c \to c'$ in C a morphism $F(f)\colon F(c) \to F(c')$ such that F preserves the additional data of a category—that is,

(1) for every $c \in \mathsf{C}$ $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$;

(2) $F(f \circ g) = F(f) \circ F(g)$.

Definition 5

A ***natural transformation*** η between two functors $F, G\colon \mathsf{C} \to \mathsf{D}$, often written as

$$\eta\colon F \to G$$

is a function $\eta\colon \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ar}(\mathsf{D})$, assigning to each object $c \in \mathsf{C}$ an arrow $\eta_c\colon Fc \to Gc$ such that for every morphism $f\colon c \to c'$ in C the following diagram commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{Ff} & Fc' \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} \\ Gc & \xrightarrow[Gf]{} & Gc' \end{array}$$

When each map η_c is an isomorphism, then η is said to be a ***natural isomorphism***.

The intuition is that a natural transformation is precisely the data that "takes a commutative diagram in F into a commutative diagram in G." This is made more precise in the following exercise.

Exercise 1

Let $\eta\colon F \to G$ be a natural transformation. Given a commutative diagram in C in the form of a functor $D\colon \mathsf{J} \to \mathsf{C}$, show that there is canonically defined natural transformation $\eta_D\colon F \circ D \to G \circ D$.

Natural transformations and functors form can be used to form a category.

Exercise 2

(a) Show that there is a category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ whose objects are functors $F\colon \mathsf{C} \to \mathsf{D}$ and whose arrows are natural transformations thereof. Show that the isomorphisms of this category are the natural isomorphisms. *[Hint: What are the identities?]*

(b) A category C is said to be ***small*** if its class of objects and morphisms are both *sets* (and thus not proper classes). Show that there is a category Cat whose objects are small categories and morphisms are functors between them. *[Hint: What are the identities?]*

(c) Show that a group G is equivalently a category BG which has a single object $*$ such that every morphism of BG is an isomorphism. *[Hint: The group G should now be the set $\hom_{BG}(*, *)$ under composition.]*

(d) Given two groups G and G', show that every functor $F\colon BG \to BG'$ determines and is determined by a group-homomorphism $G \to G'$.

(e) Define a functor $BG\colon \mathsf{Grp} \to \mathsf{Cat}$. Show that this functor is fully-faithful[a].

[a]This means that for all objects, $G, G' \in \mathsf{Grp}$, the induced map $\hom_{\mathsf{Grp}}(G, G') \xrightarrow{BG} \hom_{\mathsf{Cat}}(BG, BG')$ is an isomorphism(=bijection) of sets.

Finally, let us state the only theorem we'll need below.

Theorem

A functor $F\colon \mathsf{C} \to \mathsf{D}$ is said to be an ***equivalence*** or an ***equivalence*** of categories if there is a functor $G\colon \mathsf{D} \to \mathsf{C}$ and natural isomorphisms $FG \cong \mathrm{id}_{\mathsf{D}}$ and $GF \cong \mathrm{id}_{\mathsf{C}}$.

A functor $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence **iff** F is fully-faithful[a] and essentially surjective[b].

[a]This means that for all objects $c, c' \in \mathsf{C}$, the induced map $\hom_{\mathsf{C}}(c, c') \xrightarrow{F} \hom_{\mathsf{D}}(Fc, Fc')$ is an isomorphism(=bijection) of sets.

[b]This means that for every $d \in \mathsf{D}$, there is a $c \in \mathsf{C}$ and an isomorphism $Fc \cong d$ in D (i.e., F hits every isomorphism class in D).

Remark. There is a set-theoretic problem here we will ignore. Basically one needs a strong enough version of choice because the proof uses a choice function. But, of course, the axiom of choice is true[1], so this is irrelevant.

1.2 Characterization of Manifolds by Their Algebras of Smooth Functions

Notation 1.1. We will write $\mathsf{Man} = \mathsf{DIFF}$ throughout to emphasize we are thinking about manifolds with corners, although we understand these two things to mean the same category. We also write the ring of smooth functions on M as $C^\infty(M) = C^\infty(M, \mathbf{R})$, suppressing the $\mathbf{R}$.

The idea of Morse theory is to study manifolds using the smooth functions. How much information do the smooth function on M contain about M and its relationships with other manifolds? As it turns out, the theory of smooth manifolds is a special case of that of *locally ringed spaces*, where a manifold has as its sheaf of rings its structure sheaf—namely the sheaf of smooth functions defined on its open subsets.

Exercise 3

Let M be a smooth manifold (with corners say) and let O_M^{op} be the opposite category of its poset of open subsets. Denote the structure sheaf (by a small abuse of notation) as O_M where $\mathsf{O}_M(U) = C^\infty(U)$ for $U \in O_M^{\mathrm{op}}$.

(a) Show that the structure sheaf really is a sheaf. That is, show that the functor $U \mapsto C^\infty(U) = \mathsf{O}_M(U)$ is a sheaf $\mathsf{O}_M\colon O_M^{\mathrm{op}} \to \mathsf{Ring}$.

(b) Show that for each $p \in M$, the stalk $\lim_{U \ni p} C^\infty(U)$ is the ring of germs of smooth functions at p. Show in particular that this ring is local, with unique maximal ideal consisting of those germs of smooth functions vanishing at p.

(c) Show that the functor $\mathsf{Man}^{\mathrm{op}} \to \mathsf{LRS}$ sending $M \mapsto (M, \mathsf{O}_M)$ is a fully-faithful functor into the category of locally ringed spaces.

This fits the differential topology of smooth manifolds into the suit of algebraic geometry.

It gets better, in fact. With a little more work and elbow-grease, we can see that the theory of smooth manifolds is a special case of that of commutative $\mathbf{R}$-algebras, showing that manifolds are, in a certain sense, classified by their $\mathbf{R}$-algebras of smooth real-valued functions.

Exercise 4

Let $\mathbf{R}_k^n$ denote the model corner space $\mathbf{R}^{n-k} \times \mathbf{R}_{\geq 0}^k$. For any manifold M with corners, give $C^\infty(M)$ the $\mathbf{R}$-algebra structure arising from the ring-homomorphism $\mathbf{R} \to C^\infty(M)$ sending $r \mapsto c_r$, the constant function on M with value r.

[1] The author will brook no dissent on this point!

(a) Show that $C^\infty(\mathbf{R}^n_k) \not\cong C^\infty(\mathbf{R}^m_\ell)$ as $\mathbf{R}$-algebras unless $m = n$ and $k = \ell$.

(b) If M is connected, show that the constant functions form the unique maximal subfield of $C^\infty(M, \mathbf{R})$. What goes wrong if M is not connected? *[Hint: Show there is a unique ring-homomorphism (hence, $\mathbf{R}$-algebra homomorphism) $\mathbf{R} \to C^\infty(M)$. What can be said about the kernel of a ring-homomorphism out of a field?]*

(c) If M is not connected, say $M = \coprod M_i$ show that there is a canonical inclusion $C^\infty(M_i) \subset C^\infty(M)$ as an ideal. *[Hint: Use the preceding part.]*

(d) Show that for every ring-homomorphism $\phi\colon C^\infty(M) \to \mathbf{R}$, if $f > 0$ everywhere, then $\phi(f) > 0$ and, conversely, if $f < 0$ everywhere, then $\phi(f) < 0$.

(e) Show that for every ring-homomorphism $\phi\colon C^\infty(M) \to \mathbf{R}$, ϕ evaluated on a constant function is the value of that function. *[Hint: It is true that for every $r \in \mathbf{Q}$, ϕ takes the constant function c_q at q to $\phi(c_q) = q$ (which property of ϕ guarantees this?). Using the preceding part, conclude that $\phi(c_r) = r$ for all $r \in \mathbf{R}$ (what property does $\mathbf{Q} \subset \mathbf{R}$ have that might be relevant here?).]*

(f) Suppose M is connected. Show that every ring-homomorphism $\phi\colon C^\infty(M) \to \mathbf{R}$ is an $\mathbf{R}$-algebra homomorphism. *[Hint: Preceding part.]*

(g) Show that every ring-homomorphism $\phi\colon C^\infty(M) \to \mathbf{R}$ is evaluation at a point of M. *[Hint: Suppose this is not true and obtain a contradiction of* **(d)** *by finding a function F for which $F > 0$ but $\phi(F) = 0$. You may find it helpful to let $g\colon M \to \mathbf{R}_{\geq 0}$ be a compact exhaustion function—that is, a smooth function that is proper, which means preimages of compact sets are compact (which property of manifolds guarantees this function exists?).]*

(h) Show that the functor $\mathsf{Man}^{\mathrm{op}} \to \mathsf{Alg}_{\mathbf{R}}$ sending $M \mapsto C^\infty(M, \mathbf{R})$ is fully-faithful. If $\mathsf{CAlg}_{\mathbf{R}}$ is the subcategory of commutative $\mathbf{R}$-algebras, show that this functor lands in $\mathsf{CAlg}_{\mathbf{R}}$. *[Hint: Faithfullness is easy. To see that it is full, use the preceding part to deduce that any $\mathbf{R}$-algebra homomorphism is precomopsition with a function $f\colon M \to N$ and argue that f must be smooth.]*

Remark. If this exercise seems daunting, it may be more enlightening to show that a smooth structure on a topological manifold M may be recovered from $C^\infty(M)$. (How might you build a smooth atlas out of this?)

Even better, we can characterize the essential image of this functor and even construct its inverse equivalence out of its essential image. To do this, we have to establish some terminology and theory.

Definition 6

For concreteness, we fix $\mathbf{R}$ the field of real numbers and think of $\mathbf{R}$-algebras.

(a) The *center* $Z(A)$ of a ring A is the set $\{a \in A : \forall b \in A,\ ab = ba\}$. This is a subring of A.

(b) An $\mathbf{R}$-algebra is a ring[a] A together with ring-homomorphism $\mathbf{R} \to Z(A)$ (note that since $\mathbf{R}$ is a field, this is necessarily injective unless $A = 0$). Equivalently, an $\mathbf{R}$-algebra is a ring A with a scalar multiplication $\mathbf{R} \times A \to A$ such that for any $r, s \in \mathbf{R}$ and $a, b \in A$, $(ra) \cdot (sb) = (rs)(a \cdot b)$.

(c) A morphism $f : A \to B$ of $\mathbf{R}$-algebras is a ring-homomorphism which additionally respects the scalar product: $f(ra) = rf(a)$ for $r \in \mathbf{R}$ and $a \in A$.

(d) A *commutative* $\mathbf{R}$-algebra is an $\mathbf{R}$-algebra A such that A is a commutative ring.

(e) Say the *dual space of* $\mathbf{R}$*-points* $|A|$ of an $\mathbf{R}$-algebra A is the *set* of surjective $\mathbf{R}$-algebra homomorphisms $A \to \mathbf{R}$. We shall show later that this has a naturally defined topology in **Definition** 7.

(f) We say an $\mathbf{R}$-algebra A is *geometric* if A is a commutative $\mathbf{R}$-algebra and $\bigcap_{x \in |A|} \operatorname{Ker} x = 0$.

[a] All rings are assumed to be unital and associative.

Remark. If $A \neq 0$ is an $\mathbf{R}$-algebra, then every $\mathbf{R}$-algebra homomorphism $f : A \to \mathbf{R}$ is surjective, since, as a ring-homomorphism, $f(1_A) = 1$ and by compatibility with the scalar product, $f(r1_A) = rf(1_A) = r$. Hence, it can be shown that for $A \neq 0$, $|A| = \hom_{\mathsf{Alg}_{\mathbf{R}}}(A, \mathbf{R})$.

Remark. There is only one $\mathbf{R}$-algebra homomorphism to the zero $\mathbf{R}$-algebra and there are *no* $\mathbf{R}$-algebra homomorphisms from the zero $\mathbf{R}$-algebra because there are no ring-homomorphisms out of the zero ring other than isomorphisms.

Lemma 1

Let A be a commutative $\mathbf{R}$-algebra. Let

$$\widetilde{A} = \{\tilde{a} : |A| \to \mathbf{R} \mid \forall x \in |A|, x(a) = \tilde{a}(x)\}$$

or, in other words, those functions having the form $x \mapsto x(a)$ for some $a \in A$ and so, equivalently,

$$\widetilde{A} = \{\mathrm{ev}_a : |A| \to \mathbf{R} : a \in A\}$$

where $\mathrm{ev}_a(x) = x(a)$.

Then $\widetilde{A}$ is a commutative $\mathbf{R}$-algebra with the natural $\mathbf{R}$-algebra structure defined pointwise by

$$\begin{aligned}(\tilde{a} + \tilde{b})(x) &= \tilde{a}(x) + \tilde{b}(x) = x(a) + x(b) = x(a+b)\\ (\tilde{a} \cdot \tilde{b})(x) &= \tilde{a}(x) \cdot \tilde{b}(x) = x(a) \cdot x(b) = x(a \cdot b)\\ (r\tilde{a})(x) &= r\tilde{a}(x) = rx(a).\end{aligned}$$

The natural map

$$\tau_A\colon A \to \widetilde{A}, \qquad a \mapsto \tilde{a}$$

is a surjective map of $\mathbf{R}$-algebras. In fact, this construction if functorial and τ assembles into a natural transformation

$$\tau\colon \mathrm{id}_{\mathsf{CAlg}_{\mathbf{R}}} \to \widetilde{-}$$

which restricts to a natural isomorphism on the class of geometric algebras and thereby restricts to a natural isomorphism

$$\tau\colon \mathrm{id}_{\mathsf{GCAlg}_{\mathbf{R}}} \to \widetilde{-}.$$

Proof. All of this is obvious except the last bit, so the only thing that really needs be shown is that the natural map is injective on geometric commutative $\mathbf{R}$-algebras. For this, observe that $a \in \operatorname{Ker} \tau_A$ iff $\tilde{a} \equiv 0$ and so for all $x \in |A|$, $x(a) = 0$ and hence that $a \in \bigcap_{x \in |A|} \operatorname{Ker} x$ and we assumed this intersection is trivial. □

We now establish the following topology on the dual space of $\mathbf{R}$ points for a commutative $\mathbf{R}$-algebra.

Definition 7

The dual space of $\mathbf{R}$-points for a commutative $\mathbf{R}$-algebra A has a topology with a basis of open sets given by $\tilde{a}^{-1}(V)$ with $V \subset \mathbf{R}$ open. This is the weakest topology for which all elements $\tilde{a} \in \widetilde{A}$ are continuous. This becomes a functor $\mathsf{CAlg}_{\mathbf{R}}^{\mathrm{op}} \to \mathsf{Top}$ given on arrows $\varphi\colon A \to B$ by

$$|\varphi| : |B| \to |A|, \qquad x \mapsto x \circ \varphi.$$

Exercise 5

Let A and B be commutative $\mathbf{R}$-algebra.

(a) Verify that $|\varphi|$ really is well-defined. *[Hint: One must consider the case of the zero* $\mathbf{R}$*-algebra.]*

(b) Show that if $\varphi\colon A \to B$ is an surjective $\mathbf{R}$-algebra homomorphism, then $|\varphi|$ is an embedding.

(c) Show that if A is additionally *geometric*, then $|A|$ is Hausdorff, and so

$$|-| : \mathsf{GCAlg}_{\mathbf{R}}^{\mathrm{op}} \to \mathsf{HTop}$$

is a functor from the opposite category of geometric commutative algebras to Hausdorff spaces. *[Hint: If $x, y \in |A|$, why does there exist $f \in \widetilde{A}$ for which $f(x) \neq f(y)$?]*

(d) Combine the preceding parts to show that there is no surjective $\mathbf{R}$-algebra homomorphism $\mathsf{A} \to \mathsf{B}$ where A is geometric and $|\mathsf{B}|$ is not Hausdorff.

Exercise 6

Show that for any manifold S there is a naturally defined homeomorphism

$$\theta \colon S \xrightarrow{\cong} |C^\infty(S)|$$

given by

$$p \mapsto (f \mapsto f(p)) = \mathrm{ev}_p.$$

[Hint: That this is bijective follows from a part of the preceeding exercise.]

Definition 8

Given commutative geometric $\mathbf{R}$-algebra A and $T \subset |\mathsf{A}|$ any subset of its dual space, define the ***restriction*** $\mathsf{A}|_T$ of A to T to be the set of functions $f \colon T \to \mathbf{R}$ such that for each $p \in T$, there is a nbhd (in the subspace topology) $U \subset T$ of p and an element $\widetilde{a_U} \in \widetilde{\mathsf{A}}$ such that $f|U = \widetilde{a_U}|U$. This is a commutative $\mathbf{R}$-algebra once again in the evident pointwise fashion.

In other words, this is the set of all functions $T \to \mathbf{R}$ which are locally (in T) the restriction of an element in $\widetilde{\mathsf{A}}$ (i.e., locally given by evaluation at some element of A).

It follows that for such A commutative and geometric, there is a ***restriction homomorphism*** $\rho_T \colon \mathsf{A} \to \mathsf{A}|_T$ given by $a \mapsto \tilde{a}|T$. This is a morphism of $\mathbf{R}$-algebras

Exercise 7

Show that the restriction homomorphism is always injective. *[Hint: Show its kernel is trivial using the geometric condition.]*

Exercise 8

Let $\varphi \colon \mathsf{A} \to \mathsf{B}$ be a morphism of two geometric commutative $\mathbf{R}$-algebras, $B \subset |\mathsf{B}|$. The map

$$(|\varphi||_B)^* \colon \mathsf{A}|_{|\varphi|(B)} \to \mathsf{B}|_B, \qquad f \mapsto f \circ |\varphi||_B$$

is a morphism of $\mathbf{R}$-algebras which is an isomorphism when φ is an isomorphism.

Exercise 9

Let $T \subset M$ be any subset of a manifold and let $C^\infty(T)$ be those functions $T \to \mathbf{R}$ that are restrictions of smooth functions defined on an open nbhd of T. (See, for instance, the discussion of general notions of smoothness in the appendix). Show that $C^\infty(M)|_T = C^\infty(T)$. *[Hint: Show that $C^\infty(M)|_T$ consists of those continuous functions $T \to \mathbf{R}$ that are locally (in T) of the form $p \mapsto f(p)$ where $f\colon M \to \mathbf{R}$ is smooth. Then a theorem in Appendix A will be useful.]*

Exercise 10

Show that $C^\infty(M)$ is a complete, geometric commutative $\mathbf{R}$-algebra.

Definition 9

A commutative, geometric $\mathbf{R}$-algebra is said to be *complete* if $\rho_{|\mathsf{A}|}\colon \mathsf{A} \to \mathsf{A}|_{|\mathsf{A}|}$ is surjective and hence by the preceding exercise an isomorphism.

A complete, geometric, commutative $\mathbf{R}$-algebra A is said to be *smooth* if there exists an integer $n \geq 0$ and a countable *open* cover $\{U_i\}_{i\in\mathbf{N}}$ of the dual space $|\mathsf{A}|$ such that all the algebras $\mathsf{A}|_{U_i}$ are isomorphic to some $C^\infty(\mathbf{R}^n_k)$ where n is fixed and called the *dimension* of A and $0 \leq k \leq n$ is allowed to vary.

Example 2

If M is a smooth manifold with corners, then $C^\infty(M)$ is a complete, geometric, smooth commutative $\mathbf{R}$-algebra. This is the last exercise.

Lemma 2

Let A be a commutative, geometric $\mathbf{R}$-algebra and $A \subset |\mathsf{A}|$.

(a) $\mathsf{A}|_A$ is a subalgebra of the continuous functions $A \to \mathbf{R}$ where $A \subset |\mathsf{A}|$ has the subspace topology.

(b) The natural map

$$\mu\colon A \to |\mathsf{A}|_A| \qquad \mu(x) = \mathrm{ev}_x$$

is continuous. In fact, given $\varphi\colon \mathsf{A} \to \mathsf{B}$ a morphism of commutative, geometric $\mathbf{R}$-algebras and $B \subset |\mathsf{B}|$, TFDC:

$$\begin{array}{ccc} B & \xrightarrow{|\varphi|} & |\varphi|(B) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ |\mathsf{B}|_B| & \xrightarrow[|(|\varphi||_B)^*|]{} & \left|\mathsf{A}|_{|f|(B)}\right| \end{array}$$

(c) μ is a homeomorphism onto its image.

(d) Given $A \subset B \subset |\mathsf{A}|$,
$$(\mathsf{A}|_B)|_A = \mathsf{A}|_A\,.$$

Proof. **(a)** $\mathsf{A}|_A$ is the set of all functions $A \to \mathbf{R}$ which are locally (in A) the restriction of an element in $\widetilde{\mathsf{A}}$ (i.e., locally given by evaluation of at some element of A). It follows that $\mathsf{A}|_A$ is a subalgebra of the continuous function $A \to \mathbf{R}$ since if $f: A \to \mathbf{R}$ is locally of the form $\widetilde{a} = \mathrm{ev}_a$ for some $a \in \mathsf{A}$, then for each open set $V \subset \mathbf{R}$, $f^{-1}(V)$ is a union of sets of the form $\widetilde{a}^{-1}(V) \cap A$ and therefore is continuous for the subspace topology on A.

(b) It follows immediately that the assignment μ is continuous, since if $\widetilde{f}: |\mathsf{A}|_A| \to \mathbf{R}$ is given by evaluation at $f \in \mathsf{A}|_A$ then for $V \subset \mathbf{R}$ open, $\mu^{-1}(\widetilde{f}^{-1}(V))$ is the set of all points $g \in A$ such that $\widetilde{f}(g) = f(g) \in V$ (the notation is terrible but this is the form the evaluation takes). That is, the set $f^{-1}(V)$ which is open in the subspace topology since f is continuous on A by assumption. Hence μ is continuous.

As for the naturality condition, fixing $x \in B$ we chase the diagram

$$\begin{array}{ccc} x & \longmapsto & x \circ \varphi \\ \downarrow & & \downarrow \\ \mathrm{ev}_x & \longmapsto & (|\varphi||_B)^*(\mathrm{ev}_x) \stackrel{?}{=} \mathrm{ev}_{x\circ\varphi} \end{array}$$

and indeed $(|\varphi||_B)^*(\mathrm{ev}_x) = \mathrm{ev}_x \circ |\varphi||_B = \mathrm{ev}_{x\circ\varphi}$.

(c) μ is clearly injective since if $x_1, x_2 \in A$ are distinct points, then they are non-equal surjective $\mathbf{R}$-algebra homomorphisms $\mathsf{A} \to \mathbf{R}$. Hence, there is some $a \in \mathsf{A}$ such that $\mathrm{ev}_a(x_0) \neq \mathrm{ev}_a(x_1)$. In particular, $\mathrm{ev}_a| A \in |\mathsf{A}|_A|$ by restriction so that these points $\mu(x_1)$ and $\mu(x_2)$ must determine different homomorphisms $\mathsf{A}|_A \to \mathbf{R}$.

To see that the inverse map $\mu(A) \to A$ is continuous, consider a basis set $A \cap \widetilde{a}^{-1}(V)$ for $V \subset \mathbf{R}$ open and $\widetilde{a} \in \widetilde{\mathsf{A}}$ evaluation at a. This is mapped onto the set $\mu(A) \cap (\widetilde{a}| A)^{-1}(V)$ which is open. This shows that the inverse map is open on basis sets and therefore open and continuous.

(d) Immediate from the preceding considerations. □

Exercise 11

Suppose A is a geometric commutative $\mathbf{R}$-algebra and $\mathsf{A}|_U \cong C^\infty(\mathbf{R}^n_k)$. Show that the embedding $\mu: U \to |\mathsf{A}|_U|$ is in fact surjective and thus a homeomorphism.

[Hints: Let $i: \mathsf{A}|_U \xrightarrow{\cong} C^\infty(\mathbf{R}^n_k)$ be the isomorphism. If μ is not surjective, there exists $p \in |\mathsf{A}|_U| \setminus \mu(U)$; set $\overline{p} = |i|^{-1}(p)$. If $p \in \overline{\mu(U)}$, let $f: \mathbf{R}^n_k \setminus \{\overline{p}\} \to \mathbf{R}$ be defined by $x \mapsto 1/\|x - \overline{p}\|$ and show that $g = f\overline{i}^{-1}\mu \in \mathsf{A}|_U$ while $i(g)$ must be smooth on all of $\mathbf{R}^n$ and coincide with f on the set $|i|^{-1}(\mu(U))$, whose closure contains $\overline{p}$, deduce a contradiction. For the case $a \notin \overline{\mu(U)}$, let f be a smooth function vanishing on the closed set $|i|^{-1}(\overline{\mu(U)})$ and equals 1 on $\{\overline{p}\}$ by the smooth Urysohn lemma. Then f and the function that is identically zero must pullback by i on $\mathsf{A}|_U$ to different elements. Deduce a contradiction.]

Finally, we have our first theorem.

Theorem 1

The class of complete, geometric, smooth commutative $\mathbf{R}$-algebras characterize the essential image of the functor $\mathsf{Man}^{\mathrm{op}} \to \mathsf{CAlg}_{\mathbf{R}}$ sending $M \mapsto C^\infty(M, \mathbf{R})$. In other words, such $\mathbf{R}$-algebras are, up to isomorphism, the $\mathbf{R}$-algebras of smooth functions on a smooth manifold.

We give a somewhat "sheafy" proof of this.

Proof. Given such an $\mathbf{R}$-algebra A of dimension n, we claim that $|\mathsf{A}|$ is a second-countable, locally Euclidean and Hausdorff. The last item we have already seen. Pick countable open cover $\{U_i\}_{i\in\mathbf{N}}$ of the dual space $|\mathsf{A}|$ such that all the algebras $\mathsf{A}|_{U_i}$ are isomorphic to some $C^\infty(\mathbf{R}^n_k)$. We claim that $\mathsf{A}|_{|\mathsf{A}|}$ is the equalizer

$$\mathsf{A}|_{|\mathsf{A}|} \longrightarrow \prod_{i\in\mathbf{N}} \mathsf{A}|_{U_i} \rightrightarrows \prod_{(i,j)\in\mathbf{N}\times\mathbf{N}} \mathsf{A}|_{U_i\cap U_j}$$

where the two parallel arrows correspond to inclusions restrictions arising from $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$ and the equalizing arrow is the evident one corresponding to the inclusions $U_i \subset |\mathsf{A}|$.

To see that this is so, note that by direct construction it can be seen by direct construction that an element of the equalizer is tantamount to the data of a function $f\colon |\mathsf{A}| \to \mathbf{R}$ that restricts to each open set U_i as a continuous function and an element of $\mathsf{A}|_{U_i}$ with compatibility on overlaps, and thus is an element of $\mathsf{A}|_{|\mathsf{A}|}$. Conversely, given $f \in \mathsf{A}|_{|\mathsf{A}|}$, it is easy to see that this determines an element of the equalizer. This establishes a set-theoretic bijection between the equalizer and $\mathsf{A}|_{|\mathsf{A}|}$, that it is an $\mathbf{R}$-algebra homomorphism follows once again by inspecting the $\mathbf{R}$-algebra structure on the equalizer and observing that it is suitably pointwise as required.

But, by assumption, A is *complete*, and so the restriction map $\mathsf{A} \to \mathsf{A}|_{|\mathsf{A}|}$ is an isomorphism, so that A is the equalizer of the diagram above as well. Now, since each $\mathsf{A}|_{U_i} \cong C^\infty(\mathbf{R}^n_k)$, it follows that the composite $U_i \to \left|\mathsf{A}|_{U_i}\right| \cong |C^\infty(\mathbf{R}^n_k)|$ is in fact a homeomorphism

$$\phi\colon U_i \xrightarrow{\cong} \mathbf{R}^n_k.$$

This follows by the preceding lemma and **Exercise** 6. We claim that this implies that each $\mathsf{A}|_{U_i\cap U_j}$ is a ring of smooth functions on an open subset of some model corner space $\mathbf{R}^n_k$ (possibly with $k = 0$). In fact, we claim that $\mathsf{A}|_{U_i\cap U_j} \cong C^\infty(\phi(U_i \cap U_j))$. To see that this is so, it suffices to prove that $C^\infty(\mathbf{R}^n_k)|_{\phi(U_i\cap U_j)} \cong C^\infty(\phi(U_i \cap U_j))$ and this now follows from **Exercise** 6.

Since the equalizer above lives in $\mathsf{GCAlg}_{\mathbf{R}}$ of geometric commutative $\mathbf{R}$-algebras, we claim $|-|$ turns this particular equalizer into a coequalizer. To see this, note that there are isomorphism $\mathsf{A}|_{U_i} \cong C^\infty(\mathbf{R}^n_k)$ and that $\prod_i C^\infty(\mathbf{R}^n_{k(i)}) \cong C^\infty(\coprod_i \mathbf{R}^n_{k(i)})$. Some thought shows that this means that $|-|$ of *both* products in the equalizer become coproducts.

Hence, we have a coequalizer

$$\coprod_{(i,j)\in\mathbf{N}\times\mathbf{N}} U_i \cap U_j \rightrightarrows \coprod_{i\in\mathbf{N}} U_i \longrightarrow |\mathsf{A}|$$

where we are justified in replacing the terms $|\mathsf{A}|_T|$ by T according to the naturality result of **Lemma** 2**(b)** and the two parallel arrows are the evident inclusions $U_i \cap U_j \subset U_i$ and $U_i \cap U_j \subset U_j$, respectively. It is essentially immediate that the resulting space $|\mathsf{A}|$ is a topological manifold. One checks that this is indeed a coequalizer diagram by hand just as above.

As for its smooth structure, first note that by **Exercise** 11, since $\mu\colon U_i \to \left|\mathsf{A}|_{U_i}\right|$ is a homeomorphism, $|\rho_U|$ (where $\rho_U\colon \mathsf{A} \to \mathsf{A}|_{U_i}$ is the restriction homomorphism) must be a homeomorphism onto its image $U \subset |\mathsf{A}|$ since the composite $|\rho_U| \circ \mu$ is the inclusion $U \subset |\mathsf{A}|$. Now choose a collection of isomorphisms $\phi_i\colon \mathsf{A}|_{U_i} \xrightarrow{\cong} \mathbf{R}^n_k)$ where k is allowed to vary $0 \le k \le n$, let h_i be the the composite

$$\mathsf{A} \xrightarrow{\rho_U} \mathsf{A}|_U \xrightarrow{\phi_i} C^\infty(\mathbf{R}^n_k)$$

and and consider the family $x_i = |h_i|^{-1}$ (this makes sense because $|h| : \mathbf{R}^n \to U \subset |\mathsf{A}|$ is a homeomorphism). We claim these are smoothly compatible charts; note that $|\rho_U|$ is a homeomorphism onto its image (namely U) by **Exercise** 11.

By **Exercises** 8 and 9, the restrictions are isomorphisms

$$\phi_i|_{U_i\cap U_j}\colon\ \mathsf{A}|_{U_i\cap U_j} \xrightarrow{\cong} C^\infty(x_i(U_i\cap U_j)) \qquad \phi_j|_{U_i\cap U_j}\colon\ \mathsf{A}|_{U_i\cap U_j} \xrightarrow{\cong} C^\infty(x_j(U_i\cap U_j))$$

and so we obtain an isomorphism $\psi\colon C^\infty(x_i(U_i \cap U_j)) \to C^\infty(x_j(U_i \cap U_j))$. Hence, applying $|-|$, using the naturality given by **Exercise** 6, the resulting map (abusing notation) $|\psi| : x_j((U_i \cap U_j) \to x_i(U_i \cap U_j)$ is a diffeomorphism of open subsets of the relevant model spaces, we claim. Unwinding this, $|\psi|(\mathrm{ev}_q) = \mathrm{ev}_q \circ \psi$ and this must have the form ev_p for some $p \in x_i(U_i \cap U_j)$; if we fix the (unique) f_k for which $\psi(f_k) = x^k$, the k-th standard coordinate projection function, then $\mathrm{ev}_q \circ \psi = \mathrm{ev}_p$ where $p = (\mathrm{ev}_q\, \psi(f_1), \dots, \mathrm{ev}_q\, \psi(f_n))$. In particular, this shows that

$$|\psi|(q) = (\psi(f_1)(q), \dots, \psi(f_n)(q))$$

which is smooth, and a similar argument using ψ^{-1} shows that $\left|\psi^{-1}\right| = |\psi|^{-1}$ is smooth so that this is a diffeomorphism.

That this is independent of the choices of isomorphism ϕ_i is a simple exercise. □

Remark. In particular, $|-| : \mathsf{SmAlg}_{\mathbf{R}} \to \mathsf{Man}^{\mathrm{op}}$ from smooth, complete, geometric $\mathbf{R}$-algebras to the opposite category of smooth manifolds is an equivalence of categories for which $|C^\infty(-)| \simeq \mathrm{id}$ and $C^\infty(|-|) \simeq \mathrm{id}$.

The upshot of the discussion is the following slogan: to study manifolds, we should study their smooth functions because this already contains all of the information about the manifold and its relations with other manifolds.

2 Morse Functions

2.1 Basic Properties and First Topological Consequence

Although all smooth functions are good, $C^\infty(M)$ is an unwieldy object containing many uninteresting maps throwing away tangential information. For instance, every map $M \to \mathbf{R}$ that is constant contains no interesting information about the topology of the manifold. To get a grasp on this, we will first characterize what happens when a smooth function $f\colon M \to \mathbf{R}$ has no critical values in an interval $[a, b]$, under mild (and, in fact, generic, in a suitable sense in a space of smooth functions) hypotheses.

Notation 2.1. Given a function $f\colon M \to \mathbf{R}$, we let $M^a = f^{-1}((-\infty, a])$.

Reminder. As a consequence of the improved regular value theorem (see the appendix), whenever a is a regular value of a smooth function $f\colon M \to \mathbf{R}$, the sets $f^{-1}((-\infty, a])$ and $f^{-1}([a, \infty))$ are submanifolds of M having boundary $f^{-1}(a)$.

Recall as well that for a Riemannian manifold M and smooth function $f\colon M \to \mathbf{R}$, grad f is the unique vector field with the property that for any vector $X_p \in T_pM$ (as usual, using the identity chart on $\mathbf{R}$),

$$\langle (\operatorname{grad} f)_p \mid X_p \rangle = X_p(f) = f_{*p}(X_p).$$

In local coordinates, grad $f = \dfrac{\partial f}{\partial x^i} g^{ij} \dfrac{\partial}{\partial x^j}$.

Theorem 2

Let $f\colon M \to \mathbf{R}$ be smooth. Suppose the interval $[a, b]$ contains no critical points of f, $f^{-1}([a, b])$ is compact and $f^{-1}(a) \neq \varnothing$.

(a) Suppose $\partial M = \varnothing$. Since $f^{-1}([a, b])$ is compact, $f(f^{-1}[a, b]))$ has a maximum, say c. Then for all $d \in [c, b]$, $M^d = M^c$.

(b) Suppose $\partial M = \varnothing$. There is a diffeomorphism $M^a \cong M^b$. In fact, for any $a \leq d \leq b$, $M^a \cong M^d$.

(c) The level sets $f^{-1}(a) \cong f^{-1}(d)$ for any $a \leq d \leq c$.

(d) M^b smoothly deformation retracts onto M^a. In fact for any $a \leq d \leq b$, M^d smoothly deformation retracts onto M^a.

(e) Suppose now that M is compact with non-empty boundary ∂M and, in addition, $f\colon M \to [a, b]$ and satisfies that $f(\partial M) = \{a, b\}$. Then there is a diffeomorphism $F\colon f^{-1}(a) \times [a, b] \to M$ for which TFDC:

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & M \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle f} \\ [a,b] & = & [a,b] \end{array}$$

Warning. Every source I know of stating this theorem omits the hypothesis that $f^{-1}(a) \neq \varnothing$. The problem is that if $a \notin \operatorname{Im}(f)$ but $b \in \operatorname{Im} f$ then $f^{-1}(a) = \varnothing$ while $f^{-1}(b) \neq \varnothing$. For instance, this is the case for the embedding $f\colon \mathbf{R}_{>0} \hookrightarrow \mathbf{R}$, which has no critical points but $f^{-1}(0) = \varnothing$ while for every $\varepsilon > 0$, $f^{-1}(\varepsilon) = (0, \varepsilon]$. There is also an implicit assumption that $[a, b] \subset \operatorname{Im} f$ in all proofs.

Proof. **(a)** This is obvious.

(b) It suffices to show this for M^c and M^a. Since the condition of being full rank is an open condition, there is an open nbhd U of $f^{-1}([a, b])$ such that U contains no critical points of f and U has compact closure. Let ρ be a bump function which is identically 1 on $f^{-1}([a, b])$ and has $\operatorname{supp} \rho \subset U$ and consider the vector field

$$X = -\rho \frac{\operatorname{grad} f}{\| \operatorname{grad} f \|^2}.$$

Since $\operatorname{supp} X \subset U$ is compact, X has a global flow Φ^X and since the flow is global, Φ^X_t is a diffeomorphism of M with itself for all t.

Consider the integral curve γ_p through p. If $\gamma_p(s) \in f^{-1}([a, b])$, then

$$\frac{d}{dt} f(\gamma_p(t))\Big|_{t=s} = f_{*\gamma_p(s)}\Big(\frac{d}{dt}\gamma_p(t)\Big|_{t=s}\Big) = f_{*\gamma_p(s)}(X_{\gamma_p(s)})$$

and since $\gamma_p(s) \in f^{-1}([a, b])$,

$$f_{*\gamma_p(s)}(X_{\gamma_p(s)}) = f_{*\gamma_p(s)}\left(-\frac{(\operatorname{grad} f)_{\gamma_p(s)}}{\|(\operatorname{grad} f)_{\gamma_p(s)\|^2)}} \right)$$

and by the properties of gradients,

$$f_{*\gamma_p(s)}\left(-\frac{(\operatorname{grad} f)_{\gamma_p(s)}}{\|(\operatorname{grad} f)_{\gamma_p(s)\|^2)}} \right) = \left\langle (\operatorname{grad} f)_{\gamma_p(s)} \;\middle|\; -\frac{(\operatorname{grad} f)_{\gamma_p(s)}}{\|(\operatorname{grad} f)_{\gamma_p(s)\|^2)}} \right\rangle = -1.$$

By existence and uniqueness of solutions to ODEs, if $\gamma_p(s) \in f^{-1}([a, b])$, it must be that $f \circ \gamma_p(t + s) = -t + f(\gamma_p(s))$. The same argument shows that, more generally,

$$\frac{d}{dt} f(\gamma_p(t))\Big|_{t=s} \leq 0$$

for any $p \in M$ and $s \in \mathbf{R}$ for which the domain makes sense.

From this, it is easy to see that Φ^{c-a} sends M^c onto M^a diffeomorphically, mapping the level set $f^{-1}(c)$ onto $f^{-1}(a)$ diffeomorphically. Similarly, for any $a < d < c$, Φ^{d-a} is a diffeomorphism $M^d \cong M^a$ mapping $f^{-1}(d)$ onto $f^{-1}(a)$.

(c) Really this follows from the above and smooth invarince of the boundary (any diffeomorphism restricts to a diffeomorphism of the boundary). This establishes the claim about the diffeomorphism and the level sets.

(d) It suffices to show this for M^c. Let $H\colon [0,1] \times M^c \to M^a$ be

$$H(t,p) = \Phi^X(t \max\{f(p) - a, 0\}, p).$$

this gives a continuous deformation retract.

Now let ρ be a smooth non-negative function $\rho\colon [0,1] \times M \to [0,1]$ which is 0 precisely on the closed set $f^{-1}((-\infty, a]) \times [0,1]$ and is 1 precisely on the closed set $f^{-1}([c,\infty)) \times \{1\}$ by the strong form of the smooth Urysohn lemma. Then

$$H\colon [0,1] \times M^c \to M^a$$

given by

$$H(t,p) = \Phi^X(t\rho(t,p)(f(p) - a), p)$$

is the desired smooth deformation retract.

(e) Let

$$X = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|^2}.$$

Then X has integral curves those of grad f except reparameterized. Note that the vector field X never tangent to ∂M and is outward pointing on $f^{-1}(a)$ and inward pointing on $f^{-1}(b)$. Since M is compact, it follows that the flow for X exists wherever it makes sense and, in particular, the maximal flow domain is

$$A_X = \{(t,p) \in \mathbf{R} \times M : t \in [a - f(p), b + f(p)]\}.$$

As before, we find that for any integral curve γ of X, $\frac{d}{dt} f \circ \gamma \equiv 1$ (using the identity chart on $\mathbf{R}$ as usual). Hence, $f(\gamma(t_1)) - f(\gamma(t_0)) = t_1 - t_0$.

Define $F\colon f^{-1}(a) \times [a,b] \to M$ by

$$F(t,p) = \Phi^X(t - a, p).$$

Since $f^{-1}(a) \times [a,b]$ is compact, we only need to show that F is a smooth embedding with image all of M. We shall expand on this point in **Exercise** 13 following this theorem. For now, let us assume that it is true this is all we need to show.

The only way F can fail to be injective, due to uniqueness of integral curves, is if $\Phi^X(t-a,p) = \Phi^X(s,a,p)$. This is impossible since f increases along the integral curves of grad f and thus those of X and so F is injective. F is surjective because for each $p \in M$, we can flow backwards to $f^{-1}(a)$ from p since the integral curve through p for X has domain $[a - f(p), b + f(p)]$ and has constant speed 1.

Thus, we need only show F is an immersion to conclude. Intuitively, this is because the integral curves for Φ are transverse to the level submanifolds $f^{-1}(d)$. More precisely, note that by the ***Leibniz formula***,

$$\Phi^X_{(t,p)*}(v + \partial_t) = \Phi^X(-,p)_{t*}(\partial_t) + \Phi^X(t,-)_{*p}(v) = X_{\Phi^X(t,p)} + \Phi^X(t,-)_{p*}(v).$$

It is easy to see that $\Phi^X(t,-)|_{f^{-1}(a)}$ is a diffeomorphism onto its image—the submanifold $f^{-1}(t+a)$—since it has smooth inverse $\Phi^X(-t,-)|_{f^{-1}(t+a)}$. Hence, to show that this is an immersion (i.e., full rank), it suffices to verify the transversality condition mentioned. This follows easily since f is constant on level sets and f increases along all integral curves. □

Exercise 12

What goes wrong in the first parts above if we allow M to have boundary above?

Remark. This theorem shows that if $M \xrightarrow{f} \mathbf{R}$ has no critical points (and say is a proper map), nothing interesting is changing in the topology of of M in an interval $[a,b]$ containing no critical values of f. Later, we shall see a partial converse to this, under suitable conditions on the function f.

It is *not* true in general that if M has non-empty boundary and $f\colon M \to N$ is an injective immersion, then f is an embedding. Extra assumptions are required to make this true; one such hypothesis that was used in the proof above is that M is compact.

Exercise 13

Let $f\colon M \to N$ be a smooth injective immersion.

(a) Show that any continuous proper[a] map $g\colon M \to N$ between two manifolds is a closed map. *[Hints: The only property that is needed is that the target space is locally compact[b] Hausdorff. For a closed set F, show that $f(F)$ is closed by taking an open nbhd about any of its limit points q with compact closure (why does such a nbhd exist?) and show that $q \in f(F)$.]*

(b) If f is an open or closed map, then f is a smooth embedding.

(c) f is a proper map, then f is a smooth embedding.

(d) If M is compact, then f is a smooth embedding.

(e) If $\partial M = \varnothing$ and $\dim M = \dim N$, then f is a smooth embedding. *[Hint: Use Proposition 4 to deduce that $\operatorname{Im}(f) \subset \operatorname{Int} N$. Then use the constant rank theorem to deduce that f is a local diffeomorphism and hence open map into $\operatorname{Int} N$.]*

[a] This means that preimages of compact sets are compact.

[b] This means that every point p has an open nbhd U which contains a compact set K such that $p \in K \subset U$.

Now let us consider the hollow torus T defined as the subset of $\mathbf{R}^3$ given by

$$T = \left\{ (x,y,z) \in \mathbf{R}^3 : (2 - \sqrt{x^2+y^2})^2 + z^2 = 1 \right\},$$

which is the hollow torus of inner-radius 1 and outer radius 2. This has a projection $f\colon T \to \mathbf{R}$ sending $(x,y,z) \in T$ to z. This function is more interesting than a constant function, but is nevertheless still somehow less interesting than it could be. Notice that f has a circle of critical points at the top and bottom of T.

The problem with this projection is that it throws away *higher order* tangential information about the manifold. More precisely, let γ be a loop with constant speed parameterizing the ring of critical points on the bottom of T. It is possible to introduce smooth coordinates near each $\gamma(t)$ on T for which f looks like a smooth function $\mathbf{R}^2 \supset U \to \mathbf{R}$ depending only on the last coordinate for which $f \circ \gamma$ is constant in these coordinates. At each point $\gamma(t)$, the Hessian for f in these coordinates has the form

$$\mathrm{Hess}_{\gamma(t)} f = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$

where $* \neq 0$. This matrix does not have full rank and so has thrown away higher order tangential information. What separates the world of smooth manifolds from that of topological manifolds is the tangent bundle, so this is clearly not ideal.

Finally, as a last, attempt, let us consider the projection $T \to \mathbf{R}$ sending $(x, y, z) \mapsto x$. This function measures the "height" of T along the x-axis, and it can be shown that this function has 4 critical points, where for each the Hessian in smooth coordinates has full rank (is a non-degenerate bilinear form). This is the sort of function Morse theory concerns itself with—there is enough residual topological and smooth information at these critical points to say something interesting about the manifold.

Warning (Higher-order Partial Derivatives)**.** There is one small issue we have elided. To make sense of higher order partial derivatives, we want to differentiate along vectors. This requires, in general, a ***covariant derivative***, but as one may find in the appendix, the naive way of defining the Hessian of a smooth function is correct at any critical point. See **Lemma** 42 and **Exercise** 44 for these details.

Definition 10

If M is a smooth manifold, we say that $f\colon M \to \mathbf{R}$ is a ***Morse function*** if at every critical point $p \in \mathbf{Cr}(f)$, the Hessian $\mathrm{Hess}_p(f)$ is non-degenerate—equivalently, the Hessian has full rank at p.

Exercise 14

Let $f\colon M \to \mathbf{R}$ be smooth. Show that a critical point p for f is non-degenerate **iff** the $df\colon M \to T^*M$ intersects the zero section transversely at p (i.e., $df_{p*}(T_pM) + T_{df(p)}Z = T_{df(p)}T^*M$). *[Hint: Work in local coordinates to produce a projection*

$$p\colon\ T^*M|U \to T_p^*M$$

*and show that p is non-degenerate **iff** $0 \in T_p^*M$ is a regular value of p (why?). In the chosen coordinates, what does the Jacobian of $p \circ df$ look like?]*

Definition 11

Given f smooth and a critical point $p \in \mathbf{Cr}(f)$ that is non-degenerate, we say the ***index*** of the critical point p is the index of the bilinear form $\mathrm{Hess}_p(f)$—namely, the number of negative eigenvalues $\mathrm{Hess}_p(f)$ has, and thus by the spectral theorem the dimension of the subspace upon which this bilinear form is negative-definite.

It turns out that at non-degenerate critical points, there is a smooth chart for which f takes a particularly nice form, which essentially says that, locally, there is a chart for which f is equal to a the associated quadratic form for a particular diagonalization of $\mathrm{Hess}_p f$.

Lemma 3 (Morse)

If $f\colon M \to \mathbf{R}$ is smooth, $\dim M = m$ and f has non-degenerate critical point p of index k, then there is a coordinate system (x, U) called a ***Morse chart*** with $x\colon (U, p) \to (x(U), 0)$ a diffeomorphism onto a convex open set $x(U) \subset \mathbf{R}^m$, such that

$$f \circ x^{-1}(x^1, \ldots, x^n) = f(p) - \sum_{i=1}^{k} (x^i)^2 + \sum_{j=k+1}^{m} (x^j)^2.$$

Exercise 15

Prove the ***Morse lemma***. *[Hint: Reduce to a local argument with $p = 0$. Taylor's theorem with integral remainder form implies that $f(x) - f(0) = \sum_{i,j} a_{ij}(x^1, \ldots, x^m)x^i x^j$ where $a_{ij} = \int_0^1 (1-t)\frac{\partial^2}{\partial x^i \partial x^j} f(tx^1, \ldots, tx^m)\, dt$ and $a_{ij}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x^i \partial x^j}(0)$. Make suitable changes of coordinates.]*

Remark. The proof hinted at above is Morse's original proof. It is based on an idea of Lagrange in the case that these suitable changes of coordinates are linear functions.

An immediate corollary of the Morse lemma is that non-degenerate critical points are isolated in the set of all critical points.

Corollary 1

Non-degenerate critical points are isolated in the subset $\mathbf{Cr}(f)$. In particular, the critical points of Morse functions are isolated.

2.2 Genericity of Morse Functions

But what can be said about the existence of Morse functions and their abundance or genericity? For this, we introduce smooth function space topologies; namely, the ***weak Whitney topology*** (also called the weak, or C^∞ compact-open topology) and the ***strong Whitney topology*** (also called the strong topology).

Definition 12

Given two manifolds M and N, possibly with corners, let $C^\infty_W(M,N)$ be the set $C^\infty(M,N)$ with the topology generated by the following subbase

$$\{N^\infty_W(f;(x,U),(y,V),K,\varepsilon)\}_{f,x,y,\varepsilon}$$

where $f \in C^\infty(M,N)$, x is a chart on M, y a chart on N, $K \subset U$ is compact with $f(K) \subset V$ and $\varepsilon > 0$. Here

$$N^r_W(f,x,y,K,\varepsilon) = \text{set of } g \in C^\infty(M,N) \text{ such that}$$
$$\left|(\partial^\alpha(yfx^{-1})^j - \partial^\alpha(ygx^{-1})^j)(p)\right| < \varepsilon,\ \forall j, \forall p \in K, \forall \text{ multi-indces}, 0 \leq |\alpha| < \infty.$$

Exercise 16

Show that a sequence $(f_n)_{n\in\mathbf{N}}$ in $C^\infty(M,N)$ converges **iff** for each compact set $K \subset M$ in the domain of a chart of M, the following holds: for any charts (x,U) and (y,V) with $K \subset U$ and $f(K) \subset V$, there exists $N \geq 0$ such that for all $n \geq N$, $f_n(K) \subset V$ and in the local representation afforded by these charts, $f_n \to f$ uniformly on K and for each possible partial derivative, $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on K.

Exercise 17

Because the weak Whitney topology is (completely) metrizable (so first-countable), this condition in fact characterizes the closed sets and thus characterizes the topology. Namely, show that metric spaces are *sequential spaces*.

Definition 13

Given two manifolds M and N, possibly with corners, let $C^\infty_S(M,N)$ be the set $C^\infty(M,N)$ with the topology generated by the following base

$$\{N^\infty_S(f;\Phi,\Psi,K,\varepsilon)\}_{f,\Phi,\Psi,\varepsilon}$$

where $f \in C^\infty(M,N)$, Φ and Ψ are locally finite, countable covers of M and N by charts x_i and y_i, $K = \{K_i\}$ is a family of compact subsets $K_i \subset U_i$ and $\varepsilon = \{\varepsilon_i\}$ is a family of positive numbers. Here

$$N^\infty_S(f;\Phi,\Psi,K,\varepsilon) = \{g \in C^\infty(M,N) : \forall i \in \mathbf{N},\ g \in N^\infty_W(f;x_i,y_i,K_i,\varepsilon_i)\}.$$

Exercise 18

(a) Show that local finiteness of an open cover of a manifold M already implies that it is countable.

(b) If M is compact, show that $C^\infty_S(M, N) = C^\infty_W(M, N)$.

(c) Show that this really is a base (not a subbase) for the topology.

(d) Show that another base for the topology consists of the subset of those basis sets $N^\infty_S(f; \Phi, \Psi, K, \varepsilon)$ for which $K = \{K_i\}$ covers M. Show as well that for any subset $J \subset \mathbf{N}$, $\bigcup_{j \in J} K_j$ is closed. *[Hint: The last part will follow if K is a locally finite collection of closed sets.]*

(e) *Show that $f_n \to f$ in $C^\infty_S(M, N)$ iff there is a compact set $K \subset M$ contained in the domain of a chart such that for all but finitely many f_n, $f|\, M \setminus K = f_n|\, M \setminus K$ and on K, in any pair of compatible charts (x, U) and (y, V), for sufficiently large n that $f_n(K) \subset V$, $f_n \to f$ uniformly and $\partial^\alpha f_n \to \partial^\alpha f$ uniformly, for all partial derivatives.* [Hint: Show first that this is true for an arbitrary compact subset by taking a good compact exhaustion of M. Then show that this gives the result for K contained in the domain of a chart.]

The strong topology is not metrizable in general, and not even first-countable in general. It does have a saving grace, however.

Theorem 3

$C^\infty_S(M, N)$ is a Baire space.

Proof. Omitted. We will not need this, but a proof may be found in Hirsch's *Differential Topology*. □

Exercise 19

(a) Define a weak (resp. strong) topology on $C^r(M, N)$ for $0 \leq r < \infty$ where for $r = 0$, the weak topology corresponds to the usual compact-open topology. *[Hint: Where does r appear in the subbasis and basis sets?]*

(b) Show that the inclusion $C^{r+1}(M, N) \subset C^r(M, N)$ is continuous in the weak and strong topologies but never an embedding.

(c) Show that $C^\infty_W(M, N) = \lim_r C^r(M, N)$ in both the weak and strong topologies.

(d) Let $r \leq s$. Let $U \subset \mathbf{R}^m_k$ be open, $V \subset \mathbf{R}^n_\ell$ be open, $F \subset U$ be closed and $W \subset U$ open. If $f: U \to V$ is C^r and C^s on a nbhd of $F \setminus W$, then every nbhd

N of f in $C^r_S(U, V)$ contains a C^r map $h\colon U \to V$ which is C^s in a nbhd of K with $h| U \setminus W = f| U \setminus W$.

(e) Show that $C^s(M, N) \subset C^s(M, N)$ is a dense subset for $1 \leq r \leq s \leq \infty$ in either the strong or the weak topologies. *[Hint: Consider a basic nbhd $N^r(f; \{(x_i, U_i)\}_i, \{(y_i, V_i)\}_i, \{K_i\}_i, \{\varepsilon_i\}_i)$ with K_i a cover by* ***(d)*** *of Exercise 2.2 and let $\{W_i\}_i$ be a family of open sets with $K_i \subset W_i \subset \overline{W}_i \subset U_i$. Define a family of C^r maps by induction such that $g_0 = f$ and for $k \geq 1$, $g_k = g_{k-1}$ on $M \setminus W_k$ such that g_k is C^s on a nbhd of the closed set $\bigcup_{i=1}^k K_i$. Set $g(x) = g_{\max\{k: x \in \overline{U}_k\}}(x)$.]*

(f) Show that $C^r_W(M, N) \subset C^s_W(M, N)$ is a weak homotopy equivalence for the weak topologies for all $0 \leq s \leq r \leq \infty$. *[Hint: This is highly non-trivial. An even stronger result is that for any open subset $U \subset C^s_W(M, N)$, $U \cap C^r_W(M, N) \subset C^s_W(M, N) \subset U$ is a weak homotopy equivalence. See, for instance, this paper for more details.]*

It can be useful to consider Morse functions on manifolds with boundary too.

Definition 14

A Morse function $f\colon M \to \mathbf{R}$ is said to be ***admissible*** if $f(M) \subset [a, b]$ with $f^{-1}(a) \cup f^{-1}(b) = \partial M$ such that both a and b are regular values of f.

Remark. When $\partial M = \varnothing$ and M is compact, all Morse functions are admissible.

We shall show Morse functions are generic by showing they are dense in the strong topology; in particular, Morse functions always exist.

Remark. In a beautiful part of applied mathematics, there is a clever notion of *prevalence*. This notion is related to Takens' theorem (essentially a souped-up or spicy version of the Whitney embedding theorem) and has connections to the theory of time-delay embedding theorems, which are useful in such places as neuroscience and specifically signals-processing. For more information, one might see the papers *prevalence: a translation-invariant "almost every" on infinite-dimensional spaces* or *embedology*. This is a notion adapted to real-world examples. While Morse functions are "prevalent" in the sense of being dense—a purely topological notion of "prevalence"—they *are not* prevalent in the definition given in these papers.

We will adapt an argument from Guillemin and Pollack's Differential Topology. We will need some technical lemmas first.

Lemma 4

If $M \subset \mathbf{R}^N$ is a submanifold without boundary, then about $p \in M$, the restriction to M of some subcollection of the standard basis coordinate functions $x^{i_1}, \ldots, x^{i_m}$ of $\mathbf{R}^N$, say with $i_1 < \cdots < i_m$, may be used as smooth coordinate charts for M.

Proof. The standard coordinate basis is a linear basis, so consider their linear duals

$$(x^i)^*\colon \mathbf{R}^N \to \mathbf{R}.$$

We may think of these as the "pointwise" version of their smooth manifold duals $(x^i)^*\colon \mathbf{R}^N \to \mathbf{R}^N \times (\mathbf{R}^N)^*$, since the cotangent bundle of$\mathbf{R}^N$ is canonically trivializable $T^*\mathbf{R}^N \cong \mathbf{R}^N \times (\mathbf{R}^N)^*$, there is no ambiguity in this and on each fiber, $(x^i)^*(p)$ is the element of the linear dual $(x^i)^*$ which is the element of the dual basis as defined above.

It follows that some m of these are linearly independent when restricted to $T_p\mathbf{R}^N$, some subcollection of size m of them are linearly independent on the subspace T_pM; WLOG say $(x^1)^* \ldots, (x^m)^*$. It is certainly true that the x^i restrict to smooth functions on M, so we need only show that the function $M \ni q \mapsto (x^1(q), \ldots, x^m(q))$ has full rank at p, since then this is a local diffeomorphism by the inverse function theorem.

To see this, note that, essentially by definition, the derivative of the coordinate function $x^i\colon \mathbf{R}^N \to \mathbf{R}$ at p acts on vectors by $(x_i)_*(p)(v) = v^i = (x^i)^*(p)(v)$ and so, upon restriction to M, this acts on $T_pM \subset \mathbf{R}^N$ according to the same formula (we can and do use the identity chart on $\mathbf{R}$ as usual). This means that the map $(x^1, \ldots, x^m)\colon M \to \mathbf{R}^M$ is a local diffeomorphism at p since otherwise there is a linear dependence of the $(x^i)^*(p)$ for $i = 1, \ldots, m$, which is impossible. □

Lemma 5

Let $U \subset \mathbf{R}^m$ be open and $f\colon U \to \mathbf{R}$ be smooth. Define $F\colon \mathbf{R}^m \times U \to \mathbf{R}$ by

$$F(v, p) = f(p) + \langle v \mid p \rangle = f(p) + v \cdot p$$

where $\langle - \mid - \rangle$ is the standard inner product on $\mathbf{R}^m$. Then for almost all $v \in \mathbf{R}^m$, the function $f_v = F(v, -)$ is Morse.

Proof. Everything is Euclidean, so we may use the usual notions of calculus. Define $g\colon U \to \mathbf{R}^m$ by

$$g = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^m} \right).$$

The derivative of f_v at p is then

$$(f_v)_{p*} = g(p) + v$$

and so p is a critical point of f_v if and only if $g(p) = -v$. Note that f_v and f have the same Hessian matrix everywhere—namely g_{*x}.

Suppose now that $-v$ is a regular value for g. Then g_{*x} is non-singular whenever $g(x) = -a$. hence, every critical point of f_v is non-degenerate and so f_v is Morse. Sard's theorem now implies that $-v$ is a regular value of g for almost every $v \in \mathbf{R}^m$. □

The idea now is to use the lemmas on slices to show the following.

Proposition 1

Given M a manifold without boundary and $f\colon M \to \mathbf{R}$, there is a family of functions $F\colon \mathbf{R}^N \times M \to \mathbf{R}$ with $F(0,-) = f$ such that for almost every $v \in \mathbf{R}^N$, $f_v = F(v,-)$ is Morse.

In particular, for an embedding $M \subset \mathbf{R}^N$,

$$F(v,p) = f(p) + v \cdot p$$

viewing $p \in M \subset \mathbf{R}^N$.

Proof. WLOG suppose M is a submanifold of $\mathbf{R}^N$ for sufficiently large N. Cover M by coordinate charts U_i such that, for each, some m of the standard basis functions $x^1, \dots, x^N$ are a smooth coordinate chart on U_i by the preceding lemma. Since M is second countable, we may suppose this collection is countable.

Fixing U_i, suppose WLOG that $x^1, \dots, x^m$ form a coordinate system for U_i. We reduce to a local case, in this way. Note that for F defined as in the proposition statement, it can be shown that

For each $c \in \mathbf{R}^{N-m}$, define

$$f_{(0,c)}\colon U_i \times \mathbf{R}^{N-m} \to \mathbf{R}$$

by

$$f_{(0,c)}(p,x) = f(p) + c \cdot x.$$

Note that since the first m standard coordinate functions restrict to a chart on M, U_i is diffeomorphic to an open subset of $\mathbf{R}^m$ and $U_i \times \mathbf{R}^{N-m}$ is open in $\mathbf{R}^N$ under this diffeomorphism, so we conflate U_i with its image in $\mathbf{R}^m$.

Now define

$$F_{(0,c)}\colon (U_i \times \mathbf{R}^{N-m}) \times \mathbf{R}^m = U_i \times \mathbf{R}^N \to \mathbf{R}$$

by

$$F_{(0,c)}(p,x,b) = f(p) + \langle (b,c) \mid (p,x) \rangle = f(p) + (b,c) \cdot (p,x).$$

Thus, for each c, the preceding lemma implies that $F_{(0,c)}(-,x,b) = f_{(b,c)}$ is Morse for almost every $b \in \mathbf{R}^k$ for each $x \in \mathbf{R}^{N-m}$. Let

$$S_i = \left\{ a \in \mathbf{R}^N : f_a \text{ is not Morse on } U_i \right\}.$$

Then from what we have just shown, each slice $S_i \cap (\mathbf{R}^k \times \{c\})$ has measure 0. By Fubini's theorem, it follows that S_i has measure 0.

Thus, it is easy to see that

$$\left\{ a \in \mathbf{R}^N : f_a \text{ is not Morse} \right\} = \bigcup_{i \in \mathbf{N}} S_i$$

and the right-hand side is a countable union of measure 0 sets and therefore has measure 0. □

Corollary 2

The conclusion remains true when M is a manifold with corners.

Proof. By the flow-in theorem, M is diffeomorphic to a codimension 0 submanifold $\dot{M}$ of Int M which is a closed subset. Therefore, by the Whitney approximation theorem, f extends from $\dot{M}$ to all of Int M. Since $\dot{M}$ has codimension 0, it is easy to see that the preceding proposition implies the result immediately. □

Exercise 20

Suppose M is compact. Show that Morse functions are dense in $C^\infty(M, \mathbf{R})$ directly.

The more general assertion is as follows.

Theorem 4

Morse functions are dense in $C^\infty_S(M)$ when $\partial M = \varnothing$.

Proof. Omitted. This follows from a the multi-jet transversality theorem, once one has established some more theory. □

Exercise 21

Show that this implies Morse functions are dense in $C^\infty(M)$ for any manifold M with corners. *[Hint: An argument similar in spirit to the last corollary proves this.]*

Lemma 6

Admissible Morse functions always exist when ∂M is compact.

Proof. Take a collar in M for which the restriction of the collar to $[0, a] \times \partial M$ gives a closed embedding for all $a \geq 0$. This is always true for any collar when ∂M is compact. If ∂M is not compact, this is still true by the flow-in theorem.

Write $V_0 \coprod V_1 = \partial M$. Since the restriction of the collar to say $[0, 1/2] \times \partial M$ gives a closed embedding, there is a continuous function

$$g(p) = \begin{cases} t & p = (t, x) \in [0, 1/2] \times V_0 \\ 1 - t & p = (t, x) \in [0, 1/2] \times V_1 \\ 1/2 & \text{else,} \end{cases}$$

(it is continuous by the pasting lemma, which critically uses that the first two pieces are defined on closed subspaces) a smooth function $\tilde{g}$ which agrees with g on the closed subset $[0, 1/2] \times \partial M$ and takes values $1/4 < \tilde{g}(p) < 3/4$ outside of $[0, 1/2] \times \partial M$.

This follows by the Whitney approximation theorem. Then $\tilde{g}$ has no critical points in a nbhd of ∂M and $f^{-1}(i) = V_i$.

Embed M into $\mathbf{R}^N$ for suitably large N. Since the open unit ball is diffeomorphic to $\mathbf{R}^N$, we may suppose $M \subset \{x \in \mathbf{R}^N : \|x\| < 1\}$. We shall proceed as before, using this embedding to modify M by elements of $(\mathbf{R}^N)^*$ (linear maps $\mathbf{R}^N \to \mathbf{R}$).

Let μ be smooth, equal to 0 in $\partial M \times [0, 1/4]$ and equals 1 on in $M \setminus (\partial M \times [0, 1/2])$. Let L be any linear map $\mathbf{R}^N \to \mathbf{R}$ (i.e., an element of $\mathbf{R}^N \cong (\mathbf{R}^N)^*$). As we have seen, $\tilde{g} + L$ is a Morse function on M for almost every L.

Equip M with a metric. By compactness, $|d\tilde{g}|$ is bounded below by a *positive* number in $\partial M \times [0, 1/2]$. By taking $|L|$ sufficiently small, we can arrange that

$$|d(\tilde{g} + \mu L)| \geq |d\tilde{g}| - |d(\mu L)| > 0 \qquad \text{in } \partial M \times [0, 1/2]$$

so that $\tilde{g} + \mu L$, as can be easily seen, has no critical points therein. By denseness of Morse functions, we can arrange that $\tilde{g} + L$ is Morse (i.e., by **Proposition** 1) and thus so too is $\tilde{g} + \mu L$ where L satisfies the above inequality, thereby giving an admissible Morse function. □

Exercise 22

Prove or provide a counter-example to the following assertion: the above hold even if M is not compact. *[I do not know the answer to this. I suspect it is true and one uses a good collar afforded by the flow-in theorem and then shows that $|d\tilde{g}|$ can be arranged to be bounded below by a positive number as above. Alternatively, perhaps show that $C_S^\infty(M)^{\times 2} \to C_S^\infty(M)$ by $(f, g) \mapsto f + g$ is continuous and argue about its image to show that some suitable function can be used to perturb $\tilde{g}$ and get a Morse function.]*

2.3 Passing a Non-degenerate Critical Point

We now show that the topology changes when passing a critical point for a Morse function.

Theorem 5

Let M be compact and let $f\colon M \to [a, b]$ be an admissible Morse function having a unique critical point z of index k. Then there exists a k and an $n-k$ cell $e^k, e^{n-k} \subset M - f^{-1}(a)$ such that $e^k, e^{n-k} \cap f^{-1}(b) = \partial e^k, \partial e^{n-k}$ and there is a deformation retract of M onto $f^{-1}(b) \cup e^k$ (resp. $f^{-1}(b) \cup e^{n-k}$). Moreover, these $n-k$ cells can be chosen to intersect only at z and do so transversely.

Proof. WLOG $f(z) = 0$. It suffices to prove this for $f^{-1}[-\varepsilon, \varepsilon]$ by the preceding interval theorem applied to $[a, -\varepsilon]$ and $[\varepsilon, b]$. Let (φ, U) be a Morse chart for f about z where $\varphi(z) = (0, 0)$. Write $f\varphi^{-1} = -|x|^2 + |y|^2$ meaning the obvious thing. Set $V = \varphi(U)$. Let $0 < \delta < 1$ be so large that the rectangle

$$R = B^k(\delta) \times B^{n-k}(\delta)$$

is contained in V. Let $\varepsilon < \delta$ be small, say $\varepsilon < \delta^2/100$. Let

$$B^k = B^k(\sqrt{\varepsilon}) \times \{0\} \qquad \text{and} \qquad B^{n-k} = \{0\} \times B^{n-k}(\sqrt{\varepsilon}).$$

These are embedded cells meeting transversely at a single point and thus so too do $e^k = \varphi^{-1}B^k$ and $e^{n-k} = \varphi^{-1}B^{n-k}$. We now restrict to the case of the k-cell case, the $(n-k)$-cell case being dual.

It suffices to consider $f^{-1}[-\varepsilon, \varepsilon]$ as we have set things up, and as we commented. The proof of the retraction and all for e^k goes through for e^{n-k} by using the Morse function $-f$ instead (which has a different index, of course). So we consider the first case.

Give M a Riemannian metric that agrees with the induced metric in $\varphi^{-1}(R)$ obtained from the Euclidean metric (i.e,. a metric which on R equals $\langle \varphi_*, \varphi_* \rangle$ with $\langle \rangle$ the standard Euclidean metric. Then on $\varphi^{-1}(R)$ we have that

$$\varphi_{p*}(\operatorname*{grad}_p f) = \operatorname*{grad}_{\varphi(p)} f\varphi^{-1}$$

by computing that $g(\operatorname{grad}_p f, X) = Xf$ with this metric means that $\left\langle \varphi_{p*} \operatorname{grad}_p f, \varphi_{p*}X \right\rangle = Xf$ but on the other hand, one can check that $(\varphi_* X)(f\varphi^{-1}) = Xf$ by a computation, whence the equality.

Let

$$R_1 = B^k(\sqrt{2\varepsilon}) \times B^{n-k}(\sqrt{3\varepsilon})$$

and

$$R_2 = B^k(\sqrt{3\varepsilon}) \times B^{n-k}(2\sqrt{\varepsilon}).$$

In these coordinates, identify

$$M = f^{-1}[-\varepsilon, \varepsilon]$$

and write

$$g = f \circ \varphi^{-1}.$$

These are chosen such that $g^{-1}(\varepsilon)$ exits R_1 and R_2 at corners of these boxes.

Now, we have contrived that nothing outside of $\varphi^{-1}(R_2)$ will flow under the negative gradient flow to the critical point and in particular into R_2 before hitting $f^{-1}(-\varepsilon)$. Indeed, the integral curves in R have the form, $(x_0 e^{2t}, y_0 e^{-2t})$ interpreted appropriately. Hence, if $(x_0, y_0) \in R \setminus R_2$, then the integral curve through it along it $|x_0|$ increases and $|y_0|$ decreases and $|\operatorname{grad} f|$ has a positive lower bound in the compact manifold $M \setminus \operatorname{Int}(\varphi^{-1}R_1)$.

Since $M \setminus R_2$ is a smooth compact manifold with an admissible Morse function, it is diffeomorphic to $f^{-1}(-\varepsilon) \times [-\varepsilon, \varepsilon]$ over $[-\varepsilon, \varepsilon]$ (note that $f^{-1}(-\varepsilon)$ is a union of boundary components by our assumptions and thus is disjoint from R_2).

$$\begin{array}{ccc} M \setminus R_2 & \xleftarrow[\cong]{F} & f^{-1}(-\varepsilon) \times [-\varepsilon, \varepsilon] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ [-\varepsilon, \varepsilon] & = & [-\varepsilon, \varepsilon] \end{array}$$

Working on the right-hand side, we can scale $-\operatorname{grad} f$ by $\operatorname{pr}(p,t) = t$ so that the negative gradient flow moves each point to $f^{-1}(-\varepsilon)$ in unit time.

Inside of $g^{-1}[-\varepsilon, \varepsilon] \cap R_1$, consider the deformation

$$(x, y, t) \mapsto (x, (1-t)s(x,y)y)$$

where

$$s(x,y) = \begin{cases} 0 & |x|^2 \le \varepsilon \\ \sqrt{|x|^2 - \varepsilon} / |y| & |x|^2 \ge \varepsilon. \end{cases}$$

which is continuous; this path is the closure of the flow lines for the vector field $(0, -2y)$ in R_2. To piece these together, consider the vector field $X = (0, -2y)$, in R_2 consider the vector field $-\operatorname{grad} g = (2x, -2y)$. Let $\mu \colon M \to [0,1]$ be a smooth function with $\mu^{-1}(0) = R_1$ and $\mu^{-1}(1) = M \setminus \operatorname{Int}(R_1)$ by the strong form of the smooth Urysohn lemma.

Let

$$Y(x,y) = (2\mu(x,y)x, -2y)$$

and extend this to all of M in the evident way by declaring Y to be $-\operatorname{grad} f$ outside of the chart. Let Φ^Y be the flow for this. We now produce the deformation by moving along the closures of the flow lines of Y at constant speed, reaching $g^{-1}(-\varepsilon) \cup B^k$ in unit time where by construction the points of $g^{-1}(-\varepsilon) \cup B^k$ are stationary. □

Exercise 23

Verify Y is continuous as defined. Then show that the points of $R_2 \setminus R_1$ really deform onto $g^{-1}(-\varepsilon) \cup B^k$.

Of course, critical points of Morse functions are discrete, so it is easy to establish the following analogue of this theorem. First, we introduce some terminology.

Definition 15

Given $f \colon M \to [a,b]$ an admissible Morse function with M compact, say $\dim M = m$. Say the ***type*** of the Morse function f is $(v_0, \dots, v_m)$ where $v_k = v_k(f)$ is the number of index k critical points of f.

Corollary 3

Let M be compact. Given $f \colon M \to [a,b]$ an admissible Morse function of type $(v_0, \dots, v_m)$, there are disjoint embedded k-cells $e_i^k \subset M \setminus f^{-1}(b)$ $(1 \le i \le v_k$, $0 \le k \le m)$ such that $e_i^k \cap f^{-1}(a) = \partial e_i^k$ and M deformation retracts onto $f^{-1}(a) \cup \{\bigcup_{i,k} e_i^k\}$.

Proof. This is a minor modification of the preceding argument using the fact that the critical points of a Morse function are isolated. □

Exercise 24

Let $f\colon M \to \mathbf{R}$ be a Morse function.

(a) If M is compact and f admissible, show that there is a sequence of admissible Morse functions $f_n\colon M \to \mathbf{R}$ such that $f_n \to f$ in the weak (=strong in this case) Whitney topology such that each f_n has distinct *critical values* for its critical points. *[Hint: Let $\rho\colon [0,\infty) \to [0,1]$ be such that $\rho(0) = 1$, $\rho(t) = 0$ for $t \geq 2$ and $-1 \leq \rho'(t) \leq 0$ everywhere. In a Morse chart about $p \in \mathbf{Cr}(f)$, use this function (or a suitable modification of it) to perturb $f(p)$.]*

(b) *If M is not compact, can the same thing be arranged?*

*[I do not know if **(b)** is true!]*

3 Handle Presentation Theorem

This forms the technical heart of these lecture notes. We give a modern presentation of handle attachments and then later provide as an exercise the classical corner-smoothing procedure.

3.1 Operations on Manifolds

In this subsection, we shall only consider operations involving compact manifolds. Everything construction works more generally, but we do not obtain good invariance properties if certain manifolds are not compact because then we cannot appeal to the isotopy extension theorem.

Connect Sums

This forms the basis for all examples to come. This is the case of gluing to manifolds together by deleting the compact manifold that is a point. First we introduce the categorical prerequsite of a pushout.

Before giving the definition, which looks quite complicated, let us give some intuition. The idea is actually quite simple. When we have three objects (e.g., spaces) connected by morphisms (e.g., continuous maps)

$$C \xleftarrow{\;g\;} A \xrightarrow{\;f\;} B$$

the pushout of this data (sometimes called a *span* or *correspondence* in category theory) is precisely the object formed by gluing C and B together along A.

Example 3

Recall that $\partial D^2 = S^1$. Consider the span of the inclusions

$$D^2 \hookleftarrow S^1 \hookrightarrow D^2$$

in the category TOP of spaces. The hooks indicate that the maps are the (evident) embeddings. the pushout of this data is S^2. Similarly, the pushout (in this case, this is also called a cofiber) of the following span

$$* \longleftarrow S^1 \hookrightarrow D^2$$

($*$ is the one point space) is S^2.

In particular, we obtain a commutative diagram

$$\begin{array}{ccc} S^1 & \hookrightarrow & D^2 \\ \downarrow & & \downarrow \\ D^2 & \hookrightarrow & S^2 \end{array}$$

which is universal in a certain sense described in the definition below.

Definition 16

Fix a category C (think C = TOP spaces or C = DIFF smooth manifolds). Given a diagram in C of the form

$$\begin{array}{ccc} c_0 & \xrightarrow{f} & c_1 \\ {\scriptstyle g}\downarrow & & \\ c_2 & & \end{array} \qquad (*)$$

the *pushout* of $(*)$ (should it exist) is an object $c \in$ C along with morphisms

$$c_2 \xrightarrow{F} c \xleftarrow{G} c_1$$

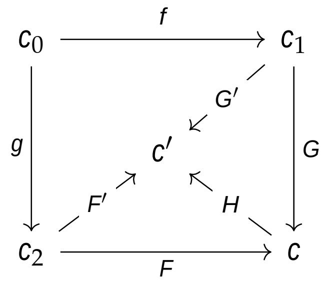
making the following diagram commutes

$$\begin{array}{ccc} c_0 & \xrightarrow{f} & c_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ c_2 & \xrightarrow[F]{} & c \end{array}$$

and satisfying the following ***universal property***. For any other commutative square

$$\begin{array}{ccc} c_0 & \xrightarrow{f} & c_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G'} \\ c_2 & \xrightarrow[F']{} & c' \end{array}$$

there is a *unique* morphism $H\colon c \to c'$ in C making TFDC



Remark. It is this universal property that makes precise the idea that we are gluing to objects together along another.

Notation 3.1. If a square

$$\begin{array}{ccc} c_0 & \xrightarrow{f} & c_1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ c_2 & \xrightarrow[F]{} & c \end{array}$$

in a category C is a pushout square, we often decorate it with the following symbol to indicate that it is a pushout square

$$\begin{array}{ccc} c_0 & \xrightarrow{f} & c_1 \\ {\scriptstyle g}\downarrow & \ulcorner & \downarrow{\scriptstyle G} \\ c_2 & \xrightarrow[F]{} & c \end{array}$$

Warning. Pushouts are not always guaranteed to exist.

Exercise 25

(a) Show that all pushouts in Set (i.e., the category of sets) exist. *[Hint: Given a span $Y \xleftarrow{f} X \xrightarrow{g} Z$ of sets, start with the coproduct $Y \coprod Z$ with the evident structure maps $Y \to Y \coprod Z \leftarrow Z$. Show that the universal property forces you to mod out by the equivalence relation on $Y \coprod Z$ generated by $f(x) \sim g(x)$.]*

(b) Show that if a pushout in TOP exists, then the by forgetting topologies it is a pushout in the underlying category of sets. *[Hint: Give any set X the discrete* *topology.]*

(c) Given a span $B \xleftarrow{f} A \xrightarrow{g} C$ in spaces, show that its pushout exists and is the quotient space $B \coprod C/\{f(a) \sim g(a)\}$.

(d) Show that while DIFF does not have all pushouts (i.e., construct a span in the category of smooth manifolds for which its pushout does not exist).

(e) Find an example of a span in DIFF $N \leftarrow M \rightarrow P$ for which the pushout in DIFF exists but does not agree with its pushout in TOP. *[Hint: Two maps out of space X into a* Hausdorff *space Y agreeing on a dense subset are equal. All manifolds are Hausdorff.]*

Clearly, then, pushouts in DIFF are not well-behaved objects. The idea to make them well-behaved is to include normal bundle data.

First we introduce some notation.

Notation 3.2. Let E be a Riemannian vector bundle over M. Given

$$\alpha\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$$

an orientation-reversing diffeomorphism (such as $x \mapsto \frac{1}{x}$), we obtain an embedding

$$\alpha_E\colon E \setminus M \to E \setminus M$$

(where M sits as the 0 section) by the assignment

$$\alpha_E(v) = \alpha(\|v\|)\frac{v}{\|v\|}.$$

When $M = *$ and $E = \mathbf{R}^m$, we simply denote this by α_m using the standard Euclidean metric.

Exercise 26

Given a pushout in TOP

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle F} \\ C & \underset{j}{\longrightarrow} & D \end{array}$$

where i is injective, as indicated by the tail.

(a) Show that

$$D = B \coprod C/\left\{c \sim fi^{-1}(c)\ \forall c \text{ such that this makes sense}\right\}.$$

[Hint: This only requires i injective.]

(b) Show that j is an injective.

(c) If i is an open (resp. closed) embedding, then j is an open (resp. closed) embedding.

Theorem 6

Fix two m-manifolds M_1 and M_2 and for simplicity suppose M_1 and M_2 are connected. Fix points $p_i \in M_i$ along with tubular nbhds (thought of as charts and therefore landing in the interiors of both manifolds) $h_i \colon \mathbf{R}^m \to M_i$. If M_1 and M_2 are orientable, assume that h_1 preserves orientation and h_2 reverses it, where $\mathbf{R}^m$ has its standard orientation.

Define a new space

$$M_1 \# M_2 = M_1 \# M_2(h_1, h_2, \alpha)$$

called their *connected sum* as the pushout

$$\begin{array}{ccc} \mathbf{R}^m \setminus \{0\} & \xhookrightarrow{h_1} & M_1 \setminus p_1 \\ {\scriptstyle h_2 \circ \alpha_m} \big\downarrow & \ulcorner & \big\downarrow \\ M_2 \setminus \{p_2\} & \hookrightarrow & M_1 \# M_2(h_1, h_2, \alpha) \end{array}$$

in the category TOP of spaces. This satisfies the following properties.

(a) $M_1 \# M_2(h_1, h_2, \alpha)$ is a topological manifold of dimension m with a unique smooth structure such that the two topological embeddings of the square above colored in blue are smooth embeddings.

(b) The pushout above is a pushout in DIFF.

(c) $M_1 \# M_2(h_1, h_2, \alpha)$ is connected when M_1 and M_2 are and $m \geq 2$.

(d) $M_1 \# M_2(h_1, h_2, \alpha)$ is orientable when M_1 and M_2 are.

(e) $\partial(M_1 \# M_2) = \partial M_1 \coprod \partial M_2$.

(f) $M_1 \# M_2(h_1, h_2, \alpha)$ is indepdent of the orientation reversing α.

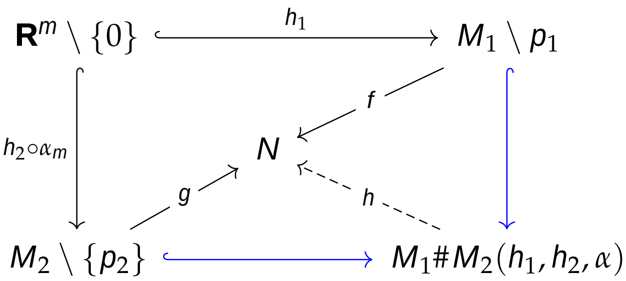
(g) When M_1 is (resp. M_2) is orientable, $M_1 \# M_2(h_1, h_2, \alpha)$ is independent of h_1 (resp. h_2) up to any other orientation-preserving (resp. orientation-reversing) embedding. When M_1 (resp. M_2) is non-orientable, then $M_1 \# M_2(h_1, h_2, \alpha)$ is independent of the choice of embedding h_1 (resp. h_2).

Proof. The pushout is easily seen to be locally Euclidean. Hence, the invariance of domain implies that the blue arrows are open maps and thus $M_1 \# M_2$ is second-countable. To show it is Hausdorff, one simply shows that if $x \in M_1 \setminus p_1$ and $y \in M_2 \setminus p_2$ are not identified, then they have disjoint nbhds by a case checking argument.

Hence, since $h_2 \alpha_m h_1^{-1}$ is an (orientation-preserving) diffeomorphism, the fact that the blue maps are embeddings implies that the smooth structures on $M_1 \setminus p_1$ and $M_2 \setminus$

p_2 are compatible inside $M_1 \# M_2$ and hence yield an (oriented) atlas for which the two blue maps are smooth embeddings. This is clearly the unique structure for which this is true.

Now consider the following diagram in TOP where each object and solid arrow belongs to DIFF.



The dashed arrow exists in TOP. To see that it is smooth, simply observe that smoothness is a local condition and the blue arrows are open smooth embeddings.

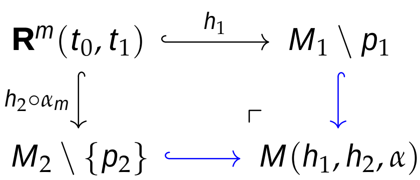
Now we verify the independence assertions. Let $t_1 \in (0, \infty)$ and let $t_0 \in [0, \infty)$ be such that $0 \leq t_0 < t_1$. Let

$$\mathbf{R}^m(t_0, t_1) = \{v \in \mathbf{R}^m : t_0 < \|v\| < t_1\},$$

and note that $\alpha_m(\mathbf{R}^m(t_0, t_1)) = \mathbf{R}^m(\alpha(t_1), \alpha(t_0))$. Note that when $t_0 = 0$, we shall write

$$\mathbf{R}^m(t_1) = \mathbf{R}^m(0, t_1).$$

Consider the pushout



it is easy to see that $M(h_1, h_2, \alpha)$ has a unique smooth structure such that blue arrows are open smooth embeddings and, furthermore, with this smooth structure it is diffeomorphic to $M_1 \# M_2(h_1, h_2, \alpha)$.

Exercise 27

Verify this. In particular, show that the above square is a pushout in DIFF. *[Hint: Then make an argument using universal properties.]*

Now we make a claim (really, an observation).

Claim. The construction $M(h_1, h_2, \alpha)$ only depends on $h_2 \alpha_m h_1^{-1}$ restricted to $h_1(\mathbf{R}^m(t_1))$ in the sense that if we had any other $h_i' \colon \mathbf{R}^m \setminus 0 \to M_i \setminus \{p_i\}$ $(i = 1, 2)$ with

$$h_1(\mathbf{R}^m(t_1)) = h_1'(\mathbf{R}^m(t_1))$$

and

$$h_2 \alpha_m h_1 | \, h_1(\mathbf{R}^m(t_1)) = h_2' \alpha_m (h_1')^{-1} \Big| \, h_1(\mathbf{R}^m(t_1))$$

then there is a diffeomorphism $M(h_1, h_2, \alpha) \cong M(h_1', h_2', \alpha)$.

This follows immediately from the alternate description of the pushout quotient relation given in **Exercise** 26.

Let β_1 be any smooth embedding $\beta_1\colon \mathbf{R}^m \to \mathbf{R}^m$ that is a shrinking in the sense of **Definition** 39 such that

$$\beta_1|\,\mathbf{R}^m(t_1) \cup \{0\} = \mathrm{id}, \quad \beta_1(\mathbf{R}^m \setminus (0)) \subset \{v \in \mathbf{R}^m : |v| < t_2\} \text{ where } t_1 < t_2.$$

Let β_2 be any smooth embedding $\beta_2\colon \mathbf{R}^m \to \mathbf{R}^m$ that is similarly a shrinking such that

$$\beta_2|\,\mathbf{R}^m(\alpha(t_1)) \cup \{0\} = \mathrm{id}, \quad \beta_2(\mathbf{R}^m \setminus (0)) \subset \{v \in \mathbf{R}^m : |v| < t_3\} \text{ where } \alpha(t_1) < t_3.$$

Since the construction above only depends on $h_2\alpha_m h_1^{-1}$ restricted to $h_1(\mathbf{R}^m(t_1))$, it follows that $M(h_1\beta_1, h_2\beta_2, \alpha)$ is diffeomorphic to $M(h_1, h_2, \alpha)$.

It follows that each $h_i\beta_i\colon \mathbf{R}^m \to M_i$ has the structure of a proper tubular nbhd about p_i. Hence, WLOG, we may suppose our tubular nbhds are proper. This allows us to show independence of the construction on the h_i.

If M_1 is not oriented, then if $h_1'\colon \mathbf{R}^m \to M$ is any other proper tubular nbhd about a point p_1', then either h_1' is orientation-preserving or there is a smooth orientation-reversing loop $\gamma\colon [0,1] \to M_1$ at p_1' which, by the isotopy extension theorem, extends to diffeotopy $K\colon M_1 \times I \to M_1$, and we therefore obtain an isotopy

$$H = (\mathbf{R}^m \times I \xrightarrow{h_1' \times I} M_1 \times I \xrightarrow{K} M_1)$$

with the property that $H(-,1)$ is a proper tubular nbhd of p_1' and the embedding with the same orientation type as h_1.

Now take any smooth path $\gamma\colon [0,1] \to M_1$ from p_1' to p_1. This once again extends to a diffeotopy $P\colon M_1 \times I \to M_1$ which moves the proper tubular nbhd $H(-,1) = K(h_1'(-),1)$ about p_1' to a proper tubular nbhd $P(K(h_1'(-),1),1)$ about p_1. Finally, by uniqueness of tubular nbhds, there is diffeotopy $\widetilde{g}\colon M_1 \times I \to M_1$ such that $\widetilde{g}(-,1) = g_0\colon M_1 \to M_1$ restricts to a linear isometry of $P(K(h_1'(\mathbf{R}^m),1),1)$ onto $h_1(\mathbf{R}^m)$.

However, this map may be such that the following diagrma *does not* commute!

$$\begin{array}{ccc} \mathbf{R}^m & = & \mathbf{R}^m \\ {\scriptstyle P(K(h_1'(-),1),1)}\downarrow & & \downarrow{\scriptstyle h_1} \\ M_1 & \xrightarrow[g_0]{} & M_1 \end{array}$$

Note that with the acquired orientations, g_0 restricts to an orientation-preserving linear isomorphism. To see this, note that orientation class is *preserved* under isotopy. Hence, the map $g_0(P(K(h_1'(-),1),1))$ has the same orientation type as h_1' and thus h_1. Hence, there is an element $A \in O(m)$ in the *identity* component of $O(m)$ such that $A \circ g_0(P(K(h_1'(-),1),1)) = h_1$ (where we only apply A to the image of $g_0(P(K(h_1'(-),1),1))$). This is critical, because it means there is a smooth path $\gamma\colon [0,1] \to O(m)$ from the identity to A. Thus, by another application of the isotopy extension theorem to $\gamma \circ g(P(K(h_1'(-),1),1))$, we obtain a diffeomorphism $g\colon M_1 \to M_1$ such that

$$g(g_0(P(K(h_1'(-),1),1))) = h_1.$$

Notation 3.3. WLOG we will simply replace the composite $g \circ g_0$ by g.

Notice, in particular, that $P(K(-,1),1))$ is a diffeomorphism $M_1 \setminus \{p_1'\} \xrightarrow{\cong} M_1 \setminus \{p_1\}$. Finally, define a map $G\colon M_1 \# M_2(h_1', h_2, \alpha) \to M_1 \# M_2(h_1, h_2, \alpha)$ by

$$G(p) = \begin{cases} g(P(K(p,1),1)), & p \in M_1 \setminus \{p_1'\} \\ p, & p \in M_2 \setminus \{p_2\}. \end{cases}$$

To see that this is well-defined, we must show that when $p \in M_2 \setminus \{p_2\}$ is in the image of $h_2\alpha_m$, then the given formula makes sense. This is because if $p = h_2\alpha_m(v)$, then it is identified with $h_1'(v) \in M_1 \setminus \{p_1'\}$ and as we saw above

$$G(h_1'(v)) = g(P(K(h_1'(v),1),1)) = h_1(v)$$

and in $M_1 \# M_2(h_1, h_2, \alpha)$, $h_1(v) = h_2\alpha_m(v)$. To see that it is smooth, simply observe that it is smooth on the open submanifolds $M_1 \setminus \{p_1'\}$ and $M_2 \setminus \{p_2\}$ and therefore smooth globally since these cover the connected sum. The same argument shows that the construction is independent of h_2 when M_2 is non-orientable.

Finally, suppose M_1 and M_2 are orientable and $h_1'\colon \mathbf{R}^m \to M_1$ is a tubular nbhd of p_1' such that h_1' and h_1 have the same orientation type. Then there is no need to do the first diffeotopy K; we may as well suppose $K = \mathrm{id}$. Doing so, the rest of the argument is identical and goes through by replacing $K = \mathrm{id}$ everywhere. But, in addition, we obtain that the map G is an orientation-preserving diffeomorphism. The same argument shows that the construction is independent of h_2 so long as h_2 is any other orientation-reversing embedding.

Exercise 28

Verify that G as described above is indeed orientation-preserving.

Finally, we can see that this is independent of α. Suppose β is any other orientation-reversing diffeomorphism $\beta\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$. Then there is an orientation-preserving diffeomorphism $g\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ which is the identity in a nbhd of 0 and in a nbhd of ∞, and such that on some interval (t_0, t_1), $\alpha|(t_0,t_1) = \beta g|(t_0,t_1)$ by **Theorem** 28. Hence,

$$\alpha_m | \mathbf{R}^m(t_0,t_1) = \beta_m g_m | \mathbf{R}^m(t_0,t_1), \qquad \text{where} \qquad \mathbf{R}^m(t_0,t_1) = \{v \in \mathbf{R}^m : t_0 < |v| < t_1\}.$$

Thus, there are diffeomorphism

$$M_1 \# M_2(h_1, h_2, \alpha) \cong M_1 \# M_2(h_1 g_m, h_2, \alpha) \cong M(h_1 g_m, h_2, \alpha) \cong M(h_1, h_2, \beta),$$

where each diffeomorphism follows from something we have shown before. As for the last diffeomorphism, this is a consequence of **Exercise** 27. □

The above construction is complicated, but its utility is that we can use arbitrary—not necessarily proper—embeddings to construct connect sums.

One might wonder about whether the connect sum can be orientable if one or even both manifolds are non-orientable. The next exercise shows this is impossible.

Exercise 29

(a) Show that every 1-dimensional manifold, possibly with corners, is orientable. *[Hint: When it does not have boundary, there are two 1-dimensional manifolds S^1 and $\mathbf{R}$.]*

(b) Show that if M_1 is non-orientable (resp. orientable), then $M_1 \setminus \{p\}$ is non-orientable (resp. orientable). *[Hint: Certain definitions of orientability in* DIFF *make this immediately obvious. The same can be shown for topological manifolds but involves algebraic topology.]*

(c) Show that if at least one of M_1 and M_2 are non-orientable, then $M_1 \# M_2$ is non-orientable. *[Hint: Use the preceding part and argue a contradiction.]*

The idea with the connect sum is that the two manifolds are joined by a tube. The following exercise makes this precise.

Exercise 30

(a) Show that there is an embedding $h\colon \mathbf{R}^m \# \mathbf{R}^m(h_1, h_2, \alpha) \to \mathbf{R}^m \times [-1, 1]$ where h_1 preserves and h_2 reverses orientation such that h_1 and h_2 are tubular nbhds of 0.

(b) Use this to deduce a similar sort of embedding for connected sum $M_1 \# M_2$ when M_1 and M_2 are orientable.

(c) What happens if we assume that h_1 and h_2 are equioriented? What happens in the non-orientable case?

Boundary Connected Sum

A Technicalities and Manifolds with Corners

A.1 General Notions of Smoothness in Local Coordinates

Important Notation and Definitions

Notation A.1. We make the following notational conventions.

$$\mathbf{R}_+ \overset{\text{def}}{=} (0, \infty)$$
$$\mathbf{R}^n_{\geq 0} \overset{\text{def}}{=} [0, \infty)^n$$
$$\mathbf{H}^n \overset{\text{def}}{=} \mathbf{R}^{n-1} \times \mathbf{R}_{\geq 0}$$
$$\mathbf{R}^n_k \overset{\text{def}}{=} \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$$

We will denote

$$i_{n,k} \colon \mathbf{R}^n_k = \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0} \to \mathbf{R}^n$$

the canonical embedding given by the evident subset inclusion for each $0 \leq k \leq n$.

Definition 17 (Smoothness on Subsets)

(a) For $A \subset \mathbf{R}^k$, a function $f \colon A \to \mathbf{R}^n$ is *smooth* if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k$ and a smooth function $\overline{f} \colon U \to N$ such that $\overline{f}\big| A = f$.

(b) Similarly, for $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_{\geq 0}$, a function $f \colon A \to \mathbf{R}^n$ is *smooth* if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k \supset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_{\geq 0}$ and a smooth function $\overline{f} \colon U \to \mathbf{R}^n$ such that $\overline{f}\big| A = f$.

(c) For a subset $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_{\geq 0}$ and a function $f \colon A \to \mathbf{R}^m_n$, we will say that f is *smooth* if for each $p \in A$, there is an open nbhd U of p in $\mathbf{R}^k \supset \mathbf{R}^{k-\ell} \times \mathbf{R}^\ell_{\geq 0}$ and a smooth function $\overline{f} \colon U \to \mathbf{R}^m$ such that $\overline{f}\big| A = f$. In other words, for the purposes of smoothness, we consider a function into $\mathbf{R}^m_n$ to be smooth **iff** it is smooth considered as a function into $\mathbf{R}^n$—in other words, f is said to be *smooth* if $i_{m,n} \circ f$ is smooth in the sense given above.

(d) We will define *manifolds with corners* in the section below. Given two such manifolds M^m and N^n, we will say a function $f \colon M \to N$ is *smooth* if for each $p \in M$, there are charts (x, U) about p and (y, V) about $f(p)$ such that the map

$$y \circ f \circ x^{-1} \colon \underbrace{x(U \cap f^{-1}y^{-1}(V))}_{\subset \mathbf{R}^m_k} \to \underbrace{y(V)}_{\subset \mathbf{R}^n_\ell}$$

is smooth in the sense just described for subsets of Euclidean space.

Basic Results

Theorem 7

Let $A \subset M$ be a subset of a manifold and $f \colon A \to \mathbf{R}^k$ be a function.

(a) f is smooth **iff** there is an open set $U \subset \mathbf{R}^k$ with $A \subset U \subset \mathbf{R}^k$ and a smooth function $\overline{f} \colon U \to N$ such that $\overline{f}\big| A = f$.

(b) If, in addition, A is assumed to be closed, then f is smooth **iff** it extends to all of $\mathbf{R}^n$. In fact, if A is any closed subset of a manifold with corners and f is smooth on A, then for any open subset containing A, there is a smooth function $\widetilde{f} \colon M \to \mathbf{R}^k$ such that $\widetilde{f}\big| A = f$ and $\operatorname{supp} \widetilde{f} \subset U$.

(c) $f: A \to \mathbf{R}^k$ is smooth **iff** f is locally the restriction of a smooth function $M \to \mathbf{R}^k$ (no assumptions on A are needed here).

Proof. This is a partition of unity argument.

(a) f admits (locally) a smooth extension in an open nbhd of each point of A by definition. Cover A by these nbhds, observe that this covering admits a locally finite open refinement and run a partition of unity subordinate to this open cover.

(b) This is a refinement of the preceding argument.

(c) This follows using point-set topological properties of manifolds and **(b)**. □

Corollary 4

The same is true if $A \subset \mathbf{R}^{k-\ell} \times \mathbf{R}^{\ell}_{\geq 0}$.

Proof. ∂A in $\mathbf{R}^{k-\ell} \times \mathbf{R}^{\ell}_{\geq 0}$ is ∂A in $\mathbf{R}^k$. Indeed, $\mathbf{R}^{k-\ell} \times \mathbf{R}^{\ell}_{\geq 0} \subset \mathbf{R}^k$ is closed, and so contains all of its limit points and hence the limit points of A in $\mathbf{R}^{k-\ell} \times \mathbf{R}^{\ell}_{\geq 0}$ is the same as the limit points of A in $\mathbf{R}^k$. □

A.2 Manifolds With Corners

Basic Definitions and Facts

Definition 18 (Model Spaces)

Consider $\mathbf{R}^n_k \subset \mathbf{R}^n$. We give this the following standard smooth structure where a smooth chart of $\mathbf{R}^n_k$ is a smooth homeomorphism onto an open subset of some $\mathbf{R}^n_\ell$ where smoothness is defined as in **Definition** 17 for subsets of Euclidean spaces. Smooth compatibility of these charts boils down to a simple exercise in point-set topology. These will be our *model spaces* or *model corner spaces* after which we pattern manifolds with corners.

Definition 19 (Manifold with Corners)

A smooth *manifold with corners* of dimension n is a second countable, Hausdorff space that is locally patterned after the spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ (k is not fixed, $k \geq 0$) with a maximal smooth atlas $\mathscr{A}$ comprised of such charts that are smoothly compatible—smooth compatibility of these charts. Smoothness of their transitions $y \circ x^{-1}$ is defined as in **Definition** 17. The definition of a smooth function between two manifolds with corners is then patterned after the notion of smoothness for functions $\mathbf{R}^m_k \to \mathbf{R}^n_\ell$ introduced above. See, specifically, **(d)** of **Definition** 17.

We shall say that a chart (x, U) for an n-manifold-with-corner M is a *boundary chart* if it is a homeomorphism from U onto an open subset of $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ such that $x(U) \cap \mathbf{R}^{n-k} \times \mathbf{R}^{k-\ell}_{\geq 0} \times \mathbf{0} \neq \varnothing$ for some $1 \leq \ell \leq k$. We shall say that a chart (x, U) for an n-manifold-with-corner M is a *corner chart* if it is a homeomorphism from U onto an open subset of $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ with $k \geq 2$ such that $x(U) \cap \mathbf{R}^{n-k} \times$

$\mathbf{R}^{k-\ell}_{\geq 0} \times \mathbf{0} \neq \emptyset$ for some $2 \leq \ell \leq k$.

This definition is equivalent to a more flexible definition. First we introduce some notation.

Definition 20

Denote by $(\mathbf{R}^n)^*$ the linear dual of $\mathbf{R}^n$ (the space of linear maps $\mathbf{R}^n \to \mathbf{R}$). Define a *k-corner subspace* of $\mathbf{R}^n$ to be any subset of the form

$$C^n_k = \{x \in \mathbf{R}^n : \lambda_1(x) \geq 0, \ldots, \lambda_k(x) \geq 0\}$$

where we require $\lambda_1, \ldots, \lambda_k \in (\mathbf{R}^n)^* \setminus \{0\}$ to bee such that $\lambda_i \neq \lambda_j$ for any $i \neq j$. If $k = 0$, we set by convention

$$C^n_k = \mathbf{R}^n.$$

Exercise 31

Show that a smooth manifold with corners of dimension n as in the definition above is equivalent to the following definition:

A smooth *manifold with corners* of dimension n is a second countable, Hausdorff space that is locally patterned after the spaces C^n_k (k is not fixed, $k \geq 0$) with a maximal smooth atlas $\mathscr{A}$ comprised of such charts that are smoothly compatible.

[Hint: It suffices (why?) to show that C^n_k is diffeomorphic[a] to $\mathbf{R}^n_k$.]

[a] In the sense that that there is a homeomorphism which is smooth with smooth inverse where smoothness is defined as in **Definition** 17.

Definition 21 (Boundary and Corners)

By abuse of notation, we shall refer to the *boundary* ∂M of a smooth manifold with corners M to be the set of all points that are mapped by some chart to the boundary of one of model spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ ($k \geq 1$) and we shall call the set of points which are mapped by some chart to the boundary of one of the model spaces $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ with $k \geq 2$ the *corner set* of M and denote it by $\angle M$.

Definition 22 (Corner Depth)

Let M be a manifold-with-corners of dimension n. For each $1 \leq k \leq n$, let $\angle_k M$ be the set of points $p \in M$ for which there is a chart (x, U), $x\colon U \to \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ such that $x(p) \in \mathbf{R}^{n-k} \times \mathbf{0} \subset \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$. We call the set $\angle_k M$ the set of *k-th order corners* or *corners of depth* k. We denote by $\operatorname{depth}_M(p)$ or simply $\operatorname{depth}(p)$ the smallest integer k for which there exists a chart (x, U) about p where $x\colon U \to \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$. We call this the *depth* of p.

Remark. The upshot of the remainder of this chapter is that *what you expect to be true is indeed true*.

The following theorem is a standard result in algebraic topology.

Theorem 8 (Topological Invariance of the Boundary)

Given a topological n-dimensional manifold with boundary M, if there is a chart (x, U) for which $x(p) \in \partial\mathbf{R}^n_{\geq 0}$, then the same is true for all other charts of M.

Proof. This is a local homology argument. By shrinking U if necessary and shifting, we may suppose $x(U)$ is an open half ball of some fixed radius $\varepsilon > 0$ centered at $x(p) = \mathbf{0} \in \mathbf{R}^n_{\geq 0}$. By excision, $H_n(M, M \setminus \{p\}) \cong H_n(U, U \setminus \{p\}) \cong H_n(x(U), x(U) \setminus \{\mathbf{0}\})$. By the LES of the pair and contractibility of $x(U)$, $H_n(x(U), x(U) \setminus \{\mathbf{0}\}) \cong H_{n-1}(x(U) \setminus \{\mathbf{0}\})$ and $x(U) \setminus \{\mathbf{0}\} \simeq S^{n-1}$ by the radial contraction onto the boundary. Hence, the local homology of p is non-trivial and evidently concentrated in degree $n-1$ with a factor of $\mathbf{Z}$. Since local homology is a homeomorphism invariant, this shows that any other chart must send p to a point with non-trivial local homology and some thought shows that the only such points lie on the boundary of $\mathbf{R}^n_{\geq 0}$ as desired. □

Theorem 9 (Smooth Invariance of Corner Points)

Let M be a manifold-with-corner.

(a) If $p \in \angle M$, then p is topologically a boundary point in the sense that there is a homeomorphism $\mathbf{R}^n_k \cong \mathbf{R}^n_{\geq 0}$ for $k \geq 1$.

(b) If $p \in \partial M$, then the defining condition is true for every chart about p in the smooth and topological case.

(c) If $p \in \angle M$, then the defining condition is true for every chart about p in the smooth case. In particular, there is no diffeomorphism $\mathbf{R}^n_{\geq 0} \not\cong \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ for any $k \geq 2$.

(d) If $i \neq j$ and $p \in \angle_i M$, then $p \notin \angle_j M$.

(e) Any diffeomorphism $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0} \to \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ preserves $\angle_k(\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0})$ for each $1 \leq k \leq n$. This lifts to manifolds with corners in the obvious way.

Sketch. The idea is that you can successively flatten the walls of $\mathbf{R}^n_k$ to get a homeomorphism $\mathbf{R}^n_k \cong \mathbf{R}^n_{\geq 0}$, but it cannot be smooth because things go "too quickly" around the origin. This can be made precise by contradiction, supposing there is a diffeomorphism $f \colon \mathbf{R}^n_k \to \mathbf{R}^n_{\geq 0}$, taking a smooth curve γ in $\partial\mathbf{R}^n_+$ passing through $f(\mathbf{0})$ at time $t = 0$ *with non-zero derivative* and then observing that $f^{-1}(\gamma)$ has a kink at time $t = 0$ and does not slow to speed 0, so could not possibly be smooth.

(c) and **(d)** are proved in essentially the same manner. The gist of it is that $\partial\mathbf{R}^n_k \setminus \angle\mathbf{R}^n_k$ is disconnected with components consisting of the boundary points of $\mathbf{R}^n_k$ for which exactly one of the coordinates $x^{n-k+1}, \ldots, x^n$ are equal to 0. □

Constant Rank Theorem

The following is adapted from Spivak.

Theorem 10

Suppose M^m and N^n are smooth manifolds (without boundary) and that $f\colon M \to N$ is smooth.

(a) If f has rank k at $p \in M$, then is some coordinate system (x, U) about p and some coordinate system (y, V) about $f(p)$ with $y \circ f \circ x^{-1}$ in the form

$$(y \circ f \circ x^{-1})(a^1, \dots, a^m) = (a^1, \dots, a^k, \psi^{k+1}(a), \dots, \psi^n(a)).$$

Moreover, given any coordinate system y, the appropriate coordinate system on N can be obtained by permuting the component functions of y.

(b) If f has rank k in a nbhd of p, then there are coordinate systems (x, U) about p and (y, V) about $f(p)$ with $y \circ f \circ x^{-1}$ in the form

$$(y \circ f \circ x^{-1})(a^1, \dots, a^m) = (a^1, \dots, a^k, 0, \dots, 0).$$

(c) If $n \leq m$ and f has rank n at p, then for any coordinate system (y, V) about $f(p)$, there is some coordinate system (x, U) about p with

$$y \circ f \circ x^{-1}(a^1, \dots, a^m) = (a^1, \dots, a^n).$$

(d) If $m \leq n$ and f has rank m at p, then for any coordinate system (x, U) about p, there is a coordinate system (y, V) about $f(p)$ with

$$y \circ f \circ x^{-1}(a^1, \dots, a^m) = (a^1, \dots, a^m, 0, \dots, 0).$$

(e) (*Equivariant Rank Theorem*) Let G be a Lie group acting on M and N and suppose the action on M is transitive. Let $f\colon M \to N$ be G-equivariant and smooth. Then f has constant rank.

Remark. Note that rank $f \leq \min\{m, n\}$. Hence, in **(c)** and **(d)**, f has full rank at p and therefore f has full rank in a nbhd of p since the condition of being full rank is an open condition.

Proof. **(a)** Fix a coordinate system (y, V) about $f(p)$ and choose some coordinate system u about p. Since $\operatorname{rank}(df_p) = k$, there is some $k \times k$ submatrix of df_p (in coordinates) whose determinant is nonzero. Thus, by performing some diffeomorphisms (i.e., permuting the coordinate functions u^i and y^i and thereby performing row/column operations) and relabeling, we can bring this $k \times k$-submatrix into the upper left-hand corner of $D(y \circ f \circ u^{-1})$:

$$\det\left(\frac{\partial(y^\alpha \circ f)}{\partial u^\beta}(p)\right) \neq 0 \qquad \alpha, \beta = 1, \dots, k.$$

Now, define

$$\begin{aligned} x^\alpha &= y^\alpha \circ f \qquad \alpha = 1, \dots, k \\ x^r &= u^r \qquad r = k+1, \dots, m. \end{aligned}$$

Then, recalling that $\dfrac{\partial(y^\alpha \circ f)}{\partial u^\beta} \stackrel{\text{def}}{=} D_\beta(y^\alpha \circ f \circ u^{-1})(u(p))$, we see that the determinant $m \times m$ matrix $\left(\dfrac{\partial x^i}{\partial u^j}(p)\right)$ is in fact

$$\det \left(\begin{array}{c|c} \left(\dfrac{\partial(y^\alpha \circ f)}{\partial u^\beta}\right)_{\alpha,\beta=1,\dots,k} & \big(D_i(y^j \circ f \circ u^{-1})(u(p))\big)_{i=k+1,\dots,m,\ j=1,\dots,n} \\ \hline 0_{k\times k} & \mathbf{1}_{(m-k)\times(m-k)} \end{array} \right) \neq 0$$

because the columns are clearly linearly independent. Unraveling what this matrix is (namely, $D_k(x^\alpha \circ u^{-1})$), it follows by the Inverse Function Theorem that $x \circ u^{-1}$ is a diffeomorphism in a nbhd of $u(p)$. Hence, $x = (x \circ u^{-1}) \circ u$ is a coordinate system in some nbhd of p in M: it will be a homeomorphism and if (z, W) were any other coordinate system about p in M, then the transition map will likewise clearly be smooth. The cases of $\partial z/\partial x$ are taken care of by noting that the Inverse Function Theorem (really the chain rule, I think) gives us a description of $\partial z/\partial x$ as $(\partial x/\partial z)^{-1}$.

Now, if $q = x^{-1}(a^1, \dots, a^m)$, then $x(q) = (a^1, \dots, a^m)$ and therefore $x^i(q) = a^i$ and hence,

$$\begin{cases} y^\alpha \circ f(q) = a^\alpha & \alpha = 1, \dots, k, \\ u^r(q) = a^r & r = k+1, \dots, m, \end{cases}$$

so

$$\begin{aligned} y \circ f \circ x^{-1}(a^1, \dots, a^m) &= y \circ f(q) \qquad \text{for } q = x^{-1}(a^1, \dots, a^n) \\ &= (a^1, \dots, a^k, \underline{\quad}). \end{aligned}$$

This is **(a)**.

(b) As above, choose coordinate systems x and v so that $v \circ f \circ x^{-1}$ has the form obtained in **(a)**. Since $\operatorname{rank}(df_p) = k$ in a nbhd of p, the lower rectangle in the $n \times m$ matrix $\left(\dfrac{\partial(v^i \circ f)}{\partial x^j}\right)$ must vanish in a nbhd of p. That is, the lower (right) rectangle of

$$\left(\begin{array}{c|ccc} \mathbf{1}_{k\times k} & & 0_{k\times(m-k)} & \\ \hline & D_{k+1}\psi^{k+1} & \cdots & D_m\psi^{k+1} \\ \times & \vdots & \cdots & \vdots \\ & D_{k+1}\psi^n & \cdots & D_m\psi^n \end{array} \right)$$

Hence, $\psi^{k+1}, \ldots, \psi^n$ are independent of $a^{k+1}, \ldots, a^m$ on said nbhd. Since the ψ^{k+i} are smooth, this means that we can write

$$\psi^r(a) = \overline{\psi}^r(a^1, \ldots, a^k) \qquad r = k+1, \ldots, n.$$

To see this, "walk along coordinate lines," use the MVT and possibly regroup—we can always walk in an open, path-connected subset of $\mathbf{R}^n$ from one point to another along coordinate lines by using compactness and a metric d to put an ε-tube around a curve connecting the two points (I think... see for instance HW 5).

Define

$$\begin{aligned} y^\alpha &= v^\alpha & \alpha &= 1, \ldots, k \\ y^r &= v^r - \overline{\psi}^r \circ (v^1, \ldots, v^k) & r &= k+1, \ldots, n. \end{aligned}$$

Since

$$\begin{aligned} y \circ v^{-1}(b^1, \ldots, b^n) &= y(q) \qquad \text{for } v(q) = (b^1, \ldots, b^n) \\ &= (b^1, \ldots, b^k, b^{k+1} - \overline{\psi}^{k+1}(b^1, \ldots, b^k), \ldots, bb^m - \overline{\psi}^n(b^1, \ldots, b^k)), \end{aligned}$$

the $n \times n$ Jacobian matrix

$$\left(\frac{\partial y^i}{\partial v^j}\right) = \begin{pmatrix} \mathbf{1}_{k\times k} & 0_{k\times(n-k)} \\ \times & \mathbf{1}_{(n-k)\times(n-k)} \end{pmatrix}$$

has nonzero determinant, clearly, as the columns are linearly independent. Therefore y is a coordinate system in a nbhd of $f(p)$ by the same reasoning as in **(a)** (i.e., diffeomorphism, etc.). Moreover, from the previous centered equation,

$$\begin{aligned} y \circ f \circ x^{-1}(a^1, \ldots, a^m) &= y \circ v^{-1} \circ v \circ f \circ x^{-1}(a^1, \ldots, a^m) \\ &= y \circ v^{-1}(a^1, \ldots, a^k, \psi^{k+1}(a), \ldots, \psi^n(a)) \\ &= (a^1, \ldots, a^k, \psi^{k+1}(a) - \overline{\psi}^{k+1}(a^1, \ldots, a^k), \ldots, \psi^n(a) - \overline{\psi}^n(a^1, \ldots, a^k)) \\ &= (a^1, \ldots, a^k, 0, \ldots, 0), \end{aligned}$$

as desired.

(c) This is basically a special case of **(a)**. Except, when $k = m$, it is unnecessary to permute the y^i (i.e., the column space), only the u^i (i.e., the rows) need to be permuted in order that

$$\det\left(\frac{\partial(y^\alpha \circ f)}{\partial u^\beta}(p)\right) \neq 0 \qquad \alpha, \beta = 1, \ldots, k.$$

(d) Since the rank of f at any point must be $\leq m$, the rank of f equals m in some nbhd of p (i.e., full rank at a point implies full rank in a nbhd). It is easier to think of the case that $M = \mathbf{R}^m$ and $N = \mathbf{R}^n$ and find the coordinate system y when we are given $x = \mathrm{id}_{\mathbf{R}^m}$—since this result is local, we don't really lose anything. Then **(b)** yields coordinate systems φ on $\mathbf{R}^m$ and ψ for $\mathbf{R}^n$ such that

$$\psi \circ f \circ \varphi^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^m, 0, \ldots, 0).$$

Even without φ^{-1}, $\psi \circ f$ takes $\mathbf{R}^m$ into $\mathbf{R}^m \times \{0\} \subseteq \mathbf{R}^n$ except—as Spivak puts it—the points of $\mathbf{R}^m$ just get moved to the wrong place in $\mathbf{R}^m \times \{0\}$. This is corrected by defining a diffeomorphism $\lambda \colon \mathbf{R}^n \to \mathbf{R}^n$. In particular,

$$\lambda(b^1, \ldots, b^n) = (\varphi^{-1}(b^1, \ldots, b^m), b^{m+1}, \ldots, b^n).$$

Then, if $\varphi^{-1}(b^1, \ldots, b^m) = (a^1, \ldots, a^m)$, we have

$$\begin{aligned} \lambda \circ \psi \circ f(a^1, \ldots, a^n) &= \lambda \circ \psi \circ f \circ \varphi^{-1}(b^1, \ldots, b^n) \\ &= \lambda(b^1, \ldots, b^m, 0 \ldots, 0) \\ &= (\varphi^{-1}(b^1, \ldots, b^m), 0, \ldots, 0) \\ &= (a^1, \ldots, a^m, 0, \ldots, 0), \end{aligned}$$

which shows that $\lambda \circ \psi$ is the coordinate system y we sought (of course, since these are smooth manifolds, the diffeomorphism λ being compatible with the maximal atlas will obviously be a chart). If we are given a coordinate system x on $\mathbf{R}^m$ other than the identity, we define

$$\lambda(b^1, \ldots, b^n) = (x(\varphi^{-1}(b^1, \ldots, b^m), b^{m+1}, \ldots, b^n),$$

and is not hard to check that $y = \lambda \circ \psi$ is the coordinate system we sought.

(e) Choose $g \in G$ such that $gp = q$ in M for any two points $p, q \in M$. By transitivity, this g exists. Since $g \cdot f = f(g \cdot -)$ (equivariance) TFDC:

$$\begin{array}{ccc} T_pM & \xrightarrow{f_{*p}} & T_{f(p)}N \\ \downarrow g_* & & \downarrow g_* \\ T_qM & \xrightarrow[f_{*q}]{} & T_{f(q)}N \end{array}$$

with the linear maps isomorphisms. Hence, f must have constant rank. □

Corollary 5

Suppose $f \colon M^m \to N^n$ has full rank at $p \in M$ and suppose that M and N have corners.

(a) Suppose $n \leq m$. For any coordinate system (y, V) about $f(p)$ (say a k-corner chart) and any coordinate system (x, U) about p and any smooth extension of $i_{n,k} \circ y \circ f \circ x^{-1}$ to a smooth function defined on an open subset of $\mathbf{R}^m$, there is a coordinate system (z, W) of $\mathbf{R}^m$ about $x(p)$ with

$$i_{n,k} \circ y \circ f \circ x^{-1} \circ z^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^n).$$

(b) Suppose $m \leq n$. For any coordinate system (x, U) about p, any coordinate system (y, V) about $f(p)$ (say a k-corner chart) and any smooth extension of $i_{n,k} \circ y \circ f \circ x^{-1}$ there is a coordinate system (z, W) about $(i_{n,k} \circ y \circ f)(p)$

with
$$z \circ i_{n,k} \circ y \circ f \circ x^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^m, 0, \ldots, 0).$$

Remark. In practice, it is convenient to drop the standard embeddings $i_{k,\ell}$ from these expressions.

Proof. Since the condition of full rank is an open condition (since the rank function is a *lower semicontinuous* function), any smooth extension of $y \circ f \circ x^{-1}$ to a function from an open subset of $\mathbf{R}^m$ into $\mathbf{R}^n$ has full rank in a sufficiently small nbhd of the original domain. We will use this in the short argument below.

(a) For *any* charts y and x, by definition of smoothness, we may suppose $y \circ f \circ x^{-1}$ is defined on an open nbhd $U \subset \mathbf{R}^m$ into $\mathbf{R}^n$ and, furthermore, since max rank is an open condition, we may suppose that f has max rank on this extension and then apply **(c)** of the constant rank theorem.

(b) This argument is entirely analogous. □

Submanifolds

Warning. The following definition is wordy and seemingly difficult to parse but the basic idea is completely tractable and that is how one should remember it. There is an easier way to phrase this but somehow I thought writing up this way would be slicker; that was dumb. We will give the idea immediately after the definition.

Definition 23 (Submanifold)

Let M be an m-dimensional manifold with corner or boundary. A subset $N \subset M$ is a *submanifold* of dimension n or an *n-dimensional submanifold* of M if the following holds.

For each point $q \in N$ there is a chart $x\colon U \to \mathbf{R}^{m-k} \times \mathbf{R}^k_{\geq 0}$ of M about q (note that necessarily $k \geq \operatorname{depth}_M(q)$ by smooth invariance of corner points) such that for each $p \in i_{m,k}(x(U \cap N))$, there is a chart (φ_p, V_p) of $\mathbf{R}^m$ about p such that for some $0 \leq \ell \leq n$,

$$V_p \cap (i_{m,k} \circ x)(U \cap N) = \varphi_p^{-1}(\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^{\ell}_{\geq 0})$$

or, equivalently,

$$\varphi_p(V_p \cap i_{m,k}(x(U \cap N))) = \varphi_p(V_p) \cap (\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^{\ell}_{\geq 0}).$$

In other words, φ_p sends $V_p \cap i_{m,k}(x(U \cap N))$ homeomorphically onto its image in $\mathbf{0}_{m-n} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^{\ell}_{\geq 0} \cong \mathbf{R}^{n-\ell} \times \mathbf{R}^{\ell}_{\geq 0}$.

Lemma 7

In the above definition, one can replace the set $\mathbf{0}_{m-n} \times \mathbf{R}^n_{\ell}$ by any permutation of the factors of the product.

Proof. Permute the components functions of φ in the definition—this permutation is a diffeomorphism. □

Remark. The idea this definition captures is relatively simple. A submanifold should be a subset that sits nicely in charts of the original manifold. This condition is too restrictive when we do not map into a full Euclidean space since we haven't allowed ourselves the room to massage a subset into a locally nice form.

Thus, the idea here is that a submanifold of a manifold with corners is a subset which can be "straightened out" locally *after* embedding the model space $\mathbf{R}^n_k$ in $\mathbf{R}^n$. Thus, in some sense, this condition is no different from the one that is encountered for manifold without boundary.

In the following definition, ∂ means the generalized boundary, as usual.

Definition 24

A ***neat submanifold*** of a manifold-with-corners N^n is a submanifold M^m of N, in the sense of being immersed and topologically embedded, such that the additoinal following data hold.

(a) $(\partial N) \cap M = \partial M$;

(b) $(\partial N) \cap \overline{M} = (\partial N) \cap M$;

(c) For every point $p \in \partial M$, $\operatorname{depth}_M(p) = \operatorname{depth}_N(p)$ and there is a (corner) chart (x, U) of N about p such that $x^{-1}(\mathbf{0} \times \mathbf{R}^{m-\operatorname{depth}_N(p)} \times \mathbf{R}^{\operatorname{depth}_N(p)}_{\geq 0}) = U \cap M$.

Remarks.

(a) **(b)** is an item of convenience in the sense that it's possible only items **(b)** and **(c)** may matter some application. For tubular neighborhoods of codimension 0 manifolds, however, **(b)** is essential, as we remark below.

(b) Sometimes people require a neat submanifold to be in addition a closed submanifold (i.e., a closed subset as well) instead of the weaker condition that $\partial M \cap \overline{N} = \partial M \cap N$. The reason why is that we may want to throw away pathological examples like $M = \{(0, y) \in \mathbf{R} \times \mathbf{R}_{\geq 0} : y > 0\}$ and $N = \mathbf{H}^2 = \mathbf{R} \times \mathbf{R}_{\geq 0}$ because M *will not* admit a tubular neighborhood!

(c) In the case of a manifold with boundary but no corners, the idea is that a neat submanifold is a submanifold that meats the boundary transversely.

(d) Observe that when $\partial N = \varnothing$, this recovers the definition of submanifold we used previously when we only discussed manifolds without boundary. The only difference is that we previously asked that it sit nicely in the first m-coordinates—we have to modify this to make notation easier.

(e) One essential difference between a neat submanifold and an ordinary submanifold is that we require the submanifold be able to be straightened out *natively* in the ambient manifold M, as opposed to straightening it out in the codomain $\mathbf{R}^n$ of some chart for M.

(f) The condition that $\operatorname{depth}_N(p) = \operatorname{depth}_M(p)$ is superfluous if we restrict ourselves only to manifolds with or without boundary. Otherwise, this guarantees that we avoid something like $M = \{(t,t,t) : t \geq 0\} \subset \mathbf{R}^3_{\geq 0}$, where M meets ∂N at a depth 3 corner point but the corresponding point in M has depth 1 (i.e., a boundary point).

Observation. For manifolds without boundary, this definition recovers the usual one since the composite of two diffeomorphisms is a diffeomorphism and so the two charts at play compose to give a single chart for the smooth structure.

Exercise 32

When working with manifolds with boundary and no corners, the condition of being a neat submanifold in the sense of satisfying **(a)** of the definition above is precisely the same as the assertion $\partial N \pitchfork M$. *[Hint: See Definition 45.]*

Example 4 (Kissing the Disk)

Let $M \cong D^2$ be the unit disk with boundary in $\mathbf{R}^2$ centered at $(x,y) = (0,1)$ and let N be the image of $(-1/2, 1/2)$ of the curve $t \mapsto (t, t^2)$. For the moment, let us forget that $N \subset \mathbf{R}^2$ and $M \subset \mathbf{R}^2$.

One can check that $N \subset M$ and that N meets ∂M tangentially at the single point $(0,0)$. We claim there is no chart (x, U) of M about $(0,0)$ such that $x(U \cap N) = x(U) \cap \mathbf{R} \times \{a\}$ in $\mathbf{R}^2_{\geq 0}$ for any $a \geq 0$. This is because, by smooth invariance of the boundary, boundary points must be sent to boundary points, so any such chart of M sends $(0,0) \mapsto \partial \mathbf{R}^2_{\geq 0}$ and similarly every other point of N in this chart must be mapped to an interior point. Moreover, since N meets the boundary of M tangentially, we are precluded from straightening N out as $\{a\} \times \mathbf{R}_{\geq 0}$.

Now let us embed this picture in $\mathbf{R}^2$ by remembering that $N \subset \mathbf{R}^2$ and $M \subset \mathbf{R}^2$. We can now imagine a chart of $\mathbf{R}^2$ that "unfurls" the boundary of the disk locally near $(0,0)$ and so sends N near $(0,0)$ onto $\mathbf{R} \times \{0\}$. Here are some words about this. The desired chart of $\mathbf{R}^2$ can be produced by sending $(x,y) \mapsto (x, y - x^2)$. This is certainly smooth and it is bijective since $(x, y - x^2) = (x_0, y_0 - x_0^2)$ if and only if $x = x_0$ and hence $y = y_0$ (from the equation $y - x^2 = y_0 - x^2$). This is invertible because the Jacobian of $(x,y) \mapsto (x, y - x^2)$ is $\begin{pmatrix} 1 & 0 \\ -2x & 1 \end{pmatrix}$ with determinant $1 \neq 0$ and so this is a bijective, smooth, locally invertible function and so it is a diffeomorphism. This sends x^2 to the line $y = 0$.

Definition 25 (Submanifold Chart)

Let M be an m-dimensional manifold with corner or boundary and let $N \subset M$ be a subset which is an n-dimensional submanifold. Given a chart $x\colon U \to \mathbf{R}^{m-k} \times \mathbf{R}^k_{\geq 0}$ such that $U \cap N \neq \varnothing$ and for which there exists a chart (φ, V) of $\mathbf{R}^m$ satisfying $V \cap (i_{m,k} \circ x)(U \cap N) = \varphi^{-1}(\mathbf{0}_{m-n} \times \mathbf{R}^n_\ell)$ as above, we say that $\varphi \circ i_{m,k} \circ x$ is a *submanifold chart* for N. As usual, we will think of the submanifold chart $\varphi \circ i_{m,k} \circ x$ as a smooth function onto an open subset of some $\mathbf{R}^n_\ell$.

Remark. This is guaranteed to exist when N is a submanifold by restricting the chart x to the open set $U \cap x^{-1}(V_p)$.

Theorem 11

Suppose $N \subset M$ is an n-dimensional submanifold with corners, where $\dim M = m$. Then N can be given the structure of a smooth manifold with corners determined by collection of submanifold charts and this makes $N \hookrightarrow M$ a smooth embedding. In particular, the corner points of N are well-defined.

Conversely, any smooth embedding $i\colon N \hookrightarrow M$ has submanifold charts in this way with the smooth structure on N determined by them and, hence, the smooth structure on N is the unique one for which the topological embedding $N \hookrightarrow M$ is an immersion. In other words, $i(N)$ is a submanifold of M and $N \to i(N)$ is a diffeomorphism.

Warning. This proof is made more difficult because of how I have defined charts. This is unfortunate. It would be easier to have slightly different (yet equivalent) definitions to make this look less notationally hideous. It is also probably easier to understand some of the arguments below if we reduce to working with model spaces.

Proof. ($\Rightarrow$) Before proceeding, we should point out that the property of being Hausdorff and second-countable are all inherited by subspaces.

The smooth structure on N is obtained by giving it the atlas (extended to a maximal atlas as usual) consisting of submanifold charts for N. $(\varphi i_{m,k} x, (i_{m,k} x)^{-1}(V) \cap U \cap N)$. To see smoothness of transitions, let us write

$$(\varphi' i_{m,k'} y)(\varphi i_{m,k} x)^{-1}) = \varphi' i_{m,k'} y x^{-1} i_{m,k}^{-1} \varphi^{-1}$$

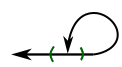
where we are now required to show that smoothness of φ^{-1} and $i_{m,k}^{-1}$ makes sense in this context. Let us consider their composite. Smoothness of $i_{m,k}^{-1}\varphi^{-1}$ means that there is a smooth extension to a function onto an open subset of $\mathbf{R}^m$, by definition. Recalling that φ is a chart of $\mathbf{R}^m$, it is clear that the smooth extension of this composite is simply φ^{-1} on its full domain. This shows, additionally, that the corners of N are well-defined. To see this, let φ and ψ be two of the charts as above. Then smoothness of $\varphi \circ \psi^{-1}$ means that $\operatorname{depth}(\psi(p)) = \operatorname{depth}(\varphi(p))$ by smooth invariance of corner points.

We should like, additionally, for N to be paracompact in the subspace topology. This follows since manifolds are ***hereditarily paracompact***. We argue this in a remark below the end of this proof. We could also appeal to the fact that every manifold is metrizable and every metric space is paracompact—since subspaces of metric spaces are metric spaces this is enough.

($\Leftarrow$) Now suppose N is a manifold with corners and $i\colon N \to M$ is a smooth embedding. Let $q \in N$ and pick a coordinate system (x, U) about q and a coordinate system (y, V) about $i(q)$ and consider the composite $y \circ i \circ x^{-1}$, which is smooth. By shrinking U and shifting things as necessary, we may suppose this is a map $x(U) \to y(i(U)) \subset \mathbf{R}^{m-k} \times \mathbf{R}^k_{\geq 0}$ and where $x(U) \subset \mathbf{R}^{n-\ell} \times \mathbf{R}^{\ell}_{\geq 0}$. In other words, WLOG we henceforth suppose $x(U) \subset V$.

Since the composite $i_{m,k} \circ y \circ i \circ x^{-1}$ is smooth, we know it extends to a function on an open subspace of $\mathbf{R}^n$ and since it is full rank, which is an open condition, we may suppose that the function has full rank on this open subspace. By **(d)** of the constant rank theorem, it follows that there is a chart (z, V_p) about each $p \in i_{m,k}(y(i(U)))$ in $\mathbf{R}^m$ such that $z_p \circ i_{m,k} \circ y \circ i \circ x^{-1}(a^1, \ldots, a^n) = (0, \ldots, 0, a^1, \ldots, a^n) \in \mathbf{R}^m$ (recall that we're being fiddly with the way coordinates go so WLOG we make them go this way). This means that $z_p \circ i_{m,k} \circ y$ *almost* constitutes a submanifold chart for $i(N)$ (after intersecting the domain with $i(N)$). It remains to show that when $z_p \circ i_{m,k} \circ y$, when restricted to $V \cap N$, or perhaps some $V' \cap N$ where $V' \subset V$ is open, has the desired form. This is where it is important that i be a topological embedding. Since i is an embedding, $i(U)$ is an open subspace of $V \cap N$, and so by definition of the subspace topology there is some W such that $W \cap N = i(U)$—we may suppose $W \subset V$ by the obvious modification and thus for the chart (W, y) we have what we want—$z_p \circ i_{m,k} \circ y$ has the right form and is a submanifold chart.

Now we consider uniqueness of the smooth structure. Let $i\colon N \to i(N)$ be a smooth embedding. Recall that the collection of all submanifold charts determines a subbase for the subspace topology on $i(N)$ and likewise determine the submanifold smooth structure on $i(N)$. We've just shown in one direction that these charts are smoothly compatible with N—namely, we just showed that $i\colon N \to i(N)$ is smooth with the charts. Now let us consider the other way around $i^{-1}\colon i(N) \to N$, which certainly exists since i is a topological embedding and so homeomorphism onto its image. This will be smooth if we can show that $x \circ i^{-1} \circ (z_p \circ i_{m,k} \circ y)^{-1}$ is smooth. This is the part where i being a topological embedding is important—we need to throw away the possibility of the immersed line $j\colon \mathbf{R} \to \mathbf{R}^2$ at right, where the map j^{-1} back to $\mathbf{R}$ from the interval indicated will necessarily discontinuous in the subspace topology. Just as before, there is some $W \subset V$ such that $W \cap N = i(U) \subset V$. Hence, for the shrunken chart (W, y), we know that $x \circ i^{-1} \circ (z_p \circ i_{m,k} \circ y)^{-1}$ has the form $(0, \ldots, 0, a^1, \ldots, a^n) \mapsto (a^1, \ldots, a^n)$ which is obviously smooth—hence, i^{-1} is smooth and therefore $i\colon N \to i(N)$ is a diffeomorphism. □



Remark. Thus, i being a topological embedding lets us exclude the possibility that some disparate piece of N intersects every open nbhd in M of $V \cap N$.

Remark (Hereditarily Paracompact)**.** All manifolds are hereditarily paracompact. According to the Wikipedia article for paracompactness, this is equivalent to having all open subspaces being paracompact. In fact, by the Whitney embedding theorem, it suffices to show that all subspaces of $\mathbf{R}^n$ are paracompact, so let $U \subset \mathbf{R}^n$ be open. In any case, there's a shortcut to this result. Any locally compact second-countable Hausdorff space is paracompact, say by **Theorem 2.6** here. The property of being second-countable and Hausdorff is hereditary. Clearly any open subspace of $\mathbf{R}^n$ is locally compact since ε-balls are precompact. Similarly for any model space $\mathbf{R}^n_k$.

Corollary 6

Fix $N \subset M$ a submanifold. A submanifold chart $y = \varphi \circ i_{m,k} \circ x$ considered as a smooth function defined on an open nbhd U of M is a diffeomorphism onto its image—in particular, $y(U)$ is a submanifold of $\mathbf{R}^n$.

Proof. The map is a smooth embedding and so by the above theorem determines a smooth structure on its image. The inverse map restricted to its image is certainly a homeomorphism and it is smooth as the map $x \circ x^{-1} \circ i^{-1}_{m,k} \circ \varphi^{-1} = i^{-1}_{m,k} \circ \varphi^{-1}$ defined on a subset of Euclidean space has smooth extension given simply by φ^{-1}. □

Theorem 12 (Universal Property of Submanifolds)

Let $S \subset N$ be a submanifold and let $i\colon S \to N$ be the inclusion. A map $f\colon M \to S$ is smooth **iff** $i \circ f$ is smooth. Say $\dim M = m$, $\dim N = n$ and $\dim S = s$.

Proof. ($\Rightarrow$) Easy since $i\colon S \to N$ is smooth. ($\Leftarrow$) Suppose $i \circ f$ is smooth. By definition of a submanifold, about each point in S, there is a nbhd V and a diffeomorphism onto its image $y\colon V \to y(V) \subset \mathbf{R}^n$, such that $y(V \cap N) = y(V) \cap (0 \times \mathbf{R}^{s-\ell} \times \mathbf{R}^{\ell}_{\geq 0})$—that is, a submanifold chart. We have concluded y is a diffeomorphism onto its image by the above corollary. Thus, in coordinates, $y \circ i \circ f \circ x^{-1}$ looks like a map onto these last s coordinates and is assumed smooth. But this has the same form as $y|\, V \cap N \circ f \circ x^{-1}$ using the submanifold chart constructed as above and, hence, $y|\, V \cap N \circ f \circ x^{-1}$ is smooth. Hence, f is smooth. □

When all manifolds in question have no corners, there is a nice criterion for detecting neat submanifolds. In the following theorem, Kosinski demands the submanifold be closed. We will not demand this. We might ask instead that $\overline{N} \cap \partial M = N \cap \partial M$ but we will not ask for this either. We will understand neatness in a slightly looser sense than our definition.

Theorem 13 (K. II.2.3)

Let $m \geq 1$ and let M be an n-manifold-with-boundary. A subset $N \subset M$ can be given the structure of a neat m-dimensional submanifold **iff** about every point $q \in N$ there is a chart (x, U) of M for which either of the two following conditions hold:

(a) If $q \in \operatorname{Int} N$, then (x, U) is a submanifold chart, and if $q \in N \cap \partial M$, then $x^{-1}(\{\mathbf{0}\} \times \mathbf{H}^n) = U \cap N$.

(b) There is a submersion $\sigma\colon U \to \mathbf{R}^{n-m}$ which is also a submersion on its restriction to $U \cap \partial M$, such that $\sigma(q) = 0$ and $\sigma^{-1}(0) = U \cap N$. (This is like a projection off of the slices containing N).

These conditions are moreover equivalent.

The resulting smooth structure is unique up to diffeomorphism. In particular, we do not need to assume N is closed or satisfies condition **(b)** of the neat submanifold definition to make this work, if we are willing to work with a particularly weak notion of a neat submanifold.

Remark. As before, to make things go as we want, we adopt the convention that submanifold charts are in the last n coordinates and 0 in the first coordinates instead of the converse. We assume $m \geq 1$ above because these conditions will not make sense for $m = 0$ (when $m = 0$ there is really nothing to prove).

Proof. If $i\colon N \subset M$ is a neat m-submanifold, then **(a)** clearly holds. Now to see that **(a)** implies a neat submanifold, all we have to do is show that the collection of neat submanifold charts **(a)** affords us gives us a differentiable structure on N. This is basically easy since, for instance, everything will come from restrictions. The (maybe?) tricky part is to see that $i\colon N \subset M$ is an immersion—note that i has full rank and hence, locally, in coordinates, any smooth extension to an open subset of $\mathbf{R}^m$ still has full rank since the condition of being full rank is an open condition (i.e., rank is lower semi-continuous) and so it follows that the usual local coordinate formulas still hold in this setting. Hence, namely, for the extension $\widehat{yix}^{-1}\colon \mathbf{R}^m \supset \widehat{U} \to \mathbf{H}^n \subset \mathbf{R}^n$—and **note** that we **must** interpret this as a function $\widehat{U} \to \mathbf{R}^n$ to obtain this extension—and so by **(c)** of the constant rank theorem, there are coordinates for which this looks like the standard inclusion $\mathbf{R}^m \to \mathbf{R}^n$ and thus i clearly is an immersion.

If $N \subset M$ is neat, then to see that **(b)** holds, note that at interior points $q \in \operatorname{Int} N$, we can take a submanifold chart (x, U) let σ be the composite $U \xrightarrow{x} \mathbf{R}^n \to \mathbf{R}^{n-m}$ the last map killing the first m coordinates thereby projecting onto the slice of N contained in this chart. This is obviously a submersion with $\sigma^{-1}(\sigma(q)) = U \cap M$. In the case that U is a boundary chart for $q \in N \cap \partial M$ which is also a neat submanifold chart, then virtually the same analysis goes through except we need something new for the boundary submersion, but this follows by virtually the same analysis as well.

Now suppose **(b)** holds. Consider any boundary chart (x, U) about $q \in N \cap \partial M$ so that $x(q) \in \partial \mathbf{H}^n$. Then we know that σx^{-1} and $\sigma x^{-1}\big|\, \partial \mathbf{H}^n$ are submersions since $m \geq 1$. Let $V \subset \mathbf{R}^n$ where V is a nbhd of $x(q)$ for which σx^{-1} extends and suppose we have extended it there so that $\sigma x^{-1}\big|\, V$ is a submersion. WLOG take V to be an open ball about $x(q)$ of constant radius, perhaps by suitably shrinking. We may therefore suppose there is a diffeomorphism $(V, V \cap \mathbf{H}^n) \cong (\mathbf{R}^n, \mathbf{H}^n)$. Since full rank is an open condition we may suppose this extension on a nbhd of $x(q)$ remains full rank and by abuse of notation we call it the same thing.

Since $m \geq 1$, $n - m \leq n - 1$ and $\dim \partial \mathbf{H}^{n-1} = n - 1$ so since by assumption the restriction of σ to $U \cap \partial M$ is a submersion, $\sigma x^{-1}\big|\, \partial \mathbf{H}^n$ is a submersion. A close inspection of **(c)** of the constant rank theorem (using the standard coordinate system as our starting chart) shows that there is a coordinate system y about $x(q)$ for which $\sigma x^{-1} y^{-1}$ is a projection onto the *first* $n - m$ coordinates and, moreover, y is a diffeomorphism of open nbhds of $0 \in \mathbf{R}^n$ mapping the upper half-plane into itself since, in particular, we may suppose $y^{n-m+1}, \ldots, y^n$ are the standard basis functions for $\mathbf{R}^n$ by close inspection of **(c)** of the constant rank theorem. This will allow us to construct a chart.

Let $g = y^{-1}x$ so that $g(U) \subset \mathbf{H}^n, \sigma g^{-1}(0 \times \mathbf{H}^m) = 0$ and $g\sigma^{-1}(0) \subset 0 \times \mathbf{H}^m$. Then (g, U) is a smoothly compatible boundary chart since we took $(V, V \cap \mathbf{H}^n) \cong (\mathbf{R}^n, \mathbf{H}^n)$ and since $g\sigma^{-1}(\mathbf{0}) = g(U \cap M) \subset \mathbf{0} \times \mathbf{H}^m, \sigma^{-1}(\mathbf{0}) \subset x^{-1}(\mathbf{0} \times \mathbf{H}^m)$ (here we have used that $y\colon (\mathbf{R}^n, \mathbf{H}^n) \cong (\mathbf{R}^n, \mathbf{H}^n)$ has as its last m coordinates the standard basis functions). The reverse inclusion $x^{-1}(\mathbf{H}^m) \subset \sigma^{-1}(\mathbf{0})$ is obvious from our assumption. Hence, **(a)** holds. The case of $p \notin N \cap \partial M$ is similar but easier.

This concludes the proof that **(a)** and **(b)** are equivalent. We now turn to considering the smooth structure and its uniqueness.

Suppose **(a)** holds. Cover N by charts satisfying **(a)**. These are all submanifold charts, as we know, and it is easy to see this defines a smooth atlas on N so that $\partial N \subset \partial M$. So we must show that that structure makes it topologically embedded and immersed—the former is obvious and the latter follows because, in coordinates, the inclusion of N into M looks like $\mathbf{0} \times \mathbf{R}^m \subset \mathbf{R}^n$ or $\mathbf{0} \times \mathbf{H}^m \subset \mathbf{H}^n$. Uniqueness of the smooth structure goes as one expects it to go. □

A.3 Whitney Theorems

Remark. All of the following material is adapted from Lee's *Introduction to Smooth Manifolds*.

Lemma 8 (Lee, 2.26)

Let M be a manifold with corners, $A \subset M$ closed, and $f\colon A \to \mathbf{R}^k$ smooth.[a] For any open nbhd U of A, there is a smooth function $\widetilde{f}\colon M \to \mathbf{R}^k$ such that $\widetilde{f}\big| A = f$ and $\operatorname{supp} \widetilde{f} \subset U$.

[a] Recall that this means that there is a smooth extension of f in an open nbhd of each point $p \in A$.

Proof. This is a partition of unity argument. □

Warning. If A is not closed, then we have no control over the boundary behavior and this will therefore fail in general. For example, consider $1/x$ defined on the set $(0, 1] \subset \mathbf{R}$—we cannot extend this at 0. However, it is still possible to smoothly extend functions defined on a subset A to an open subset containing A.

Theorem 14 (Whitney Approximation Theorem for Functions)

Let M be a manifold with corners and $F\colon M \to \mathbf{R}^k$ continuous. Given any positive continuous function $\delta\colon M \to \mathbf{R}$, there is a smooth function $\widetilde{F}\colon M \to \mathbf{R}^k$ that is δ-*close* to F—that is $\left|F(x) - \widetilde{F}(x)\right| < \delta(x)$ for all $x \in M$. If F is smooth on a closed subset $A \subset M$, then $\widetilde{F}$ can be chosen such that $\widetilde{F}\big| A = F| A$.

Proof. Partition of unity argument along with the lemma above. □

Corollary 7

If M is a manifold with corners and $\delta\colon M \to \mathbf{R}$ a continuous function, then there is a smooth positive function $\varepsilon\colon M \to \mathbf{R}$ with $0 < \varepsilon(x) < \delta(x)$ for all $x \in M$.

Proof. Apply Whitney approximation to construct a smooth $e\colon M \to \mathbf{R}$ such that $\left|e(x) - \frac{1}{2}\delta(x)\right| < \frac{1}{2}\delta(x)$. □

Remark. This gives an easy way to construct the smooth function used in the proof of the collar nbhd theorem for smooth manifolds.

Theorem 15 (Whitney Approximation Theorem)

Let N be a manifold with corners, M is a manifold *without* boundary and let $F\colon N \to M$ be continuous. Then F is homotopic to a smooth map $\widetilde{F}\colon N \to M$. If F is already smooth on a closed subset $A \subset N$, then the homotopy can be taken relative to A (this means that the homotopy is fixed on A).

Remark. It will turn out that dropping the relative homotopy assumption makes this go through for manifolds M with boundary, but perhaps not necessarily with corners.

Corollary 8

Suppose M has no boundary and we are given a homotopy $H\colon N \times I \to M$ between smooth maps $f, g\colon N \to M$. Then there is a *smooth* homotopy $\widetilde{H}\colon N \times I \to M$ between f and g such that H and $\widetilde{H}$ are themselves homotopic rel $N \times \partial I$.

Proof. Let $A = N \times \partial I$ be a closed subset and note that H is already smooth on it. The Whitney approximation theorem tells us that there exists a smooth homotopy $\widetilde{H}$ satisfying the properties we want. □

Remark. In particular, this shows that for a manifold M with empty boundary, the homotopy groups of M may defined in the *smooth* category by taking $A = * \times I$ where $* \in S^n$ is a chosen basepoint.

Corollary 9

If N is a manifold with corners, M has no boundary, $A \subset N$ is closed and $f\colon A \to M$ is smooth, then f has a smooth extension to N **iff** it has a continuous extension to N.

Proof. Whitney approximation! □

Here's an example of what goes wrong when M has boundary and we insist the homotopy be fixed on a closed subset.

Example 5 (Lee, 6-7)

Let $F\colon \mathbf{R} \to \mathbf{H}^2$ by $t \mapsto (t, |t|)$, $A = [0, \infty)$. Then no such homotopy fixed on A exists.

To get this to work for manifolds with boundary, but without corners, we need to construct a smooth "flowing in" map $R\colon M \to \overbrace{\operatorname{Int} M \subset M}^{\iota}$ and a smooth homotopy $H\colon M \times I \to M$ satisfying the following properties: H is a smooth homotopy from $\iota \circ R$ to id_M and the restriction of H to $\operatorname{Int} M \times I$ gives a smooth homotopy from $R|\operatorname{Int} M$ to $\mathrm{id}_{\operatorname{Int} M}$.

We shall prove the following later, after we construct collars.

Theorem 16 (Lee 9.26, Flow-in)

Let M be a manifold with boundary and possibly corners. Let $\iota\colon \operatorname{Int} M \hookrightarrow M$ be the inclusion.

(a) Suppose M has no corners. There exists a collar nbhd $C\colon \mathbf{R}_{\geq 0} \times \partial M \hookrightarrow M$ such that for any $a \in (0, \infty)$, the subset $M(a) \stackrel{\text{def}}{=} M \setminus C([0, a) \times \partial M)$ is a properly embedded submanifold of M and, furthermore, for any such a there exists a proper[a] smooth embedding $R_a\colon M \hookrightarrow \operatorname{Int} M$ with image $M(a)$ such that the composites $\iota \circ R$ and $R \circ \iota$ are smoothly homotopic to the identity maps—in particular, ι is a homotopy equivalence. In particular, we shall construct a collar of M all of whose restrictions to intervals $[0, a]$ are closed collars.

(b) If M has corners, there exists a proper smooth embedding $R\colon M \to \operatorname{Int} M$ such that the composites $\iota \circ R$ and $R \circ \iota$ are homotopic to the identity maps—in particular, ι is a homotopy equivalence.

(c) In each case, the image of M in its interior is a closed subset (because the map is proper) and when M has no corners, there is a strong isotopy between the identity map and the map R_a.

[a] This means that the preimage of compact sets is compact.

Remark. An injective immersion that is proper is an embedding with closed image. This is a consequence of a theorem in the chapter **Point-Set Results**.

Theorem 17 (Whitney Approximation Theorem)

Let N be a manifold with corners, M a manifold *with* boundary or corners and let $F\colon N \to M$ be continuous. Then F is homotopic to a smooth map $\widetilde{F}\colon N \to M$.

Proof. With the flow-in theorem in hand, we see that

$$R \circ F \qquad : \qquad N \xrightarrow{F} M \xrightarrow{R} \operatorname{Int} M$$

is smoothly homotopic to a map G by the standard Whitney approximation theorem. Let $\iota\colon \operatorname{Int} M \to M$ be the inclusion. Then the flow back homotopy gives a homotopy $\iota \circ G \simeq \iota \circ R \circ F \simeq F$, so $\iota \circ G\colon N \to M$ is a smooth map homotopic to F. □

Theorem 18

Let M and N be manifold with corners. If $F, G\colon N \to M$ are homotopic, then they are smoothly homotopic.

Proof. Let R be the flow-in constructed above. Then $R \circ G$ and $R \circ F$ are homotopic smooth maps from N into $\operatorname{Int} M$, so they are smoothly homotopic. Thus we have smooth homotopies $F \simeq \iota RF \simeq \iota RG \simeq G$ as desired. Obviously smooth homotopy is an equivalence relation so we're good. □

A.4 Collars and Boundaries

Lemma 9

Let M be a manifold-with-boundary. Then TM is a manifold-with-boundary and, in particular, $\partial TM = T\partial M$.

Proof. This is essentially the vector bundle construction lemma, Lee 10.6, and is not hard to see directly. The bundle charts are the same, they are still $(x^1, \ldots, x^n, \partial_1, \ldots, \partial_n)$ and so we see we only run into issues when the chart x in question is a boundary chart. □

Lemma 10

Let M be a manifold-with-boundary. Then, in coordinates, for every $p \in \partial M$, $T_p\partial M \subset T_pM$ consists of the vectors with last coordinate 0.

Proof. This is easiest to see with curves. □

Definition 26

Let M be a manifold-with-boundary and $p \in \partial M$. It is easy to see that one may still take $T_p M$ to be the vector space of derivations of germs of smooth functions. Moreover, $T_p M$ has a distinguished class of ***inward pointing*** vectors, defined as those vectors with a strictly positive last coordinate. This definition is invariant under choice of coordinates. One similarly defines ***outward pointing*** vectors.

Remark. We might be tempted to define $T_p M$ in terms of smooth curves, but this seems to require annoying modifications—we must allow ourselves to consider *smooth* curves with domain $(-\varepsilon, 0]$ and $[0, \varepsilon)$ (really just one by symmetry) to make sense of this. There is a geometric interpretation of inwards pointing vectors in terms of smooth curves.

Exercise 1. The above definitions are invariant under choice of coordinates and can be detected using curves (in the appropriate sense) and derivations.

Definition 27 (Collar)

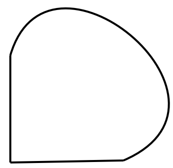
A ***collar*** of a manifold-with-boundary M is an embedding $i\colon \partial M \times [0,1) \to M$ (or equivalently an embedding $\partial M \times \mathbf{R}_{\geq 0} \hookrightarrow M$) such that $i|_{\partial M \times \{0\}}$ is the canonical inclusion of $\partial M \subset M$. In particular, a collar is a ***neat submanifold*** (see above for the definition). Say a ***closed collar*** is the restriction of the embedding of an open collar to $i\colon \partial M \times [0,a] \to M$ for some $a > 0$ with closed image. A closed collar always contains a collar, but we cannot guarantee that every collar may be restricted to a closed collar.

Example 6

Let $M = \mathbf{H}^2 \setminus \{\mathbf{0}\}$. Then the collar $\mathbf{R} \setminus \{0\} \times [0,1) \hookrightarrow M$ does not contain a closed collar for any a.

Remark. There is an evident way to fix this. We shall see that every manifold admits an open collar all of whose restrictions to closed intervals are closed collars.

Warning. While it might be tempting to try and define collars for manifolds with corners, we run into a serious issue with smoothness. Namely, consider the (filled) teardrop. This is a smooth manifold with corners of dimension 2. But its boundary could not possibly be a manifold with corners with its subspace topology, because it has a singularity! This is basically because, as remarked before, the boundary of a manifold with corners *does not* have a smooth structure unless there is no corner set. However, if we were content to work *outside* some category of smooth manifolds, then we strongly suspect that collars will exist in some modified sense and the same argument will work.



Example A.2. The manifold $\mathbf{H}^2 \setminus \{0\}$ never has a closed collar.

Proposition 2

A collar $i\colon [0,1) \times \partial M \hookrightarrow M$, is an open submanifold of M. A closed collar (if it exists) is a closed submanifold. In particular, they are open (resp. closed) maps.

Proof. The latter part is essentially immediate since the embedding is a closed codimension 0 submanifold by definition. We consider the first part.

The invariance of domain implies that any embedding between manifolds with empty boundary of the same dimension is an open map, since it amounts to giving an injective map from an open subspace of $\mathbf{R}^n$ into itself sends the subspace to another open subspace, and being an open map is a local property when the map in question is injective. Hence, on the interior of the collar $(0,1) \times \partial M$, at least, the map C an open map. We can cheat for points on the boundary. Fix a coordinate nbhd for the boundary of $[0,1) \times \partial M$. In coordinates, we might as well assume the map looks like an embedding $\mathbf{H}^n \supset U \to \mathbf{H}^n$. We can then extend this to a smooth map $\mathbf{R}^n \supset \widehat{U} \to \mathbf{R}^n$. Since collar map is an embedding of full rank, this is an open condition and so we may assume the extended map has full rank. This means that in a nbhd of p the map is a local embedding and therefore by invariance of domain an open map. But this means that its restriction to U is open by inspecting what the subspace topology does. □

Remark. It is important to note in the last step here that we are really showing that the restriction of the extension to the map between the relevant subspaces is an open map. In general, an embedding between $\mathbf{H}^n \to \mathbf{R}^n$ will not be an open or closed map, it will only be so onto its image, so we are really some extra information here.

Note that neatness is essentially automatic since the only points to worry about from the definition are the boundary points and we gave ourselves the entire boundary!

Remark. We will prove these always exist. First we need a few lemmas. We will go about this in the most natural way to prove it, at least I think. Another way to prove it is to use tubular neighborhoods by embedding the manifold in $\mathbf{R}^N$ for big enough N (here we simply mean an immersion and topological embedding). Kupers takes this approach in his differential topology lecture notes.

Say a vector field on a manifold-with-boundary or corners M is an *inward pointing vector field* if for all $p \in \partial M$, X_p points inward.

Lemma 11

Let M be a manifold-with-boundary or corners of dimension n. Then there exists an inward pointing vector field X on M.

Proof. This is a partition of unity argument where we stipulate that on a non-boundary coordinate patch U_α, $X_\alpha = \dfrac{\partial}{\partial x^n}$, and on a coordinate patch for a corner with order k, we set $X_\alpha = \sum_{i=n-k}^{n} \frac{\partial}{\partial x^i}$. Then we set $X = \sum_\alpha \rho_\alpha X_\alpha$. It is easy to see that X_p is inward pointing since only boundary charts intersect the boundary. □

The idea is to flow in along this vector field.

Remark. It is important to point out that the flow for an inward pointing vector field exists and is smooth. The proof is a variation upon the usual argument which we sketch below.

Theorem 19 (Collar Neighborhood Theorem)

Let M be a manifold-with-boundary of dimension n.

(a) M has a collar. In addition, for a collar $C\colon [0,1) \times \partial M \to M$, the complement of $C(a) = \mathrm{Im}(C|\,[0,a) \times \partial M)$ is closed. In particular, a collar is an open submanifold and the collar map is an open map.

(b) Suppose $N \subset M$ is a neat submanifold. Then we can find a collar for M that restricts to a collar for N.

(c) If ∂M is compact, then any collar of M and any $a > 0$ restricts to a compact and hence closed collar on the interval $[0,a]$.

Remark. We give two proofs. The first will be for **(a)** and the second for **(b)**, which implies **(a)**. The first I learned from some notes by Sander Kupers. The second may be found in Hirsch's book. For **(b)**, the idea is roughly that we can find a fat enough covering of N by neat submanifold charts and then cover the rest of M by charts that never meet ∂N. It is worth pointing out that *we do not* need to assume $\overline{N} \cap \partial M = \partial N$ and *we do not* need to assume N is closed for this argument to work.

Note that **(b)** does not follow from **(a)**. The problem is that an inward pointing vector for ∂M may not correspond to an inward pointing vector for ∂N in general.

We prove **(a)** and **(c)** together first.

Proof. **(c)** This follows from **(a)** by restricting a collar $\mathbf{R}_{\geq 0} \times \partial M \to M$ to $[0,1] \times \partial M \to M$, using the fact that $[0,1] \times \partial M$ is compact to show that the map is proper and therefore an embedding. See, for instance, **Exercise** 13.

(a) Let X be an inward pointing vector field on M and consider the ODE on M given by $\dot{\gamma} = X(\gamma)$ with initial condition $\gamma(o) = p \in M$. In coordinates, this locally has the form $y' = f(t, y(t))$ where $f(t, y(t)) = y(t)$ and this is Lipschitz continuous in the dummy variable $y(t)$ so that the Picard-Lindelöf theorem applies (and one can easily check that transitions preserve solutions). Kosinski I.6.3 shows that the flow exists and, because of the time tube argument for flows extending to a global flow, we know that in general the valid times for the flow may taper off to 0 unless the manifold is compact. So let A be the maximal flow domain about $M \times \{0\}$ in $M \times \mathbf{R}$, and let the flow be Φ. Let $\mathrm{U} = A \cap (\partial M \times \mathbf{R})$ and note that this is open in $\partial M \times \mathbf{R}_{\geq 0}$. Then for $(q, 0) \in \mathrm{U}$, $\Phi_{*,(q,0)}(\partial_i, r \cdot d/dt) = \partial_i + rX^n(q)$ and so clearly is an isomorphism between tangent spaces $T_{(q,0)}\mathrm{U} \to T_q M$, since we have arranged that $X^n \neq 0$ for any

$q \in \partial M$. We used the fact that $X(q)$ only has component in the inwards direction from the construction above.

Note that Φ maps the boundary $\partial(\partial M \times \mathbf{R}_{\geq 0}) = \partial M \times \{0\}$ to the boundary of M. Hence, by **(d)** of the constant rank theorem $\Phi|\,\mathsf{U}$ is a local diffeomorphism, we claim. Of course, one might rightly worry that the constant rank theorem does not apply because of the boundary and, in particular, the part about the map being a local diffeomorphism. To see that things work out, note that this certainly applies for $\Phi|\,(\mathsf{U} \cap (\partial M \times (0,\infty)))$. To see that this plays nicely with the boundary, it is enough to inspect what the corollary of the constant rank theorem says carefully.

Exercise 33

Verify this by appealing to **Corollary** 5. *[Hint: Choose a boundary chart x about $p \in \partial M$ and consider the charts $y = x$ on the target and $(x|\,(\mathbf{R}^{n-1} \times \{0\}) \times t)$ on the domain where $t = \mathrm{id}_{\mathbf{R}_{\geq 0}}$ is the identity chart. Then apply* ***(b)*** *of the corollary. What has to be true about the chart z in relation to $y = x$?]*

Observation. We can glue the local inverses together once we know that Φ is injective on an open subspace of U consisting of the union of the nbhds upon which Φ is invertible, showing that Φ is an embedding on that nbhd.

One way to see the existence of such a nbhd is to observe that the integral curves of a flow are unique and thus intersect only if they are the same integral curve. Thus, some thought using uniqueness of integral curves shows that the only possible issue is if two points on the boundary of ∂M are connected by an integral curve—this is impossible since our vector field points inwards everywhere, so no integral curves exists between two distinct points of ∂M. A more high-powered way to see the existence of such a nbhd is the tubular neighborhood trick.

Thus, we may also suppose WLOG that Φ is an embedding on U, perhaps by shrinking it first—note that U will always contain $\partial M \times \{0\}$. (It is clearly an embedding.) Suppose we have a smooth function $\varepsilon\colon \partial M \to (0,\infty)$ such that $(q, \varepsilon(q)) \in \mathsf{U}$ for the moment. Then $c\colon \partial M \times [0,1) \to M$ by $(p,t) \mapsto \Phi(p, t\varepsilon(p))$ is an embedding that is neat on $[0,1)$. It is certainly smooth because everything in sight is smooth and to show it is an embedding it suffices to show that $(p,t) \mapsto (p, t\varepsilon(p))$ is an embedding into $\mathsf{U} \subset \partial M \times \mathbf{R}_{\geq 0}$ since Φ is an embedding on U by hypothesis now. This function is also certainly smooth and injective. It has differential $(\mathrm{id}, ?)$ into $\partial M \times \mathbf{R}_{\geq 0}$ so it suffices to determine the differential of $(p,t) \mapsto t \cdot \varepsilon(q)$. In coordinates, the matrix for this will be $1 \times (n+1)$ or a row vector of length $n+1$ and it is clear that this will be (using the identity chart on the time part) $(t\partial_1\varepsilon \ \cdots \ t\partial_n\varepsilon \ \varepsilon(q))$. Since $\varepsilon(q) > 0$ for all q, this will always have full rank. Hence, the differential is componentwise $(\mathrm{id}, \text{full rank})$ and so is clearly an isomorphism. It therefore remains to construct ε.

The construction of ε can be done in an ad hoc fashion as a partition of unity argument in ∂M or by appealing to **Corollary A.2.4**. We proceed to do this with an ad hoc construction.

Construction. Note that every $q \in \partial M$ has a coordinate nbhd U such that $U \times [0, \varepsilon_U) \in \mathsf{U}$ where $\varepsilon_U > 0$. Pick an open cover of ∂M by charts $\{U_\alpha\}_{\alpha \in J}$ such that for each

$\alpha \in J$, there exists $u_\alpha > 0$ such that $\{q\} \times [0, u_\alpha] \subset U$ for all $q \in U_\alpha$. To see this exists, simply shrink everything as needed, using the fact that, locally (perhaps after extending things), flows exist for a uniform time parameter.

Exercise 34

Verify this. *[Hint: You will probably find Theorem I.5.2 from Spivak's first Differential Geometry volume about existence of flows useful.]*

WLOG we may suppose by paracompactness that $\{U_\alpha\}_{\alpha \in J}$ is locally finite.

Let I_α be the (finite) set of $\beta \in J$ such that $U_\alpha \cap U_\beta \neq \varnothing$Let $I_{\alpha,2}$ be the (finite) set of all $\gamma \in J$ for which there exists $\beta \in J$ such that $U_\alpha \cap U_\beta \neq \varnothing$ and $U_\beta \cap U_\gamma \neq \varnothing$ (i.e., "second-order" intersection). Note that $I_\alpha \subset I_{\alpha,2}$. Let $N_\alpha = \max\left\{\#(I_\beta) : \beta \in I_{\alpha,2}\right\}$.

Observation. For each $\beta \in I_\alpha$, $N_\beta \geq \#(I_\alpha)$ since $\alpha \in I_{\beta,2}$ and, in particular, $\alpha \in I_\beta$.

Set

$$t_\alpha = \min\left\{u_\beta / \max\left\{N_\alpha^2, N_\beta^2\right\} : \beta \in I_{\alpha,2}\right\}.$$

Running the partition of unity subordinate to $\{U_\alpha\}$, we put

$$\varepsilon = \sum \rho_\alpha t_\alpha.$$

For $q \in U_\alpha$, we now wish to show that for each $\gamma \in I_\alpha$, $\varepsilon(q) < u_\gamma$. Fix now $\gamma \in I_\alpha$. Then for each $\beta \in I_\alpha$, $t_\beta \leq u_\gamma / \max\left\{N_\beta^2, N_\gamma^2\right\}$. Then

$$\varepsilon(q) \leq \sum_{\beta \in I_\alpha} t_\beta \leq \sum_{\beta \in I_\alpha} u_\gamma / \max\left\{N_\beta^2, N_\gamma^2\right\} = \#(I_\alpha) u_\gamma \sum_{\beta \in I_\alpha} 1 / \max\left\{N_\beta^2, N_\gamma^2\right\} \leq \#(I_\alpha) u_\gamma / \#(I_\alpha) = u_\gamma$$

hence, $\varepsilon(q) \leq u_\beta$ for all $\beta \in I_\alpha$ so we've achieved our goal, ε is smooth into where this is an embedding.

□

Here's a slightly different and more terse proof for **(b)**.

(b). Cover $\partial N \subset \partial M$ by neat submanifold charts in M with image coordinate balls of radius 2, say $\{(z_i, V_i)\}_{i \in I}$. WLOG we may assume this collection is locally finite by paracompactness since manifolds are hereditarily paracompact. Let U be the union of the restriction of each neat submanifold chart (z_i, V_i) to the coordinate balls of radius 1—call the resulting chart (z_i, U_i). Let F be the union of the restrictions of the U_i to the closed coordinate balls of radius $3/4$ and note by local finiteness F is closed.

In the coordinates of the neat submanifold charts, the last coordinate points inward for *both* N and M. We must be prudent about how we extend this covering. Roughly, we need to preserve everything sufficiently closed to $p \in \partial N$. For this, we use F. Indeed, we just need to find nbhds separating p and F and this amounts to saying that a manifold is a regular space. Thus, we may find a sufficiently small boundary chart (x, V) about p such that $V \cap (U \cap \partial M) = \varnothing$.

Cover the rest of ∂M by such charts and then observe that $M \setminus F$ is open and cover it by charts contained in it. Now we construct a partition of unity subordinate to this open cover where we use the radius 1 charts constructed in the first paragraph. Let $X = \sum \rho_\alpha X_\alpha$ where X_α is, in coordinates, $\frac{\partial}{\partial x^m}$ the last coordinate. Then for any $p \in \partial N$, X_p is inward pointing, being a sum of inward pointing vectors for N and M and similarly for any $p \in \partial M$. This is a consequence of the above construction.

Let $W_1 \subset M \times \mathbf{R}_{\geq 0}$ be the open subset on which the flow of X is defined, call the flow Φ, and let $W \subset \partial M \times \mathbf{R}_{\geq 0}$ be $W_1 \cap \partial M \times \mathbf{R}_{\geq 0}$. Then since W_1 is open, W is open in $\partial M \times \mathbf{R}_{\geq 0}$. We must shrink W to yet another open subset to make things work out. Begin by noting that for $q \in \partial M$ and working in one of our neat submanifold charts about this point, $\Phi_{*(q,0)}(\partial_i + r \cdot d/dt)$ can be computed as

$$\begin{aligned}(\partial_i + r\frac{d}{dt})(x^j \circ \Phi) = (\partial_i + r\frac{d}{dt})(x^j \circ \Phi) = (\partial_i + r\frac{d}{dt})\Phi^j \\ = \partial_i \Phi^j + r\frac{d}{dt}\Phi^j = \partial_i \Phi^j + r\left.\frac{d}{dt}\gamma_q^j\right|_{t_0} = \partial_i \Phi^j + r\dot{\gamma}_q^j(0) \\ = \partial_i \Phi^j + r\dot{\gamma}^j_{\Phi(q,t_0)}(0) = \partial_i \Phi^j + rX_q^j = \partial_i + X_q^j\end{aligned}$$

where we have used the group law to deduce this for the X term and since $\Phi(-,0) = \mathrm{id}$, so the directional derivative ∂_i of id at $(q,0)$ is still ∂_i. It follows easily that $\Phi_{*(q,0)}$ has full rank. Hence, even though we have boundary from $\mathbf{R}_{\geq 0}$, the inverse function theorem implies that this is a local diffeomorphism and thus we may shrink W to an open subset where $\Phi_{*(q,t)}$ has full rank and by the same argument as above—following the remark—we may suppose that Φ is injective on W.

As above, we can construction an embedding $\partial M \times [0,1) \hookrightarrow W$ and now the desired collar map is

$$\partial M \times [0,1) \hookrightarrow W \xrightarrow{\Phi} M$$

since everything in sight here has full rank. The open part follows as before.

We now want to show that we can restrict this to a collar for N. At this point, we might worry that Φ may shoot W out of N, despite pointing into N, so we need to shrink W yet again. To fix this, let U be the union of the boundary charts in our open cover and let $W' = W \cap \Phi^{-1}(U)$. Redoing the above construction with W' in place of W gives us a collar that restricts as a consequence of the delicate construction of our given open cover. Essentially, restricting to $W \cap \Phi^{-1}(U)$ makes us shoot into points of only U—by working in the nice submanifold coordinates, for points $p \in \partial N$, we see that we are simply flowing vertically inward for both N and M in U.

Openness of the restricted collar is the same argument as usual. □

Let us call such a function ε as above a smooth ***shrinking function***.

Lemma 12

Shrinking functions exist.

This lemma should be interpreted appropriately.

Corollary 10

Every open nbhd of ∂M contains a collar.

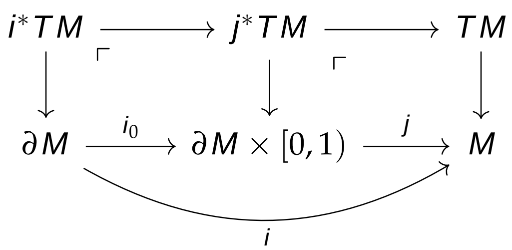
Proof. An open nbhd U of ∂M is an open submanifold and, in particular, it is neat submanifold-with-boundary, so the same argument applies to show a collar exists. □

Although we didn't need the collar nbhd theorem to show the following, it makes it particularly straightforward and easy to see.

Corollary 11

Suppose M is orientable. Then $TM|\partial M \cong T\partial M \oplus \underline{\mathbf{R}}$ where as usual $\underline{\mathbf{R}}$ is the trivial bundle over ∂M with fiber $\mathbf{R}$. In particular, the normal bundle of ∂M in M is trivial.

Proof. Let $i\colon \partial M \to M$ be the inclusion and let $j\colon \partial M \times [0,1) \to M$ be a collar nbhd so that $j|\partial M \times \{0\} = i$. First note that $TM|\partial M \cong i^*TM$. The collar neighborhood is an open submanifold of M and has tangent bundle diffeomorphic to $T\partial M \times \underline{\mathbf{R}}$ over $\partial M \times [0,1)$ and, as before, this is diffeomorphic to j^*TM. The collar has a submanifold (and note that the condition of being a neat submanifold is transitive) $\partial M \times \{0\}$. By pasting pullbacks we get the following rectangle with every rectangle a pullback



where $j^*TM \cong T\partial M \times \underline{\mathbf{R}}$ as we said above. Hence, we must compute $i_0^* j^* TM$. Of course, one sees immediately that this is what we described. □

Remark. To identify the normal bundle $\nu_{\partial M}$ with $\underline{\mathbf{R}}$, one can simply use a partition of unity argument and a collar to produce a Riemannian metric on M which is a product metric in a nbhd of ∂M. Say we make it the product metric at least on $[0,1/4)$ by covering M with open sets that only intersect the collar at $[1/4,1) \times \partial M$. This can be done using coordinate balls whose closure in M is compact.

For this next corollary, it helps to know that M is orientable **iff** TM is orientable as a vector bundle over M. First, we make a definition.

Definition 28 (Induced Orientation)

Let M be an orientable manifold with boundary (but not corners) of dimension n. Then ∂M inherits an ***induced orientation*** from M. The natural way of specifying this for which Stokes' theorem has a nice form is the ***outward pointing first convention***. Namely, for each $p \in \partial M$, we define an orientation class for $T_p\partial M$ by declaring a tuple of vectors $(v_1, \ldots, v_{n-1}) \in T_p\partial M$ to be in this orientation class iff for each outward pointing vector (hence, any outward pointing vector) $w \in T_pM$, $(w, v_1, \ldots, v_{n-1})$ defines a positively oriented basis in T_pM. One could similarly make a definition by using the ***inward pointing first convention*** but we do not need this.

Of course, we must check that these actually define an orientation.

Corollary 12

Let M be an orientable manifold with boundary. Then ∂M inherits a natural orientation by the ***outward pointing first convention***. Namely, for each $p \in \partial M$, we define an orientation class for $T_p\partial M$ by declaring a tuple of vectors $(v_1, \ldots, v_n) \in T_p\partial M$ to be in this orientation class iff for each outward pointing vector (hence, any outward pointing vector) $w \in T_pM$, $(w, v_1, \ldots, v_n)$ defines a positively oriented basis in T_pM.

Proof. This is straightforward using the definitions. □

B Proof of the Flow-in Theorem

Before we begin, we provide a remark and state a theorem.

Remark. The idea here is to show that the smooth flow of an inwards pointing vector field is locally an open map and an embedding in a nbhd of each point $p \in \partial M \times \mathbf{R}_{\geq 0}$. Since we have less access to the tools of calculus—since ∂M is not a manifold—we have to get our hands dirty and work directly with the flow in coordinates and crack open the standard existence and uniqueness results about flows to get what want.

The proof of this theorem is not actually difficult, just extremely tedious. The standard result about flows that we want is **Theorem I.5.2** from Spivak's first *Differential Geometry* volume. We also want to know that the flow in the aforementioned theorem is sufficiently smooth; this is not proved in Spivak's book, but on page 145, Spivak indicates that it is smooth. A proof may be found in Lang's *Introduction to Differential Manifolds* (2nd ed.) or in Lang's *Real and Functional Analysis* (3rd ed.), pp. 371 – 379. Spivak claims the latter proof is easier to digest. For convenience, we state this result in the form that we will need it.

Theorem (Spivak, Thm. I.5.2)

Let $f\colon U \to \mathbf{R}^n$ be any function of class C^k with $k \geq 1$, where $U \subset \mathbf{R}^n$ is open. Let $x_0 \in U$. There exists $a > 0$ and constants $K, L > 0$ with the following properties.

(1) The closed ball $\overline{B}_{2a}(x_0)$ of radius $2a$ and center x_0 is contained in U;

(2) $|f| \leq L$ on $\overline{B}_{2a}(x_0)$;

(3) f is Lipschitz continuous with Lipschitz constant K on $\overline{B}_{2a}(x_0)$.

Choose $b > 0$ such that

(4) $b \leq a/L$;

(5) $b < 1/K$.

Then for each $x \in \overline{B}_a(x_0)$, there is a unique $\gamma_x \colon (-b, b) \to U$ such that

$$\dot{\gamma}_x(t) = f(\gamma_x(t)) \quad \text{and} \quad \gamma_x(0) = x.$$

Furthermore, the map

$$\alpha \colon \overline{B}_a(x_0) \times (-b, b) \to U \qquad (p, t) \mapsto \gamma_p(t)$$

is of class C^k.

We have stated this in a restricted generality. Here the remarks needed to see that this is implied by what is in Spivak's book which we leave as an exercise.

Exercise 35

Show that any C^1 function is locally Lipschitz, which guarantees the flow α is at least continuous. Show that this implies that any C^1 function is Lipschitz on compact sets. More generally, show that any locally Lipschitz function on a locally compact metric space is Lipschitz continuous on compact subsets.

We begin with a lemma.

Lemma 13

Let M be a manifold with corners. An inwards-pointing vector field on M exists.

Remark. Inwards-pointing means precisely the analogous thing for manifolds that have corners. Namely, in coordinates

$$x \colon U \xrightarrow{\cong} \mathbf{R}^n_k = \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0},$$

an inwards pointing vector at a point $p \in \partial \mathbf{R}^n_k$ is a vector whose last k-coordinates are all positive. Invariance under coordinate change follows by considering a curve whose initial velocity vector is an inwards pointing vector and observing that coordinate changes cannot make the curve leave the chart.

Proof. This is exactly the same as the proof for manifolds-with-boundary, except we must argue that invariance of inwards-pointing vectors in corner charts, and that argument is itself exactly analogous to the case of manifolds-with-boundary—that is, we use smooth invariance of corner points of depth k.

In particular, we can always assume the corner charts are diffeomorphism $x \colon U \to \mathbf{R}^n_k = \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ for some k. In this case, the inwards-pointing vector field on $\mathbf{R}^n_k$ may be constructed as $X = (0, \ldots, 0, 1, \ldots, 1)$ where the last k-coordinates are 1. $\square$

An immediate consequence of this lemma is the following.

Corollary 13

The restriction $TM|\partial M$ of the tangent bundle TM to the subspace ∂M splits as a Whitney sum of a vector bundle of rank $m-1$ over ∂M and a trivial line bundle:

$$TM|\partial M \cong H_X \oplus \underline{\mathbf{R}}.$$

Proof. The bundle $TM|\partial M$ is the pullback bundle

$$\begin{array}{ccc} TM|\partial M & \longrightarrow & TM \\ \downarrow & & \downarrow \\ \partial M & \hookrightarrow & M \end{array}$$

so it is certainly a vector bundle. Moreover, if X is an inwards-pointing vector field on M, then X is a global non-vanishing section of $TM|\partial M$ which implies the splitting into the "horizontal" subbundle H_X and the trivial line bundle $\underline{\mathbf{R}}$. □

We should also get a better characterization of corner points of depth greater than or equal to 2.

Lemma 14

In a manifold with corners M of dimension m, the subset $\partial_1 M$ of ∂M consisting of those points which haves corner depth 1 (have a nbhd diffeomorphic to $\mathbf{R}^{m-1} \times \mathbf{R}_{\geq 0}$) is an open dense subset of ∂M (but not necessarily open in M). In fact, the subset of proper corner points have measure 0 and are a closed subset. Moreover, corners of maximal depth m are a discrete set.

Proof. It is easy to see $\partial_1 M$ is open in ∂M, since in ∂M each point of corner depth 1 has a nbhd of points of the same corner depth, essentially by definition. To see that it is dense, cover ∂M by a countable collection of corner charts. In each such chart U, the subset

$$\left\{(x^1, \ldots, x^m) \in \mathbf{R}^m_k : x^{i_1} = x^{i_2} = 0 \text{ for some } i_1 \neq i_2\right\}$$

is the complement of $\partial_1 M$ in this chart (implicitly using smooth invariance of corner points); and this subset satisfies that the closure of $\partial_1 M$ in U is everything. Thus, similarly, since each of these has measure 0, their countable union does too.

The last assertion is obvious by smooth invariance of corner points. □

Lemma 15

Fix a manifold M of dimension m with corners. Then M admits a smooth collar neighborhood. That is, M admits an embedding

$$C \colon \partial M \times \mathbf{R}_{\geq 0} \hookrightarrow M$$

having the satisfying properties.

(a) C is an open map that further restricts to the identity on ∂M.

(b) C is a homeomorphism onto its image. Moreover, there is a a splitting of $TM|\partial M \cong H_X \oplus \underline{\mathbf{R}}$ such that C induces a fiberwise injective map $C_*\colon H_X \times \underline{\mathbf{R}} \to TM$.

Proof. Cover ∂M by corner charts $\{(x_\alpha, U_\alpha)\}_{\alpha\in J}$. WLOG suppose $\{U_\alpha\}_{\alpha\in J}$ is locally finite. Let $\{\rho_\alpha\}_{\alpha\in J}$ be a partition of unity for $\bigcup U_\alpha$ subordinate to the evident covering. Using these, we may construct an inwards-pointing vector field X on M with support $\operatorname{supp} X \subset V$. One can show that the flow for X exists in the manner expected and is smooth[2] in the manner described in the statement of the lemma. More precisely, the flow map $\Phi^X\colon A_X \to M$ is smooth in the sense that for each $(p,t) \in A_X$, there is a nbhd U of (p,t) in $M \times \mathbf{R}$ and a smooth map $\phi\colon U \to M$ for which Φ^X is the restriction of ϕ—this follows from how one builds flows for manifolds with boundary or corners.

Using the flow of X as in the proof of the collar nbhd theorem, we may find an open nbhd U of $\partial M \times \{0\}$ inside the subset $\partial M \times \mathbf{R}_{\geq 0}$ upon which the flow is defined.

Claim. $\partial M \times \mathbf{R}_{\geq 0} \cap A_X$ is open in $\partial M \times \mathbf{R}_{\geq 0}$.

The maximal integral curves starting at $p \in \partial M$ are easily seen to be embeddings $[0,a) \to M$ since if the integral curve ever became stationary, then it must have been constant to begin with by a uniqueness of integral curves argument. Now, given $(p,t) \in \partial M \times \mathbf{R}_{\geq 0} \cap A_X$, the remarks above show that there is an open nbhd about (p,t) in $M \times \mathbf{R}$ for which the flow exists exists—at least after passing to coordinates and extending X to be defined on an open nbhd in Eucldiean space and then appealing to the usual results about existence and uniqueness of flows—and in particular, for which this flow exists in such a way that the flow stays inside of the image of M in our chosen coordiantes. This is so because we are guaranteed that $\Phi^X(p,t) \in \operatorname{Int} M$ whenever $t > 0$. Some thought then shows that, indeed, $\partial M \times \mathbf{R}_{\geq 0} \cap A_X$ is open in $\partial M \times \mathbf{R}_{\geq 0}$.

Remark. If the vector field were not everywhere inward-pointing, then we are not guaranteed that the flow will move points into M when we work in coordinates and extend the vector field X. For instance, consider the flow of an outwards-pointing vector field—at boundary points, the flow can only exist for times $t \leq 0$.

We claim that the flow map is an embedding in a nbhd of each point $p \in \partial M$ and, moreover, an open map on some nbhd of ∂M. This will follow from a more delicate (but still entirely usual) argument about existence and uniqueness of flows. We spell this out explicitly.

Pick $p \in \partial M$ and pick a chart about it, say (x, U_0) where $x\colon U_0 \to \mathbf{R}^m_k$ is a diffeomorphism onto an open nbhd U_0 of 0 in the model corner space. In coordinates, we may extend the vector field X to an open nbhd U of $x(U_0)$ in $\mathbf{R}^m$. Since X is inwards-pointing, it is easy to see that $X|\,(x(U_0) \cap \mathbf{R}^m_k)$ is inwards pointing. By the usual results on flows, there is a smooth flow defined on an open nbhd V of $x(p) \in U \subset \mathbf{R}^m$ and a

[2]Essentially because one suitably extends things to construct the flow in coordinates and then restricts.

$\varepsilon > 0$ such that the flow F of X is defined as a smooth map

$$F\colon V \times (-2\varepsilon, 2\varepsilon) \to U$$

where, as usual, $F(-,t)$ is a diffeomorphism onto its image for each $t \in (-\varepsilon, \varepsilon)$. It is possible to, moreover, assume that V has compact closure contained inside of U and so we do this. (See, for instance, Theorem I.5.2 from Spivak's first *Differential Geometry* volume.) We assume that this is so. WLOG we additionally suppose that the flow F is defined on $\overline{V} \times (-2\varepsilon, 2\varepsilon)$, perhaps by shrinking V first, if necessary.

Using the chart (x, U_0), the flow for X on M is constructed by suitably restricting and then lifting this flow back to M. Since $F(-,t)$ is a diffeomorphism, it follows easily that $F(-,t)|\,(V \cap U_0)$ is a diffeomorphism onto its image. Note that since X is inwards pointing,

$$\operatorname{Im}(F|\,(V \cap U_0) \times [0, \varepsilon)) \subset \mathbf{R}^m_k.$$

Now let $W = V \cap \partial \mathbf{R}^m_k$ and let

$$\Psi = F|\, W \times [0, \varepsilon).$$

We claim that Ψ is a topological embedding into $\mathbf{R}^n_k$ and, in fact, an open map.

First we note that Ψ is injective as a consequence of uniqueness of integral curves. So to show it is a topological embedding. For this, we claim we may reduce to show that the restriction of F to

$$\overline{W} \times [-\varepsilon, \varepsilon]$$

is a proper map, where

$$\overline{W} = \overline{V} \cap \partial \mathbf{R}^m_k \subset \partial \mathbf{R}^m_k$$

and where, from our assumptions, $\overline{V}$ is compact. Clearly from our assumptions F is defined and, by the same argument above, is injective on $\overline{W} \times [-\varepsilon, \varepsilon]$. It is enough to show this is a proper map (see **Exercise** 13), so if this is proper, then it is a topological embedding and, hence, it restricts a topological embedding on its subspace $W \times [0, \varepsilon)$. This now follows because the domain $\overline{W} \times [-\varepsilon, \varepsilon]$ is compact, so Ψ is a topological embedding.

Now we show Ψ is an open map. For this, it is enough to show that

$$\Psi' = F|\, W \times (-\varepsilon, \varepsilon)$$

is an open map into U, we claim. First let us show that Ψ' is an open map.

Note that, after forgetting about smoothness, we may suppose WLOG that $\partial \mathbf{R}^m_k = \mathbf{R}^{m-1}$ and thereby suppose W is homeomorphic to an open subspace of $W \subset \mathbf{R}^{m-1}$. Then Ψ' is a continuous injective map $W \times (-\varepsilon, \varepsilon) \to \mathbf{R}^m$ and hence by invariance of domain, Ψ' is an open map.

Now we claim that we may suppose Ψ is an embedding in the sense of item **(c)** in some open nbhd of $\partial M \times \mathbf{R}_{\geq 0}$. This follows by working in local coordinates in the extension of Ψ as above and then arguing the same as in the proof of the collar nbhd theorem. Since the details are essentially identical, we omit them.

Since X is an inward-pointing vector field, it is not hard to see that $(\Psi')^{-1}(\mathbf{R}^m_k) = W \times [0, \varepsilon)$. It follows that $\operatorname{Im}(\Psi') \cap \mathbf{R}^m_k = \operatorname{Im} \Psi$ which, by definition of the subspace topology, is consequently open in $\mathbf{R}^n_k$ and hence open in $U \supset x(U_0)$ and it follows that Ψ is an open map as well.

It is easy to see that Ψ lifts to an open map $x^{-1}(V) \times [0,\varepsilon) \to M$, where $x^{-1}(V) \subset \partial M$ is open. It is moreover smooth in the sense we have just described and an injective open map in a nbhd of p and so a topological embedding. Since we can do this everywhere, by using a partition of unity argument on a locally finite covering, we can show that there is an open nbhd U of $\partial M \times \times \{0\}$ in $\partial M \times \mathbf{R}_{\geq 0}$ and an open injective map (hence topological embedding) $\mathsf{U} \to M$ that restricts to the identity on ∂M and is smooth in the sense that it is locally the restriction of a smooth map. This is an argument that is also tedious. It is analogous to what we did in the proof of the collar nbhd theorem but see also **Theorem 9.20** in Lee's *Introduction to Smooth Manifolds*.

As for the smoothness part, this is essentially automatic from the construction, in that each map $x^{-1}(V) \times [0,\varepsilon) \to M$ is locally the restriction of a smooth map defined on a nbhd U of this set in M. It is worth pointing out by the last lemma, since $\partial_1 M \subset \partial M$ is an open dense subset, it follows easily that $\partial_1 M$ acquires the structure of a submanifold without boundary and $C\colon \partial_1 M \times \mathbf{R}_{\geq 0} \cap \mathsf{U} \to M$ is a smooth embedding that is an open map.

Finally, the same argument as in the collar nbhd theorem lets us assume C is an embedding $\partial M \times \mathbf{R}_{\geq 0} \to M$ by suitably shrinking. It suffices, clearly, to do this for $[0,1)$ instead and doing so we can assume the shrinking function shrinking $\partial M \times [0,1)$ into U is smooth in the sense we are using by working in coordinates about points in the boundary and using a similar local existence and uniqueness argument for flows as above. Performing the same construction as above, we may find a function $\varepsilon\colon \partial M \to \mathbf{R}_{>0}$ that is locally the restriction of a smooth function such that $\{p\} \times [0,\varepsilon(p)) \subset \mathsf{U}$ for all p. See the proof of the collar nbhd theorem for how to construct such a function. The only difference is that one uses charts of M and then restricts the resulting function to ∂M and we can just as well assume this extends to all M as a smooth function $\varepsilon\colon M \to \mathbf{R}_{>0}$. We claim that the function

$$(p,t) \mapsto C(p,\varepsilon(p)t)$$

is a collar C' in that C' is an open embedding and smooth in the sense we have just described. For the open embedding part, we claim the map

$$\iota\colon \partial M \times [0,1) \ni (p,t) \mapsto (p,\varepsilon(p)t) \in \mathsf{U}$$

is an open embedding that is smooth in the sense we are using the word in. For smoothness, observe that since ε locally extends, so too does ι by the same formula—it is easiest to see this by passing to an open rectangle—except possibly taking image in $M \times \mathbf{R}_{\geq 0}$. Hence, ι is locally the restriction of a smooth map. Moreover, that smooth map is easily seen to give a smooth embedding

$$M \times [0,1) \to M \times \mathbf{R}_{\geq 0} \qquad (p,t) \mapsto (p,\varepsilon(p)t),$$

by the same formula (some care must be used at the boundary). Hence, the restriction is likewise an embedding onto its image. Invariance of domain implies it is an open map on its interior. The same sort argument as in **Exercise** 33 then shows that it is open and an embedding everywhere. It follows that the restriction

$$\partial M \times [0,1) \to \mathsf{U} \subset \partial M \times \mathbf{R}_{\geq 0}$$

is an open map and embedding; the embedding part is obvious, the open part by the definition of the subspace topologies and the fact that $M \times [0,1) \to M \times \mathbf{R}_{\geq 0}$ is an open embedding.

We now give a collar $C\colon \partial M \times [0,1) \to M$ the smooth structure inherited from that of its image—an open submanifold of M. This is constructed by passing back charts. The uniqueness of the smooth structure follows from **Theorem** 37. □

Theorem (Lee 9.26, Flow-in)

Let M be a manifold with boundary and possibly corners. Let $\iota\colon \operatorname{Int} M \hookrightarrow M$ be the inclusion.

(a) Suppose M has no corners. There exists a collar nbhd $C\colon \mathbf{R}_{\geq 0} \times \partial M \hookrightarrow M$ such that for any $a \in (0,\infty)$, the subset $M(a) \stackrel{\text{def}}{=} M \setminus C([0,a) \times \partial M)$ is a properly embedded submanifold of M and, furthermore, for any such a there exists a proper[a] smooth embedding $R_a\colon M \hookrightarrow \operatorname{Int} M$ with image $M(a)$ such that the composites $\iota \circ R$ and $R \circ \iota$ are smoothly homotopic to the identity maps—in particular, ι is a homotopy equivalence. In particular, we shall construct a collar of M all of whose restrictions to intervals $[0,a]$ are closed collars.

(b) If M has corners, there exists a proper smooth embedding $R\colon M \to \operatorname{Int} M$ such that the composites $\iota \circ R$ and $R \circ \iota$ are homotopic to the identity maps—in particular, ι is a homotopy equivalence.

(c) In each case, the image of M in its interior is a closed subset (because the map is proper) and when M has no corners, there is a strong isotopy between the identity map and the map R_a.

[a] This means that the preimage of compact sets is compact.

We start with **(b)**, since **(a)** is really a special case. The first order of business will be establishing the manifolds with corners admit a collar in a suitable sense.

Proof. Fix a collar $C\colon \partial M \times \mathbf{R}_{\geq 0} \to M$ as above. We claim that we may assume that C has full rank everywhere in the sense that

By modifying the construction of the inwards pointing vector field suitably, it is possible to assume that, for some choice of complete metric on M, that $|X|$ is bounded. For instance, choose a covering of ∂M by charts (x_i, U_i) in which each point p of $\bigcup_i U_i$ is in at most $m+1$ charts. This is possible to arrange because the covering dimension of an m-dimensional manifold is the same as its dimension. Then construct an inwards pointing vector field as usual and in each chart (x_i, U_i) modify the vector field X_i constructed so that $|X_i| < 1$ in U_i. This is possible to do using a suitable scaling function that is smooth. Using a partiton of unity to piece together the X_i, it is easy to see that the resulting vector field X has $|X|$ bounded above for this covering. By standard arguments, this means the flow of X is global in a suitable sense for manifolds with corners. In particular, the flow map Φ^X has domain containing $\partial M \times \mathbf{R}_{\geq 0}$.

One can then easily verify that $\Phi^X|\partial M \times \mathbf{R}_{\geq 0}$ is a homeomorphism onto its image by showing that it is an open map; one shows that by showing that it sends sufficiently small open sets to open sets. The argument is entirely analogous to how we showed the collar map is an open map for manifolds with corners—namely, one invokes invariance of domain in coordinates after suitably extending things. In particular, note that ∂M is at least locally homeomorphic to open subsets of Euclidean space—this is because there are homeomorphisms $\mathbf{R}^n_k \xrightarrow{\cong} \mathbf{R}^n$ sending the boundary to the boundary. Since $\partial M \times \mathbf{R}_{\geq 0} \to M$ is injective and continuous, by invariance of domain, it is an open map on $\partial M \times (0, \infty)$ since the restriction to $\partial M \times (0, \infty)$ maps into the interior of M. The fact that it is an open map in a nbhd of $\partial M \times \{0\}$ now follows from precisely the analysis from above.

Note that for this flow, the domain of the flow contains $M \times \mathbf{R}_{\geq 0}$. The flow map restricts to a map $\partial M \times \mathbf{R}_{\geq 0} \to M$. We claim that this map is an immersion everywhere with respect to the product bundle $H_X \times \underline{\mathbf{R}} \to \partial M \times \mathbf{R}_{\geq 0}$ so that, in particular, the flow map induces a map
$$H_X \times \underline{\mathbf{R}} \to TM.$$
To see this, simply note that this map is the evident restriction of the differential
$$\Phi_*\colon TM \times \underline{\mathbf{R}} \to TM \qquad M \times \mathbf{R}_{\geq 0} \to M.$$

Now, the only way the induced map $H_X \times \underline{\mathbf{R}} \to TM$ could fail to be injective at a point $(p, s) \in \partial M \times \mathbf{R}_{\geq 0}$ is if some vector $v \in (H_X)_p$ is either mapped to zero or is mapped to the vector $\Phi_{*(p,s)}(\partial/\partial t) = X_p(\gamma_p(s))$ where γ_p is the integral curve of X starting at p.

To see that it is an immersion everywhere, note that this is so for induced map on the bundle $TM|\partial M \to TM$ We claim that the map $M \times \{t\} \to M$ sending $(p, t) \mapsto \Phi^X(p, t)$ is a likewise a diffeomorphism onto its image. It is certainly smooth, so we wish to verify it is a topological embedding and thus a smooth embedding and therefore a homeomorphism onto its image. For this, we show the map is an open map. When $t = 0$ this is obvious so suppose $t > 0$. To see this, note that the restriction $\operatorname{Int} M \times \{t\} \to \operatorname{Int} M \subset M$ is injective so by invariance of domain the map is open. To see that the map is open in a nbhd of $\partial M \times \{t\}$ in $M \times \{t\}$, note that from what we showed above, it follows that $\partial M \times \{t\} \to M$ is an open map onto its image.

This is essentially a consequence of **Corollary** 5 and the fact that for $t > 0$, the map has image in the interior of M and for $t = 0$ the map is the identity.

WLOG we suppose C is this collar.

Now let $f\colon M \to \mathbf{R}_{>0}$ be a smooth compact exhaustion function, that is, a smooth map such that $f^{-1}((-\infty, c])$ is compact for all c. This can always be arranged. Let
$$W = \{(p, t) \in \partial M \times \mathbf{R}_{\geq 0} : f(C(p, t)) > f(p) - 1\}.$$
This is an open nbhd of $\partial M \times \{0\}$ in $\partial M \times \mathbf{R}_{\geq 0}$. Using a partition of unity as in the proof of the collar nbhd theorem, we may construct a smooth positive function $\varepsilon\colon \partial M \to \mathbf{R}_{>0}$ such that $(p, t) \in W$ whenever $0 \leq t < \varepsilon(x)$. Define
$$\widetilde{C}\colon \partial M \times \mathbf{R}_{\geq 0} \to M \qquad \widetilde{C}(p, t) = C(p, \varepsilon(p)t).$$

Then $\widetilde{C}$ is still an open injective map and smooth embedding in the sense of the lemma above. Indeed, the map $\partial M \times \mathbf{R}_{\geq 0} \to \partial M \times \mathbf{R}_{\geq 0}$ given by $(p,t) \mapsto (p, \varepsilon(p)t)$ is clearly smooth in that it is the restriction of a smooth map $G \colon M \times \mathbf{R}_{\geq 0} \to M \times \mathbf{R}_{\geq 0}$ defined by the same formula and one easily verifies that, moreover, G is a diffeomorphism having smooth inverse

$$(p,t) \mapsto (p, (1/\varepsilon(p))t).$$

This then provides a new collar $\widetilde{C}$ where $\widetilde{C}$ is a collar in the sense we defined it above. We do not demand that the map $\partial M \times \mathbf{R}_{\geq 0} \to \partial M \times \mathbf{R}_{\geq 0}$ respect the smooth structure inherited from the collar C on the left-hand side and the collar $\widetilde{C}$ on the right-hand side and vice-versa.

Observe that, by construction,

$$f(\widetilde{C}(p,t)) > f(p) - 1 \qquad \forall (p,t) \in \partial M \times \mathbf{R}_{\geq 0}.$$

We show that for all $a \in \mathbf{R}_{>0}$, the set

$$M_a = \widetilde{C}(\partial M \times [0,a])$$

is closed.

Suppose p is a boundary point of M_a so that there is a sequence $(p_i, t_i)_i$ in $\partial M \times [0,a]$ such that $\widetilde{C}(p_i,t_i) \to p$. Then $f(\widetilde{C}(p_i,t_i))$ is bounded by convergence and so $f(p_i) < f(\widetilde{C}(p_i,t_i)) + 1$ likewise is bounded. Since ∂M is closed in M, $f|\partial M$ is also an exhaustion function, and so the sequence $(p_i)_i$ lies in some compact subset $K = f^{-1}((-\infty, c]) \cap \partial M$ of ∂M. Hence, there is a subsequence $(p_{i_j})_j$ which converges to a point p_0 of K. Similarly, since $[0,a]$ is compact, the correspondingly indexed subsequence $(t_{i_j})_j$ of $(t_i)_i$ has itself a convergent subsequence. WLOG we may suppose it is this subsequence and therefore $(p_{i_j}, t_{i_j})_j \to (p_0, t_0)$ converges by compactness $K \times [0,a]$. It follows that $p = \widetilde{C}(p_0,t_0) \in \widetilde{C}(\partial M \times [0,a])$ is therefore closed.

This shows that $\widetilde{C}$ is a collar satisfying some additional nice properties. $\widetilde{C}$ is still a collar in the sense that it is an open map, one should carefully note. WLOG we may suppose

$$C = \widetilde{C}$$

to simplify notation henceforth. It follows quickly from this last thing that the preimages of compact subsets under the collar map $\widetilde{C} = C$ are compact.

For each $a \in \mathbf{R}_{\geq 0}$, let $C(a) = C(\partial M \times [0,a))$ and let $M(a) = M \setminus C(a)$, which is a closed subspace by the above considerations. We claim that $M(a)$ is a submanifold with corners diffeomorphic to M. To see this, observe that $M(a)$ is simply the image of the subset $M \times \{a\} \subset M \times \mathbf{R}_{\geq 0}$ under the flow map, and we have seen that $(p,a) \mapsto \Phi^X(p,a)$ is a diffeomorphism onto its image. Since its image is, additionally, closed, this is in fact a *proper* smooth embedding.

Let $\psi \colon \mathbf{R}_{\geq 0} \to [\frac{1}{3}, \infty)$ be an increasing diffeomorphism that is the identity for all $s \geq \frac{2}{3}$ and define $R \colon M \to \operatorname{Int} M$ by

$$R(p) = \begin{cases} p, & p \in \operatorname{Int}(M(2/3)) \\ C(x, \psi(s)), & p = C(x,s) \in \operatorname{Im} C. \end{cases}$$

These two definitions are the identity on the open set $\mathrm{Im}(C) \setminus C(2/3)$, where they overlap and so R is smooth by the pasting lemma. This map is a diffeomorphism onto the closed subspace $M(1/3)$ and so it is a proper map and thus a smooth embedding of M into $\mathrm{Int}(M)$.

Define $H\colon M \times I \to M$ by

$$H(p,t) = \begin{cases} p, & p \in \mathrm{Int}(M(2/3)) \\ C(x, ts + (1-t)\psi(s)), & p = (x,s) \in \mathrm{Im}\, C. \end{cases}$$

This scaling is manifestly smooth when ∂M is a manifold (e.g., when M has no corners) since the smooth structure on $\mathrm{Im}\, C$ is precisely that of $\partial M \times \mathbf{R}_{\geq 0}$. It is at least continuous when M has corners.

As for the last part, properness implies the map is closed. W

□

Exercise 36

Is scaling smooth for the smooth structure coming from the collar? *[I do not know how to show this, but it is worth pointing out that the smooth structure is pinned down on $\partial_1 M \times \mathbf{R}_{\geq 0}$ as the product smooth structure and $\partial_1 M \times \mathbf{R}_{\geq 0}$ is a dense open subset of $\partial M \times \mathbf{R}_{\geq 0}$.*

Exercise 37

Let M be a manifold with corners and give M a complete Riemannian metric[a] and suppose X is a vector field on M for which $|X|$ is bounded in the given Riemannian metric.

(a) Show that if M has corners, then any vector field on M that is everywhere tangent to $\partial_1 M$ must be 0 at corner points of depth greater than 1. *[Hint: You may find the second lemma of this section useful for more than just the notation $\partial_1 M$.]*

(b) Show that every maximal integral curve γ of X with initial condition $\gamma(0) = p$ is defined on a connected interval $J(x)$ which is a closed subset of $\mathbf{R}$.

(c) If X is tangent to ∂M everywhere, then the maximal flow domain of X is $M \times \mathbf{R}$.

(d) If the vector field is inwards-pointing, show that the maximal flow domain contains $M \times \mathbf{R}_{\geq 0}$.

[a] This always exists in a conformal class of any Riemannian metric using a suitable compact exhaustion function.

C Bundles, Normal Bundles, Tubular Neighborhoods

C.1 Bundle Potpourri

Remark. Many of the following proofs are adapted or reproduced from Brian Conrad's thorough differential geometry handouts.

Proposition 3

Let B be a paracompact Hausdorff space and $p\colon E \to B$ be a vector bundle. Then E admits a metric (i.e., inner product).

Proof. Define $E^* \otimes E^*$ as before and define S^2E^* as before. Construct local sections $\omega\colon U_\alpha \to S^2E^*|_{U_\alpha}$. $\omega(x) = \sum_{ij} \omega_{ij}(x)(\ell_i(x) \otimes \ell_j(x))$ (in general). Set $\omega(x) = \sum_i \ell_i(x) \otimes \ell_i(x)$. Then ω is positive definite. Partition of unity $\{\lambda_i\}$. Convex linear combination (adds to 1, not negative) $\sum \lambda_i \omega_i$ for positive definite ω_i. Since this is a convex linear combination of positive definite forms, the resulting function is positive definite. □

Remark. Paracompact Hausdorff is equivalent to the statement that every open cover admits a subordinate partition of unity.

Lemma 16

Let $p\colon E \to B$ be a vector bundle of (as we always implicitly assume) finite rank. Then the dual bundle $E^\vee$ exists and there is a natural isomorphism of bundles $E^{\vee\vee} \cong E$. Moreover, $E^\vee \cong \mathrm{Hom}(E, \underline{\mathbf{R}})$.

Proof. $E^\vee$ is constructed as in the vector/fiber bundle construction lemma. To show that $E^{\vee\vee} \cong E$ naturally, we simply let $E_p^{\vee\vee} \cong E_p$ be the natural double duality isomorphism for FDVSs. On trivializations, this is basically just $U \times \mathbf{R}^{\vee\vee} \to U \times \mathbf{R}$.

For the next part, pick a trivialization U for E. Then $\mathrm{Hom}(E, \underline{\mathbf{R}})$ on U has trivialization given essentially by doing φ^{-1*}—that is, on fibers it is $\mathrm{Hom}(E_p, \mathbf{R}) \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$. □

Theorem 20

Let $f\colon E' \to E$ be a morphism of smooth vector bundles over M. The function $p \mapsto \dim \mathrm{Ker}\, f_p$ is locally constant **iff** there is a covering of M by open sets U_i such that $E'|\, U_i$ admits a trivializing frame containing a subset whose specialization in each fiber over each point $p \in U_i$ is a basis of $\mathrm{Ker}\, f_p$ (i.e., a subset of the collection of specified local sections on U_i are at each point a basis for the kernel).

Proof. ($\Leftarrow$) This is obvious. ($\Rightarrow$) WLOG we may assume the U_i are path-connected. Admitting a trivializing frame is the same as saying the U_i are trivializing, we remark. Since we have assumed local constant-ness, we may assume that for all $p \in U_i$, dim Ker $f_p = d$. Let $\{s'_i\}$ and $\{s_j\}$ be trivializing frames with $1 \leq i \leq n'$ and $1 \leq j \leq n$ so that $r = n' - d$ is the common rank of the maps f_p on U_i. We can write

$$f(s'_j) = \sum_i a_{ij} s_i$$

since the s_i are a local frame where $a_{ij} \colon U_i \to \mathbf{R}$ are smooth functions. For each $p \in U_i$, since f_p has rank r (i.e., for all $p \in U_i$, rank $f_p = r$). It follows from standard linear algebra that an $r \times r$ submatrix of $(a_{ij}(p))$ has full rank (i.e., is invertible), call it $A(p)$ where A is the function which is this *particular* submatrix at all points. Since rank is a lower semi-continuous function, the set of points $q \in U_i$ for which rank $A(q) > r - 1$ is open. Hence, we can cover U_i by open sets for which some submatrix satisfies this property—say we cover U_i by U_α for which a submatrix A_α is invertible and let I_α and J_α be the sets of indices picking out A_α in (a_{ij}).

Fix α and restrict attention to U_α. WLOG suppose that the upper left $r \times r$ matrix of (a_{ij}) is A_α, perhaps by rearranging indices. Since (a_{ij}) has rank r on U_α, it is easy to see that the first r columns of (a_{ij}) span the image of (a_{ij}) at each point—basically this is because a linear dependency among the full column vector would imply a linear dependency for A_α which is impossible because A_α is invertible. Hence, for each $j > r$ and $p \in U_\alpha$, there is a unique linear combination in E'_p

$$f(s'_j)(p) = \sum_{k=1}^{r} c_{kj}\, f(s'_k)(p) = \sum_{k=1}^{r} \sum_{i=1}^{n'} c_{kj} a_{ik}(p) s_i(p).$$

Of course, also, by linear independence of the s_i everywhere, we must have that

$$a_{ij}(p) s_i(p) = \sum_{k=1}^{r} c_{kj} a_{ik}(p) s_i(p)$$

or in other words

$$a_{ij}(p) = \sum_{k=1}^{r} a_{ik}(p) c_{kj}.$$

This gives a system of n' equations for fixed j by varying i. Since $(a_{ij})_{1 \leq i,j \leq r}$ is invertible everywhere, *Cramer's Rule* allows us to solve for each c_{kj} uniquely such that all of these n' equations are satisfied. In particular, Cramer's rule tells us that each c_{kj} is a rational function of the $a'_{ij}s$ with denominator the determinant polynomial which is non-vanishing by assumption. So these are all smooth.

Hence, we get d sections

$$v_j = s'_{j+r} - \sum_{k=1}^{r} c_{k,j+r} s'_k$$

with $1 \leq j \leq d$ such that $v_j(p) \in \mathrm{Ker}(f|_p)$ for all $p \in U_\alpha$. One sees this since we just showed for $j > r$ that $f(s'_j) = \sum_{k=1}^{r} c_{kj} f(s'_k)$ and f is linear on each fiber so this means that $f(s'_j) - f(\sum_{k=1}^{r} c_{kj} s'_k) = 0$ and so $s'_j - \sum_{k=1}^{r} c_{kj} s'_k$ is in the kernel of f at each point but $s'_j - \sum_{k=1}^{r} c_{kj} s'_k \neq 0$ by linear independence of the s'_i.

By inspection, the d vectors v_j are linearly independent essentially because if $j \neq j'$ then v_j has a factor of s'_{j+r} whereas $v_{j'}$ has a factor of $s_{j'+r}$. Hence, dimension considerations force $v_1, \ldots, v_d$ to span $\operatorname{Ker} f|_p$ at each point $p \in U_\alpha$.

Finally, consider the n' sections $s'_1, \ldots, s'_r, v_1, \ldots, v_d$. By construction, for each $p \in U_\alpha$, $f(s'_1(p)), \ldots, f(s'_r(p))$ are a basis for the image of $f|_p$ whereas $v_1(p), \ldots, v_d(p)$ are a basis for its kernel. Hence, together they form a basis for E'_p by dimension considerations and the Rank-Nullity theorem. □

Corollary 14

Let $f\colon E \to E'$ be a bundle surjection over B, then $p \mapsto \operatorname{Ker} f_p$ is locally constant **iff** $\operatorname{Ker} f$ is a subbundle of E.

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) We have local trivializing frames by the preceding theorem. □

Corollary 15

If $f\colon E \to E'$ is a bundle surjection then $\operatorname{Ker} f$ is a subbundle of E.

Theorem 21

Let $f\colon E \to E'$ be a smooth bundle map over B. Then f is a bundle isomorphism **iff** it is a fiberwise (linear) isomorphism.

Proof. ($\Rightarrow$) This is obvious.

($\Leftarrow$) Choose local coordinates about a point $b \in B$—say with the same trivializing open set U WLOG—with trivializations g and h for E and E', respectively, and consider the composite $U \times \mathbf{R}^n \xrightarrow{h^{-1}fg} U \times \mathbf{R}^n$—we must show this is a diffeomorphism.

Since f is smooth, $h^{-1}fg$ is smooth. Working in the evident local frame, we see that this map is therefore given by mapping $(b,x) \mapsto (b,y)$ with

$$y^i = \sum_j a_{ij}(b)x^j$$

and, furthermore, that the non-singular matrix $(a_{ij}(b))_{i,j}$ varies smoothly with b. The formula for the inverse matrix involves dividing by the determinant and the cofactors of the given matrix—these are all polynomial in the entries of (a_{ij}) and thus is smooth on the same domain. Denote the inverse matrix by $(A_{ij}(b))$. Then

$$g^{-1}f^{-1}h(b,y) = (b,x) \qquad x^i = \sum_j A_{ij}(b)y^j$$

and this is smooth since A_{ij} depends smoothly on b as we have just argued. □

Reminder. Recall that a subbundle of a vector bundle $p\colon E \to B$ is a subspace $E' \subset E$ such that for all $p \in B$

(a) $E'_p \subset E_p$ is equipped with the natural vector subspace structure coming from E_p;

(b) $E'_p \subset E_p$ has rank constant k (at least, say, on each connected component of E if we really want to include that possibility).

We also demand that $p\colon E' \to B$ has the structure over a vector bundle over B. If we forget to say smooth before subbundle, we will probably mean a smooth subbundle, which is a subbundle that is also a submanifold of E.

Lemma 17

Let $p'\colon E' \to B$ and $p\colon E \to B$ be smooth vector bundles over B of rank n' and n respectively and let $i\colon E' \to E$ be a smooth bundle morphism which is injective on fibers (a bundle monomorphism).

(a) Then $i(E')$ is a smooth subbundle of E. In particular, i is a closed embedding and immersion (hence, a submanifold inclusion) and i locally looks like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} = U \times \mathbf{R}^n$.

(b) If $f\colon E'_1 \to E$ is a bundle map over B with $f(E'_1) \subset i(E')$, then there is a unique smooth bundle map $\phi\colon E'_1 \to E'$ over B such that $i\phi = f$. If f is a fiberwise isomorphism, then ϕ is a smooth bundle isomorphism. In particular, the subset $i(E') \subset E$ uniquely determines the pair (E', i) up to a unique smooth bundle isomorphism.

We shall do this by showing that there are local trivializations determined by frames such that n' of the local sections lie entirely in E' entirely and constitute a frame for E'—we then extend this to a local frame for E.

Proof. **(a)** i is obviously injective. We will first show that i is a closed immersion. Let U be a common trivialization of E' and E perhaps by shrinking things enough. We may also suppose U is path-connected. Restricting to U, we may suppose that the bundles in question are both trivial. Henceforth we assume the bundles over B are trivial.

Pick trivializing frames $\{s'_k\}$ and $\{s_j\}$. There is an $n \times n'$ matrix (a_{jk}) such that $i_p s'_k(p) = \sum_j a_{jk}(p) s_j(p)$ where $a_{jk}\colon B \to \mathbf{R}$ are smooth. This has rank n' at all points i is injective on all fibers. It is a standard linear algebra fact that at each point $p \in B$, an $n' \times n'$ submatrix of $(a_{jk}(p))$ is invertible. Since rank is lower semi-continuous, this is an open condition. Hence, we can once again pass to smaller (connected) neighborhood, say $V \subset U$ on which the *same* $n' \times n'$ submatrix of (a_{jk}) is invertible at all points. Hence, we might as well assume that the bundles are trivial and, furthermore, that upper left $n' \times n'$ submatrix of (a_{jk}) is everywhere invertible on B (perhaps after rearranging indices).

Notation C.1. Denote is'_k the function $i(s'_k)$ for each $1 \le k \le n'$.

Denote this submatrix by $A(p)$ at each point $p \in B$.

Observation. The $n \times n$ matrix (call is M) of smooth functions representing $\Sigma = (is'_1, \ldots, is'_{n'}, s_{n'+1}, \ldots, s_n)$ in the basis of the s_j's has upper left $n' \times n'$ submatrix A. Furthermore the upper right $n' \times (n - n')$ submatrix is 0, the lower right $(n - n') \times (n - n')$ submatrix is the identity matrix.

These observations imply that the matrix M is invertible at all points $p \in B$—for instance, expanding the determinant along the last column each time will reduce us to computing $\det A$ so that $\det M = \pm \det A$. It follows that $M(p)$ is a basis for the vector space over p for each $p \in B$. In particular, Σ comprises a trivializing frame.

The bundle morphism i in the bundle charts determined by $\{s'_k\}$ and Σ is then

$$(p, (v_1, \ldots, v_{n'})) \mapsto (p, (v_1, \ldots, v_{n'}, 0, \ldots, 0)).$$

It is easy to see from this description that i is an immersion and an embedding. To see that $\operatorname{Im}(i)$ is closed, let $v \in E \setminus \operatorname{Im}(i)$ and say it lies over the fiber over $p \in B$. In coordinates, this looks like $v \in V \times \mathbf{R}^n \setminus \mathbf{R}^{n'} \times \mathbf{0}$ and from this description it is clear that the complement is open so that $\operatorname{Im}(i)$ is closed.

Since i is a closed injective immersion, it is a proper injective immersion. Proper injective immersions are exactly the submanifold inclusions with closed image. Since the given map is in fact closed, this is already satisfied.

(b) Once we build ϕ uniquely as a bundle map, then when f is a fiberwise isomorphism we will have that ϕ is a bundle map that is bijective on fibers and hence a bundle isomorphism. Uniqueness of ϕ follows immediately since i is a fiberwise injection. As for existence of ϕ, it is certainly a set map that is fiberwise linear. To check smoothness, it is enough to work locally. By **(a)**, we may assume i is locally the standard inclusion. Then we are reduced to showing that the smooth map $U \times \mathbf{R}^{n'_1} \to U \times \mathbf{R}^n$ which lands in the submanifold $U \times \mathbf{R}^{n'} \subset U \times \mathbf{R}^n$ is smooth, and this is clear even without using the universal property of submanifolds because of the niceness of the standard smooth structure on Euclidean space. □

Lemma 18

Let $E \to B$ be a vector bundle of rank n and let $E' \subset E$ be a fiberwise subset having constant dimension n'. Then E' is a subbundle of rank n' over B **iff** there is a covering $\{U_i\}_{i \in I}$ of B by trivializing open sets such that over each U_i there exists a vector bundle E''_i and bundle isomorphisms $\varphi_i\colon E'|U_i \oplus E''_i \cong E|U_i$ satisfying that the composite $E'|U_i \to E'|U_i \oplus E''_i \cong E|U_i$ is the inclusion map over U_i.

Remark. The idea is take local frames for E' and E and apply linear algebra to see that at a point p there is a basis for the fiber E_p that contains the frame for E' evaluated at p. Then we use calculus to show this holds in fact holds locally.

Proof. ($\Rightarrow$) We can construct frames for both bundles $\{s'_i\}$ and $\{s_j\}$ over a small enough trivializing nbhd U. Fix $p \in U$. Then some subcollection of the s_j's append to $\{s'_i\}$ to construct a linearly independent set at p, WLOG say $s_{n'+1}, \ldots, s_n$.

The $n \times n'$ matrix (a_{jk}) of smooth functions satisfying $s'_k = \sum a_{jk} s_j$ has rank n' everywhere and therefore has an $n' \times n'$ invertible submatrix at p, which we may suppose after rearranging indices is the block $(a_{jk})_{1 \leq j,k \leq n'}$. This is an open condition so let $p \in V \subset U$ be open where this block is invertible. On V it follows that the matrix of coefficients for $\{s'_1, \ldots, s'_{n'}, s_{n'+1}, \ldots, s_n\}$ in terms of the $\{s_j\}$ has upper left $n' \times n'$ block $(a_{jk})_{1 \leq j,k \leq n'}$ (perhaps after rearranging), upper right $n' \times (n - n')$ block 0 and lower right $(n - n') \times (n - n')$ block the identity matrix. Hence, this matrix is invertible and so is invertible locally on $p \in V' \subset V \subset U$ and so furnishes a frame.

This construction gives us a trivialization for which $E'|V' \cong V' \times \mathbf{R}^{n'} \times \mathbf{0} \subset V' \times \mathbf{R}^n \cong E|V'$. Let $E'' = V' \times \mathbf{0} \times \mathbf{R}^{n-n'}$. That $E'|V' \oplus E'' \cong E|V'$ in the desired manner follows by

$$E'|V' \oplus E''|V' \cong (X \times \mathbf{R}^{n'}) \oplus (X \times \mathbf{R}^{n-n'}) \cong X \times (\mathbf{R}^n \oplus \mathbf{R}^{n-n'}) \cong X \times \mathbf{R}^n \cong E|V'$$

where in the first isomorphism we used the local frame $\{s'_1, \ldots, s'_{n'}\}$ on E' over V' to construct the isomorphism, noting that $E''|V' = X \times \mathbf{R}^{n-n'}$, and in the last isomorphism we used the inverse of the trivialization afforded by $\{s'_1, \ldots, s'_{n'}, s_{n'+1}, \ldots, s_n\}$. This obviously respects the inclusion in the sense that the composite $E'|V' \to E'|V \oplus E''|V' \cong E|V'$ is the inclusion.

($\Leftarrow$) The conditions here imply that E' has the structure of a smooth vector bundle since smoothness is local and it is clearly subbundle from the condition here as well. □

Corollary 16

If $E' \subset E$ is a subbundle of $p\colon E \to B$ where E' has rank n' and E has rank n, then there are bundle charts of E covering B such that $\varphi_i\colon (p^{-1}(U_i), p^{-1}(U_i) \cap E') \cong (U_i \times \mathbf{R}^n, U_i \times \mathbf{R}^{n'} \times \mathbf{0})$.

Proof. We constructed these charts above. □

Corollary 17

Let $E' \subset E$ be a subbundle of rank n' of the vector bundle $p\colon E \to B$ of rank n. Then the ***quotient bundle*** $E/E' \to B$ exists.

Proof. Using the charts above, we may fix and consistently use the obvious isomorphism $\mathbf{R}^n/\mathbf{R}^{n'} \times \mathbf{0} \cong \mathbf{R}^{n-n'}$ sending a vector to the element defined by its last $n - n'$ coordinates. Define E/E' to be fiberwise the quotient E_b/E'_b. Pick bundle charts U_i for E such that $p^{-1}(U_i) \cap E'$ maps under the trivialization to $U_i \times \mathbf{R}^{n'} \times \mathbf{0}$ and let $q\colon E/E' \to B$ be the obvious projection. We topologize $q^{-1}(U_i)$ by declaring the isomorphism of sets $q^{-1}(U_i) \cong U_i \times \mathbf{R}^{n-n'}$ induced by $p^{-1}(U_i) \cong U_i \times \mathbf{R}^n \to U_i \times \mathbf{R}^n/\mathbf{R}^{n'} \cong U_i \times \mathbf{R}^{n-n'}$ by the universal property of the quotient to be a *homeomorphism*. By giving U_i the inherited smooth structure, we can pull back the smooths structure on $U_i \times \mathbf{R}^{n-n'}$ to give $q^{-1}(U_i)$ a smooth structure. We generate topologies/take maximal atlases everywhere. □

Corollary 18

Every subbundle of rank k of a real bundle $p\colon E \to B$ of rank n over a paracompact Hausdorff space B admits a complement. In particular, if $E_1 \subset E$ is a subbundle, then $E/E_1 \cong E_1^\perp$ (non-canonically, I think) for any choice of metric on E. In particular, $E \cong E_1 \oplus E_1^\perp$ and $E/E_1 \cong E_1^\perp$.

Proof. Let $E_1 \subset E$ be a subbundle over B. Fix a metric g and let $E_1^\perp$ be its fiberwise orthogonal complement. One can check that $E_1^\perp$ is a subbundle and that $E \cong E_1 \oplus E_1^\perp$. Denote $q\colon E/E_1 \to B$ and $q_1\colon E_1^\perp \to B$ the bundle projections (the latter being the restriction of p to $E_1^\perp$ and the former being defined in essentially the same manner). Note further that we can give the bundle $q_1\colon E_1^\perp \to B$ the *same* trivializations as q and as p. For E/E_1, the trivializations are defined as above.

There is a fiberwise isomorphism $E_1 \oplus E_1^\perp \to E$ by sending vectors to their sum. Note that this sends the obvious subbundle $E_1 \oplus 0$ to the subbundle E_1 diffeomorphically, clearly. To see that this is smooth, note that in coordinates this looks like $U \times \mathbf{R}^k \times \mathbf{R}^{n-k} \to U \times \mathbf{R}^n$ sending $(p, v, w) \mapsto (p, v + w)$ and this is certainly smooth. To get this description, we just have to observe that local frames for E_1 and $E_1^\perp$ yield a local frame for their direct sum as well as for E. Since this is smooth and bijective, it is a diffeomorphism.

The last thing to check is that $E_1 \oplus E_1^\perp / E_1 \cong E_1^\perp$, since it surely must be that $E_1 \oplus E_1^\perp / E_1 \cong E/E_1$ because the isomorphism given above preserves the copies of E_1. Define $E_1 \oplus E_1^\perp \to E_1^\perp$ by sending $(p, v, w) \mapsto (p, w)$. This descends to the desired fiberwise quotient as a function. The description of the quotient given above immediately shows that it is smooth with little effort. □

Before this next theorem, we need an easy auxiliary result.

Proposition 4

If $f\colon M \to N$ is smooth and $q \in \partial N$ is a regular value for f, then $f^{-1}(q) \subset \partial M$. More generally, for smooth $f\colon M \to N$ with $\dim N \leq \dim M$ where f has maximal rank at $p \in M$, if $f(p) \in \partial N$ then $p \in \partial M$.

Proof. For f to even have a regular point p, we must have that $\dim N \leq \dim M$. We will prove parts in one go, since nothing we do below will depend on q being a regular value of f, only that $f(p) = q$ and f has maximal rank $\dim N \leq \dim M$ at p.

Let $q \in \partial N$ be a regular value and suppose for a contradiction there is a point $p \in f^{-1}(q) \cap \mathrm{Int}(M)$. Then we can take a small enough coordinate nbhd about p such that the coordinate chart is strictly Euclidean.

At this point, we assume WLOG that we are working in the following coordinate set up where our map has the form

$$\mathbf{R}^m \supset U_0 \xrightarrow{f} \mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0} \subset \mathbf{R}^n. \qquad (*)$$

By **(c)** of the constant rank theorem, there is a chart for $\mathbf{R}^m$ about $p \in U_0$, call it (x, U) with $U \subset U_0$ such that the new coordinate form of $(*)$ looks like a projection

$U \to \mathbf{R}^n$ killing the last $m-n$ coordinates. We now wish to analyze what happens to U under this projection.

Observation. Note that we only modify the domain chart, so the image of f in the coordinates of (x, U), $U \subset U_0 \subset \mathbf{R}^m$, remains unchanged and so is still a subset of $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$.

Since $U \subset \mathbf{R}^m$ is open and the projection $\mathbf{R}^m \to \mathbf{R}^n$ is an open map, it follows from the above description of f in the coordinates of (x, U) that $f|U$ is an open map *into* $\mathbf{R}^n$, since it is an open map in the coordinates of (x, U). By the observation, regardless of the domain chart, f takes U into $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$. But the open subsets of $\mathbf{R}^n$ *contained in* $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$ are precisely the ones that miss $\partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0})$, since there is no open subset V of $\mathbf{R}^n$ for which $W = V \cap \partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}) \neq \varnothing$ is open, and this is because W its own boundary in that case.

Thus, $f(p) = q$ could not possibly be in ∂N since, in coordinates, it misses $\partial(\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0})$. This is the first part. As mentioned at the beginning, this has secretly shown the second part. $\square$

Warning. I do not think this analysis can be extended further because we would have to extend from an open subset of $\mathbf{R}^{m-\ell} \times \mathbf{R}^\ell_{\geq 0}$ to an open subset of $\mathbf{R}^m$ and we can no longer guarantee that the extension stays in $\mathbf{R}^{n-k} \times \mathbf{R}^k_{\geq 0}$.

Remark. The contrapositive of the above is that for smooth f with maximal rank $\dim N$ at p, $p \in \operatorname{Int}(M)$ implies that $f(p) \in \operatorname{Int}(N)$. So for a submersion, it appears to be possible that boundary points in M are send to interior points in N.

Corollary 19

There is no surjective smooth submersion $f\colon M \to N$ for $\partial M = \varnothing$ and $\partial N \neq \varnothing$.

Proof. Easy consequence of the above. $\square$

Theorem 22

Let $\pi\colon E \to B$ be a smooth vector bundle and let $i\colon Z \hookrightarrow E$ be a closed injective immersion such that $Z \cap E_b$ is a linear subspace for all $b \in B$ whose dimension is locally constant as a function of $b \in B$ (in particular, $Z \cap E_b \neq \varnothing$ for any $b \in B$). Suppose in addition the following properties hold.

(i) Locally, i can be made to look like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^n$.

(ii) For every $z \in Z$, there is a smooth local section about $\pi(z) \in B$ of π such that $\pi(z) \mapsto z$.

Then Z admits a unique structure as a smooth vector bundle over B for which i is a subbundle inclusion. If Z has no boundary, then B has no boundary and the local assumptions are automatically satisfied.

Proof. WLOG B is connected. Then Z inherits a linear structure on its fibers and we must only check there are compatible local trivializations for this structure. By the universal property of submanifolds (a closed injective immersion is an embedding, after all), the zero section $B \to E$ which lands in Z is smooth into Z. Thus, $B \xrightarrow{0} Z \xrightarrow{\pi} B$ is smooth and the identity and so by the chain rule $Z \to B$ is a submersion. It is surjective from our assumptions.

When Z has no boundary, there can be no smooth surjective submersion onto B unless $\partial B = \varnothing$ as well by the above. Hence, by the constant rank theorem, the desired local sections will clearly exist. Thus, for each $b \in B$, if $X_1, \ldots, X_{n'}$ is a basis of Z_b, then we can find, locally, smooth sections $s_1, \ldots, s_{n'}$ such that $s_i(b) = X_i$ for each i.

Having shown the above, we return to the general case. By shrinking, we may suppose the problem is local and thus we may suppose the bundle E is trivial over B, say

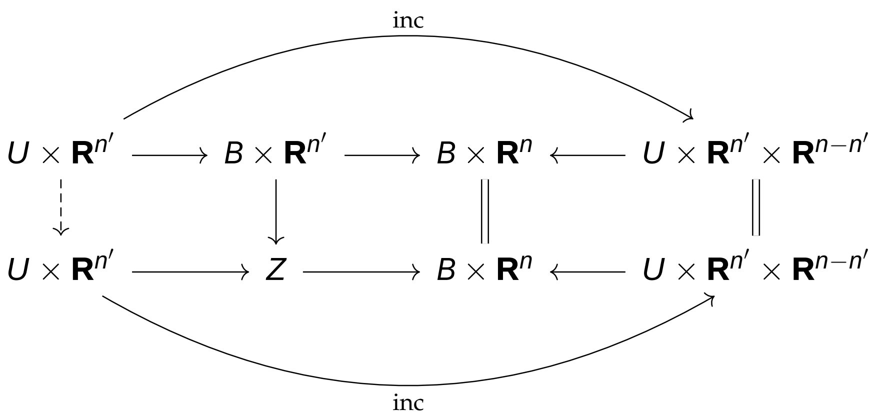
$$E = B \times \mathbf{R}^n.$$

Let the $s_1, \ldots, s_{n'}$ be as in the paragraph above for some $b \in B$. At b, these smooth sections $s_1, \ldots, s_{n'}$ form an $n \times n'$ matrix of rank n' and therefore there is an $n' \times n'$ submatrix which is invertible and thus invertible on a nbhd of b. Shrinking again we may suppose that this is globally invertible and thereby suppose the sections $s_1, \ldots, s_{n'}$ are fiberwise linearly independent *for all* $b \in B$.

Now consider the map $S\colon B \times \mathbf{R}^{n'} \to B \times \mathbf{R}^n$ defined by

$$(b, r_1, \ldots, r_{n'}) \mapsto (b, r_1 s_1(b), \ldots, r_{n'} s_{n'}(b)).$$

This is clearly smooth and fiberwise injective with $\operatorname{Im}(S) = Z$ and defines a subbundle essentially because we have constructed the sections s_i. Thus, S is a closed immersion and thus also an embedding. Since $i\colon Z \to E$ is another map with the same properties and same image. Hence, there are unique continuous maps $Z \to B \times \mathbf{R}^{n'}$ and $B \times \mathbf{R}^{n'}$ which factor i and S through each other. To show that this is smooth, we simply use the fact that, locally, each map $Z \to B \times \mathbf{R}^n$ and $B \times \mathbf{R}^{n'} \to B \times \mathbf{R}^n$ look like the standard inclusion $U \times \mathbf{R}^{n'} \to U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} \cong U \times \mathbf{R}^n$. Essentially, TFDC:



which, by projecting, $U \times \mathbf{R}^{n'} \times \mathbf{R}^{n-n'} \to U \times \mathbf{R}^{n'}$ shows that the dashed arrow is simply the identity. Similarly in the other direction for $Z \to B \times \mathbf{R}^{n'}$. This shows that the two maps $Z \to B \times \mathbf{R}^{n'}$ and $B \times \mathbf{R}^{n'}$ are in fact smooth and fiberwise linear. They necessarily inverse to each other, so this establishes the desired fiberwise linear isomorphism. What we have actually shown (un-reducing all of our assumptions) is that Z has local trivializations, as desired. □

Theorem 23

Over paracompact Hausdorff spaces, all short exact sequences of bundles split, but as usual the splitting is not natural. In particular, in the smooth category, the splitting is additionally smooth.

Proof. We just need access to partitions of unity. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of bundles (i.e., fiberwise short exact). We construct a section $s\colon B \to A$ of $i\colon A \to B$. Pick a local trivialization of A and extend this to a local trivialization of B in such a way that the trivialization has A sit as $\mathbf{R}^k \times \mathbf{0}$ in $\mathbf{R}^n$ —this exists as we have seen. The section is obvious then. Doing this locally everywhere by a partition of unity argument, we must show that the resulting thing is a global left inverse. One can do this with careful analysis.

Now we must show that this implies that B splits. This follows by showing that $A \oplus B/A \cong B$, which can be done. □

Remark. Alternatively, equip the bundle B with a Riemannian metric by a partition of unity argument and take the orthogonal complement of A in B. The argument fails in the holomorphic category because we need not have a holomorphic partition of unity. This is what separates the world of smooth manifolds from the world of complex manifolds—the latter are much more rigid objects.

Warning. Kernels are only guaranteed to exist in the category of vector bundles when we take the kernel of an epimorphism. See Hirsch's book on page 93.

Definition 29

A ***(linear) sphere bundle*** (resp. ***(linear) disk bundle***) is a fiber bundle in which every fiber is (homeomorphic to) the standard metric (i.e., unit) sphere (resp. metric disk) in Euclidean space having structure group the orthogonal group.

Lemma 19

A (smooth) vector bundle (of rank n) $E \to B$ is the same thing as a fiber bundle $F \to E \to B$ with structure group $GL_n(\mathbf{R})$ and a (smooth) $GL_n(\mathbf{R})$-equivariant isomorphism $F \cong \mathbf{R}^n$ for all $p \in B$.

If B is paracompact Hausdorff, then a (smooth) vector bundle (of rank n) $E \to B$ is additionally the same thing as a (smooth) vector bundle with structure group $O(n)$ which is the same thing as a fiber bundle $F \to E \to B$ with structure group $O_n(\mathbf{R})$ and a (smooth) $O_n(\mathbf{R})$-equivariant isomorphism $F \cong \mathbf{R}^n$.

Proof. For the first part, the inclusion $\subset$ is clear from the trivializations. For $\supset$, make F into a vector space by pulling back the vector space structure on $\mathbf{R}^n$. We can then define new trivializations by composing with the isomorphism $F \cong \mathbf{R}^n$: $\psi_j \colon q^{-1}(U_i) \cong U_i \times F \cong U_i \times \mathbf{R}^n$. Define a vector space structure on E_p by fixing a trivialization about p and pulling back the vector space structure from any trivialization. The choice of trivialization does not matter up to isomorphism of vector spaces. To see this, begin by letting $p \in U_i \cap U_j$. Then the transition functions relate the homeomorphisms/diffeomorphisms $\psi_j \colon E_p \cong \mathbf{R}^n$ and $\psi_i \colon E_p \cong \mathbf{R}^n$ by a linear isomorphism, since $F \cong \mathbf{R}^n$ is $GL_n(\mathbf{R})$-equivariant. The claim, then, is that the two induced structures on E_p are isomorphic, and this is clear because pulling back this structure means that the two structures will themselves be related by an element of $GL_n(\mathbf{R})$. Thus, for the trivialization $\psi_i \colon q^{-1}(U_i) \cong U_i \times \mathbf{R}^n$, we have for $p \in U_i \cap U_j$ and E_p the structure coming from the index j that $\psi_i | E_p$ is still linear since it becomes linear after post-composition with $t_{ji}(p) = t_{ij}(p)^{-1} \in GL_n(\mathbf{R})$, which is a linear isomorphism and so forces $\psi_i | E_p$ to be.

For second part, give the vector bundle a (smooth) metric and on each trivialization let $e_1^i, \dots, e_n^i \colon U_i \to U_i \times \mathbf{R}^n$ be a (smooth) orthonormal frame for the metric. Let the transition functions now be defined by letting $t'_{ij}(p)$ be the change of basis matrix taking $(e_1^j(p), \dots, e_n^j(p)) \mapsto (e_1^i(p), \dots, e_n^i(p))$. This is clearly smooth and the resulting vector is still isomorphic to the one with the old t_{ij} via the identity map. The last part is analogous to the above. □

Proposition 5

Over a paracompact Hausdorff base space, a real vector bundle of rank n having structure group $O(n)$ determines and is determined by linear sphere bundles and linear disk bundles. That is, these notions are "the same."

Proof. Strictly speaking, this follows by the equivalence of categories $\mathrm{Bun}_{O(n)}^{\mathbf{R}^n} \simeq \mathrm{Prin}_{O(n)} \simeq \mathrm{Bun}_{O(n)}^{S^{n-1}}$ and similarly for linear disk bundles. □

Lemma 20

Let V and W be vector bundles over X. Then $\mathrm{Hom}(V, W) \cong V^* \otimes W$ and if V and W have common rank n, then the subset $\mathrm{Iso}(V, W)$ is a fiber bundle over X with typical fiber $\mathrm{GL}_n(\mathbf{R})$ and $\Gamma(\mathrm{Iso}(V, W)) \cong \{\text{bundle isos } V \cong W\}$.

Proof. A section $X \to \mathrm{Iso}(V, W)$ is a choice of isomorphism $V_p \to W_p$ for all $p \in X$. We must show that this determines an isomorphism of bundles. In a nbhd of $U \subset X$, this is a section $U \to U \times \mathrm{GL}_n(\mathbf{R}))$ and is therefore determined by $f_U \colon U \to \mathrm{GL}_n(\mathbf{R})$. Such a map determines at each $p \in U$ a map $\mathbf{R}^n \to \mathbf{R}^n$ and so an assignment $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$ given by $(p, v) \mapsto (p, f_U v)$ which is therefore as continuous or smooth as f_U is. We worked locally and these all glue. □

Lemma 21

Let V and W have the same rank. Then $\mathrm{Iso}(V, W)$ is an open subset of $\mathrm{Hom}(V, W)$

Proof. In the trivializations, this looks something like $U \times \mathbf{R}^{n^2}$ and the isos are the matrices of full rank which is an open condition. □

C.2 Some Further Recollections on Bundles

Lemma 22

Let $f, g\colon M \to \mathbf{R}$ be functions from a manifold into $\mathbf{R}$ and let $0 \leq k \leq \infty$. If $f_1 + \cdots + f_n = h$ is C^k and $f_1, \ldots, f_{n-1}$ are C^k, then f_n is C^k.

Proof. $f_n = h - (f_1 + \cdots + f_{n-1})$ and so must be C^k since h and the sum $f_1 + \cdots + f_{n-1}$ are. □

Proposition 6

Let $p\colon E \to B$ and $p'\colon E' \to B$ be vector bundles of rank n and let $f\colon E \to E'$ be a smooth map that is a linear isomorphism on each fiber. f is then a bundle isomorphism—that is, it is a diffeomorphism over B.

Proof. In bundle coordinates, f looks like a map $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$ by $(p, v) \mapsto (p, f_p(v))$ for f_p the bundle coordinate version of the relevant linear isomorphism. Define f^{-1} by $(p, v) \mapsto (p, f_p^{-1}(v))$. Let $A\colon U \to \mathrm{GL}_n(\mathbf{R})$ be such that $A(p)v = f_p(v)$ so that f is $(p, v) \mapsto (p, A(p)v)$.

Claim. The action $(p, v) \mapsto A(p)v$, $U \times \mathbf{R}^n \to \mathbf{R}^n$, is smooth. Therefore the adjoint of A is smooth into $\mathrm{GL}_n(\mathbf{R})$, which is equivalent to saying that A is smooth into $\mathbf{R}^{n^2}$ and hence equivalent to saying that the component functions of A are smooth.

For convenience, we will write A for $A(p)$. Since $\mathrm{GL}_n(\mathbf{R})$ is an open subset of $\mathbf{R}^{n^2}$, being the preimage under det of $\mathbf{R} \setminus \{0\}$, smoothness into $\mathrm{GL}_n(\mathbf{R})$ is equivalent to smoothness into $\mathbf{R}^{n^2}$. Recall that we are in bundle coordinates $U \times \mathbf{R}^n \to U \times \mathbf{R}^n$—WLOG suppose U is the domain of a chart on B perhaps by shrinking if necessary. Observe that smoothness of $(p, v) \mapsto (p, A(p)v)$ means that the assignment $(p, v) \mapsto A(p)v$, $U \times \mathbf{R}^n \to \mathbf{R}^n$, is smooth. This is because finite products exist in the category of manifolds. In particular, fix $v_0 = (\delta_j^i)$. Then $U \times \{v_0\} \to \mathbf{R}^n$ is smooth since $U \times \{v_0\}$ is a submanifold of $U \times \mathbf{R}^n$. This map is then $(p, v_0) \mapsto (A_{1i}, \ldots, A_{ni})$ and so for this to be smooth in $\mathbf{R}^n$, each component must be smooth. Now the map $A\colon U \to \mathrm{GL}_n(\mathbf{R})$ is simply the map $p \mapsto (A_{ij}(p))$ and by the above observation that $\mathrm{GL}_n(\mathbf{R})$ is open in $\mathbf{R}^{n^2}$, this is smooth because each component is smooth.

Claim. The inversion $(-)^{-1}\colon \mathrm{GL}_n(\mathbf{R}) \to \mathrm{GL}_n(\mathbf{R})$ is smooth.

The inverse of matrix has entries rational functions which in the (i,j) spot has numerator a polynomial in the various relevant entries for the relevant minor and has numerator the determinant of the matrix. Since $\det\colon \mathrm{GL}_n(\mathbf{R}) \to \mathbf{R}$ is smooth and non-vanishing, the denominator is a smooth and non-vanishing function, so everything checks out.

Putting this together, the function defined in bundle coordinates as $(p,v) \mapsto (p, A^{-1}(p)v)$ is smooth, it is well-defined since we have defined it in bundle coordinates locally, and it is clearly inverse to the given map. $\square$

Lemma 23

Let $p\colon E \to B$ be a smooth rank n vector bundle. Let $\underline{\mathbf{R}}$ be the trivial rank 1 bundle over B. Then the bundle maps $m\colon \underline{\mathbf{R}} \oplus E \to E$ and $+\colon E \oplus E \to E$ are smooth, where this is the Whitney sum.

Proof. These are the Whitney sums of the bundles. Let U be a trivializing nbhd for E, which we can assume exists by shrinking if necessary any trivializing nbhd. The resulting trivialization of $\underline{\mathbf{R}} \times E$ is then simply the one sending $(p,r,v) \mapsto (p,r,\Phi_p(v))$ where $\Phi\colon p^{-1}(U) \to U \times \mathbf{R}^n$ is the trivializing diffeomorphism. The first map in coordinates is given by $U \times \mathbf{R} \times \mathbf{R}^n \to U \times \mathbf{R}^n$ sending $(p,r,v) \mapsto (p,rv)$. This is basically a diagram chase since for $p \in B$ $m_p(r,v_p) = rv_p \in E_p$ since the trivializations respect vector space operations. This map is further in coordinates $\mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^m \times \mathbf{R}^n$ by $(p,r,v) \mapsto (p,rv)$. The multiplication map $\mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is clearly smooth. For the second map, one argues as before and notes that addition $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is clearly smooth. $\square$

Lemma 24 (Lee, 10.19)

Let $p\colon E \to B$ be a smooth vector bundle of rank n and let $U \subset B$ be an open neighborhood. Denote $\widetilde{e}_i\colon U \to U \times \mathbf{R}^n$ the i-th standard section $p \mapsto (p, e_i)$. For any smooth local frame $\{s_1, \dots, s_n\}$ on U, there exists a diffeomorphism—in fact trivialization—$\Psi\colon p^{-1}(U) \to U \times \mathbf{R}^n$ such that $\Psi^{-1} \circ \widetilde{e}_i = s_i$. Hence, smooth sections over an open set U determine a smooth bundle trivialization and conversely.

Proof. We will define Ψ^{-1} and show it is fiberwise linear and a diffeomorphism, justifying the inverse notation. Define $\Psi^{-1}(p,(v_1,\dots,v_n)) = \sum_i v_i s_i(p)$ and note that this is certainly fiberwise linear! To show this is smooth, we only need to check that the operation of summing is smooth on $p^{-1}(U)$. This is true since for any $V \subset U$ a trivializing open nbhd with Φ the trivialization, Φ is a diffeomorphism linear on each fiber and so commutes with the sum and hence $\sum_i v^i s_i(p) = \Phi^{-1}\Phi(\sum_i v^i s_i(p)) = \Phi^{-1}(\sum_i \Phi(v^i s_i(p)))$ and the fiberwise sum on $U \times \mathbf{R}^n$ is smooth as part of the definition of a smooth vector bundle from the above. Thus, if Ψ^{-1} is smooth, then Ψ is a smooth local trivialization and clearly we have $\Psi^{-1} \circ \widetilde{e}_i = s_i$.

It is clear that Ψ^{-1} is a bijection since the s_i form a frame, so to show it is a diffeomorphism, it suffices to show it is a local diffeomorphism. Let $V \subset U$ be a trivializing open nbhd as above. If we can show that $\Phi \circ \Psi^{-1}\big| V \times \mathbf{R}^n$ is a diffeomorphism of $V \times \mathbf{R}^n$ with itself, then since Φ is a diffeomorphism, we will have that Ψ^{-1} is a diffeomorphism $V \times \mathbf{R}^n \to p^{-1}(V)$. Now, $\Phi \circ s_i$ is smooth as a composite of smooth functions. Hence, in coordinates $\Phi(s_i(p)) = (p, (\sigma_1^i(p), \ldots, \sigma_n^i(p)))$ and the σ_i must be smooth in p for this function to be smooth. Thus,

$$\Phi \circ \Psi^{-1}(p, (v_1, \ldots, v_n)) = \Phi(\sum_i v_i s_i(p)) = (p, (\sum_i v_i \sigma_1^i(p), \ldots, \sum_i v_i \sigma_n^i(p))) = \sum_i \Phi(v_i s_i(p))$$

which is smooth as the sum operation is smooth as soon as we know that the sum operation is smooth and we do know this (essentially the last equality). What's happening here is that the smooth matrix $(\sigma_j^i)_{i,j}$ is at each point p the matrix $(\sigma_j^i(p))_{i,j}$ which transforms something in the ordered basis $(s_1(p), \ldots, s_n(p))$ for E_p to something in the standard basis for $\mathbf{R}^n$. In other words, this is a change of basis matrix and it is therefore invertible. Thus, $\Phi \circ \Psi^{-1}(p, (v_1, \ldots, v_n)) = (\sigma_j^i(p))(v_1, \ldots, v_n)^t$ the matrix multiplication—this is smooth because the matrix multiplication just gives polynomials in smooth functions. It follows that the inverse is given by $(\Phi \circ \Psi^{-1})^{-1}(p, (w_1, \ldots, w_n)) = (\sigma_j^i(p))^{-1}(w_1, \ldots, w_n)^t$ and since (σ_j^i) is everywhere invertible, its determinant is always non-zero and smooth, so the inverse matrix is a smooth function being a rational function of smooth functions where the denominator never vanishes. □

Remark. Nothing we used above relied on using $\mathbf{R}^n$ for the typical fiber. We could just as well have consider complex vector bundles with typical fiber $\mathbf{C}^n$.

Exercise 38

Does the above hold for the real (or complex) quaternions?

Corollary 20

If an open nbhd $U \subset B$ admits a smooth local frame for $p \colon E \to B$ a smooth vector bundle of rank n, then U is a trivializing open nbhd.

Corollary 21

A smooth local trivialization is equivalent to a smooth local frame by sending $v \in E_p$ to $(v_1, \ldots, v_n)$ where $\sum_i v_i s_i(p)$.

Proof. This just deconstructs what the construction above did. □

Corollary 22

Let $\pi \colon E \to B$ and $\pi' \colon E' \to B$ be smooth vector bundles of rank n and n' respectively with say $\dim B = m$. Let $f \colon E \to E'$ any fiberwise linear function (not assumed to be continuous or anything). Then f is smooth **iff** each point $p \in B$ is contained in the domain of a smooth local frame $\mathscr{F}$ such that f sends each section in $\mathscr{F}$ to a smooth function.

Proof. The direction $(\Rightarrow)$ is trivial since f is fiberwise linear, so let $s_1, \ldots, s_n$ be smooth sections of the first in a nbhd of a point that form a frame and let $\sigma_i = f \circ s_i$ and suppose the σ_i are smooth. Then in the trivialization constructed from the smooth local frame $\mathscr{F}$, we know this is $U \times \mathbf{R}^n \to (\pi')^{-1}(U)$ by

$$(p, (v_1, \ldots, v_n)) \mapsto \sum_i v_i s_i(p) \mapsto \sum_i v_i \sigma_i(p).$$

Note that we have used the fact that f is fiberwise linear to pull the coefficients out at the last step—this is evidently an indispensable assumption.

Let s_i' be local frame for E' on this same nbhd (perhaps by shrinking). Since the σ_i are smooth, $\sigma_i = \sum_{k=1}^{n'} c_{ik} s_k'$ where the c_{ik} are smooth real-valued functions. Thus, this can be written

$$(p, (v_1, \ldots, v_n)) \mapsto \sum_i \sum_{k=1}^{n'} v_i c_{ik}(p) s_k'(p) = \sum_{k=1}^{n'} \left(\sum_{i=1}^{n} v_i c_{ik}(p) \right) s_k'(p)$$

Hence, in the local trivializations afforded to us by these frames as we constructed above, the assignment is

$$(p, (v_1, \ldots, v_n)) \mapsto (p, (\sum_{i=1}^{n} v_i c_{i1}(p), \ldots, \sum_{i=1}^{n} v_i c_{in'}(p))).$$

This is smooth because each of the components are smooth. Indeed, using a chart for U, this is basically just

$$((x_1, \ldots, x_m), (v_1, \ldots, v_n)) \mapsto ((x_1, \ldots, x_m), (\sum_{i=1}^{n} v_i c_{i1}(x_1, \ldots, x_n), \ldots, \sum_{i=1}^{n} v_i c_{in'}(x_1, \ldots, x_n)))$$

All mixed partial derivatives with respect to each coordinate $x_1, \ldots, x_n, v_1, \ldots, v_n$ clearly exist and are always smooth, clearly. □

Corollary 23

Let $\pi \colon E \to B$ be a smooth vector bundle over B and $f \colon E \to \mathbf{R}$ a map that is linear on each fiber. Then f is smooth **iff** f sends some smooth local frame in a neighborhood of every point to smooth functions $B \to \mathbf{R}$.

Proof. f is the composite $E \to \mathbf{R} \times B \to \mathbf{R}$ where the last map is the projection and is therefore smooth and the first map sends $v \in E_p$ to $(f(v), \pi(p))$ which is smooth precisely if f is smooth (since π is assumed to be smooth). This reduces us to the case above for the map (f, π) where it suffices to show that (f, π) satisfies the conclusions of the preceding corollary and surely it does. □

C.3 Normal Bundles & Tubular Neighborhoods

Normal Bundles

Reminder. Recall that we have seen that $E'/E \cong E^\perp$.

Definition 30

Let $f\colon M \to N$ be an immersion. Denote $\nu_f = (f^*TN)/TM$ the *normal bundle of the immersion* f. Here, the quotient by TM occurs via the identification of TM with its image in TN. When f is an embedding of M into N, we denote this by ν_M.

Remark. Recall that $f^*TN = \{(p,v) \in M \times TN : f(p) = \pi_N(v)\}$.

Lemma 25

If N is a Riemannian manifold, then ν_f may be taken to be the subbundle of

$$f^*TN = \{(p,v) \in M \times TN : f(p) = \pi_N(v)\}$$

consisting of all pairs (p,v) where $v \in T_pM^\perp$ (identifying T_pM with its image).

Proof. Exercise. Should be similar to the proof that $E'/E \cong E^\perp$. □

Theorem 24

Let $f\colon M \to N$ be an immersion. Then $f^*TN \cong TM \oplus \nu_f$.

Proof. Use a metric. Define $TM \oplus \nu_f \to f^*TN$ by sending $(p,v,w) \mapsto (p, v+w)$. This is smooth and a fiberwise isomorphism so it is a diffeomorphism. □

Remark. Everything above ought to hold for manifolds with boundary.

Exponential Map and Shrinking

Taken from Riemannian Geometry class notes. All manifolds are without boundary.

Reminder. Recall that for a Riemannian manifold M with $\dim M = n$, we call $\gamma_{p,v}$ the geodesic having $\dot\gamma(0) = v$ and $\gamma(0) = p$. In coordinates, the geodesic equation is $\ddot\gamma^\ell(t) + \Gamma^\ell_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t) = 0$ for $1 \le \ell \le n$, where $\Gamma^\ell_{ij} = \frac{1}{2}g^{\ell k}(g_{ik,j} + g_{jk,i} - g_{ij,k})$. More concisely, this is $D_t\dot\gamma(t) = 0$, where D_t is the covariant derivative along γ.

Proposition 7 (Naturality of geodesics)

Let M and $\widetilde{M}$ be two Riemannian manifolds and $\varphi : M \to \widetilde{M}$ a Riemannian isometry. If $p \in M$ and γ is a geodesic on M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v \in T_pM$, then $\widetilde{\gamma} := \varphi \circ \gamma$ is a geodesic on $\widetilde{M}$ such that $\widetilde{\gamma}(0) = \varphi(p)$ and $\dot{\widetilde{\gamma}}(0) = \varphi_*(v)$.

Remark. Note that the geodesic equation $D_t\dot{\gamma}(t) = 0$ is a *non-linear* differential equation.

Lemma 26

There exists a unique vector field G on TM whose integral curves are of the form $t \mapsto (\gamma(t), \dot{\gamma}(t))$ where γ is a geodesic. The flow of G is called the ***geodesic flow***.

Proof. The geodesic equations are in local coordinates $\ddot{x}^\ell + \Gamma^\ell_{ij}\dot{x}^i\dot{x}^j = 0$. We reduce this to a first order equation by introducing the variable $y^k = \dot{x}^k$. Then in bundle coordinates for TU, a solution to the geoedesic equation $t \mapsto (x^1(t), \ldots, x^n(t), \dot{x}^1(t), \ldots, \dot{x}^n(t))$ satisfies the system of first order equations

$$\begin{cases} \dot{x}^k = y^k & 1 \le k \le n, \\ \dot{y}^k = -\Gamma^k_{ij}y_iy_j & 1 \le k \le n. \end{cases}$$

where, here, this is in terms of the coordinates afforded by the trivializing frame $(x^1, \ldots, x^n, \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. By standard results, there is a flow for this (the centered equations just above) pinned down by the usual specification. We recall that the flow is obtained by piecing together the integral curves, and it is unique by uniqueness of integral curves as usual—in particular, the integral curves are geodesics where the geodesic through (p, v) is precisely $\gamma_{p,v}$. □

Corollary 24 (Local Existence and Uniqueness)

Suppose $\partial M = \varnothing$. Let $p_0 \in M$ and $u_0 \in T_{p_0}M$. Then there exists $\varepsilon_0 > 0$ and an open neighborhood $U_0 \subset TM$ of (p_0, u_0) with the following properties:

1. For any $(p, u) \in U_0$, there exists a unique geodesic $\gamma_{p,u} : (-\varepsilon, \varepsilon) \to M$ such that $\gamma_{p,u}(0) = p$ and $\dot{\gamma}_{p,u}(0) = u$.
2. The map $\gamma_{\cdot,\cdot}(\cdot) : U_0 \times (-\varepsilon_0, \varepsilon_0) \to M$ defined by $((p, u), t) \mapsto \gamma_{p,u}(t)$, is smooth.

Proof. This follows by consideration of the properties that flows have. □

Remark. This has an analogous phrasing when $\partial M \neq \varnothing$, just changing words slightly.

Corollary 25

Fix $s \in \mathbf{R}$. If $\gamma_{p,sv}(1)$ exists, then $\gamma_{p,v}(s)$ exists and $\gamma_{p,tv}(1) = \gamma_{p,v}(s)$. In particular, $\gamma_{p,sv} = \gamma_{p,v}(s \cdot -)$.

Proof. If $s = 0$, then we can check by hand that this is true. So suppose $s \neq 0$. In local coordinates, one checks that $\gamma_{p,v}(s \cdot -)$ is the solution to the IVP for

$$\ddot{\gamma}^\ell(t) + \Gamma^\ell_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \qquad 1 \leq \ell \leq n$$

subject to the initial conditions $\dot{\gamma}(0) = sv$ and $\gamma(0) = p$. This is because we can divide through by the common factor of s^2. Hence, uniqueness forces our hand. ☐

Set

$$O_p \overset{\text{def}}{=} \{v \in T_pM : \gamma_{p,v}(t) \text{ is defined for all } t \in [0,1]\} \subset T_pM.$$

Notice that by the preceding, there exists $\delta > 0$ such that $B^{T_pM}_\delta(0_p) \subset O_p$ (an open ball). It will turn out that O_p is open and that $O = \bigcup_{p\in M} O_p$ are both open.

Definition 31

For $p \in M$, define the *exponential map* at p as $\exp_p : O_p \to M$ by $v \mapsto \gamma_{p,v}(1)$. When $\partial M \neq \varnothing$, we must restrict to inward pointing vector for this to make sense.

Remarks.

1. For p fixed, the map $\exp_p$ is C^∞ (for instance by smoothness of flows).
2. For $t \in \mathbf{R}$ and $v \in T_pM$ such that $tv \in O_p$, we have $\exp_p(tv) = \gamma_{p,tv}(1) = \gamma_{p,v}(t)$.

Proposition 8

Let $\dim M = n$. The differential map $d\exp_p(0_p)$ is the identity where we understand $T_0T_pM \cong \mathbf{R}^n$ and $T_pM \cong \mathbf{R}^n$.

Proof. Pick $v \in T_pM$. Since $\gamma_{p,tv}(1) = \gamma_{p,v}(t)$, we have

$$d\exp_p(0_p)(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv) = \frac{d}{dt}\Big|_{t=0} \gamma_{p,v}(t) = \dot{\gamma}_{p,v}(t)\Big|_{t=0} = v.$$

☐

Corollary 26

On a neighborhood of $0_p \in T_pM$, the exponential map $\exp_p$ is a diffeomorphism onto its image in M (with suitable modification to inward pointing vectors when $\partial M \neq \varnothing$).

Proof. This follows from the inverse function theorem since $d\exp_p(0_p)\colon T_{0_p}O_p \to TM$ is an isomorphism. When $\partial M \neq \emptyset$ and $p \in \partial M$, we restrict ourselves to inwards pointing vectors. Smoothness of $\exp_p$ about 0_p implies it has a smooth extension in an open nbhd of $\mathbf{R}^n$ (in coordinates) and thus since full rank is an open condition, WLOG that it is full rank on this nbhd. The same argument now works. □

Lemma 27

Suppose $\partial M = \emptyset$. exp is smooth on an open subset of O. In particular, O is open in TM, O_p is open in T_pM, and exp is smooth on O. When $\partial M \neq \emptyset$, exp is at least smooth.

Proof. Suppose $\dim M = n$. Let G denote the geodesic flow, which we assume is maximal, as always—let $A \subset \mathbf{R} \times M$ denote the maximal flow domain, which we know is a subset of $\mathbf{R} \times TM$ which is open when $\partial M = \emptyset$. Thus, when M has no boundary, the set $TM_1 = \{(p,v) \in TM : (1,p,v) \in A\}$ is open in TM. In particular, if $(p,v) \in TM_1$, then $(p,v) \in TM_t$ for all $t \in [0,1]$ since one constructs the maximal flow domain as the union of the maximal integral curves. We can therefore write the exponential function on its maximal domain of definition as the composite $TM_1 \xrightarrow{(1,\mathrm{id})} A \xrightarrow{G} TM \xrightarrow{\pi} M$. All functions in sight are smooth and TM_1 is open in TM. Now observe that $TM_1 = O$ which is therefore open and moreover that $TM_{1,p} = TM_1 \cap T_pM = \left\{(p,v) \in T_pM : (1,p,v) \in A\right\} = O_p$ is open in T_pM in the subspace topology—the subspace topology on T_pM is equivalent to the topology it inherits from being diffeomorphic with $\mathbf{R}^n$. When $\partial M = \emptyset$, local existence at least tells us that the exponential is smooth locally and thus globally by the same composite argument where now TM_1 is simply a subset and not a submanifold. In other words, the geodesic flow still makes sense and we simply compose it with the projection—we must restrict ourselves to inwards pointing vectors of course on the boundary.

□

Corollary 27

Consider the map $E\colon O \to M \times M$ given by $(p,v) \mapsto (p, \exp_p(v))$. Then for each $p \in M$,

$$dE((p,0_p))\colon T_{(p,0_p)}TM \to T_{(p,p)}(M \times M)$$

is nonsingular.

Proof. Let (x,U) be chart about p in M. Note that any basis $\left.\frac{\partial}{\partial dx^i}\right|_{(p,0_p)}$ has for $1 \leq i \leq m$ $\left.\frac{\partial}{\partial dx^i}\right|_{(p,0_p)} = \left.\frac{\partial}{\partial x^i}\right|_p$ essentially by definition. Equipping the codomain with the basis induced by the chart $x \times x$, we see that the matrix of $dE((p,0_p))$ must have the form $\begin{pmatrix} \mathrm{id}_{m\times m} & 0_{m\times m} \\ X & Y \end{pmatrix}$ as the projection is independent of ∂_j for $j \geq m+1$. On the other hand, for $m+1 \leq i \leq 2m$, we already know that $d\exp_p(0_p)$ is the identity by the above. Hence, $Y = \mathrm{id}_{m\times m}$. Hence, in coordinates, we must have

$$dE((p,0_p)) = \begin{pmatrix} \mathrm{id}_{m\times m} & 0_{m\times m} \\ X & \mathrm{id}_{m\times m} \end{pmatrix}$$

which is upper triangular and therefore invertible. Hence, for each $p \in M$, $dE((p,0_p))$ is non-singular. □

Theorem 25 (Naturality exponential map)

Let M and $\widetilde{M}$ be two Riemannian manifolds, $\Phi : M \to \widetilde{M}$ be a Riemannian isometry and p a point in M. Denote by $\exp^M$ and $\exp^{\widetilde{M}}$ the exponential maps of M and $\widetilde{M}$, respectively. Then

$$\exp^{\widetilde{M}}_{\Phi(p)} \circ \Phi_* = \Phi \circ \exp^M_p .$$

Theorem 26

Let M and $\widetilde{M}$ be two Riemannian manifolds, and $\Phi_1, \Phi_2 : M \to \widetilde{M}$ be two Riemannian isometries. If there exists $p \in M$ such that $\Phi_1(p) = \Phi_2(p)$ and $d\Phi_1(p) = d\Phi_2(p)$, then $\Phi_1 \equiv \Phi_2$.

Proof. Exercise. (*Hint:* Prove that the set where the two isometries agree is both open and closed.) □

C.4 Polar Decomposition and Hermitian Bundles

Definition 32

Fix a choice for CAT. Given a complex vector bundle $p\colon E \to B$, a ***Hermitian metric*** on E is a section $h\colon B \to \overline{E} \otimes_{\mathbf{C}} E^*$ of the bundle $\overline{E} \otimes_{\mathbf{C}} E^* \to B$ such that at each point $p \in B$, the h_p is a complex inner product that is conjugate linear in the first coordinate.

Here, E^* is constructed as usual and has fiber over b F_b^* the $\mathbf{C}$-linear dual of F_b. The bundle $\overline{E}$ has the same underlying fiber bundle $E^* \to B$ but with $\mathbf{C}$-action on the fibers defined as follows. For $a + ib \in \mathbf{C}$ and $v \in F_p^*$, $(a+ib)v = av - ibv$.

Definition 33

Given a real vector bundle $p\colon E \to B$, a ***Riemannian metric*** on E is a section $h\colon B \to E^* \otimes_{\mathbf{R}} E^*$ of the bundle $E^* \otimes_{\mathbf{R}} E^* \to B$ such that at each point $p \in B$, h_p is an inner product. We demand this be smooth when everything in sight is smooth.

Proposition 9

Let ξ denote the rank $n \geq 1$ k-vector bundle $p\colon E \to B$ over a paracompact Hausdorff space with $k = \mathbf{R}$ or $\mathbf{C}$. Then there exists a bundle atlas for ξ with transitions in $O(n)$ when $k = \mathbf{R}$ and $U(n)$ when $k = \mathbf{C}$.

Proof. Using a partition of unity argument, we may equip the vector bundle with a metric in the real case and a hermitian metric in the complex case. The point is that convex linear combinations of positive-definite symmetric forms is positive definite and symmetric and, similarly, a convex linear combination of positive-definite hermitian forms is positive-definite hermitian.

Take any bundle atlas for ξ and consider the standard sections for each fiber given by $(s_1, \ldots, s_n)$ where $s_i\colon U \to p^{-1}(U)$ is defined by using the associated trivialization $\varphi\colon p^{-1}(U) \cong U \times \mathbf{R}^n$ (or $U \times \mathbf{C}^n$) as $p \mapsto \varphi^{-1}(p, e_i)$ everywhere. Even in the case of the hermitian inner product, the same Gram-Schmidt algorithm works and we may apply this algorithm to the sections associated to the trivializations to obtain everywhere orthonormal sections $\bar{s}_1, \ldots, \bar{s}_n$. In particular, using Lee 10.19, we may modify the trivialization to obtain a CAT trivialization that sends $\bar{s}_i \mapsto \bar{e}_i$ in the real case. In the complex case, a careful inspection of Lee 10.19 or **Lemma** 24 reveals nothing particular to $\mathbf{R}^n$ was used so this carries over to complex case just as well.

Having done this, we must check that the resulting transition functions are CAT and lie in $O(n)$ (resp. $U(n)$). For CAT = DIFF, smoothness follows in the real case from an argument from before and in the complex case by the analogous argument applied to $\mathrm{GL}_n(\mathbf{C})$. On overlaps, the transitions are $(p, v) \mapsto (p, g_{ij}(p)v)$ and we claim that $g_{ij}(p) \in O(n)$ (resp. $U(n)$). The corresponding transition coming from the trivializations we just constructed, call it φ_{ij}, sends $(e_1, \ldots, e_n)$ to an ordered orthonormal basis $(s_1^i(p), \ldots, s_n^i(p))$ for the relevant fiber. There a unique $O(n)$ (resp. $U(n)$) change-of-basis matrix A that sends this ordered orthonormal basis to the one corresponding to ordered orthonormal basis constructed the chart φ_j, say $(s_1^j(p), \ldots, s_n^j(p))$. Since φ_j is linear, $\varphi_j(A(s_1^i(p), \ldots, s_n^i(p))) = (e_1, \ldots, e_n) = A\varphi_j((s_1^i(p), \ldots, s_n^i(p)))$ and hence, one then deduces that $g_{ij}(p) = A^{-1} \in O(n)$ (resp. $U(n)$). $\square$

Remark. The uniqueness of A in the real case is apparent since it is completely specified by the requirement that $As_k^i = s_k^j$ for each $1 \leq k \leq n$. To check $A \in O(n)$ (resp. $A \in U(n)$), it is easier to forget about bases and simply check that $\langle Av \mid Aw \rangle = \langle v \mid w \rangle$, and this is certainly true because A took and orthogonal basis to an orthogonal basis.

Lemma 28

The space of symmetric (resp. hermitian) matrices is diffeomorphic to $\mathbf{R}^{n(n+1)/2}$ (resp. $\mathbf{R}^{n(n+1)/2+n(n-1)/2} = \mathbf{R}^{n^2}$ and the symmetric positive-definite (resp. hermitian positive-definite) matrices is an open submanifold thereof. The diffeomorphism is defined for hermitian matrices by decomposing such a matrix into $X + iY$ where X is a real symmetric matrix and Y is an antisymmetric (i.e., skew-symmetric) matrix.

Proof. Note that every hermitian matrix A is the sum $X + iY$ with X a real symmetric matrix and Y a real antisymmetric (i.e., skew-symmetric) matrix and in fact this decomposition is unique in the obvious way by taking real and imaginary parts which we can express as $X = \frac{1}{2}(A + A^t)$ and $Y = \frac{1}{2i}(A - A^t)$, since $A^* = A$ and $A^* = \overline{A^t}$. Hence, the hermitian matrices are diffeomorphic to $\mathbf{R}^{n(n+1)/2+n(n-1)/2} = \mathbf{R}^{n^2}$ and the positive-definite hermitian matrices are an open submanifold of the hermitian matrices and thus an open subhmanifold of $\mathbf{R}^{n^2}$, we claim.

Now, the characteristic polynomial of any hermitian matrix A depends smoothly on A as a function from hermitian matrices into the vector space of degree at most n real polynomials (since the roots of the characteristic polynomial for a hermitian matrix are all real) and therefore the roots depend smoothly on A. If all roots are positive, this then remains true under small perturbations of A and thus of the characteristic polynomial, whence the conclusion. In the real case, the symmetric matrices are a submanifold of $\mathbf{R}^{n^2}$ diffeomorphic to $\mathbf{R}^{n(n+1)/2}$ and the positive-definite symmetric matrices are an open submanifold by precisely the same sort of argument. □

The following is a form of ***polar decomposition*** for matrices.

Theorem 27 (Polar Decomposition)

Given a matrix $M \in \mathrm{GL}(n)$, there is a unique symmetric positive definite (hermitian) matrix S such that $M = OS$ with $O \in O(n)$ ($O \in U(n)$ for complex things). Moreover, S depends smoothly on M and hence the matrix O is unique and depends smoothly on M as well.

Proof. A symmetric positive definite matrix is invertible (its determinant is positive), so the smoothness of O will follow by setting $O = MS^{-1}$ (S^{-1} will of course depend smoothly on M). This likewise shows that O is unique. Let us assume that the unique factorization exists. By the spectral theorem, there is a unitary (resp. orthogonal) matrix P such that $P^{-1}SP$ is diagonal. Suppose we can show that P depends smoothly on S. One then sets $S = (M^*M)^{1/2}$ (M^*M is positive-semidefinite and hermitian, the first by observing that $z^*M^*Mz = (Mz)^*Mz$ and the second by the obvious argument) and so has a unique positive-definite hermitian square root. Then one sets $O = MS^{-1}$.

We claim that the matrix square root (sending a positive-definite hermitina matrix to its unique positive-definite square root) is smooth on the space of positive-definite hermitian matrices. To see this, we must show $A \mapsto A^2$ is a diffeomorphism of the space of positive-definite hermitian matrices with itself. Indeed, A^2 is positive-definite and hermitian since $(A^2)^* = (AA)^* = A^*A^*$ and $A^* = A$ for a hermitian matrix so this is clear and it is positive-definite as a consequence of the spectral theorem. If we had $A^2 = B^2$ then there is a unitary matrix P such that $P^{-1}A^2P = P^{-1}B^2P$ as symmetric matrices. Hence, $(P^{-1}AP)^2 = (P^{-1}BP)^2$. Since A has an orthonormal basis of eigenvectors, A^2 has the same orthonormal basis of eigenvectors up to scaling. The same is true for B. Hence, for P to act as a change of basis on A^2 diagonalizing A, it must also do so for A. Hence, $P^{-1}AP$ is diagonal and thus so too is $P^{-1}BP$. Positive-definiteness forces us to conclude $A = B$. This map is obviously smooth and it is obviously surjective.

To see that it is an immersion, note that conjugation by a unitary matrix is an automorphism of the space of positive-definite hermitian matrices. Let $P \in U(n)$ be such that at $P^{-1}AP$ is diagonal where A is positive-definite and hermitian. Then the rank of this self map at A is the rank of this self map at the diagonal matrix $P^{-1}AP$ by the chain rule. If we know this, we are done. □

Exercise 39

Fill in the last step of this proof. *[Hint: Since everything is a submanifold of a Euclidean space, it is enough to work with coordinates in that Euclidean space.]*

Remark. This goes through in the real case basically verbatim. Positive-definite hermitian matrices are a submanifold since they are determined by the upper triangle of the matrix entries and thus clearly sit in the prescribed way!

We will use these results later.

D Isotopy and Diffeotopy

Definition 34

An *isotopy* is a smooth map $h\colon M \times I \to N$ such that for each $t \in I$, $h(-,t)$ is an embedding of M in N.

Definition 35

An *strong isotopy* is a level-preserving embedding $h\colon M \times I \to N \times I$.

Theorem 28

Every strong isotopy is an isotopy. If M is compact, then every isotopy $M \times I \to N$ is a strong isotopy, but the converse is not true in general.

Remark. The converse statement is a common error in textbooks! A counter-example is furnished in a short paper by Hansjörg Geiges. It is always true that a level-preserving embedding $M \times I \to N \times I$ defines a (smooth) isotopy but the converse that the track of an isotopy yields an embedding is definitely not true in general.

Proof. Let $h\colon M \times I \to N \times I$ be a level-preserving embedding. For fixed t, since h is an embedding, $h_t\colon M \times \{t\} \to N \times I$ is an embedding of a submanifold. Since it takes image in $N \times \{t\}$ which is a submanifold of $N \times I$, it is a submanifold of $N \times \{t\}$. Now forget the ts. When M is compact, strongness follows since the map is injective of full rank and proper (by compactness of the domain). □

We used the following.

Lemma 29

If $L \subset M \subset N$ where M is a submanifold of N and L is a submanifold of M, then L is a submanifold of N.

Proof. Since $M \to N$ is an immersion and homeomorphism onto its image, it follows that $L \to N$ is a homeomorphism onto its image (since $L \to M$ is a homeomorphism onto its image) and since $L \to M$ is an immersion, so too is $L \to N$. An alternative argument uses submanifold charts but is trickier to see. □

Definition 36

A ***normalized isotopy*** is a strong isotopy $h\colon M \times I \to N \times I$ which extends to a level preserving embedding $h\colon M \times \mathbf{R} \to N \times \mathbf{R}$ such that for $t \leq 0$, $h(p,t) = h(p,0)$ and for $t \geq 1$, $h(p,t) = h(p,1)$.

Proposition 10

Every strong isotopy $h\colon M \times I \to N \times I$ admits a normalization.

Proof. Let

$$k(x) = \begin{cases} e^{-(2(x-\frac{1}{2})-1)^{-2}} e^{-(2(x-\frac{1}{2})+1)^{-2}} & x \in (0,1) \\ 0 & x \notin (0,1). \end{cases}$$

This is a modified version of the classic bump function which is found in Spivak's book. Then

$$l(t) = \int_0^t k(x)\,dx \Big/ \int_0^1 k(x)\,dx$$

is 0 for $t \leq 0$, is positive for $t > 0$, is 1 for all $t \geq 1$ and increasing on $(0,1)$. In particular, $l\colon \mathbf{R} \to [0,1]$ is smooth.

Given h a strong isotopy, normalize it by setting $H(p,t) = (\mathrm{pr}_N h(p,l(t)),t)$. This is what we want if we can show it is a smooth embedding. This is still an embedding, we claim. To see this, consider the composite

$$M \times \mathbf{R} \xrightarrow{\mathrm{id}_M \times (l,\mathrm{id}_\mathbf{R})} (M \times I) \times \mathbf{R} \xrightarrow{h \times \mathrm{id}_\mathbf{R}} (N \times I) \times \mathbf{R}.$$

Each of these is an embedding, we claim. The first is an embedding since $(l,\mathrm{id}_\mathbf{R})$ is clearly an embedding—the tangent vector never maps to zero, clearly, and the image is just the graph of the function and so a homeomorphism onto its image—so it is a product of embeddings and thus an embedding. As for the latter map, h is known to be an embedding and so this is a product of embeddings.

Hence, the whole composite $M \times \mathbf{R} \to N \times I \times \mathbf{R}$ is an embedding and, in particular, $(l,\mathrm{id}_\mathbf{R})\colon \mathbf{R} \to I \times \mathbf{R}$ is an embedding and one easily checks that this implies

$$M \times \mathbf{R} \xrightarrow{h \times (l,\mathrm{id}_\mathbf{R})} (N \times I) \times \mathbf{R} \xrightarrow{\mathrm{pr}_N \times \mathrm{id}_\mathbf{R}} N \times \mathbf{R}$$

is an embedding and that this composite is H. □

Using this we can show the following.

Theorem 29

An isotopy from f to g is equivalently a level-preserving map $H\colon M \times \mathbf{R} \to N \times \mathbf{R}$ such that for each t $H_t\colon M \to N$ is an embedding and for $t \leq 0$ $H(p,t) = f(p)$ and for $t \geq 1$ $H(p,t) = g(p)$ and equivalently a level-preserving map $G\colon M \times I \to N \times I$ such that each $G_t\colon M \to N$ is an embedding.

Proof. Use the function/construction above for the first part. The second part is obvious. □

Theorem 30 (Isotopy Concatenation)

Given isotopies $H\colon f \to g$ and $G\colon g \to h$, there is an isotopy $K\colon f \to h$.

Proof. Use the function/construction above for $M \times \mathbf{R} \to N \times \mathbf{R}$. One may worry about the concatenation point, but by being stationary between joining the two isotopies, the resulting function is smooth and then some thought shows that we may let this stationary period disappear. We have to perform this twice as fast, then but that is no issue. □

Definition 37

Let $f,g\colon M \to N$ be two embeddings. An ***ambient isotopy*** between f and g is an isotopy $F\colon N \times I \to N$ such that $F(p,0) = p$ and $F(f,1) = g$. Some thought on what this means shows that it is an isotopy extending an isotopy between f and g.

Definition 38

The *support of an isotopy* $H\colon M \times I \to N$ is

$$\operatorname{supp}(h) = \operatorname{Cl}_M\left(\{p \in M : h(p,t) \neq h(p,s) \text{ for some } t,s \in I \text{ with } t \neq s\}\right).$$

Remark. An isotopy $h\colon M \times I \to N$ with compact support is precisely an isotopy which is stationary outside of a compact subset of M. A useful way to think about the support of the isotopy is through its adjoint as the closure of the set of points $p \in M$ that map to non-constant paths in N.

In particular, a point $p \in \partial \operatorname{supp}(h)$ for $\operatorname{supp}(h)$ compact is such that for all t, $h(p,t) = h(p,0)$ since otherwise there is some $t_0 > 0$ for which $h(p,t_0) \neq h(p,0)$ and thus this is true for (q,t) in a nbhd about (p,t_0) since by continuity h for every nbhd of $h(p,t_0)$, there is a nbhd of (p,t_0) which maps into this nbhd, so by Hausdorfness we can choose a separating nbhd for $h(p,t_0)$ and $h(p,0)$ and hence, the nbhd guaranteed to exist by continuity about (p,t_0) maps into two disjoint sets.

Theorem 31

An isotopy with compact support is a strong isotopy.

Proof. Let $h\colon M \times I \to N \times I$ be an isotopy and let $K = \operatorname{supp}(h)$ its support. Then $h|K \times I\colon K \times I \to N \times I$ is a strong isotopy. In fact, for $K^c \times I = M \times I \setminus K \times I$, which is an open subset of $M \times I$, $h\colon K^c \times I \to N \times I$ is an embedding we claim—this is because this map is constant with each slice being the same embedding. In fact, ∂K is a closed set and for each $p \in \partial K$, $h(p,t) = h(p,0)$ by the above remark, so it follows easily that $h\colon \operatorname{Int}(K)^c \times I \to N \times I$ is an embedding, where $K^c \times I \cup \partial K \times I = \operatorname{Int}(K)^c \times I$. Since h is an embedding on two closed sets whose union is all of $M \times I$, it is an embedding on I. Indeed, simply think about showing that this map sends closed sets to closed sets in the subspace topology. $\square$

Theorem 32

Let $f\colon E \to F$ be an isomorphism between two smooth Riemannian (resp. hermitian) vector bundles over a base manifold M. Then there is an isometry $g\colon E \to F$ and in fact a strong isotopy (diffeotopy, even) from f to g.

Proof. WLOG suppose $F = E$. The two metrics can be represented by two families of orthonormal bases of each tangent space—this is just saying that there is a reduction of structure group. An automorphism of E is locally given by matrices M_p such that M_p is simply the change of basis matrix. This automorphism is then an isometry **iff** these matrices are all orthogonal matrices. (We should use trivializations here that have the same open sets and choose our trivializations to be isometries as well.)

Writing $M_p = O_p S_p$ as above, the matrices O_p define an isometry over $p^{-1}(U)$ (perhaps by carrying it back to this). Working in an overlapping trivializing chart, the same procedure produces a compatible map since the transition functions are change of bases corresponding to the image of the given bases in the fibers under the two trivializations. One sees this using uniqueness of the polar decomposition and some thought.

Let $M_p(t) = O_p(tI_n + (1-t)S_p)$. The set of positive-definite hermitian (resp. symmetric) matrices is convex, so the same argument as above shows that this defines the desired parametrized family of automorphisms $H \colon E \times I \to E$ (and it is certainly smooth with the obvious extension). This is a strong isotopy because $E \times I \to E \times I$ is a diffeomorphism with inverse given by using $M_p(t)^{-1}$. □

Theorem 33

For every $\varepsilon > 0$, for every continuous section s of a Riemannian bundle that is smooth on a closed subset K, s can be approximated by a smooth section t agreeing with s on K such that, fiberwise, $\|s(p) - t(p)\| < \varepsilon$.

Proof. Standard approximation argument using partition of unity. □

Definition 39

Let $\varepsilon \colon M \to \mathbf{R}_{>0}$ be smooth. Then a ε-*shrinking* of a Riemannian bundle $\pi \colon E \to M$ over M is the smooth map $\tilde{\varepsilon} \colon E \to E$ given by

$$v \mapsto \varepsilon(\pi(v)) \frac{v}{\sqrt{1 + \|v\|^2}}$$

For simplicity, assume now that M is connected. Then, more generally, a *shrinking* of a vector bundle $\pi \colon E \to M$ is any fiber-preserving smooth embedding $\varepsilon \colon E \to E$ with the following properties.

(1) If $E_p = \pi^{-1}(p)$, then $\varepsilon(E_p) \subsetneq E_p$ is a proper subset.

(2) If $w, v \in E_p$ are collinear, then $\varepsilon(w)$ and $\varepsilon(v)$ are collinear in E_p.

(3) $\mathrm{Im}(\varepsilon)$ has the structure of an open-disk subbundle of E. In other words, it is a subset of E with the structure of fiber bundle with typical fiber an open disk of unit radius in $\mathbf{R}^k$ centered at 0 and structure group $O(k)$ where k is the rank of the bundle with trivializations inherited by restriction from those of E.

Lemma 30

Fix a Riemannian vector bundle $\pi \colon E \to M$ with metric g_0 and let $\varepsilon \colon M \to \mathbf{R}_{>0}$ be smooth. Then the above ε-shrinking procedure really does afford an open (linear) disk bundle. In particular, we may modify g_0 within its conformal class to obtain a metric g such that the ε-shrinking is the open unit disk-bundle of E with metric g.

Proof. Suppose the rank of the bundle is k. Then in the conformal class of g_0, we may modify it to a new metric $g = \frac{1}{\varepsilon} \cdot g_0$ (this is well-defined since $\varepsilon > 0$ everywhere). Note that the image of the ε-shrinking procedure has fiberwise image $\{v \in E_p : g_0(v, v) < \varepsilon(p)\}$ and so, equivalently, $\{v \in E_p : g(v, v) < 1\}$. Using the metric g, the Gram-Schmidt procedure allows us to construct trivializations $\varphi\colon \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbf{R}^k$ that are fiberwise isometries where $\mathbf{R}^k$ acquires the standard Euclidean inner product, thereby reducing the structure group of the bundle to transition functions all lying in $O(k)$. It follows easily that the transitions respect the unit open-disk bundle for the metric g and thus the image of the ε-shrinking procedure. □

Now let us justify that the shrinking terminology is what we expect it to be.

Lemma 31

If $\varepsilon\colon E \to E$ is a shrinking of E where $\pi\colon E \to M$ has rank k. Then in for any Riemannian metric on E, the following hold.

(i) If $v \in E_p$ (the fiber over p) is non-zero, then $\varepsilon(v) \neq 0$.

(ii) For any trivialization $\varphi\colon \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbf{R}^k$ that is a fiberwise isometry and any $v \in E_p \setminus \{0\}$, $\frac{\varepsilon(t\varphi(v))}{dt} < 0$.

(iii) If $v, w \in E_p$ are collinear and $|v| < |w|$, then $|\varepsilon(v)| < |\varepsilon(w)|$.

Proof. Using a fixed choice of Riemannian metric on E, we may assume this bundle has structure group $O(k)$ with trivializations fiberwise isometries. Note that ε restricts to an embedding $E_p \to E_p$ for each p.

(a) For the first, this is because only 0 is collinear with every other vector, and ε is required to preserve this.

(b) We implicitly use that ε sends one-dimensional subspaces to one-dimensional subspaces now. Fix $v \in E_p \setminus \{0\}$. Now for any trivialization $\varphi\colon \pi^{-1}(U) \xrightarrow{\cong} U \times \mathbf{R}^k$ that is a fiberwise isometry, $\frac{\varepsilon(t\varphi(v))}{dt} = 0$ for some t where $v \neq 0$, this would contradict that $\varepsilon\colon E_p \to E_p$ is an embedding and thus has full rank. On the other hand, *some* t satisfies that $\frac{\varepsilon(t\varphi(v))}{dt} < 0$ we claim. Indeed, by invariance of domain, $\varepsilon(E_p)$ is open in E_p, an open subspace of E_p containing the one-dimensional line $\text{span}(v)$ cannot inherit the structure of an open-disk bundle from E since there is no metric on E for which every vector in $\text{span}(v)$ has finite length. Hence, $\frac{\varepsilon(t\varphi(v))}{dt} < 0$ for all t since otherwise by continuity some t would have $\frac{\varepsilon(t\varphi(v))}{dt} = 0$ which we have seen is impossible.

(c) Some thought shows this follows immediately from **(b)**. □

Thus, a shrinking really is a function that shrinks vectors inwards towards the origin.

Lemma 32

Suppose $\partial M = \varnothing$ and let X be a vector field on $M \times \mathbf{R}$ with t-coordinate $\frac{\partial}{\partial t}$ (where t is the identity coordinate system on $\mathbf{R}$). If X admits a global flow, then X induces a strong isotopy of the identity map of M via $(p, t) \mapsto \Phi((p, 0), t)$. This is in fact a diffeotopy since it has the evident inverse for each t.

Remark. Suppose M is a compact manifold. Then a vector field $\widetilde{X}$ on M has a global flow. It follows (abusing notation) that the vector field $X = \widetilde{X} + \frac{\partial}{\partial t}$ admits a global flow on $M \times \mathbf{R}$ since X has bounded velocity on $M \times \mathbf{R}$ for any product Riemannian metric on $M \times \mathbf{R}$ (i.e., $\sup_{(p,t)} \|X_{(p,t)}\| < \infty$). See Hirsch's remarks on flows and bounded velocity in his book—the boundedness condition guarantees the existence of a global flow.

Theorem 34 (Isotopy Extension Theorem)

Let $f\colon M \to N$ be an embedding, let $K \subset M$ a compact subset and let $G\colon M \times \mathbf{R} \to N \times \mathbf{R}$ a strong isotopy of f. Suppose $G(K \times I) \cap (\partial N) \times I = \varnothing$. Then there is an isotopy of the identity map on N, $H\colon N \times \mathbf{R} \to N \times \mathbf{R}$, such that for all $x \in K$, $H(f(x), t) = G(x, t)$. In particular, this isotopy has compact support and is thus strong and is in fact a diffeotopy. Furthermore, for any appropriate nbhd U of $G(K \times I)$ with compact closure, we may suppose this diffeotopy has support contained in $\operatorname{pr}_N \overline{U}$ which is compact.

The same, moreover, is true if we suppose instead that N has boundary and no corners and we suppose $G(K \times I) \subset (\partial N) \times I$. It follows that the diffeotopy H restricts to a diffeotopy of ∂N and $\operatorname{Int} N$, separately.

Proof. We start with the first case. Let $X = DG(\partial t)$. Note that X has t-coordinate in the obvious product chart simply ∂t. Since $G(M \times \mathbf{R}) \subset N \times \mathbf{R}$ is a submanifold, this vector field is locally extendable. Let $B = G(K \times [0, 1])$. This is compact and closed and so X extends over $N \times \mathbf{R}$ to a vector field Y agreeing with X on B and WLOG vanishing outside of some nbhd U of B where we assume $\overline{U}$ is compact and does not intersect the boundary of N at any time (compactness allows us to arrange this). Let

$$Z = (Y - t\text{-coordinate of } Y) + \partial t$$

so that the t-coordinate of Z in the obvious sort of product chart is ∂t and $Z|B = Y|B = X|B$.

Let V be an open set with compact closure containing $\overline{U}$ such that $\overline{V} \cap \partial N = \varnothing$. Then the integral curves originating outside V have the form $(x_0, t + t_0)$ until they reach U. Since $\overline{U}$ is compact, is a $\delta > 0$ for which all solutions starting outside of V are defined for $|t| < \delta$. For instance, we may suppose V and U are product nbhds to see this. Furthermore, since $\overline{V}$ is compact, there is an $\varepsilon > 0$ such that all solutions starting in V are defined for time $|t| < \varepsilon$. Indeed, this is the tube lemma applied to the slice $\overline{V} \times \{0\} \subset \operatorname{Int} N \times \mathbf{R}$ for the open nbhd $A_Z \cap \overline{V} \times \mathbf{R}$ containing this slice, where A_Z is the flow domain of Z restricted to $\operatorname{Int} N$. This implies that the flow domain for Z in $\operatorname{Int} N$ contains a tube about $\operatorname{Int} N \times \{0\}$ and thus has a global flow in $\operatorname{Int} N$ by the usual argument as in the existence of global flows for vector fields on compact manifolds.

Thus, there is an isotopy $\operatorname{Int} N \times \mathbf{R}$ of the identity map of $\operatorname{Int} N$ by the preceeding **Lemma** and some thought gives us the other piece of the conclusion. All that is left is to extend this to an isotopy $N \times \mathbf{R} \to N \times \mathbf{R}$. What we have so far is constructed as in the above **Lemma** and since Z has only ∂t coordinate outside U, it is easy to see this isotopy is the identity outside of U and thus extends in the evident way.

The isotopy so constructed has compact support, we claim. Note that the vector field Z on $N \times \mathbf{R}$ has trivial projection onto TN outside of the nbhd U of B where $\overline{U}$ is compact. Thus, the support of the isotopy is contained in $\overline{\operatorname{pr}_N U}$ clearly and since $\operatorname{pr}_N \overline{U}$ is compact (continuous image of a compact set) and thus closed, we get

$$\operatorname{pr}_N \overline{U} = \overline{\operatorname{pr}_N \overline{U}} \supset \overline{\operatorname{pr}_N U}$$

and so the support is compact. This implies, as we have seen, that it is a strong isotopy.

Now consider the latter case. Let $X = DG(\partial t)$. Note that X has t-coordinate in the obvious product chart simply ∂t. Since $G(M \times \mathbf{R}) \subset N \times \mathbf{R}$ is a submanifold, this vector field is locally extendable. Let $B = G(K \times [0,1])$. This is compact and closed and so X extends over $N \times \mathbf{R}$ to a vector field Y agreeing with X on B and WLOG vanishing outside of some nbhd U of B where we assume $\overline{U}$. Let

$$Z = (Y - t\text{-coordinate of } Y) + \partial t$$

so that the t-coordinate of Z in the obvious sort of product chart is ∂t and $Z|B = Y|B = X|B$.

Now the horizontal piece of Z, namely $Z = (Y - t\text{-coordinate of } Y)$, is tangent to $\partial(N \times \mathbf{R}) = (\partial N) \times \mathbf{R}$ at all times and has compact support. It is easy to see that this is a vector field for which the flow is global just as above. The flow for the vertical piece of Z is likewise globally defined. We can piece these together to form the flow for Z in the evident way. The same argument about compact support works showing the diffeotopy is strong. As for the part about restriction, this follows by smooth invariance of the boundary. Alternatively, a point in $\operatorname{Int} N$ at time 0 can never flow into ∂N since Z is always tangent to ∂N at all times, so this contradicts uniqueness of integral curves. □

Definition 40

Let $\gamma \colon S^1 \to M$ be a loop in an m-manifold M. Let $\widetilde{M}$ be the orientation double-cover of M. We say that γ is ***orientation-preserving*** if $\gamma^* \widetilde{M}$ is a trivial $\mathbf{Z}/2$-bundle and is ***orientation-reversing*** if $\gamma^* \widetilde{M}$ is a non-trivial $\mathbf{Z}/2$-bundle.

Remark. Since the orientation double-cover is a regular covering space and thus a principle $\mathbf{Z}/2$-bundle, there are only two $\mathbf{Z}/2$-bundles over S^1.

Remark. The orientation double cover is always itself oriented—when the base is oriented, then it is the trivial (disconnected) double cover and we give opposite orientations to each piece. Viewing γ as a class in some fundamental group of M (M is connected, of course, as always), then γ being orientation-preserving is the same as saying that the action of γ on the covering space is orientation-preserving (for either choice of orientation, of course).

Warning. Double check a few things in the next proposition.

Proposition 11

If M is orientable, all smooth loops are orientation preserving. If M is not orientable, there is an orientation-reversing loop through any point. There are also orientation-preserving loops if M is not orientable. When M is not orientable, the orientation-preserving loops form a subgroup of index 2 in $\pi_1(M)$ (as usual M is connected) which is thus a normal subgroup.

Proof. M is orientable **iff** it has disconnected orientation double cover (the trivial $\mathbf{Z}/2$-principal bundle), in which case this is clear. Now suppose M is non-orientable and let γ be the projection of a path between the two points in the fiber of $\widetilde{M} \to M$ over p and let $\gamma\colon S^1 \to M$ be the induced loop with $0 \in S^1$ mapping to p obtained by post-composition with the bundle projection of the path. If $\gamma^*\widetilde{M}$ is trivial, then over the basepoint 0 we get a $\mathbf{Z}/2$-equivariant map that maps one point to each fiber. On the other hand, restricting to one piece of this gives a loop in $\widetilde{M}$ covering γ and this is clearly impossible. A loop in $\widetilde{M}$ will always pass to an orientation-preserving loop.

The point here is that $\widetilde{M}$ is a regular covering space with two sheets, so the image of $\pi_1(\widetilde{M})$ in $\pi_1(M)$ is a normal subgroup of index 2. □

Corollary 28 (Disc Theorem, Palais)

Let N be a manifold with corners that is connected for simplicity. Let $f, g\colon D^k \to N$ be two embeddings of the closed disk in N. If $k = n$ and N is orientable, suppose further that f and g are both orientation preserving or orientation reversing. Then f and g are ambient isotopic.

In particular, the following hold.

(a) If $\mathrm{Im}(f) \cup \mathrm{Im}(g) \subset \mathrm{Int}(N)$, then this isotopy may be assumed to be a diffeotopy with compact support.

(b) If ∂N is compact, then this isotopy may be assumed to have compact support.

(c) If $f = g$ on a disc $D^m \subset D^k$ and $\mathrm{Im}(f) \cup \mathrm{Im}(g) \subset \mathrm{Int}(N)$, then this isotopy may be assumed to be stationary on D^m.

(d) The same is true of embeddings of the open disks as long as the embeddings extend to $\mathbf{R}^k$—more precisely, so long as the embeddings are restrictions of embeddings of closed disks of strictly larger radius than 1.

Warning. We repeatedly and implicitly use the isotopy extension theorem throughout the proof. We silently suppress the assumption of the open disk case when we apply the isotopy extension theorem.

Proof. If f and g intersect the boundary of N, we can use the We first show the closed-disk case reduces to the case of embeddings of open disks. We can then knock both cases (half-disks and non half-disks) out in one go. For a closed disk, by shrinking just a little, we find that the embedding is isotopic to the "restriction" to the closed disk of radius $1/2$ inside D^k viewed as the map

$$\widetilde{f}\colon D^k \xrightarrow{1/2} D^k \xrightarrow{f} N,$$

(i.e., smoothly shrinking the radius 1 disk into the radius $1/2$ disk). This means that $\widetilde{f}$ admits an extension to an embedding of the open disk B of radius $4/3 = 1 + 1/3$ by the same formula as $\widetilde{f}$, call this $\widetilde{\widetilde{f}}$. We can do this for any two maps f and g. An isotopy between $\widetilde{\widetilde{f}}$ and $\widetilde{\widetilde{g}}$ will restrict to an isotopy between f and g. Since the domains of $\widetilde{\widetilde{f}}$ and $\widetilde{\widetilde{g}}$ are diffeomorphic to open balls of radius 1, knowing the result for open balls of radius 1 suffices.

If $\partial N \neq \varnothing$, then there is an isotopy of N retracting a collar of N diffeomorphically say into the interior of N. If ∂N is compact, we may suppose this is a diffeomorphism generated by an isotopy $\partial N \times \mathbf{R}_{\geq 0} \times I \to \partial N \times \mathbf{R}_{\geq 0}$ between the identity and the map sending $[0,1] \subset \mathbf{R}_{\geq 0}$ to $[1/2, 1] \subset \mathbf{R}_{\geq 0}$ diffeomorphically and in such a way that the isotopy is constant outside of $\partial N \times [0,1]$ and so has compact support. Assuming the other assertions are true, we are now free to apply them so this reduces to the case of embeddings into the interior.

We consider the case of non-half disks first.

For $\partial N = \varnothing$ and open disks, take a smooth path between the centers of the disks $f(0)$ to $g(0)$ and extend this to an ambient isotopy which we may assume is a diffeotopy with compact support (fixing the boundary in the case $\partial N \neq \varnothing$ since by shrinking we may assume these disks do not have closure intersecting the boundary up to isotopy and so WLOG). Call the images of the open disks fD and gD and call the image of fD under this isotopy $K_1 fD$. Since the final map is a diffeomorphism of N with itself, it is an open map, so some open nbhd of $f(0)$ gets mapped into an open nbhd of $g(0)$. WLOG we assume it contains all of fD perhaps by shrinking first. Call the last map of the isotopy of the disk so far constructed H_1 so that we can declare $H_1 D = K_1 fD$.

Suppose for the moment that N is oriented. Then f and g have the same orientation type for the standard orientation on the open disk, then since ambient isotopies (and in particular ambient diffeotopies) are orientation preserving, being homotopic to the identity map, the orientations of the final map of fD into gD is necessarily orientation preserving between the disks with their orientations inherited from f and g (which are both the same orientation type). By pre-composing with f, this gives an isotopy from f to an embedding into g, perhaps after shrinking $K_1 fD$ (these are both still necessarily open nbhds of $g(0)$ by invariance of domain so we can do this), and these all preserve orientation type since this is an isotopy and thus the map onto the image of g has the same orientation type as f and g.

By shrinking we may assume g has image in the domain of a chart and that these coordinates $g(0) = 0$. WLOG we may likewise suppose by shrinking, perhaps, that $H_1D = K_1fD \subset gD$ (these are both still necessarily open nbhds of $g(0)$ by invariance of domain and the fact that we are doing ambient isotopies so we can do this).

When $k = n$, then, equipped with the induced orientations, in coordinates, there is no reason that f and g (and thus H_1 and g) need to have the same orientation type when we make only the assumption that N is *non*-orientable. However, we may take an orientation-reversing loop in the manifold and isotope H_1 around this so that the orientations then agree. Thus, WLOG suppose that in this chart, then, when $k = n$, then both f and g preserve orientation. When N is orientable, this follows from the additional hypothesis on f and g.

Whatever the case on the dimensions,g is, in coordinates, isotopic to the linear map given by its differential at 0. This isotopy is the ***standard isotopy*** $D^k \times I \to \mathbf{R}^n$ given by

$$(x,t) \mapsto \begin{cases} t^{-1}g(tx) & 0 < t \leq 1 \\ Dg_0x & t = 0 \end{cases}$$

which is smooth essentially by ***Hadamard's lemma*** applied to $g(tx)$, for which we may write

$$g(tx) = g(0) + \sum_i tx^i h_i(tx) = \sum_i tx^i h_i(tx)$$

with h_i satisfying some relevant properties and x^i the i-th coordinate. Note that Dg_0 has rank k.

When $k = n$, then Dg_0 is in the identity path-component of $\mathrm{GL}_n(\mathbf{R})$ based on our assumptions and there is an isotopy from this map to the identity map by taking a smooth path from Dg_0 to the identity matrix in this Lie group. Similarly in these coordinates H_1 admits an isotopy to the identity linear map of the same form by our assumptions. These isotopies act via $(x,t) \mapsto \gamma(t)x$ where $\gamma(t) \in \mathrm{GL}_n(\mathbf{R})$.

When $k < n$, we first isotope f and g to linear maps of the form above. Then we extend these to all of $\mathbf{R}^k$ by the same formula for the subspace $\mathbf{R}^k \times 0 \subset \mathbf{R}^n$ and then we choose the remaining $n-k$ coordinates by picking a basis of $\mathbf{R}^n$ for which the first k coordinates span the image of Dg_0 (resp. Df_0) and the last $n-k$ simply extend this to a basis. We choose this such that there is an evident linear extension (automorphism even) which we write suggestively as $\mathbf{R}^k \times \mathbf{R}^{n-k} \to \mathbf{R}^k \times \mathbf{R}^{n-k}$ having positive determinant, which we can arrange by choosing the signs of the last $n-k$ coordinates appropriately where they map in this basis as $e_{k+i} \mapsto \pm e_{k+i}$. The reasoning of above applied to these maps shows they are isotopic and they then restrict to the isotopy of the maps desired.

Now consider the boundary case. We can retract a collar nbhd of N into N and therefore we may assume WLOG that the embeddings are actually into the interior of N. WLOG by shrinking suppose f and g take image in an interior chart o N. Smoothness then means that near 0, each of f and g extend locally—hence, we may suppose they are defined in an open ball of radius $\varepsilon > 0$ about 0 in $\mathbf{R}^k \supset D^k$. By shrinking the disk, we may even suppose that f and g are then defined on the interior of this open ball. This reduces us to the case above, since an isotopy of these extensions restricts to an isotopy of the open half-disks and we know the latter exists. □

Exercise 40

Formulate and prove an extension for half-disks of the above theorem. Break this into two interesting cases:

(a) Make no other assumptions. *[Hint: Flow inwards. Assume you may construct this as an embedding.]*

(b) Suppose ∂N is compact.

Corollary 29

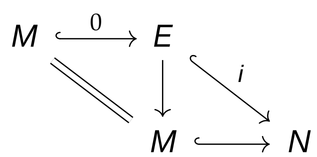
If $f_i, g_i \colon D^k \to M$ ($i = 1, \ldots, n$) are embeddings of closed disks (resp. open disks that are restrictions of embeddings of strictly larger closed disks) such that the f_i have disjoint images (resp. have disjoint closure of images) and the g_i have disjoint images (resp. have disjoint closure of images). When M is orientable and $k = n$ we assume in addition that the orientation type of f_k and of g_k are the same for each $1 \leq k \leq n$. Then there is an is a diffeotopy of M bringing each f_i to g_i.

Proof. As before, the case of open disks suffices. By the above there is a diffeotopy bringing f_1 to g_1, call the resulting diffeomorphism H_1. This diffeotopy keeps the closures of the f_i and the g_i respectively disjoint. By modifying the proof of the isotopy extension theorem we are using, using the fact the closures of the disks are disjoint, we can construct a diffeotopy of M taking $H_1 f_2$ to g_2 while keeping the closure of $g_1(D^n)$ fixed. This repeats. We may have to retract into the interior of N first but this is no problem. □

E Tubular Neighborhoods

Definition 41

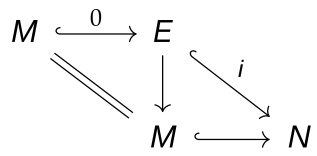
A *tubular neighborhood* of a submanifold $M \subset N$ is a vector bundle $E \to M$ of rank $\dim N - \dim M$ and a smooth commutative diagram



with i an embedding and 0 is the zero section. It is sometimes easier to think of a tubular nbhd as a subset of N equipped with extra structure.

A *closed tubular neighborhood* of a submanifold $M \subset N$ is a disk bundle

$E \to M$ of rank $\dim N - \dim M$ with structure group $O(\dim N - \dim M)$ and a smooth commutative diagram



with i an embedding and 0 is the zero section. It is sometimes easier to think of a tubular nbhd as a subset of N equipped with extra structure.

Remark. When $\partial M = \varnothing$, a closed tubular neighborhood is a submanifold with boundary with closed image. This is because the interior of each disk (thus, the interior of the disk bundle viewed as a manifold) is a Euclidean space and the structure group preserves the interior. Hence, it may be equipped with the structure of a vector space (perhaps after fixing once and for all a diffeomorphism of the interior with $\mathbf{R}^{\dim N - \dim M}$ which thus gives a tubular neighborhood in the preceding sense. When M as boundary, it has corners.

Definition 42

A *proper tubular neighborhood* of a submanifold $M \subset N$ is a tubular nbhd obtained by a shrinking of another.

Remark. Closed and proper tubular nbhds correspond in a certain sense.

Proposition 12

If a closed tubular nbhd is the disk bundle of some tubular nbhd, then the interior of that closed tubular nbhd is a proper tubular nbhd. A proper tubular nbhd is always interior of a closed tubular nbhd obtained by taking its closure.

Restricting to closed tubular nbhds that arise as disk bundles of tubular nbhds, closed tubular nbhdds and proper tubular nbhds correspond. Here we restrict to $M \subset \operatorname{Int} N$ and those closed tubular nbhds of M that do not intersect the boundary of the ambient manifold.

Remark. One might think we need to restrict to proper tubular nbhds whose closure does not intersect the boundary of the ambient manifold, but any such tubular nbhd is not proper to begin with because it cannot properly extend—the boundary point is necessarily a limit in the fibers away from the zero section since each point of M has an open nbhd not intersecting ∂N (since every point of N satisfies the same). Thus this cannot properly extend. Similarly, closed tubular nbhds will not intersect the boundary of N in this set up because they too must extend.

Proof. We may as well suppose $\partial N = \varnothing$. It is clear that the closure of a proper tubular nbhd T gives the structure of a closed tubular nbhd since a proper tubular nbhd is precisely a shrinking of a tubular nbhd and there its closure in N is its closure in the image of the total space E in N since $\dim E = \dim N$ and $T \subset E$ is a fiberwise proper subset has the structure of an open-disk bundle, by definition.

Conversely, given a closed tubular nbhd of the sort above, there is an implicit metric floating around for which its interior is precisely a proper tubular nbhd. These procedures are manifestly inverse to each other on the class considered. □

The situation is better when M is compact.

Proposition 13

If $M \subset \operatorname{Int} N$ is a compact submanifold, then any closed tubular nbhd extends to a tubular nbhd. Hence, every proper tubular nbhd extends.

Proof. Let $\pi\colon E \to M$ be a disk bundle with structure group $O(n-m)$ as in the definition of a closed tubular nbhd. Let $p\colon E' \to M$ be the associated bundle with fiber $\mathbf{R}^{n-m}$ so that E is (at the very least up to isomorphism) the unit disk bundle of E' for a choice of metric on the bundle.

Let $\rho(t)$ be a bump function $\mathbf{R} \to \mathbf{R}$ that is 0 for $t \le 0$, 1 for $t \ge 1$ and increasing on $(0,1)$. Given a closed tubular nbhd $i\colon E \hookrightarrow N$, let $H\colon E \times \mathbf{R} \to N$ be

$$H(v,t) = i(1-\rho(t)/2)v).$$

This is indeed an isotopy since the scaling map is an embedding, clearly.

Since E is compact, this extends by the isotopy extension theorem to a strong diffeotopy $h\colon N \times \mathbf{R} \to N$.

Let $\lambda\colon \mathbf{R} \to \mathbf{R}$ be a smooth function with $\lambda < 1$ everywhere, $d\lambda/dt > 0$ everywhere and $\lambda(t) = t/2$ for $t \in [0/1]$. For example,

$$\lambda(t) = \frac{1}{2}\int_0^t (1 + (e^{-x} - 1))\rho(x-1)\,dx.$$

Now we extend the embedding $H_{1/2}$ to a tubular nbhd by letting ψ be

$$\psi\colon E' \to N$$

by

$$\psi(v) = i(\lambda(\|v\|)v/\|v\|).$$

This is an embedding because i is, once we know that

$$v \mapsto \lambda(\|v\|)v/\|v\|$$

is a well-defined embedding $E \to E$, note that, near the 0 element, $\lambda(\|v\|) = 1/2\|v\|$, so that the zero vector maps to itself in the limit so that this is well-defined, understood suitably. To see that it is an embedding, one simply notes that it is an injective map of constant (full) rank and so a local diffeomorphism that is injective and so a diffeomorphism.

Finally, one easily verifies that

$$j = h_{1/2}^{-1} \circ \psi,$$

is the extended tubular nbhd. □

Theorem 35 (Existence, I)

Let $M \subset N$ be a submanifold where $\partial N = \varnothing$ or $M \cap \partial N = \varnothing$.

(a) The exponential map restricted to $\nu_N M$ is defined on an open nbhd of the zero-section of the normal bundle $\nu_N M$ in N and furthermore exp is an embedding on this nbhd onto a submanifold of N containing M as a submanifold M (in particular, as the zero section).

(b) When $\partial M = \varnothing$, this submanifold is in fact open in N.

(c) In particular, tubular nhbds exist in the cases considered, even for submanifolds with corners.

Remark. When the codimension of M in N is 0, then the normal bundle of M in N has rank 0 and thus is the unique rank 0 vector bundle over M up to diffeomorphism.

Proof. Identify $\nu_N M$ as a subbundle of $TN|\,M$. One shows exp has full rank on the zero section of $\nu_N M$ and thus in a nbhd of the zero-section. Then one applies the tubular neighborhood trick—this requires either one of the assumptions $\partial N = \varnothing$ or $M \cap \partial N = \varnothing$. For the latter case we can just as well suppose $\partial N = \varnothing$. When $\partial M = \varnothing$, this nbhd is open since then $\nu_N M$ is manifold without boundary and so the invariance of domain applies. ☐

Corollary 30

If $f\colon M \to N$ is an embedding with the relation between M and N as above ($\partial N = \varnothing$ or $M \cap \partial N = \varnothing$). Then f extends to an embedding of $\nu_N M$ (strictly speaking, the normal bundle to the immersion f) in N and when $\partial M = \varnothing$ this is an open nbhd of $f(M)$ in N.

Proof. Let ε be a suitable shrinking function taking $\nu_N M$ into itself as in the ε-shrinking procedure such that the image of this shrinking is contained in the nbhd of the zero section upon which the exponential map is defined. ☐

Lemma 33 (Hadamard's Lemma)

Let $U \subset \mathbf{R}^m$ be a convex (or even star convex) open nbhd about a point $p \in \mathbf{R}^m$ and $f\colon U \to \mathbf{R}$ a smooth function. Then

$$f(x) - f(0) = \sum_i a_i(x)(x^i - p^i)$$

for smooth functions $a_1(x), \ldots, a_m(x)$ with

$$a_i(p) = \frac{\partial f}{\partial x^i}(p).$$

In particular, when $p = 0$,

$$f(x) = \sum_i a_i(x)x^i \qquad a_i(0) = \frac{\partial f}{\partial x^i}(0).$$

Proof. By the chain rule and the fundamental theorem of calculus (integrating a 1-dimensional line),

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx + (1-t)p)\, dt = \int_0^1 \sum_i \frac{\partial f(tx + (1-t)p)}{\partial x^i}(x^i - p^i)\, dt \\ &= \sum_i \left(\int_0^1 \frac{\partial f(tx + (1-t)p)}{\partial x^i}\, dt \right)(x^i - p^i). \end{aligned}$$

So set $a_i(x) = \int_0^1 \frac{\partial f(tx+(1-t)p)}{\partial x^i}\, dt$. The last part now follows by differentiating. □

Remark. This lifts to manifolds by taking charts with convex image. When the manifold has boundary or corner, then a smooth function $f \colon M \to \mathbf{R}$ in a corner chart extends to an open subset of $\mathbf{R}^m$ and open subsets of $\mathbf{R}^m$ are locally convex and so up to a suitable modification the same is true.

This next result implies that the vector bundle structure on tubular nbhds is unique for $\partial M = \varnothing$ and, moreover establishes that any two tubular nbhds are isotopic.

Theorem 36 (Uniqueness, I)

Let $M \subset N$ and suppose $\partial M = \varnothing$. Let F^1 be a tubular nbhd of M in N and let $F^0 \to M$ be a bundle of rank $k \leq \dim N - \dim M$. Then there is an isotopy G of the inclusion $F^0 \subset N$ such that $G_t(p) = p$ for $p \in M$ and G_1 is a linear map $F^0 \to F^1$ of rank k on each fiber.

We view these as submanifolds of N throughout and as vector bundles over M interchangeably throughout the proof.

Proof. We start with a claim.

Claim. There is a ε-shrinking of F^0 to a bundle E^0 such that for every $p \in M$ there is a nbhd U in M and a trivializing chart $V \subset M$ for F^1 such that $E^0_U \subset F^1_V$ (this denotes the restriction of each bundle viewed as subsets of N).

Fix choices of metrics for both bundles F^0 and F^1.

Let V be a trivializing chart for F^1 in M, say a submanifold chart perhaps by shrinking (where we mean that the restriction F_V is trivial) about $p \in M$. Let U be a nbhd with compact closure contained in V and containing p. Giving F^0 a metric, there is an $\varepsilon_p > 0$ such that all ε_p-discs in the fibers of F^0 over U are contained in F^1_V by a compactness argument and, **critically**, by using that F^1_V is open in N—here we are viewing this containment inside of N and we remark that it is not necessarily a fiberwise inclusion. F^1_V is open in N since F^1 is open in N by a preceding result as M has no boundary.

Remark. If F_V^1 were not open (e.g., in certain cases where $\partial M \neq \varnothing$) then this would fail miserably—just draw a picture in the case of a 1-dimensional submanifold with boundary sitting in $\mathbf{R}^2$ with the vector spaces over a boundary point orthogonal to each other! Something slightly different is true however and we will show that next.

Now, cover M by such charts and WLOG assume it is locally finite and the charts of this open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ have corresponding numbers ε_α. Taking a partition of unity subordinate to this open cover say $\{\rho_\alpha\}$, with a little bit of effort, say by replacing each ε_α by

$$\varepsilon'_\alpha = (\min\{\varepsilon_\beta : U_\alpha \cap U_\beta \neq \varnothing\})/(\#\{\beta \in \Lambda : U_\alpha \cap U_\beta \neq \varnothing\} + 1),$$

we can construct a smooth positive function $\varepsilon\colon M \to \mathbf{R}_+$ such that $\varepsilon(p) < \varepsilon_\alpha$ whenever $p \in U_\alpha$ by setting $\varepsilon = \sum \varepsilon'_\alpha \rho_\alpha$. Then a ε-shrinking of F^0 results in a bundle E^0 satisfying the properties of the claim.

The shrinking map is itself isotopic to the inclusion $F^0 \to N$ so it suffices to construct an isotopy of the inclusion $\iota\colon E^0 \to F^1$ to a linear non-degenerate map $E^0 \to F^1$. Note that $\iota\colon E^0 \to F^1$ is now an embedding since it is an embedding into N with image contained in the submanifold F^1.

For $t \neq 0$, define

$$G_t(v) = \frac{1}{t}\iota(tv).$$

When $t = 0$, we define G_0 as follows. If U and V are as in the claim, then $\iota\colon E_U^0 \to F_1^V$ is given in local (bundle) coordinates by

$$v = (x,y) \mapsto (f(x,y), g(x,y)) \in \mathbf{R}^m \times \mathbf{R}^{n-m} \qquad (x,y) \in \mathbf{R}^m \times \mathbf{R}^k$$

where $f(x,0) = x$ and $g(x,0) = 0$. WLOG we may suppose that these coordinates are a convex open set (when M has boundary, which are not supposing, we can still do this by extending the function on the corner piece). Then in these coordinates (or at least locally in these coordinates), by Hadamard's lemma, we may write $g = (g_1, \ldots, g_{n-m})$ with

$$g_i(x,y) = \sum_j a_j^i(x,y)y^j \qquad \text{where} \qquad a_j^i(x,0) = \frac{\partial g_i}{\partial y^j}(x,0).$$

for each $i = 1, \ldots, n-m$ and $j = 1, \ldots, k$. Therefore, canceling ts

$$G_t(x,y) = (f(x,ty), \sum_j a_j^1(x,ty)y^j, \ldots, \sum_j a_j^{n-m}(x,ty)y^j)$$

which is now very manifestly smooth, even in consideration of the boundary coming from the interval I, and also now well-defined for all t since the ys are part of the fiber coordinates.

Now G_0 maps the fiber of E^0 at x into the fiber of F^1 at x by a linear map given by the matrix $J = (a_j^i(x,0))$. The Jacobian $J(\iota)$ has rank $m+k$, being an embedding and for $p \in M \subset E^0$, it has the form (in bundle coordinates)

$$J(\iota) = \begin{pmatrix} I_m & * \\ 0 & J \end{pmatrix}$$

so J must be of rank k and thus that G_0 is as desired. It is clear by construction that this is stationary on M. □

Remark. In the case of boundary, we have to extend things appropriately to apply the preceding lemma. This is involved, but ultimately works out as we shall see. The easier generalization is for collars, which we do next.

Corollary 31

Every two tubular nbhds of a submanifold $M \subset \operatorname{Int} N$ with $\partial M = \varnothing$ are isomorphic as bundles and have isotopic embeddings in N.

There is also a version that applies to collars.

Theorem 37 (Uniqueness of Collars, I)

Let $M = \partial N$ where N may have corners and let F^0 and F^1 be two open collars of M. Then F^0 is isotopic to F^1 by an isotopy that is stationary on M. In particular, this shows that $\operatorname{Im}(F^0)$ is diffeomorphic to $\operatorname{Im}(F^1)$ by an embedding fixed on ∂M and hence that the smooth structure on $\partial M \times \mathbf{R}_{\geq 0}$ making it a manifold with corners is unique.

Proof. Open collars of $M = \partial N$ are in particular open subspaces. Multiplication by non-negative numbers is allowed in $\mathbf{R}_{\geq 0}$, so the proof goes through without change—thinking about collars as a sort of trivial "half" tubular nbhd—and yields an isotopy $G(p,s)$ for $(p,s) \in M \times \mathbf{R}_{\geq 0}$ where G_1 is the inclusion of F_1 and G^0 is the inclusion of F^0 except scaled by first mapping $(p,s) \mapsto (p, a(p)s)$ where $a(p)$ is a smooth positive function on M (i.e., the shrinking)—this last map is isotopic to the inclusion of F^0.

The only thing that needs verification is that the we can find a shrinking function $\varepsilon\colon \partial M \to (0,1]\mathbf{R}_{>0}$ which is suitably smooth for the smooth structure given to the collar when M has corners, by which we mean that we require

$$(p,t) \mapsto (p, \varepsilon(p)t) \qquad \partial M \times \mathbf{R}_{\geq 0} \to \partial M \times \mathbf{R}_{\geq 0}$$

to be smooth for the smooth structure inherited from its collar map $C\colon \partial M \times \mathbf{R}_{\geq 0} \to M$.

For this, it is enough to assume the collar has a particular and convenient form, we claim. This follows essentially because we can the cocatenate isotopies.

For this, note that we may WLOG suppose the collar F^0 arose from a vector field X. By modifying the construction of the inwards pointing vector field suitably, it is possible to assume that, for some choice of complete metric on M, that $|X|$ is bounded. For instance, choose a covering of ∂M by charts (x_i, U_i) in which each point p of $\bigcup_i U_i$ is in at most $m+1$ charts. This is possible to arrange because the covering dimension of an m-dimensional manifold is the same as its dimension. Then construct an inwards pointing vector field as usual and in each chart (x_i, U_i) modify the vector field X_i constructed so that $|X_i| < 1$ in U_i. This is possible to do using a suitable scaling function that is smooth. Using a partiton of unity to piece together the X_i, it is easy to see that the resulting vector field X has $|X|$ bounded above for this covering. By standard arguments, this means the flow of X is global in a suitable sense for manifolds with corners. In particular, the flow map Φ^X is already defined on $\partial M \times \mathbf{R}_{\geq 0}$. One can then easily verify that $\Phi^X|\partial M \times \mathbf{R}_{\geq 0}$ is a homeomorphism onto its image by showing that it is an open map; one shows that by showing that it sends sufficiently small open sets to open sets. The argument is entirely analogous to how we showed the collar map is an open map for manifolds with corners—namely, one invokes invariance of domain in coordinates after suitably extending things.

Note that for this flow, the domain of the flow contains $M \times \mathbf{R}_{\geq 0}$. We claim that the map $M \times \{t\} \to M$ sending $(p,t) \mapsto \Phi^X(p,t)$ is a diffeomorphism onto its image. It is certainly smooth, so we wish to verify it is a topological embedding and thus a smooth embedding and therefore diffeomorphism onto its image. This is essentially a consequence of **Corollary** 5 and the fact that for $t > 0$, the map has image in the interior of M and for $t = 0$ the map is the identity.

With these details out of the way, we claim that we can find a smooth ε-shrinking of this particular collar using the smooth structure coming from its image. For this, note that we may use the flow to flow backwards smoothly. Thus, cover the image of the collar $C(M \times \mathbf{R}{\geq}\, 0)$ by a locally finite collection of open subsets $\{U_i\}$ such that for each open subset there is a smooth function $\varepsilon_i \colon U_i \to \mathbf{R}_{\geq 0}$ such that for each $p \in U_i$, $\Phi^X(p, -\varepsilon_i(p)) \in F^1$. By an argument similar to the one given above, we may find a smooth function $\varepsilon \colon M \to \mathbf{R}_{\geq 0}$ such that $\Phi^X(p, \varepsilon(p)) \in F^1$ for all $p \in F^0$. Using this collar

that it is possible to construct such a $\varepsilon \colon \partial M \to [0,1)$ which is the restriction of a smooth map $M \to [0,1)$. We claim that the above composite then must be smooth. notice that we may first construct ε

Suppose we have constructed a shrinking function that is at least continuous. This can be done as above. Since the collar is homeomorphic to its image, we may equip them both with a product metric where ∂M inherits a metric from M and $\mathbf{R}$ gets its usual metric. By a suitable application of the Whitney approximation theorem,

observe that it is possible to construct this function in such a way that it is locally the restriction of a smooth function defined on an open nbhd of M. This requires an argument similar to the one given in the **Theorem** B about local flows, but this can be done.

We claim this means it is smooth with respect to the structure the collar inherits from its image in M and for this, all that is required is that we show the assignment

$$(p,t) \mapsto C(p, \varepsilon(p)t)$$

is smooth with respect that structure smooth. It is at least continuous. Note that $C_*(\partial_t)$ is a smooth vector field on the open submanifold $\mathrm{Im}(C)$ and that its flow Φ restricts precisely to the map C. Hence, this assignment is locally in $\mathrm{Im}(C) \times \mathbf{R}_{\geq 0}$ the restriction of a smooth function. This forces the assignment to be smooth

To see that this is so, note that the collar C This is so because we have required collars of manifolds with corners to be smooth in the sense that the collar embedding $C \colon \partial M \times \mathbf{R}_{\geq 0} \to M$ is locally the restriction of a smooth map defined in an open nbhd U of (p,t) in $M \times \mathbf{R}_{\geq 0}$; ε may be supposed, perhaps after shrinking to a open rectangle

$V \times W$, to be smooth on $U = V \times W$ by the formula $(p, s) \mapsto (p, \varepsilon(p)s)$, and this is indeed smooth since ε admits a smooth extension. This means the assignment above is smooth since □

This result, as mentioned, can be extended with some effort to submanifold that have boundary. The point as that as long as the manifold with boundary is a submanifold of a submanifold *without boundary*, then it admits a tubular nbhd with an extension that is an open subset of N (and thus say Int N) and therefore the shrinking trick above will still work.

Theorem 38

Let $j\colon M \times \mathbf{R}_{\geq 0} \to N$ be an embedding where M and N have corners and where $\overline{\operatorname{Im}(j)} \subset \operatorname{Int}(N)$. Then j extends to an embedding of M in an open nbhd V of $M \times \mathbf{R}_{\geq 0}$ into Int N. In particular, this open nbhd V is diffeomorphic to $M \times (-1, \infty)$ by a map that is the identity on $M \times \mathbf{R}_{\geq 0}$.

As usual, we assume M and N are path-connected. The idea is to construct a suitable vector field and use its flow to find the embedding.

Proof. Since $\overline{\operatorname{Im}(j)} \subset \operatorname{Int} N$, we may as well suppose WLOG that $\partial N = \varnothing$. If $\overline{\operatorname{Im}(j)} \cap \partial N \neq \varnothing$, then certain nbhds in Int N of $\operatorname{Im}(j)$ would not be nbhds of $\operatorname{Im}(j)$ in N and things would go wrong.

Let $\dfrac{d}{dt}$ be the evident vector field on $M \times \mathbf{R}_{\geq 0}$. Then since j is an embedding, we get a vector field on the subset $M \times \mathbf{R}_{\geq 0}$ given by $j_* \dfrac{d}{dt}$. We can arrange to extend this to a vector field X having support in an open subset of $M \times \mathbf{R}_{\geq 0}$ by a partition of unity argument such that $X = j_* \dfrac{d}{dt}$ on $M \times \mathbf{R}_{\geq 0}$, we claim.

The idea is to take any locally finite open cover of $M \times \mathbf{R}_{\geq 0}$ say by submanifold charts $\{U_i\}_{i \in I}$ whose image is an open rectangle of the appropriate type (see below) and appending an open set U to this cover such that $\mathsf{U} \cap M \times \mathbf{R}_{\geq 0} = \varnothing$ and such that $(\bigcup U_i)^c \subset \mathsf{U}$. Running the usual partition of unity argument, if we through away the function ρ_{U} associated to U, then we obtain a vector field $\widetilde{X}$ that is X on $M \times \mathbf{R}_{\geq 0}$ and vanishes outside of an open nbhd of $M \times \mathbf{R}_{\geq 0}$ such that no integral curve of $\widetilde{X}$ starting at a point $p \in j(M \times \{0\})$ has closure a loop. We must only construct suitable vector fields on suitable nbhds of $M \times \mathbf{R}_{\geq 0}$ at this point.

Keep X as above. Take a submanifold chart (x, U) about $p \in M \times \mathbf{R}_{\geq 0} \subset N$ with path-connected U, $x(U \cap M \times \mathbf{R}_{\geq 0}) = x(U) \cap \mathbf{R}^{m+1-k} \times \mathbf{R}^k_{\geq 0} \times \mathbf{0}$. Then in coordinates

$$Y = X|_{(U \cap M \times \mathbf{R}_{\geq 0})}$$

is always tangent to this submanifold of Euclidean space. Since $x(U) \subset \mathbf{R}^n$ is open (since this is the interior of N), it contains an open rectangle. Thus, WLOG, we may suppose each chart in this open cover satisfies that $x(U) = V_1 \times V_2$ with V_1 and V_2 path-connected where $V_1 \subset \mathbf{R}^{m+1} \times 0$ is such that $x(U \cap M \times \mathbf{R}_{\geq 0}) \subset V_1$ sits as a quadrant and is thus path-connected. Working in these coordinates, since Y is smooth, it extends to an open nbhd of $x(U) \cap \mathbf{R}^{m+1-k} \times \mathbf{R}^k_{\geq 0} \times \mathbf{0} \subset V_1$ in $\mathbf{R}^{m+1}$ where it is non-zero (this also requires the evident continuity argument). WLOG suppose it is all of V_1. Now extend Y to all of $x(U) = V_1 \times V_2$ by defining $Y_{(a,b)} = X_{(a,0)}$ (it only depends on the $\mathbf{R}^{m+1}$ coordinate). This is smooth since it is the composite of smooth functions $V_1 \times V_2 \xrightarrow{\text{pr}} V_1 \xrightarrow{Y} \mathbf{R}^{m+1}$—note that V_1 and V_2 are path-connected.

Covering $M \times \mathbf{R}_{\geq 0}$ by such charts, we may WLOG assume this collection is locally finite. Then, running the associated partition of unity argument described above, we then have a smooth extension $\widetilde{X}$ of X. We claim that the vector field $\widetilde{X}$ has integral curves starting at $p \in j(M \times \{0\})$ whose closure (of their image on their maximal domain) is not a loop.

To see this, first set

$$M_0 = j(M \times 0)$$

for convenience and take $p \in M_0$. Observe that $\widetilde{X}_p = X_p$ which is an inward pointing vector by construction and thus that the integral curve $\gamma_p^{\widetilde{X}}(t)$ has time derivative X_p an inward pointing vector. Since $\bigcup \operatorname{supp} \rho_i = \operatorname{supp} \sum \rho_i$ (by local finiteness) is contained in $\bigcup U_i$ (and contains $\operatorname{Im} j$), this integral curve does not have closure a loop we claim. If it did, then $\lim_{t\to\infty} \gamma_p^{\widetilde{X}}(t) \in M_0$, but then some thought shows that the chart we used to construct $\widetilde{X}$ about p must also intersect M_0 and $M \times [s, \infty)$ for all $s > 0$ and this will contradict the assumption that the component V_1 of this chart is connected.

Now, the flow of $\widetilde{X}$ as a vector field on N is defined on an open nbhd of $N \times 0$ in $N \times \mathbf{R}$. Moreover, for each $p \in M_0$ and $t \geq 0$, $\Phi^{\widetilde{X}}(p,t) = j(p,t)$ and so in particular the flow domain $A_{\widetilde{X}}$ contains at least the interval $[0, \infty)$ about each $p \in M_0$. By our construction, the integral curves for points $p \in M_0$ for the vector field $-\widetilde{X}_p$ have maximal intervals (a_p, b_p) such that $\gamma_p^{-\widetilde{X}}([0, b_p))$ is contained in a compact set, by construction and thus have a limit point by continuity and compactness, call it b. If $-\widetilde{X}_b \neq 0$, then there is an integral curve through this $\gamma_b^{-\widetilde{X}}$ and these two integral curves may be concantenated smoothly since local existence and uniqueness about b implies that this integral curve agrees with γ_p after adjusting domains and this contradicts maximality. Hence, it must be that $-\widetilde{X}_b = 0$ and that $b \in \partial(\operatorname{supp} \widetilde{X})$.

In any case, there is an open nbhd

$$V \supset M_0 \times \{0\}$$

of

$$M_0 \times \{0\}$$

in the flow domain $A_{\widetilde{X}}$ such that each time interval in V about $p \in M_0$ is of the form $(-b_p, \infty)$.

Since integral curves are either the same or disjoint, we know that $(p,t) \mapsto \Phi^{\widetilde{X}}(p,t)$ is an injective immersion on V. We must show it is also an embedding and so we show it is an open map onto its image and for this, since it is injective, it suffices to check that it is open on small enough basis elements for the topology.

Let $U_1 \times U_2$ be an open nbhd of $(p,t) \in V$ where U_1 is open in M and U_2 open in $\mathbf{R}$. Then U_1 is the intersection with M of some open subset $U_1' \subset N$ and we may suppose, perhaps by shrinking, that $A_{\widetilde{X}} \supset U_1' \times U_2$ where $A_{\widetilde{X}}$ is the flow domain of $\widetilde{X}$. Suppose we know that we may suppose $\Phi^{\widetilde{X}}$ is injective on $U_1' \times U_2$ by shrinking if necessary. Then $\Phi^{\widetilde{X}}$ restricted to $U_1' \times U_2$ is a diffeomorphism onto its image since U_1' is open and for appropriate t (perhaps shrinking) we know that $p \mapsto \Phi^{\widetilde{X}}(p,t)$ is a diffeomorphism from U_1' onto its image with inverse $p \mapsto \Phi^{\widetilde{X}}(p,-t)$ by the properties of flows. Indeed, this implies that each image is open and therefore the union of its images are open. One then verifies that $\Phi^{\widetilde{X}}(U_1 \times U_2) \cap \Phi^{\widetilde{X}}(U_1' \times U_2) = \Phi^{\widetilde{X}}(U_1' \times U_2) \cap \Phi^{\widetilde{X}}(V)$ and therefore $U_1 \times U_2$ maps to an open set and, in particular, all sufficiently small open rectangles about each point (p,t) in V map open subsets of the image and thus this is an open map.

Now, there exists by the usual arguments (such as in my proof of the collar nbhd theorem) a shrinking function $\varepsilon\colon M \to (0,1]$ such that $M \times (-1,\infty) \to V$ is a diffeomorphism onto an open subset of $M \times \mathbf{R}_{\geq 0}$ in V extending the evident inclusion $M \times \mathbf{R}_{\geq 0} \to V$. WLOG, suppose ε lands in

$$\varepsilon\colon M \to (0,\frac{1}{2}].$$

We construct such an ε as in the collar nbhd theorem so that $(p,t) \mapsto (p,\varepsilon(p)t) \in V$ as usual. By a suitable and obvious algebraic modification we can just as well write this as $(p,t) \mapsto (p,t-\varepsilon(p)t)$ (in other words replace the $\varepsilon(p)$ in this new equation by $(1-\varepsilon(p))$ for $\varepsilon(p)$ as in the old equation). Then mollify this function to fix it on $\mathbf{R}_{\geq 0}$ by using some modification of the smooth function $x \mapsto l(x)$, call it $L(x)$, constructed as in Spivak's book. The function $L(x)$ satisfies the following properties.

(a) $L(x)$ is 0 for all $x \leq -1$;

(b) $L(x)$ is 1 for $x \geq 0$;

(c) $L(x)$ is increasing and positive on $(-1,0)$.

To see what this needs to be, let $L(p,t)$ be some function and let us demand that for $t \in (-1,0]$,

$$t - L(p,t)(1-\varepsilon(p))t \leq t - (1-\varepsilon(p))t$$

This becomes for $t \neq 0$

$$L(p,t) \geq 1$$

and we impose the additional obvious constraint that for $t \neq 0$,

$$L(p,t) < \frac{1}{1-\varepsilon(p)}$$

which checks out since by our assumptions $\frac{1}{1-\varepsilon(p)} > 1$ always and so we want

$$1 \leq L(p,t) < \frac{1}{1-\varepsilon(p)}$$

and we should hope additionally that as $t \to 0$, $L(p,t) \to \dfrac{1}{1-\varepsilon(p)}$. Since

$$\frac{1}{1-\varepsilon(p)} = 1 + \frac{\varepsilon(p)}{1-\varepsilon(p)}$$

this suggests we take

$$L(p,t) = 1 + L(t)\frac{\varepsilon(p)}{1-\varepsilon(p)}$$

with $L(t) = L(x)$ as above. Then certainly this is always ≥ 1 and it is also always $\leq \dfrac{1}{1-\varepsilon(p)}$ since $0 \leq L(t) \leq 1$ on the relevant interval. It is well-defined because we assumed $\varepsilon < 1/2$ always and it is clearly smooth.

In particular then, we obtain a diffeomorphism

$$M \times (-1,\infty) \to V \qquad (p,t) \mapsto (p, t - \left(1 + L(t)\frac{\varepsilon(p)}{1-\varepsilon(p)}\right)(1-\varepsilon(p))\,t)$$
$$= (p, t - (1-\varepsilon(p) + L(t)\varepsilon(p))t).$$

with everything in the right slot denoting multiplication, we claim. Note that since $\varepsilon(p) > 0$ always, this makes sense and is smooth and we have arranged as above that this maps into V so we are good. This simplifies further of course to

$$(p,t) \mapsto (p, \varepsilon(p)(t - L(t)t)).$$

That this assignment $M \times (-1,\infty) \to V$ a diffeomorphism onto its image in V and that its image is open follows easily. □

Corollary 32

If $i\colon M \to N$ is an embedding with $\overline{\operatorname{Im} i} \subset \operatorname{Int} N$ and $\partial M \neq \varnothing$, then there is a smooth manifold $\overline{M}$ containing M as a submanifold such that $\partial\overline{M} = \varnothing$ and an embedding $j\colon \overline{M} \to \operatorname{Int}(N)$ extending i.

As usual we will conflate embeddings with subsets and so on.

Proof. Once again, WLOG $\partial N = \varnothing$. Let $\iota\colon \partial(M) \times \mathbf{R}_{\geq 0} \to M$ be a collar nbhd. Then we can extend $i|\,\partial M \times \mathbf{R}_{\geq 0}$ to an embedding $\bar{i}\colon \partial M \times (-1,\infty) \to N$. An easy modification—with a little bit of thought about the definition of submanifold charts—of the proof above (namely the construction of locally finite collection of submanifold charts) allows us to assume that for each point $(p,t) \in \partial M \times (-1,0)$ has an open nbhd U in N such that $U \cap (M \setminus (\partial M \times \mathbf{R}_{\geq 0})) = \varnothing$ in N.

Exercise 41

Verify this. *[Hint: This argument is similar to the one given above that no integral curve starting at a point in the boundary has closure a loop.]*

Construct a new submanifold of N and call it $\overline{M}$ as $\overline{M} = \mathrm{Im}(\bar{i}) \cup \mathrm{Im}(i)$. The construction above—with the same little bit of thought about the definition of submanifold charts—allows us to assume that there are submanifold charts about each (p,t) in the extended collar of $\mathrm{Im}(\bar{i})$ which do not intersect $M \setminus (\partial M \times \mathbf{R}_{\geq 0})$. For points $p \in \mathrm{Cl}_M(\partial M \times \mathbf{R}_{\geq 0})$, there are submanifold charts intersecting only M in N (otherwise this contradicts the above construction). Finally for points in $M \setminus (\partial M \times \mathbf{R}_{\geq 0})$ there are submanifold charts intersecting only M (otherwise we obtain the same contradiction in the preceding sentence). This shows that the subset $\overline{M} \subset N$ is a submanifold. It has empty boundary since every point belongs to $\mathrm{Int}(M) \subset N$ (a diffeomorphic image) or to $\partial M \times (-1, \infty)$ (a diffeomorphic image) and so the submanifold charts have no boundary (we may have to shrink them for $\mathrm{Int}(M)$ but we can do this say by using a metric and arguing about balls). □

Corollary 33 (Uniqueness, II(a))

Let $M \subset N$ and suppose $\partial M \neq \varnothing$ and M has no corners. Let $\overline{M}$ be a submanifold without boundary extending the inclusion of M into N. Let F^1 be a tubular nbhd of $\overline{M}$ in N and let $F^0 \to M$ be a bundle of rank $k \leq \dim N - \dim M$. Then there is an isotopy G of the inclusion $F^0 \subset N$ such that $G_t(p) = p$ for $p \in M$ and G_1 is a linear map $F^0 \to F^1|\, M$ of rank k on each fiber as bundles over M.

Proof. WLOG $\partial N = \varnothing$. The tubular nbhd of $\overline{M}$ is open in N, so we can shrink the fibers of a tubular nbhd of M into it as before. The same argument now goes through, taking care to address boundary charts by extending functions to convex sets to apply the lemma needed. □

Corollary 34 (Uniqueness, II(b))

Let $M \subset \mathrm{Int}\, N$ and suppose $\partial M \neq \varnothing$ and M has no corners. Let F^0 and F^1 tubular nbhds of M in N. Then there is an isotopy G of the inclusion $F^0 \subset N$ such that $G_t(p) = p$ for $p \in M$ and G_1 is an isomorphism of bundles $F^0 \to F^1$.

Proof. Fix a tubular nbhd F of $\overline{M}$. Then $F|\, M$ is a tubular nbhd of M. There is then an isotopy between the inclusion $F^0 \to N$ and an isomorphism $\phi\colon F^0 \cong \overline{F}|\, M$ and similarly an isotopy between $F^1 \subset N$ and an isomorphism $\psi\colon F^1 \cong \overline{F}|\, M$. Concatenating these and smoothing them into $[0,1] = I$ gives the result since the over all composite at the end $F^0 \to F^1$ is simply $\psi^{-1}\phi$ which is a linear isomorphism. □

Theorem 39 (Uniqueness, III)

If M is compact, $M \subset \mathrm{Int}\, N$, and F^0 and F^1 are either both proper or closed tubular nbhds of M in N both arising from existing tubular nbhds, then there is a strong ambient isotopy H_t of the identity map of N that keeps M fixed and such that $H_1|\, F^0$ is an isomorphism $F^0 \to F^1$. If F^0 and F^1 have metrics, we may suppose in addition that $H_1|F^0$ is an isometry.

Here, we give F^0 and F^1 the metrics induced by their ε-shrinkings in the proper case and in the closed case we that the map at the end is a morphism of bundles with the same orthogonal group as their structure group.

Proof. Let F^0 and F^1 be proper and ε-shrinkings of E^0 and E^1 respectively WLOG into the unit disk bundles of E^0 and E^1 (if these are not the unit disk bundles, that's fine, nothing changes actually). From the above, there is an isotopy H of $E^0 \subset N$ to an isomorphism $H_1 \colon E^0 \to E^1$. We have seen that all such bundle maps are themselves strongly isotopic to isometries so there is a strong isotopy G_t of H_1 to an isometry G_1. Let K be H followed by G smoothly concatenated. The unit disk bundles of E^0 and E^1 are compact since M is compact and so K restricted to this subset of N admits an extension to an isotopy of the identity map of N. This then gives an isometry of F^0 and F^1 in their induced metrics in the proper case and also gives a morphism of the corresponding unit disk bundle that is a morphism of bundles with structure group $O(n-m)$.

As for strong-ness of this isotopy, the isotopy extension theorem constructs an isotopy with compact support and thus is strong as we have seen. □

There is also an analogous result for collars.

Theorem 40 (Uniqueness of Collars, II)

If $M = \partial N$ is compact and we have two open collars of M that are proper in the obvious sense, then they are ambient isotopic and strongly so.

Proposition 14

If $M \subset N$ is neat, then $\nu_{\partial N}\partial M \cong \nu_N M|\partial M$ naturally.

Definition 43

We suppose M and N have no corners. A tubular neighborhood of a neat submanifold $M \subset N$ is said to be a ***neat tubular neighborhood*** if $F \cap \partial N$ is a tubular nbhd of ∂M in ∂N and F is a tubular nbhd of M in N.

Proposition 15

A neat tubular neighborhood is an open set and a neat submanifold.

Proof. We claim $F \subset N$ is a submanifold of the same dimension which is moreover neat. Indeed, viewing this as an embedding of the normal bundle of M into N, we know what its boundary is and neatness follows very simply by observing that since for each $p \in F \cap \partial N = \partial F$ (as a submanifold), thus since the embedding $F \to N$ has full rank, in coordinates, this looks like a smooth map of full rank between $\mathbf{H}^n \to \mathbf{H}^n$. Extending this to a map of full rank (open condition) $\mathbf{R}^n \to \mathbf{R}^n$, this becomes an open map since it is then a local embedding and we can use invariance of domain. Therefore its restriction $\mathbf{H}^n \to \mathbf{H}^n$ must have been an open map to begin with, some thought shows. In particular, this since $\operatorname{Int}(F) \to N$ is an open map, this shows that nbhds of the boundary of F also map to open sets and hence that $F \subset N$ is an open map and hence that $F \subset N$ is open in N. Knowing this, neatness is automatic because for any $p \in \partial F$, there a boundary chart of N whose domain lies entirely in F by openness and so it is therefore a neat submanifold chart as desired. □

Theorem 41 (Existence, II)

Every neat submanifold of a manifold with boundary N has a neat tubular neighborhood.

Proof. Give N a Riemannian metric on N that is a product metric in a collar neighborhood $\partial N \times \mathbf{R}_{\geq 0}$ of ∂N. This means that if $\nu M | \partial N$ is identified with the normal bundle of ∂M in ∂N, then geodesics corresponding to normal vectors of ∂M and issued at points of ∂N will stay in ∂N (because M intersects the boundary of N transversely by neatness). Therefore the same sort of argument in the usual tubular neighborhood theorem still works without change. It will yield a neat tubular nbhd of M in N. □

Definition 44 (Tubular Nbhds of Submanifolds of the Boundary)

Let $M \subset \partial N$ where N has no corners and M may have corners. Note that

$$\nu_N M \cong \nu_{\partial N} M \oplus \nu_{\partial N} N \cong \nu_{\partial N} M \oplus \underline{\mathbf{R}}.$$

A ***tubular neighborhood*** of $M \subset \partial N$ in N is an embedding $\nu_{\partial N} M \times \mathbf{R}_{\geq 0} \to N$ extending an embedding of $\nu_{\partial N} M \to \partial N$ witnessing $\nu_{\partial N} M$ as a tubular nbhd of M in ∂N. The definition of a proper tubular nbhd of this sort is unchanged.

Theorem 42 (Existence, III)

If $M \subset \partial N$ where N has no corners and M boundary but no corners, then M has a tubular nbhd.

Proof. Let $C \colon \partial N \times \mathbf{R}_{\geq 0} \to N$ be a collar. Given $\nu_{\partial N} M \to \partial N$ a tubular nbhd of M in ∂N, this extends by the embedding $C | (\nu_{\partial N} M \times \mathbf{R}_{\geq 0})$. □

Theorem 43 (Uniqueness, IV)

If M is compact neat submanifold of N with boundary and F^0 and F^1 are either both neat proper or neat closed tubular nbhds of M in N both arising from existing neat tubular nbhds, then there is a strong ambient isotopy H_t of the identity map of N that keeps M fixed and such that $H_1 | F^0$ is an isomorphism $F^0 \to F^1$. If F^0

and F^1 have metrics, we may suppose in addition that $H_1|F^0$ is an isometry.

Here, we give F^0 and F^1 the metrics induced by their shrinking maps in the proper case and in the closed case we that the map at the end is a morphism of bundles with the same orthogonal group as their structure group.

Proof. First we apply the proof of the third uniqueness result to the tubular nbhds resulting by intersecting the boundary and then extend this to a diffeotopy of all of N by the isotopy extension theorem. Thus, WLOG, $F^0|\partial M \subset F^1$ is a linear subspace.

One then applies the proof of the third uniqueness result again to the interior of N and the interior of the resulting tubular nbhds to conclude. Strictly speaking, we would concatenate the diffeotopies if we forgot about the WLOG above. □

F Transversality and Regular Value Theorems

F.1 Regular Value Theorems for Manifolds With Boundary

Before we begin with the regular value theorem, let us introduce an auxiliary lemma and use it to prove a proposition.

Lemma 34

Let M be a smooth manifold without boundary and let $g\colon M \to \mathbf{R}$ be smooth. Suppose g has regular value 0 and $g^{-1}(0) \neq \varnothing$. Then $g^{-1}([0,\infty))$ is a submanifold with boundary $g^{-1}(0)$ and dimension equal to that of M. In particular, the submanifold charts for $g^{-1}(\mathbf{R}_{\geq 0})$ can be chosen in such a way that $g^{-1}(\mathbf{R}_{\geq 0})$ sits as $\mathbf{H}^m \subset \mathbf{R}^m$ without further straightening—these submanifold charts would exhibit $g^{-1}(\mathbf{R}_{\geq 0})$ as a neat submanifold in a different context.

Proof. Since 0 is a regular value of g, $g^{-1}(0)$ is a codimension one submanifold of M by the usual constant rank theorem. We have that $g^{-1}((0,\infty))$ is an open submanifold being open in M. We only need to check that there is a smooth structure on this and that we have submanifold charts. Really the only issue is with the boundary. Each $p \in g^{-1}(0)$ admits a submanifold chart for $g^{-1}(0)$ and we must show we can make this a submanifold chart for $g^{-1}(\mathbf{R}_{\geq 0})$.

A submanifold chart exists for each $p \in g^{-1}(0)$, say (x, U), such that $U \cap g^{-1}(0) = x^{-1}(\mathbf{R}^{m-1} \times \{0\})$. We want to show that $g^{-1}((0,\varepsilon)) \cap U$ sits in this chart as $\mathbf{H}^m$. With respect to the given chart, since $g|g^{-1}(0)$ is constant and $g^{-1}(0) \subset \mathbf{R}^{m-1} \times \{0\}$, g has trivial derivatives in the directions lying in the $\mathbf{R}^{m-1} \times \{0\}$ subspace. Hence, in these coordinates, for each $p \in g^{-1}(0)$, $g_{*p} = (0,\ldots,0,v)$ for some $v \neq 0$, $v \in \mathbf{R}$—and so in any chart, $v > 0$ or $v < 0$ since by the IVT it will otherwise be 0 somewhere—since 0 is a regular value, forcing $v \neq 0$. Therefore suppose in our chart $v > 0$. Then our coordinates, each $q \in x(U)$, $q = (q^1,\ldots,q^m)$, with $q^m > 0$ has $g(q) > 0$. Hence, $U \cap g^{-1}(\mathbf{R}_{\geq 0}) \subset x^{-1}(\mathbf{H}^m)$ as desired. This is a submanifold chart because the boundary of $g^{-1}(0)$ already sits neatly in the chart and we do not need to do any more straightening. □

Lemma 35

Let M have boundary but no corners and let $f\colon M \to N$ be smooth, $\dim M = m$, $\dim N = n$. No point $q \in \partial N$ can be a regular value for *both* f and $f|\partial M$ unless $f^{-1}(q) = \varnothing$ or $n = 0$.

Proof. Suppose this is not the case; that is, suppose $q \in \partial N$ is a regular value for both f and f^{-1}. We will see in **Proposition** 4 of the appendix that $f^{-1}(q) \subset \partial M$ is forced if q is a regular value for f so we might as well assume this; the idea is that **(c)** of the constant rank theorem only forces us to modify coordinates in the domain and so in coordinates the image of f remains unchanged and a point-set argument implies that its image then lies in the interior of N in suitable coordinates coming from the constant rank theorem. We will show that this is contradicted under our assumption.

Now, f_{*p} surjects $T_pM \to T_qN$ and so has kernel dimension $m - n \geq 1$ (with the inequality following from our assumption that $f|\partial M$ has q as a regular value so that $m - 1 \geq n$). We claim that this implies there is a vector $v \in T_pM$ for which $f_{*p}(v) = 0$ but $v \notin \operatorname{Ker} f|\partial M_{*p}$; indeed, this follows by dimensional considerations since $\dim \operatorname{Ker} f_{*p} = m - n$ whereas $\dim(\operatorname{Ker} f|\partial M_{*p}) = m - 1 - n$. Therefore v is an outward or inward pointing vector—that is, it has a component in the outward or inward direction.

Working in coordinates (x, U) and (y, V) and extending f to a function of maximal rank on an open nbhd about $x(p) = 0$ in $\mathbf{R}^m$, there is a coordinate system (x', U'), $x(p) = 0 \in U'$ of $\mathbf{R}^m$ by **(c)** of the constant rank theorem for which f looks like a projection $\mathbf{R}^m \to \mathbf{R}^n$, say projecting onto the first $m - n$ coordinates. We casually identify vectors for these Euclidean spaces with vectors in the naive sense. Let w be the image of the vector v in this coordinate system. In these coordinates, f_* is the block diagonal matrix that is $I_{m-n\times m-n}$ in the upper-left corner and 0 everywhere else. Hence, for f_* to have vanishing derivative in the direction of w, w must be a linear combination of the last n coordinates of $\mathbf{R}^m$ and therefore, in particular, $f(rw) = \mathbf{0}$ for all sufficiently small $r \in \mathbf{R}$ for which rw remains in U' (since open subsets of Euclidean spaces are locally convex and since f is a projection).

Claim. It is not hard to see that for small enough r with one of either $r \geq 0$ or $r \leq 0$, rw remains in the image of $x(U)$ under the diffeomorphism taking us to the coordinates in which f is a projection.

This is because the open half-ball $x(U)$ has interior an open subset of Euclidean space and so contains all points sufficiently close to $x(p) = 0$ in the upper half-plane and thus all points sufficiently close to $x(p)$ with last coordinate positive (i.e., an inward pointing component, under the casual identification of points of $\mathbf{R}^n$ with vectors). Hence, $f^{-1}(0)$ must contain points not lying in ∂M and this is a contradiction. □

Exercise 42

Can this be generalized to when M has corners? *[I'm virtually certain the answer is the affirmative but I didn't write the proof with that case in mind so I have not not thought* *about it and do not want to think about it.]*

We can get a feel for what's going on here by the following corollary, which essentially implies that what goes wrong is dimensional when q is a regular value of f but not the restriction $f|\partial M$.

Corollary 35

Let M have boundary and no corners as above. If $q \in \partial N$ is a regular value for f, then for each $p \in f^{-1}(q)$, $\operatorname{Ker} f_{*p} \subset T_p\partial M$.

Remark. We will see in **C** of the appendix that $f^{-1}(q) \subset \partial M$ is forced if q is a regular value for f; we are using that here.

Proof. Suppose $\operatorname{Ker} f_{*p} \not\subseteq T_p\partial M$ and let $V = \operatorname{Ker} f_{*p} \cap T_p\partial M$. Since q is a regular value for f, f_{*p} has rank $n = \dim N$ and $\dim \operatorname{Ker} f_{*p} = m - n$ and $\dim V \le m - n - 1$. Working in a boundary chart, one deduces $V = \operatorname{Ker}(f|\partial_M)_{*p} \subset T_p\partial M \subset T_pM$. By the rank-nullity theorem, $\dim V + \operatorname{rank}(f|\partial_M)_{*p} = m - 1$ and therefore

$$\operatorname{rank}(f|\partial_M)_{*p} = m - 1 - \dim V \ge m - 1 - m + n + 1 = n$$

but also $\operatorname{rank}(f|\partial_M)_{*p} \le n$ since $\dim T_qN = n$ so in fact

$$\operatorname{rank}(f|\partial_M)_{*p} = n$$

so q is a regular value for $f|\partial M$. This contradicts the above lemma. □

Theorem 44 (Improved Regular Value Theorem)

Let M and N be smooth manifolds with boundary but no corners of dimension m and n, respectively and let $f : M \to N$ be smooth. If $q \in N$ is a regular value of both f and $f|\partial M$, then $f^{-1}(q)$ is a neat submanifold of M of codimension n (i.e., $\dim f^{-1}(q) = m - n$).

Remark (Improvement)**.** This theorem is an improvement over the usual regular value theorem because of the boundaries. In the usual regular value theorem, we consider only manifolds without boundary, and the theorem is essentially an immediate consequence of the constant rank theorem. Namely, in the usual case, for every point $p \in f^{-1}(q)$, the constant rank theorem guarantees that there are charts (x, U) and (y, V) about p and q respectively for which f looks like standard projection $\operatorname{pr} : \mathbf{R}^m \to \mathbf{R}^n$ onto the last (or first, it doesn't matter by rearranging things) n coordinates. This proves that $\operatorname{pr}^{-1}(0) = x(f^{-1}(q) \cap U) = \mathbf{R}^{m-n} \times 0$ which means x is a submanifold chart for $f^{-1}(q)$.

Remark (Additional Comments)**.** For $q \in N$ to be a regular value of f means that for all $p \in f^{-1}(q)$, $\operatorname{rank}(df_p) = \dim N$, and this forces $\dim N \le \dim M$. We must throw out the vacuous case in this theorem which is why we additionally stipulated that $f^{-1}(q) \ne \varnothing$.

For our assumptions, it will turn out that for $q \notin \partial N$, $\dim N \leq \dim M - 1$ if $\partial N \cap f^{-1}(q) \neq \emptyset$ and $\dim N \leq \dim M$ if $\partial N \cap f^{-1}(q) = \emptyset$. For $q \in \partial N$ it will turn out we only need $\dim N \leq \dim M$ because $f^{-1}(q) \subset \partial M$ in this case and it is furthermore not possible for q to be a regular value of both f and $f|\partial M$. This follows from the preceding lemma.

Proof. Note that WLOG we may assume that $m \geq n$. We may suppose that $n \geq 1$ since when $n = 0$, N is a disjoint collection of points and so $f^{-1}(q)$ is simply a component of M. Now consider the case $\dim M = \dim N$. Then $f^{-1}(q)$ is necessarily a collection of isolated points. Indeed, in coordinates, after extending f while keeping it maximal rank (an open condition), we could write the extension of f in yet another set of coordinates as the identity map by the constant rank theorem. This means that f must be injective in a nbhd of p. We may therefore assume $m > n$. In particular, this means we may suppose $m > n$ and $n \geq 1$ and so $m > n \geq 1$. We have seen that for q to be a regular value of both f and $f|\partial M$, under our hypotheses, it must be that $q \in \operatorname{Int} N$, so we may assume this.

We begin by supposing $p \in f^{-1}(q)$ is not in ∂M. Then $f^{-1}(q)$ is a submanifold in a nbhd of p. This is because, in coordinates, we may write this locally as a projection from an open subset of $\mathbf{R}^m$ onto $\mathbf{R}^n$, say killing off the first $m - n$ coordinates, with no other words needed. Hence, if $p = (a^1, \ldots, a^m)$ in this coordinate system, then this is clearly a submanifold chart for $f^{-1}(q)$ about p since all points of the form $(x^1, \ldots, x^{m-n}, a^{m-n+1}, \ldots, a^m)$ are sent to the image of p under f in these coordinates.

This takes care of the points not in the boundary of M. Next, we must consider points in the boundary of M and verify neatness as well.

Now consider the case $p \in \partial M \cap f^{-1}(q)$. Pick charts (x, U_0) and (y, V_0) such that $x(p) = 0$ and $y(q) = 0$ and set

$$x(U_0) = U \qquad y(V_0) = V.$$

We have a smooth map $U \to \mathbf{R}^n$, the coordinate form of f, with U open in $\mathbf{H}^m$ the upper half-space which we may extend to an open subset $\widetilde{U} \subset \mathbf{R}^m$ and get $\widetilde{f} \colon \widetilde{U} \to \mathbf{R}^n$. Since max rank is an open condition, we may suppose this extension has max rank. It follows that $\widetilde{f}^{-1}(0)$ is a submanifold of $\mathbf{R}^m$ of codimension n (i.e., of dimension $m - n$). WLOG suppose U is an open unit coordinate ball in $\mathbf{H}^m$ and $\widetilde{U}$ is the completion of it to a full open unit coordinate ball in $\mathbf{R}^m$—we can arrange for this by shrinking; the point is that we want $\widetilde{f}$ to agree with f on the boundary $\partial \mathbf{H}^m \subset \mathbf{R}^m$.

Let $\pi \colon \widetilde{f}^{-1}(0) \to \mathbf{R}$ be the projection onto the m-th coordinate and recall that this coordinate for any boundary chart is the outward/inward pointing direction. We claim this has regular value 0—i.e., $\widetilde{f}^{-1}(0)$ has non-trivial tangent vectors in the x^m-direction.

Suppose this was not the case. Then the tangent space to $\widetilde{f}^{-1}(0)$ at 0 (i.e., $x(p)$) would lie completely in some collection of n of the direction $\frac{\partial}{\partial x^i}$ where $i \neq m$ and so $\widetilde{f}^{-1}(0)$ lies in a subset of the first $m - 1$ coordinates. But, restricting to these coordinates, one easily verifies that (abusing notation)

$$(f|\partial \mathbf{H}^m) = (f|\partial M)^{-1}(0) = (\widetilde{f}\big|\partial \mathbf{H}^m)^{-1}(0)$$

as a consequence of how we constructed $\widetilde{U}$ and U (see above). Hence, we have (working in coordinates) $(f|\partial\mathbf{H}^m)^{-1}(0) = \widetilde{f}^{-1}(0) \cap \partial\mathbf{H}^m$ (i.e., those points with $x^m = 0$). Since q is a regular value for $f|(U \cap \partial M)$ (and in fact for $f|\partial M$), this submanifold must have dimension $m - n - 1$; but if 0 is not a regular value of the m-th coordinate projection map, then in fact $T_0(f|\partial M)^{-1}(0) \subset T_0\partial\mathbf{H}^m$ and therefore it is a submanifold of dimension $m - n$. This is a contradiction.

Now, $f^{-1}(0) = \pi^{-1}(\mathbf{R}_{\geq 0})$ and by the **Lemma**, $\pi^{-1}(\mathbf{R}_{\geq 0})$ is a submanifold of $\widetilde{f}^{-1}(0) \subset \widetilde{U}$ contained in U with boundary $\pi^{-1}(0)$—that is, $\widetilde{f}^{-1}(0) \cap U = f^{-1}(0)$. Thus, $f^{-1}(0)$ admits reasonable submanifold charts in $\widetilde{f}^{-1}(0)$ and has codimension 0 therein. We also know that $f^{-1}(0)$ is a submanifold of U since U is a submanifold of $\widetilde{U}$ for the obvious reasons (consider how we constructed U and $\widetilde{U}$). It remains to show that it is *in addition* neat.

The only trouble arises for points in $\pi^{-1}(0)$, so let (α, U_α) be a submanifold chart for $\pi^{-1}(0)$ in $\widetilde{f}^{-1}(0)$. Then (after rearranging) $U_\alpha \cap \pi^{-1}(0) = \alpha^{-1}(\mathbf{0} \times \mathbf{R}^{m-n-1} \times \{0\})$. Since $i\colon \widetilde{f}^{-1}(0) \to \widetilde{U}$ is an embedding between manifolds without boundary, **(d)** of the constant rank theorem guarantees that there is a chart (β, V_β) such that (after rearranging) $\beta i \alpha^{-1}(a^1, \ldots, a^{m-n}) = (0, \ldots, 0, a^1, \ldots, a^{m-n})$. The reasoning of the preceding **Lemma** shows us that $\pi^{-1}(\mathbf{R}_{\geq 0})$ must sit as the collection of points in the image having the form $(0, \ldots, 0, a^1, \ldots, a^{m-n-1}, v)$ where either $v \geq 0$ for all such a^i or $v \leq 0$ for all such a^i. □

Remark. This proof can be simplified.

F.2 Transversality

Here is the basic concept of transversality.

Definition 45

Let K, L and N be manifolds with corners and let $f\colon K \to N$ and $g\colon L \to N$ be smooth maps. Then we say that f *is transversal to* g, denoted by $f \pitchfork g$, if whenever we have $f(k) = g(l) = p$, we have $f_* T_k K + g_* T_l L = T_p N$. We can also say that *$f$ and g are transverse*.

More generally, given $f\colon M \to N$ and a submanifold $P \subset N$ and a subset C of X, we say that f is *transversal to P on a subset C* of X if the transversality condition

$$f_* T_p M + T_{f(x)} P = T_{f(x)} Y$$

is satisfied for every $p \in C \cap f^{-1}(P)$. We write this as $f \pitchfork Z$ on C or, maybe even more concisely, as $f \pitchfork_C Z$.

Remark. If $f(K) \cap g(L) = \varnothing$, then transversality holds vacuously.

Remark. The only way, it seems, to get good results for transversality, at least with little effort, is to assume neatness in some places. Essentially, the issue is that the regular value theorem, as we know how to prove it, is insensitive to the corners or boundaries. Basically, the argument one wants to use relies upon not having extra structure floating around on M. It is possible to compensate for this by imposing additional constraints on the map f to get an analogous result for manifolds with boundary. A neat submanifold is assumed to only have corner points of depth k match up with the corner points of depth k in the ambient manifold, and this assumption eliminates the extra data needed to make certain arguments go. Another issue is that the regular value theorem only makes sense in the category DIFF and if M has corners then ∂M is not smooth.

Lemma 36

Let $f\colon K \to M$ and $g\colon L \to M$ be smooth. Then $f \pitchfork g$ **iff** $f \times g\colon K \times L \to N \times N$ is transverse to the diagonal $\Delta_N \subset N \times N$.

Proof. This is simply a matter of unraveling definitions and a small computation. □

The proof of the following theorem is adapted from Hirsch's book—the picture below is likewise adapted from Hirsch.

Theorem 45

Let M^m and N^n be smooth manifolds with boundary of dimension m and n, respectively. Let $A \subset N$ be a k-dimensional neat submanifold of N. If $f\colon M \to N$ is smooth and $f \pitchfork A$ and $f|\partial M \pitchfork A$, then $f^{-1}(A)$ is a neat submanifold of codimension $n-k$ (i.e., dimension $m-n+k$) with $\partial f^{-1}(A) = f^{-1}(A) \cap \partial M$.

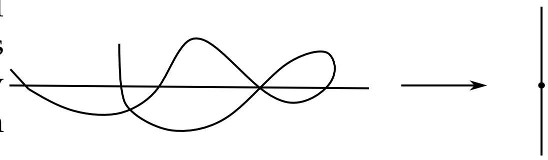
Remark. If A has no boundary, A is not automatically neat. If $\partial M = \varnothing$, then the condition $f|\partial M \pitchfork A$ is vacuously true. One can say something about normal bundles but we defer that to the proposition below.

Proof. Either $\partial A = \varnothing$ or $A \cap \partial N = \partial A$. First consider the interior points of A. These are points which, by definition, also lie in the interior of N. In particular, $A \setminus \partial A$ is a smooth boundary-less manifold and $N \setminus \partial N$ is too. Since $A \cap \partial N = \partial A$, we may choose our submanifold chart about for each $q \in A \setminus \partial A \cap \mathrm{Im}(f)$ to be an *interior chart* of N and, perhaps by shrinking, we may suppose our submanifold chart (y, W) about q has image a product nbhd $y(W) = U \times V \subset \mathbf{R}^k \times \mathbf{R}^{n-k}$ such that $y(A \cap W) = U \times \mathbf{0}$. Pick coordinates, (x, Z) about $p \in f^{-1}(q)$ in M with Z so small that $f(Z) \subset W$, so we don't have to worry about intersecting things. To avoid breaking into cases, suppose $x(Z) \subset \mathbf{H}^m$ is open but we do not specify whether (x, Z) is a boundary chart or not. Transversality of f to A then becomes transversality of $y \circ f \circ x^{-1}$ to $U \times \mathbf{0}$ and transversality of $f|\partial M$ to A then similarly becomes transversality of $y \circ f \circ x^{-1}\big|(Z \cap \partial M)$ to $U \times \mathbf{0}$. The first of these is equivalent to the assertion that the composite

$$g : \mathbf{H}^m \supset x(Z) \xrightarrow{f \circ x^{-1}} W \xrightarrow{y} U \times V \xrightarrow{\mathrm{pr}} V \subset \mathbf{R}^{n-k}$$

has regular value $\mathbf{0}$ and the latter that $g|\partial \mathbf{H}^m$ has regular value $\mathbf{0}$. Transversality of $f|\partial M$ to A then becomes transversality of $y \circ f \circ x^{-1}\big|(Z \cap \partial M)$ to $U \times \mathbf{0}$.

This shows that $g^{-1}(\mathbf{0})$ is a *neat* submanifold of $x(Z)$ having codimension $n-k$ (i.e., dimension $m-n+k$) as a consequence of the regular value theorem proved above. But x is a diffeomorphism from Z onto $x(Z)$, so $f^{-1}(A\cap W)$ must be a submanifold of $f^{-1}(W)\subset M$. Now suppose $q\in\partial A$ and so by neatness of A, $q\in A\cap\partial N$. Since A is neat, we may replace our target chart (y,W) by a neat submanifold chart for $q\in A\cap\partial N$. Then the same argument above goes through after replacing V by an open $V\subset\mathbf{H}^{n-k}$ intersecting the boundary and switching the roles of U and V (we like the last coordinate to be positive for boundary charts).



This shows that every point in $f^{-1}(A)$ has a submanifold chart. To see the assertion about the boundary of $f^{-1}(A)$ and neatness, one simply unpacks the notation and cases above. □

Warning. The situation is not so nice when we ask about the general case. The following extremely simple example is due to Lars Tyge Nielsen in his paper *Transversality and the Inverse Image of a Submanifold With Corners*.

Let $M=\mathbf{H}^2$, $N=\mathbf{R}^2$ and $A=\{x\in\mathbf{R}^2 : \|x\|\le 1\}$ the unit disk and $f\colon M\to N$ is the inclusion. Then $f\pitchfork A$, $f\pitchfork\partial A$, $\partial f\pitchfork A$ and even more $\partial f\pitchfork\partial A$, but $f^{-1}(A)$ is a manifold with corners, being the set of points $\{(x,y)\in\mathbf{R}^2 : \|(x,y)\|\le 1,\ y\ge 0\}$ and thus we leave the category of manifolds with boundary but no corners.

Remark. A more refined sort of notion of transversality is needed to generalize the preceding theorem. A nice treatment of such a generalization is given in the preceding paper. Another such generalization may be found in Pavel Hájek's master's thesis, *On Manifolds with Corners*. The upshot of this discussion is that you really don't want to have to think about such cases unless you absolutely have to.

Here is a different proof reproduced from Kosinski's book.

Proposition 16

Let $V\subset N$ be a neat submanifold of dimension r and $f\colon M\to N$ smooth. If $f\pitchfork V$ and $f|\partial M\pitchfork V$, then $W=f^{-1}(V)$ is a neat submanifold of M such that $\operatorname{codim}_M(W)=\operatorname{codim}_N(V)$ and $\nu_M W\cong f^*\nu_N V$.

Proof. Let $p\in W$ and $q=f(p)$. Given a neat submanifold chart (y,Z) about q for A, we may suppose $y(Z)=A\times B$ with B open in $\mathbf{R}^r_{\ge 0}$ with $0\in A$. Then the composite

$$h\colon Z\xrightarrow{y} A\times B\xrightarrow{A}\subset\mathbf{R}^{n-r}$$

is a submersion with $h^{-1}(0)=Z\cap V$. Then $f^{-1}(Z)=f^{-1}h^{-1}(0)$ and $(hf)_*$ and $(hf|\partial M)_*$ are surjective from our assumptions. It follows that W is a neat submanifold of M of the stated codimension by **Theorem A.2.9**.

The normal bundle of V in N, $\nu_N V$, be identified with the quotient $TN|V/TV$. Let $\pi\colon TN|V \to \nu_N V$ be the quotient map and consider the composite $g = \pi \circ f_*|(TM|W)$; let $d = \dim \operatorname{Ker} g$ and note that since g is surjective, $m - d \geq \operatorname{codim} V$ and so $d \geq \dim W$, but since $TW \subset \operatorname{Ker} g$, $d \geq \dim W$ and so $d = \dim W$ and so dimensional considerations force $\operatorname{Ker} g = TW$. Hence, $f\colon W \to V$ induces a map bundle map $TM|W/TW \cong \nu_M W \to \nu_N V \cong TN|V/TV$ covering the map $f\colon W \to V$. By the pullback theorem for bundles, this forces $\nu_M W \cong f^*\nu_N V$ as desired. ☐

Corollary 36

Let M^m have boundary and no corners and let $K, L \subset M$ be neat submanifolds of dimensions k and ℓ, respectively. If $K \pitchfork L$ and $\partial K \pitchfork L$, then $K \cap L$ is a neat submanifold of M of dimension $k + \ell - m$. Moreover, $K \cap L$ is a neat submanifold of both K and L of dimension $k + \ell - m$ and the boundary of $K \cap L$ is $\partial K \cap \partial L$.

Remark. Since K and L are submanifolds, $\dim K, \dim L \leq m$ ($k, \ell \leq m$) and since they are transverse, $\dim K + \dim L \geq \dim M$ ($k + \ell \geq m$) because for all $p \in K \cap L$, $T_pK + T_pL = T_pM$. When K and L *both* have boundary, then this inequality tightens to $k + \ell - 1 \geq m$ because we assumed $\partial K \pitchfork L$.

Proof. Let $f\colon K \to M$ be the embedding of K into M. Since $f \pitchfork L$ and $f|\partial K \pitchfork L$, it follows by the the preceding that $f^{-1}(L)$ is a neat submanifold of K of dimension $k - m + \ell = k + \ell - m$ (i.e., of codimension $m - \ell$). Similar reasoning with f simply an embedding of K in the general case shows that $f^{-1}(L)$ is a submanifold of K of dimension $k - m + \ell = k + \ell - m$ with boundary $f^{-1}(\partial L) \cup (\partial K \cap f^{-1}(L))$—since f is an inclusion, $f^{-1}(\partial L) = K \cap \partial L$ and $\partial K \cap f^{-1}(L) = \partial K \cap K \cap L = \partial K \cap L$.

We only have to show in the first case that the neat embedding f restricts to a neat embedding $f\colon K \cap L \to M$.

The result now follows from the following claim, whose proof is exemplary of the utility of thinking locally.

Claim. If $A \subset B \subset C$ and B is neat in C and A is neat in B, then A is neat in C (neatness forces $\dim A \geq 1$ when $\partial A \neq \varnothing$). In particular, there is a neat submanifold chart for A in C which is also a neat submanifold chart for B in C.

Say $\dim A = i$, $\dim B = j$ and $\dim C = k$. We make some reductions. Consider the boundary first. Pick a neat submanifold chart for B in C, call it (y, V), and suppose $a \in \partial A \cap V$. Using this chart, we may reduce to the Euclidean case where we suppose, in particular, that $C = \mathbf{H}^k$, $B = \mathbf{0} \times \mathbf{H}^j$ and $A \subset B$ is neat—we may make this assumption by shrinking to a subset diffeomorphic to the open unit half-ball in $\mathbf{H}^k$ via our chart and then using the evident radial diffeomorphism. We have thus reduced to the case that $A \subset \mathbf{0}_{k-j} \times \mathbf{H}^j \subset \mathbf{H}^k$ with A neat in $\mathbf{0} \times \mathbf{H}^j$.

Suppose WLOG $0 \in A$ is our new a. Pick a neat submanifold chart (x, U) for A about 0 in $\mathbf{0} \times \mathbf{H}^j$ and suppose U is an open half-ball in $\mathbf{H}^j$ about 0 of radius ε. Then $x\colon U \to \mathbf{H}^j$ is an embedding for which $x(U \cap A) = x(U) \cap \mathbf{0}_{j-i} \times \mathbf{H}^i \subset \mathbf{H}^j$.

We can now extend this to a chart for $\mathbf{H}^k$ having domain B_ε the open half-ball of radius ε in $\mathbf{H}^k$ about 0 as follows. For $a = (a^1, \ldots, a^{k-j}, a^{k-j+1}, \ldots, a^k) \in B_\varepsilon$, we define a chart (z, B_ε) by $a \mapsto (a^1, \ldots, a^{k-j}, x(a^{k-j+1}, \ldots, a^k))$. Since $z = (\mathrm{pr}, x)$ on its domain, where pr is the projection onto the first $k - j$ coordinates, it is clearly a diffeomorphism. The inverse is $z^{-1} = (\mathrm{pr}, x^{-1})$ which is likewise smooth. Thus, this is a chart and moreover $z(V \cap A) = y(U \cap A) = x(U \cap A) = x(U) \cap \mathbf{0}_{k-j} \times \mathbf{H}^j \subset \mathbf{H}^k$ as desired.

When $a \in \mathrm{Int}\, A$ and thus is likewise an interior point of B and C, the procedure is the same, mutatis-mutandis. □

Proposition 17

Let K, L and M be manifolds where only K may have boundary. If $f\colon K \to M$ and $g\colon L \to M$ satisfy $f \pitchfork g$ and $f|\partial K \pitchfork g$, then $K \times_M L$ is a smooth manifold with boundary $\partial K \times_M L$ satisfying the evident universal property in the smooth category and, in particular, it is a smooth submanifold of $K \times L$.

Proof. We give the proof when $\partial K = \varnothing$. The graphs $G(f)$ and $G(g)$ of f and g are submanifolds of the evident space and thus so too is their product. One notes that $G(f) \times G(g) \pitchfork \{(a, p, b, p) \in K \times M \times L \times M\}$ so that

$$G(f) \times G(g) \cap \{(a, p, b, p) \in K \times M \times L \times M\}$$

is a submanifold by the preceding theorem. Then since the projection $K \times M \times L \times M \to K \times L$ is a smooth open map and since its restriction to

$$G(f) \times G(g) \cap \{(a, p, b, p) \in K \times M \times L \times M\}$$

is an injective immersion, it follows that this map is a embedding and thus the submanifold assertion follows. The universal property is easily verified. □

Remark. The case when $\partial K \neq \varnothing$ proceeds similarly invoking the same theorem.

The following is adapted from Bredon's book. It makes more precise the intuition that transversal submanifolds intersect generically.

Theorem 46

Let $N^n, S^s \subset M^m$ be neat submanifolds with $N \pitchfork S$. If $m \leq n + s$, then there is a chart (x, U) about each $p \in N \cap S$ such that either $U \cap N$ is represented by the first n coordinates and $U \cap S$ by the last s coordinates or $U \cap N$ is represented by the first $n - 1$ coordinates and the last coordinate and $U \cap S$ by the last s coordinates, depending on whether or not $p \in \partial M$.

Proof. Let $P = N \cap S$ which, by the above, is a neat submanifold. Let $p \in P$. We can find (perhaps by shrinking) charts $x\colon U \to \mathbf{H}^m$, $y\colon U \to \mathbf{H}^m$ and $z\colon U \to \mathbf{H}^m$ about p exhibiting N, S and P, respectively, as neat submanifolds of M (we make no assumptions yet about the images of these charts in the upper half-plane to avoid breaking into cases as much as possible). In particular, then, by projecting, there are maps (submersions, even) $f\colon U \to \mathbf{R}^{m-n}$, $g\colon U \to \mathbf{R}^{m-s}$ such that $f^{-1}(0) = U \cap N$,

$g^{-1}(0) = U \cap S$ and similarly a smooth map $h\colon U \to \mathbf{H}^{n+s-m}$ projecting onto $N \cap S$ such that $h^{-1}(0) = U \cap P$. This is essentially **Theorem A.2.9**.

Consider the map

$$\varphi\colon U \to \mathbf{R}^{m-n} \times \mathbf{R}^{n+s-m} \times \mathbf{R}^{m-s} \supset \mathbf{R}^{m-n} \times \mathbf{H}^{n+s-m} \times \mathbf{R}^{m-s} \qquad q \mapsto (f(p), h(p), g(p)).$$

A computation shows that for $p \in U \cap P$, φ_{*p} is surjective and thus by dimensional considerations an isomorphism.

If $p \notin \partial M$, then φ is a local diffeomorphism at p and thus perhaps by restricting the desired chart. If $p \in \partial M$, φ is still a local diffeomorphism and so gives us the desired chart after shuffling the copy of $\mathbf{R}_{\geq 0}$ to the end. □

F.3 Homotopy and Transversality

The power of transversality lies in its genericness among smooth maps. We will approach this through the notion of stability to avoid defining smooth function spaces for the moment. If we were to define the ***weak Whitney topology*** (i.e., the ***smooth compact-open topology***) and prove the genericness theorem for those function spaces, then the following would be an immediate consequence since for compact domain, the weak and strong Whitney topologies agree.

Theorem 47 (Submanifold Stability)

Fix smooth manifolds K, L and M of dimensions k, ℓ and m, respectively where all manifolds are connected. Suppose $L \subset M$ is a neat submanifold which is a closed subset. Let $f\colon K \to M$ such that $f \pitchfork g$.

- **(a)** If K is compact, then for any smooth homotopy of $H\colon K \times I \to M$ of f, there is a sufficiently small $\varepsilon > 0$ such that for all $0 \leq \delta < \varepsilon$, $H_\delta\colon K \to M$ is such that $H_\delta \pitchfork L$.
- **(b)** If K is compact and we assume additionally that K and L have no corners and $\partial f \pitchfork L$, then we may suppose in addition that $\partial H_\delta \pitchfork L$ in this interval.
- **(c)** Compactness of K may be dropped in **(a)** and **(b)** if we only require a perturbation to exist on an open say connected nbhd U of $K \times 0 \subset K \times I$. In this case, $K_t = \{k \in K : (k,s) \in U \text{ for all } 0 \leq s < t\}$ is an open submanifold of K and for each $0 \leq s < t$, $H_s| K_t \pitchfork L$.

Proof. **(a)** We note that transversality of f with a (boundary-less) submanifold L is a local condition—about each point $p \in L$, there is a nbhd U of M and a submersion by $\sigma\colon (U, p) \to (\mathbf{R}^{m-\ell}, 0)$ such that $\sigma|(U \cap \partial M)$ is also a submersion and such that $\sigma^{-1}(0) = U \cap L$ by **Theorem A.2.9**. Transversality of f to L at p implies that $\sigma \circ f$ has regular value 0 (vacuous when $p \notin \operatorname{Im}(f)$). In other words, transversality translates locally into a submersion condition.

Let H be any smooth homotopy of f. Since K is compact, it suffices to prove that every point $(k, 0) \in K \times I$ has a nbhd U in $K \times I$ such that for all $(k,t) \in U$, $(H_t) \pitchfork L$ at k or $H_t(k) \notin L$. The set $H^{-1}(M \setminus L)$ is open since L is closed and it contains

$K \setminus f^{-1}(L) \times \{0\}$, so for each $k \in K$, there is a nbhd $U = U_K \times U_I$ (WLOG a product open set) of k in $K \times I$ such that $H(U) \cap L = \varnothing$. So we consider the points $k \in K$ that do intersect L in some nbhd of $(k, 0)$.

By neatness of L, each $q \in L$ has a nbhd V in M of q and a submersion $\sigma \colon (V, q) \to (\mathbf{R}^{m-\ell}, 0)$ such that $\sigma^{-1}(0) = V \cap L$ by **Theorem A.2.9** (choose a neat submanifold chart and shrink it to product to project off of the slice $V \cap L$). Transversality of f to L at q implies that $\sigma \circ f$ is a submersion at p whenever $f(p) = q$ (and by assumption $q \in \operatorname{Im} f$). The condition of being a subermsion is an open condition, so there is a nbhd U (once again WLOG a product open set say by shrinking) of $(p, 0)$ for which H is a submersion on U.

Finally, by compactness of K, we may cut down this collection to a finite subcollection then take the smallest interval length from 0 appearing, say $[0, \varepsilon)$.

(b) Repeat the argument above all the way up to the end of the third paragraph. The extra assumptions mean that each $q \in L$ has a nbhd V in M of q and a submersion $\sigma \colon (V, q) \to (\mathbf{R}^{m-\ell}, 0)$ such that $\sigma^{-1}(0) = V \cap L$ is a submersion by **Theorem A.2.9** as above. Transversality of f to L at q implies that $\sigma \circ f$ is a submersion at p whenever $f(p) = q$ (and by assumption $q \in \operatorname{Im} f$). The condition of being a subermsion is an open condition, so there is a nbhd $U_p = U_K \times U_I$ (once again WLOG a product open set say by shrinking) of p for which H is a submersion on U; if $p \in \partial K$, we assume that $U_K = U_{\partial K} \times U_0$ in the coordinates of a collar nbhd WLOG. Transversality of ∂f to L means that $\sigma \circ \partial f$ is a submersion at p whenever $p \in \partial M$ and $f(p) = q$; since transversality is open condition, there is a nbhd $U'_p = U'_{\partial K} \times U'_I$ of $(p, 0)$ in $\partial M \times I$ (WLOG a product nbhd again) such that ∂H is a submersion on U'. For each $p \in \partial K$, replace U_p by $(U_{\partial K} \cap U'_{\partial K}) \times (U_I \cap U'_I)$. Running the same argument as above, we build the desired interval.

(c) Simply run the same arguments as above to produce a covering of an open nbhd of $K \times \{0\}$ which WLOG is connected. Then the preimage of $[0, t)$ under the projection $K \times I \to I$ maps U to an open set (projections are open maps) and therefore the preimage of the restriction to U is an open subset of $K \times I$ contained in U consisting of all points (k, s) where H is defined on $[0, t)$ for the point k. The projection of this preimage to K is then open subset (hence, open submanifold) of K which we called K_t (projections are open maps). The two assertions about transversality are clear by the construction. □

This implies the following stability result.

Corollary 37 (Map Stability)

Let K, L and M be manifolds where we assume $\partial M = \varnothing$ and all manifolds are connected. Let $K \xrightarrow{f} M \xleftarrow{g} L$ be smooth maps such that $f \pitchfork g$.

- **(a)** If K is compact, then for any smooth homotopy H of f, there is a $\varepsilon > 0$ such that for all $0 \le \delta < \varepsilon$, $H_\delta \pitchfork g$.
- **(b)** If K is not compact, then the same conclusion is true in an open say connected nbhd of $K \times 0 \subset K \times I$.

Proof. Replacing M by $M \times L$, we may replace L by the neat embedding $(g, \mathrm{id})\colon L \to M \times L$ (its graph); if $\partial M \neq \varnothing$, then (g, id) would not be a neat embedding. Observe that $\mathrm{Im}((g, \mathrm{id}))$ is a closed subspace of $M \times L$ since the graph of any continuous function is closed. Finally, replace K by the compact manifold $K \times L$ and f by $f \times \mathrm{id}\colon K \times L \to M \times L$. Then $f \pitchfork g$ becomes transversality of $f \times \mathrm{id}$ with the neat submanifold $L = \mathrm{Im}((g, \mathrm{id}))$. Since $\partial M = \varnothing$, $\partial(M \times L) = M \times \partial L$. Therefore $\mathrm{Im}((g, \mathrm{id}))$ is a neat submanifold of $M \times L$ (this fails if $\partial M \neq \varnothing$).

Let H be a homotopy of f. Then we may consider $\widetilde{H}$ a homotopy of $f \times \mathrm{id}$ by letting $\widetilde{H}\colon K \times L \times I \to M \times L$ be the homotopy $(k, l, t) \mapsto (H(k, t), l)$. Even if K is not compact, the hypotheses of the preceding theorem are met.

By the theorem, there is an open nbhd U of $K \times L \times 0$ in $K \times L \times I$ (WLOG connected) upon which $\widetilde{H}$ preserves transversality $(K \times L)_t$ (which is an open submanifold of $K \times L$ by the arguments made in the preceding theorem, we recall) to $\mathrm{Im}((g, \mathrm{id}))$. Consider the following commutative diagram

$$
\begin{array}{ccc}
K \times I & \xrightarrow{(\mathrm{id}, q) \times \mathrm{id}} & K \times L \times I \\
\uparrow & & \downarrow \widetilde{H} \\
K \times \{0\} & \xrightarrow{(f, q)} & M \times L \\
\downarrow f & & \downarrow \mathrm{pr}_M \\
M & = & M
\end{array}
$$

and call the resulting composite map $H\colon K \times I \to M$.

Now, $U \cap K \times \{q\} \times I$ is an open nbhd of $K \times 0 \subset K \times I$. If K is not compact, we are done. Indeed, consider the open submanifold of K_t consisting of all point $k \in K$ for which $(k, t) \in U \cap (K \times \{q\} \times I)$. A simple argument shows that for $s \in [0, t)$, $H_s | K_t \pitchfork g$. Otherwise, if K is compact, then the tube lemma then implies that this open set contains a tube $K \times [0, \varepsilon)$ and once again a simple argument shows that $H_t \pitchfork g$ for all $0 \leq t < \varepsilon$. □

Remark. The following results are taken from Guillemin and Pollack's book. There are likely formulations for neat submanifolds too and maybe even more general setups.

Theorem 48 (Transversality Theorem)

In this theorem, no manifold is permitted to have corners.

Let $F\colon X \times S \to Y$ is smooth where only X has boundary and $Z \subset Y$ is a submanifold without boundary. If F and $F|\partial(X \times S)$ are transversal to Z, then for almost every $s \in S$, $F(-, s)$ and $F(-, s)|\partial X$ are transversal to Z. More generally, if $g\colon Z \to Y$ is any map where we again assume $\partial Z = \partial Y = \varnothing$, then if F and $F|\partial(X \times S)$ are transversal to g, then for almost every $s \in S$, $F(-, s)$ and $F(-, s)|\partial X$ are transversal to g.

Theorem 49 (Transversality Homotopy Theorem)

In this theorem, no manifold is permitted to have corners.

For any smooth $f: X \to Y$ where $\partial Y = \varnothing$ and any boundary-less submanifold Z of Y, there is a smooth map $g: X \to Y$ such that both $g \pitchfork Z$ and $g|\partial X \pitchfork Z$ and f is smoothly homotopic to g. More generally, given a map $g: Z \to Y$ where $\partial Z = \partial Y = \varnothing$, there is a smooth map $h: X \to Y$ such that both $h \pitchfork g$ and $h|\partial X \pitchfork g$ and f is smoothly homotopic to h.

Proof. Embed Y in $\mathbf{R}^m$ and let νY be its normal bundle therein and let $r: \nu Y \to Y$ be the retraction (or bundle projection, depending on you think about it) onto Y. Define $h: \nu Y \to \mathbf{R}^m$ by $h(y, v) = y + v$. Then h is regular at every point of $Y \subset \nu Y$ as the zero section and maps Y onto itself diffeomorphically. The tubular nbhd trick shows that this is then true in an open nbhd of $Y \subset \nu Y$ and thus we use this (plus perhaps a sort of ε-shrinking argument similar to what we have done in previous proof) to build a nice ε-nbhd of Y in $\mathbf{R}^m$ where ε is a smooth function on Y; call this Y^ε. Then we define with S the open unit ball in $\mathbf{R}^m$, $F: X \times S \to Y$ by $F(x, s) = r(f(x) + \varepsilon(f(x))s)$. Then $F(x, 0) = f(x)$ and for fixed x, $s \mapsto f(x) + \varepsilon(f(x))s$ is a submersion $S \to Y^\varepsilon$ and so $s \mapsto F(x, s)$ is a submersion. Both F and $F|\partial(X \times S)$ are submersions then. It follows that F and $F|\partial(X \times S)$ are transversal to any boundary-less submanifold Z of N and hence for almost all $s \in S$, $F(-, s)$ and $F(-, s)|\partial X$ are transversal to Z. Finally, f is homotopic to each such map by the evident straight-line homotopy which is clearly smooth. The same proof will work for the latter version as well. □

Theorem 50 (Extension Theorem)

In this theorem, no manifold is permitted to have corners.

Let $Z \subset Y$ be a closed subset and a submanifold without boundary of N where $\partial N = \varnothing$, let $C \subset X$ be a closed subset of X and let $f: X \to Y$ be a smooth map with $f \pitchfork Z$ on C and $f|\partial X \pitchfork Z$ on $C \cap \partial X$. Then there exists a smooth map $g: X \to Y$ smoothly homotopic to f such that $g \pitchfork Z$, $g|\partial X \pitchfork Z$ and on a nbhd of C we have $g = f$. The same assertion is true if we only assume that $h: Z \to Y$ is a map for which $h(Z)$ is closed in Y, in which the conclusion is that $g \pitchfork h$, $g|\partial X \pitchfork h$ and on a nbhd of C we have $g = f$.

The idea is to use the preceding theorem. Since ∂X is always closed in X, we also have the following corollary.

Corollary 38

In this corollary, no manifold is permitted to have corners.

Given $f: X \to Y$ such that $\partial f: \partial X \to Y$ is transversal to a submanifold Z which is closed in Y and $\partial Z = \partial Y = \varnothing$, then there is a smooth map $g: X \to Y$ smoothly homotopic to f such that $\partial g = \partial f$ (in fact $g = f$ on a nbhd of the boundary) and $g \pitchfork Z$.

For the next corollary, see here.

Corollary 39

Consider only boundary-less manifolds. If $f, f' \colon X \to Y$ are homotopic smooth maps with $f \pitchfork g$ and $f' \pitchfork g$ where $g \colon Z \to Y$ has closed image, then the pullback by f and g and the pullback by f' and g are bordant.

Proof. We may assume the maps are smoothly homotopic by standard approximation theorems, call this $F \colon X \times I \to Y$. Then there is a smoothly homotopic homotopy $F' \colon X \times I \to Y$ which is transversal to g for which the pullback by F' and g is the desired bordism as consequence of the extension theorem. □

F.4 Construction of Perturbations From Tubular Nbhds

Proposition 18

Let $f \colon M \to E$ where $\pi \colon E \to B$ is a smooth fiber bundle with typical fiber F. If $F_q = \pi^{-1}(q)$, then $f \pitchfork F_q$ **iff** q is a regular value of πf.

Proof. This isn't hard to see working in a local trivialization about q. □

Corollary 40

If $f \colon M \to W \times V$ is smooth, then there is a dense set of points $q \in V$ such that $f \pitchfork W \times \{q\}$.

Proof. This follows from the above along with Sard's theorem. □

Corollary 41

Let $\pi \colon E \to B$ be a smooth fiber bundle. Let $V \subset E$ be a submanifold. Then V is a smooth section of the bundle **iff** V intersections every fiber F_q transversely in a single point $s(q)$.

Proof. $(\Rightarrow)$ This is clear. $(\Leftarrow)$ There is the evident map $s \colon B \to E$ inverse to $\pi|\, V$. By the above, $\pi_*|\, TV$ is surjective and so since $\dim V = \dim B$ this map is an isomorphism. It follows from the constant rank theorem that the inverse s of $\pi|\, V$ is smooth (namely one shows it is a local diffeomorphism). □

Lemma 37

All smooth vector bundles over smooth manifolds are stably trivial. That is, if ξ is a smooth vector bundle over a manifold M, then there is another vector bundle η such that $\xi \oplus \eta$ is isomorphic to the trivial vector bundle over M.

Proof. Suppose rank $\xi = k$. This is an application of Hirsch's globalization theorem to find an embedding of ξ in a trivial vector bundle $M \times \mathbf{R}^s$ where $s \geq k + \dim M$. Then one defines η to be the evident orthogonal complement. □

Lemma 38

Consider a fiber bundle ζ and projection $\pi\colon E \to B$. Let $f\colon M \to B$ and $g_1\colon V \to E$. Then in the commutative diagram

$$\begin{array}{ccccc} E(f^*\zeta) & \xrightarrow{f_1} & E(\zeta) & \xleftarrow{g_1} & V \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle g} & \\ M & \xrightarrow[f]{} & B & & \end{array}$$

where $g = \pi g_1$. If $f_1 \pitchfork g_1$, then $f \pitchfork g$.

Remark. It may not appear obvious but this does hold for smooth manifolds with corners.

Proof. Suppose $f(p) = g(q)$. There is a point $p_1 \in E(f^*\zeta)$ such that $f_1(p_1) = g_1(q)$ and $\pi_1(p_1) = p$. Apply π_* to the transversality condition. note that π_* and $(\pi_1)_*$ are surjective and use commutativity of the square to deduce that

$$T_{f(p)}E = \pi_* T_{f_1(p_1)}E(\zeta) = (\pi f_1)_* T_{p_1}E(f^*\zeta) + (\pi g_1)_* T_q V = f_* T_{\pi_1(p_1)}M + g_* T_q V,$$

as desired. □

Theorem 51

Let ξ be a vector bundle over V with total space $E(\xi) = E$ and projection $p\colon E \to V$. Let $f\colon M \to E = E(\xi)$ be a smooth map. Then there is a (smooth) section $s\colon V \to E$ such that $f \pitchfork s$. In particular, s may be arbitrarily close to the zero section (but not necessarily the zero section).

Remark. This likewise works for any class of smooth manifolds. The assumption that the fiber is a vector space may not be able to be removed since we are using stable triviality of such bundles.

Proof. If ξ is trivial, say $V \times \mathbf{R}^k$, then as we have seen there is a dense set of points $q \in \mathbf{R}^k$ such that $f \pitchfork V \times \{q\}$. In particular, this set is non-empty so choosing any such q, the section $s\colon V \to E$ sending $v \mapsto (v, q)$ proves the theorem for trivial bundles.

In general, let η be a complement to ξ so that $\zeta = \xi \oplus \eta$ is trivial, say $E(\zeta) \cong V \times \mathbf{R}^N$. There is then a natural projection $\pi\colon E(\zeta) \to E$ which is a vector bundle projection. This gives the square of the diagram in the preceeding lemma. Since $E(\zeta)$ is trivial, we may find a section g_1 transverse to f_1. Indeed, as we have seen, since $f_1\colon E(f^*\zeta) \to V \times \mathbf{R}^N$, there is a dense set of points $q \in \mathbf{R}^N$ such that $f_1 \pitchfork V \times \{q\}$. We thus let $g_1\colon V \to E(\zeta) \cong V \times \mathbf{R}^N$ be the section $v \mapsto (v, q)$ for such a $q \in \mathbf{R}^N$.

At this point, we would be done if we know that $g = \pi g_1$ is a section of ξ. This follows since g_1 is a section of $\zeta = \eta \oplus \eta$ and the commutativity of the diagram

$$\begin{array}{ccc} E(\zeta) & \longrightarrow & V \\ {\scriptstyle\pi}\downarrow & \nearrow & \\ E & & \end{array}$$

where all maps are projections of vector bundles. This digram commutes since it is the morphism of vector bundles over V given by the obvious maps

$$\begin{array}{ccc} \eta \oplus \eta & \longrightarrow & 0_V \\ \downarrow & \nearrow & \\ \eta & & \end{array}$$

where $0_V = V \times \mathbf{R}^0 \cong V$.

Note that since this set of points in $\mathbf{R}^N$ is dense, we may take q to be as close to $0 \in \mathbf{R}^N$ as we like. □

The following are now taken from Bredon.

Corollary 42

Let M and V be manifolds, let $f\colon M \to N$ is smooth and $g_0\colon V \hookrightarrow N$ a neat submanifold embedding where V and N have boundary and no corners. Then there is an arbitrarily small isotopy h of the embedding g_0 such that $f \pitchfork h_1$ and $\partial f \pitchfork \partial h_1$. In particular, this isotopy is strong.

Proof. Let T be a neat tubular nbhd of V in N. This is then open in N, as we have seen. Hence, $W = f^{-1}(T)$ is an open submanifold of M—such a submanifold is necessarily neat, we remark, since it has codimension 0 and around any point $p \in f^{-1}(T) \cap \partial M$ it contains an open nbhd of p.

Now, we only need to deform g_0 inside of V to be transverse to $f|W\colon W \to T$. Applying the preceding theorem, a section $s\colon V \to T$ exists arbitrarily close to V as the zero section of T for which $f|W \pitchfork s$. The desired isotopy is then the evident fiberwise isotopy $(v, t) \mapsto s(v)t$. This isotopy is strong (i.e., the track $V \times I \to T \times I$ is a level-preserving *embedding*) for the obvious reasons. □

Corollary 43

Let $V \subset N$ be a compact submanifold and U be an open nbhd of V in N and $f\colon M \to N$ a smooth map. Then there is a strong isotopy h of N that is the identity outside of U and such that $f \pitchfork h_1(V)$.

Proof. By shrinking, we may find a proper tubular nbhd of V in U and a section s of it transverse to f □

Lemma 39

If $p\colon E \to B$ is an orientable vector bundle of rank $n \geq 1$ and $i\colon X \to B$ is an embedding, then the induced bundle $i^*p\colon i^*E \to X$ formed by the pullback is orientable.

Proof. Since i is an embedding, one easily verifies that there is bundle isomorphism $i^*E \cong p^{-1}(X) = E|X$. This is verified topologically by universal properties and one then checks that the homeomorphism given is in fact a bundle isomorphism by recalling how the vector space structure is defined on the fibers of i^*E.

We therefore give each fiber $p^{-1}(x)$ the orientation μ_x is had originally. Fix a trivializing open nbhd U in B of a point $x \in X$. Then $U \cap X$ is a trivializing open nbhd in X. Moreover, one quickly verifies that $p^{-1}(X) \supset p^{-1}(U \cap X) \hookrightarrow p^{-1}(U) \cong U \times \mathbf{R}^n$ is therefore orientation preserving or orientation reversing everywhere, and so $i^*E \cong p^{-1}(X)$ is orientable in the obvious way. □

It once again helps to know the definition of orientability of a vector bundle over M.

Theorem 52

Fix $n \geq 1$. Let $N \subset M$ is be a submanifold of an orientable manifold with corners M and suppose $\dim N = \dim M - 1$ (i.e., a hypersurface). Then N is orientable **iff** the normal bundle of N is trivial.

Remark. M being orientable is surely needed since the Möbius band M is not orientable and $\partial(M \times [0,1)) \cong M$ is not orientable, where $\dim \partial M = \dim(M \times [0,1)) - 1$.

Proof. ($\Leftarrow$) Suppose the normal bundle of N is trivializable. It follows that $TM|N \cong TN \oplus \underline{\mathbf{R}}$. Since M is orientable, $TM|N = TN \oplus \underline{\mathbf{R}}$ is orientable, we claim, and this follows from the preceding lemma. The other lemma now shows that TN must be orientable and hence N is orientable. ($\Rightarrow$) Is N is orientable, then TN is orientable. Hence, $0 \to TN \to TN \oplus \nu_N \to \nu_N \to 0$ is a SES of vector bundles and the middle one is orientable once again because M is orientable and we have an isomorphism $TM|N \cong TN \oplus \underline{\mathbf{R}}$. Hence, ν_N must be orientable. But the only orientable line bundle is trivial, so we conclude. □

G Point-Set Results

G.1 Miscellany

Theorem 53 (May, Thm 7.4.1)

Let $p\colon E \to B$ be a map and $\mathscr{U}$ be a numerable open cover of B. Then p is a Hurewicz fibration **iff** $p\colon p^{-1}(U) \to U$ is a Hurewicz fibration for all $U \in \mathscr{U}$.

Proof. Omitted. There are two typos in May's proof. u_j should be $u_j = \sum_{i=1}^{j} \gamma_{T_i}(\beta) / \sum_{i=1}^{q} \gamma_{T_i}(\beta)$ and $s(e, \beta)$ should be $s(e, \beta)(0) = e$. □

Corollary 44

Every numerable fiber bundle is a Hurewicz fibration.

Proof. For an element U of a numerable open cover by trivializing open sets, it suffices to show in the coordinates of the trivialization that $U \times F \to U$ is a Hurewicz fibration. Of course, the dashed lift in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & U \times F \\ {\scriptstyle i_0}\big\downarrow & \nearrow & \big\downarrow \\ X \times I & \xrightarrow[H]{} & U \end{array}$$

always exists and can be taken to be the map $(H, g \circ \mathrm{pr}_X)$. Hence, the previous theorem allows us to conclude. □

Theorem 54

Every fiber bundle $E \to B$ is a Serre fibration.

Proof. Omitted. □

Theorem 55 (Lee, A.57)

A proper continuous map to a locally compact Hausdorff space is a closed map.

Proof. We show that for $f\colon X \to Y$ continuous and proper and $C \subset X$ closed, $f(C)^c$ is open. Since Y is LCH, each $y \in f(C)^c$ has an open nbhd V containing y that is precompact (open set whose closure is compact). So $K = f^{-1}(\overline{V})$ is compact as f is proper and so $C \cap K$ is a closed subset of the compact space K and so is compact in K and, hence, also X. Hence, $f(C \cap K) = f(C) \cap \overline{V}$ is compact. Since Y is Hausdorff, it is also closed. Hence, $V \setminus (f(C) \cap \overline{V}) = V \setminus f(C)$ is an open nbhd of y not intersecting $f(C)$. □

G.2 Submanifolds are Locally Closed

Definition 46

Say a subspace $A \subset X$ is *locally closed* if it A is a closed subspace of an open subspace V of X.

Lemma 40

Let $A \subset X$. TFAE:

(a) A is locally closed.

(b) Each $p \in A$ has an open nbhd $U \subset X$ such that $A \cap U$ is closed in U.

(c) A is open in its closure $\overline{A}$.

Proof. (a) $\Rightarrow$ **(b)** $A \subset V \subset X$. The nbhd if V since $V \cap A = A$ is closed in V.

(b) $\Rightarrow$ **(c)** Let U_p be a nbhd of $p \in A$ asserted to exist. Then $\mathrm{Cl}_{U_p}(U_p \cap A) = U \cap \mathrm{Cl}_X(A)$ since if $x \in \mathrm{Cl}_{U_p}(U_p \cap A)$, then every nbhd of x in U contains points of A and therefore since U_p is open $x \in \overline{A}$, which is the non-trivial inclusion. Since $U_p \cap A$ is closed in U_p, it follows that $U_p \cap A = U_p \cap \overline{A}$ and so $U_p \cap A$ is a nbhd of p in the subspace topology on $\overline{A}$. Since p was arbitrary, $A \subset \overline{A}$ is open in the subspace topology.

(c) $\Rightarrow$ **(a)** Since $A \subset \overline{A}$ is open in the subspace topology, there is an open subspace U of X such that $U \cap \overline{A} = A$. □

Theorem 56

Submanifolds are locally closed.

Proof. Let $N^n \subset M^m$ be a submanifold. By **(b)** above, this is a local problem, so fix $p \in N$. Then there is a chart (x, U) of M about p which, for convenience, we assume $x\colon U \to \mathbf{R}^m$ is a diffeomorphism onto an open subspace of some $\mathbf{R}^{m-k} \times \mathbf{R}^k_{\geq 0} \subset \mathbf{R}^m$ and we assume $x(U)$ is an open ball, as well as a straightening diffeomorphism $\varphi\colon V \to \mathbf{R}^m$ where we may as well assume $x(U) \subset V$, where V is open in $\mathbf{R}^m$. Then $\varphi x(U \cap N) = \varphi x(U) \cap \mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_{\geq 0} \subset \mathbf{R}^m$. But this is closed in $\varphi x(U)$ since its complement is

$$\varphi x(U) \cap \varphi x(U \cap N)^c = \varphi x(U) \cap (\varphi x(U) \cap \mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_{\geq 0} \subset \mathbf{R}^m)^c = \varphi x(U) \cap (\mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_{\geq 0})^c$$

and $\mathbf{0} \times \mathbf{R}^{n-\ell} \times \mathbf{R}^\ell_{\geq 0}$ is closed so its complement is open, and therefore the intersection is an open subset in $\varphi x(U)$. This shows that U is an open nbhd of $p \in N$ for which $N \cap U$ is closed in U. We conclude by **(b)**. □

Remark. The preceding theorem allows us to throw away the closed hypothesis in many assertions in the literature. It can be useful to pair this with the corollary of the following theorem. Note that we phrase it differently from Kosinski, however, because it seems that his statement is not quite correct.

G.3 Tubular Neighborhood Trick

In order to prove the following theorem in the smooth case, we need the following auxiliary lemma.

Lemma 41

If $f : M \to N$ is a local diffeomorphism and $C \subset M$ is a submanifold for which $f|C$ is a homeomorphism onto its image, then $f(C)$ is a submanifold of N and hence $f|C$ is a diffeomorphism onto its image.

Proof. This is an exercise in definitions. Since $f|C$ is a homeomorphism onto its image, it is a topological embedding. We therefore only need to verify that it is an immersion, and this follows because the property of being an immersion is local and f is locally a diffeomorphism. □

The following theorem is taken from Daniel Tausk's notes, Lemma 8.12, where it is proved carefully.

Theorem 57 (Tubular Neighborhood Trick)

If $f : X \to Y$ is a local homeomorphism where Y is hereditarily paracompact and Hausdorff and f is a homeomorphism on a subspace $C \subset X$, then f is a homeomorphism on a nbhd U of C.

This can be upgraded to DIFF as follows. If $f : X \to Y$ is a local diffeomorphism which is a homeomorphism on a submanifold $C \subset X$, then f is a diffeomorphism on a nbhd U of C.

Since closed subspaces of a paracompact Hausdorff spaces are themselves paracompact, the proof admits minor modifications showing the following.

Corollary 45

If $f : X \to Y$ is a local homeomorphism where Y is paracompact Hausdorff and f is a homeomorphism on a subspace $C \subset X$ such that $f(C)$ is closed in Y, then f is a homeomorphism on a nbhd U of C.

Remark. We have already shown that manifolds are hereditarily paracompact.

Proof. First, let us agree on some ad hoc terminology. For an open subset V of X, we will call the map $f|V$ a *chart for* f if $f|V$ is a homeomorphism onto its image. We will let $C' = \overline{f(C)}$. Now, the trickiest part of this is showing that a nbhd of $f(C)$ of the correct form exists.

Claim. For each point of $x \in C$ and nbhd U in X of x, there is a nbhd $V \subset U$ of x such that $f(V \cap C) = f(V) \cap f(C)$.

Since $U \cap C$ is open in C, $f(U \cap C)$ is open in $f(S)$. Hence, there is an open subset $A \subset Y$ such that $f(U \cap S) = A \cap f(S)$. Let $V = U \cap f^{-1}(A)$. Then V is an open nbhd of x contained in U and trivially we have $f(V' \cap C) \subset f(V') \cap f(C)$. On the other hand,

$$f(V) \cap f(C) \subset A \cap f(C) = f(U \cap C) = f(V \cap C).$$

For the last equality, observe that $V \subset U$ so $V \cap C \subset U \cap C$, while on the other hand, $U \cap C \subset f^{-1}(A)$ (basically just apply f^{-1} to $f(U \cap C) = A \cap f(C)$) so that by intersecting both sides of $U \cap C \subset f^{-1}(A)$ with U and C, we obtain $U \cap C \subset U \cap f^{-1}(A) \cap C = V \cap C$ and so $f(U \cap C) \subset f(V \cap C)$ and therefore have equality.

Note that a local homeomorphism that is injective is a homeomorphism. Therefore it suffices to find an open set $Z \subset X$ containing C such that $f|\,Z$ is injective. For each $x \in C$, let

$$f_x = f|\,U'_x \colon U'_x \to V'_x$$

be a local homeomorphism. By the claim, we may assume WLOG that $f(U'_x \cap C) = V'_x \cap C$. Let $Y_0 = \bigcup_{x \in C} V'_x$. Then this is open and paracompact Hausdorff since Y is hereditarily paracompact and Hausdorff. Therefore $\{V'_x\}$ admits a locally finite open refinement, say $\{V_i\}_{i \in I}$ (the family $\{V_i\}_{i \in I}$ is locally finite *in* Y_0).

For each index i, choose $x \in C \cap V_i$ such that $V_i \subset V'_x$ and set

$$U_i = f_x^{-1}(V_i) = (f|\,U'_x)^{-1}(V_i) \subset U'_x,$$

which is open since Y_0 is open and therefore its open subsets are open in Y. Then

$$f_i = f|\,U_i \colon U_i \to V_i$$

is a local homeomorphism and

$$f(U_i \cap C) = V_i \cap f(C).$$

This latter thing follows because f_x is a homeomorphism and therefore

$$f_x^{-1}(V_i \cap f(C)) = f_x^{-1}(V_i \cap f_x(C)) = f_x^{-1}(V_i) \cap f_x^{-1} f(C) = U_i \cap C.$$

Since paracompact Hausdorff spaces are normal, the shrinking lemma guarantees a locally finite open refinement of the V_i on the same index set, say $\{W_i\}$ with $W_i \subset V_i$ such that $\mathrm{Cl}_{Y_0}(W_i) \subset V_i \subset V'_x$. For each $i \in I$, let

$$Z_i = f_i^{-1}(W_i).$$

Then $Z_i \subset U_i \subset U'_x$ is open in X and, by abuse of notation, $f_i = f|\,Z_i \colon Z_i \to W_i$ is a homeomorphism. Once again, since f_x is a homeomorphism, we have that

$$f(Z_i \cap C) = W_i \cap f(C).$$

Now we claim that

$$C \subset \bigcup_{i \in I} Z_i.$$

Indeed, for $x \in C$, there exists $i \in I$ such that $f(x) \in W_i$ and therefore $f(x) \in W_i \cap f(C) = f(Z_i \cap C)$; it follows that there exists $y \in Z_i \cap C$ with $f(y) = f(x)$ but since $f|C$ is injective, $x = y$, proving the claim.

For each $x \in C$, let

$$I_x = \left\{ i \in I : f(x) \in \mathrm{Cl}_{Y_0}(W_i) \right\}.$$

Since the closed cover $\left\{\mathrm{Cl}_{Y_0}(W_i)\right\}$ is locally finite as $\overline{W}_i \subset V_i$ and $\{V_i\}$ is locally finite in Y_0 so $\#(I_x) < \infty$. Moreover, $I_x \neq \varnothing$ from the above.

Keep $x \in C$. If $i \in I_x$, then from what we have shown,

$$f(x) \in \mathrm{Cl}_{Y_0}(W_i) \cap f(C) \subset V_i \cap f(C) = f(U_i \cap C),$$

and so since $f|C$ is injective, $x \in U_i$ and, in particular

$$x \in \bigcap_{i \in I_x} U_i,$$

and this holds for all $x \in C$.

Let us find an open nbhd G_x of $f(x)$ in Y_0 with the following properties:

(a) for each $i \in I$, $G_x \cap W_i \neq \varnothing$ **iff** $i \in I_x$;

(b) $G_x \subset f(\bigcap_{i \in I_x} U_i)$.

Such a set G_x can be defined by

$$G_x = \underbrace{(Y_0 \setminus \bigcup_{i \in I \setminus I_x} \mathrm{Cl}_{Y_0}(W_i))}_{(\mathbf{a})} \cap \underbrace{f(\bigcap_{i \in I_x} U_i)}_{(\mathbf{b})}.$$

We claim that G_x is open in Y_0 (and hence Y). Since f is an open map and $\#(I_x) < \infty$, $f(\bigcap_{i \in I_x} U_i)$ will be open in Y_0 and hence Y. Since $\left\{\mathrm{Cl}_{Y_0}(W_i)\right\}$ is locally finite and the union of any collection of locally finite sets is closed, $Y_0 \setminus \bigcup_{i \in I \setminus I_x} \mathrm{Cl}_{Y_0}(W_i)$ is open in Y_0 and hence Y—therefore G_x is open in Y_0 and hence Y. Note that for *any* locally finite collection of sets, the closure operator distributes over the union, which is where the penultimate assertion comes from.

Let $G = \bigcup_{x \in S} G_x$ and let $Z = f^{-1}(G) \cap \bigcup_{i \in I} Z_i$. Then G is open in Y_0 and hence Y and therefore Z is open in X. Moreover, $S \subset Z$ since $C \subset \bigcup_{i \in I} Z_i$ and clearly $f|Z \colon Z \to G$. Since Z is open and f is a local homeomorphism, $f|Z$ is a local homeomorphism. It therefore suffices to show it is injective to complete the proof.

Let $x, y \in Z$ with $f(x) = f(y)$. Pick indices $i, j \in I$ with $x \in Z_i$ and $y \in Z_j$. Now, $f(x) = f(y) \in G_z$ for some $z \in C$ so $f(x) \in G_z \cap W_i$ and $f(y) \in G_z \cap W_j$ and therefore $i, j \in I_z$ by property **(a)**. Property **(b)** implies $G_z \subset f(U_i \cap U_j)$ and therefore there exists $p \in U_i \cap U_j$ with $f(x) = f(p) = f(y)$. But since f is injective on U_i and on U_j individually, f is injective on $U_i \cap U_j$. Therefore $x = p = y$.

Observe that everything we did above made no explicit mention of whether we worked in TOP or DIFF. Indeed, because smoothness is a local property, everything still goes through in the smooth category. □

H Connections and Differential Geometry

We start with a useful theorem.

Theorem 58

Let E and E' be total spaces of smooth vector bundles over M. A map $f\colon \mathit{\Gamma}(E) \to \mathit{\Gamma}(E')$ is $C^\infty(M)$-linear **iff** there exists a smooth bundle map $F\colon E \to E'$ such that for any $s \in \mathit{\Gamma}(E)$,

$$f(s) = F \circ s.$$

In particular, it follows that f only depends on the value of the section at a point and, in fact, given f, F is defined on $v \in E_p$ by $F(v) = f(\tilde{v})(p)$ where $\tilde{v}$ is any smooth global section of E which is v at p.

The proof of this theorem is an extremely typical argument. As usual, being C^∞-multilinear (multi$=1$ in this case) implies that everything is determined pointwise.

Definition 47

Let $p\colon E \to M$ be a smooth vector bundle. A *connection* on E is a map

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that for $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$,

(a) $\nabla_X s$ is $C^\infty(M)$-linear in X and **R**-linear in s;

(b) (*Leibniz rule*) if $f \in C^\infty(M)$, then $\nabla_X(fs) = (Xf)s + f\nabla_X s$ or, equivalently, $\nabla_X(fs) = (df)(X)s + f\nabla_X s$.

We say a section $s \in \Gamma(E)$ is *flat* if $\nabla_X s = 0$ for all $X \in \mathfrak{X}(M)$.

Remark. Equivalently, $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$. Since $V^* \otimes W \cong \mathrm{Hom}(V, W)$ for finite-dimensional vector spaces, one can show that this is equivalently $\nabla\colon \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$ and by the preceding, since an element of the bundle $\mathrm{Hom}(TM, E)$ is exactly the same as a map $TM \to E$, it is exactly the same as a $C^\infty(M)$-linear map $\mathfrak{X}(M) \to \Gamma(E)$; if we fix $x \in \Gamma(E)$, $X \mapsto \nabla_X s$ is precisely a $C^\infty(M)$-linear map $\mathfrak{X}(M) \to \Gamma(E)$, which is why these are equivalent.

In this formulation, a connection is an **R**-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that $\nabla(fs) = df \otimes s + f\nabla s$.

Exercise 2. $\Gamma(E_1 \otimes E_2) \cong \Gamma(E_1) \otimes_{C^\infty(M)} \Gamma(E_2)$.

Theorem 59

Connections exist.

Proof. Any convex linear combination of connections is a connection. Use a partition of unity. □

Recall the following.

Notation H.1. Let $T^k_\ell(M)$ denote tensor fields of type $\binom{k}{\ell}$. That is, smooth sections of $(T^*M)^{\otimes k} \otimes (TM)^{\otimes \ell}$.

Theorem 60

There is a bijective correspondence between $C^\infty(M)$-multilinear maps $A\colon \mathfrak{X}(M)^{\times n} \times \mathfrak{X}^{*\times m} \to C^\infty(M)$ and type $\binom{n}{m}$ tensor fields. In particular, A depends only on the vectors and covectors pointwise.

Corollary 46

$C^\infty(M)$-multilinear alternating maps $A\colon \mathfrak{X}(M)^{\times n} \to C^\infty(M)$ are in natural bijective correspondence with $\Omega^n(M)$.

Remark. This theorem implies that for fixed s, we may define $\nabla_X s = p \mapsto \nabla_{X_p} s$ since ∇ is $C^\infty(M)$-multlinear in X. This allows us to define a connection as a function which associates to every vector $v \in T_pM$ and section $s \in \Gamma(E)$ a smooth section $\nabla_v s$ such that ∇ is linear over $\mathbf{R}$ in v, satisfies that

$$\nabla_v(fs) = f(p)\nabla_v s + v(f)s_p$$

for $f \in C^\infty(M)$ and is such that if $X \in \mathfrak{X}(M)$, then $p \mapsto \nabla_{X_p} s$ is a smooth section of E.

In particular, $\nabla_X s(p)$ depends only locally on s and pointwise on X.

We construct an analogy of this now.

Definition 48

Let E be a vector bundle over M. Then by an E-*valued* k-*form* with mean a section of $(\Lambda^k T^*M) \otimes E$. We denote by $\Omega^k(M;E)$ the collection of such forms.

Theorem 61

$C^\infty(M)$-multilinear alternating maps $A\colon \mathfrak{X}(M)^{\times n} \to \Gamma(E)$ are in natural bijective correspondence with $\Omega^n(M;E)$.

Theorem 62

The difference between two connections is a tensor field and in particular an element of $\Omega^1(M;\mathrm{End}(E))$.

Proof. $(\nabla^1 - \nabla^2)(fs) = f(\nabla^1 - \nabla^2)(s)$ so this is now $C^\infty(M)$-multilinear and hence, $\nabla^1 - \nabla^2$ is a $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(T^*M \otimes E)$. Alternatively, we may view $\nabla^1 - \nabla^2 \colon \mathfrak{X}(M) \to \Gamma(E^* \otimes E) \cong \Gamma(\mathrm{End}(E))$ by noting that a multilinear map $\Gamma(E) \to \Gamma(T^*M) \otimes \Gamma(E)$ is which is $C^\infty(M)$-linear still and thus corresponds to an element of $\Omega^1(M; \mathrm{End}(E))$. ☐

Remark. This says that the space of connections is an affine space for $\Omega^1(M; \mathrm{End}(E))$.

We are most interested in the case where $E = TM$.

Proposition 19

Given a connection ∇ TM, there is a unique operator, for each $X \in \mathfrak{X}(M)$,

$$A \mapsto \nabla_X A$$

from smooth tensor fields to smooth tensor fields preserving all types such that

(a) $\nabla_X f = X(f)$ (differentiation);

(b) $\nabla_X Y$ is the vector field given by the connection ∇;

(c) $A \mapsto \nabla_X A$ is linear over $\mathbf{R}$;

(d) $\nabla_X(A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B)$ (Leibniz);

(e) For any contraction C, $\nabla_X \circ C = C \circ \nabla_X$.

Moreover each $\nabla_X A$ is $C^\infty(M)$-linear in X. Thus, as before, we may show that this depends only locally on A and pointwise X and therefore define $p \mapsto \nabla_{X_p} A$ a smooth tensor field of the same type as A defined by

$$\nabla A(p)(X_{1p}, \dots, X_{kp}, X_p) = \nabla_{X_p} A(X_{1p}, \dots, X_{kp})$$

where, by a preceding theorem, it suffices to define this pointwise on vectors.

In fact, if S has type $\binom{k}{\ell}$, then ∇S is a tensor field of type $\binom{k+1}{\ell}$.

Finally, for any one 1-forms ω, we will have that $\nabla_X \omega$ is given on vector fields Y by

$$\nabla_X(\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y),$$

Proof. The point is that for a 1-form ω, we must have

$$X(\omega(Y)) = \nabla_X(\omega(Y)) = \nabla_X(C(\omega \otimes Y) = C((\nabla_X \omega) \otimes Y + \omega \otimes \nabla_X Y),$$

and this is

$$\nabla_X(\omega)(Y) + \omega(\nabla_X Y),$$

and so we must have for any Y

$$\nabla_X(\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y),$$

which by a preceding theorem pins down the 1-form $\nabla_X \omega$ precisely. The general case follow easily from this since a general tensor field is a sum of functiosn times tensor products of vector fields and 1-forms (covector fields). $\square$

Corollary 47

For $f \in \Omega^0(M) = C^\infty(M)$, $\nabla f = df$, where df is the differential $df\colon M \to T^*M$ given in local coordinates by $\frac{\partial f}{\partial x^i} dx^i$.

Definition 49

The ***Levi-Civita connection*** on a Riemannian manifold M is a connection ∇ on TM which satisfies the following two properties specifying it uniquely.

(a) ∇ *preserves the metric* $\nabla g = 0$ or, equivalently, $X(\langle Y \mid Z \rangle) = \langle \nabla_X Y \mid Z \rangle + \langle Y \mid \nabla_X Z \rangle$;

(b) ∇ is ***torsion-free***, meaning the torsion tensor vanishes

$$[X, Y] - \nabla_X Y + \nabla_Y X = [X, Y] - (\nabla_Y X - \nabla_X Y) = 0.$$

Remark. The first condition has the geometrically pleasing interpretation that it guarantees that parallel transport preserves the lengths of vectors.

The second condition is tantamount to saying that the derivatives of two vector fields along each other *commutes as much as possible*. The bracket $[X, Y]$ can be shown to measure the infinitesimal failure of flowing along Y for time h, then X for time h then Y for a time $-h$ and then X for a time $-h$ to return to the initial point and that, in fact, this is the complete obstruction for X and Y to commute at some point. Spivak's first volume has beautiful discussion on this point in chapter 5.

When two vector fields X, Y commute locally about some point p, they may be used to locally construct coordinates $\frac{\partial}{\partial x^1} = X$ and $\frac{\partial}{\partial x^2} = Y$, and the prototype for directional derivatives commuting are those coming from coordinates. Hence, the failure of the covariant derivatives to commute should be precisely the failure for the two vector fields to commute since otherwise their covariant derivatives along each other may be identicial but X and Y cannot be used to introduce local coordinates.

Definition 50

Given a connection ∇, define the ***Hessian*** of a smooth function $f\colon M \to \mathbf{R}$ as $\mathrm{Hess}(f) = \nabla\nabla f$. Since f is a 0-form, $\nabla f = df$, $\mathrm{Hess}(f)$ is the type $\binom{2}{0}$ tensor field ∇df.

Exercise 43

If ∇ is the Levi-Civita connection for M a Riemannian manifold, then

$$\operatorname{Hess}(f)(X,Y) = \langle \nabla_X \operatorname{grad} f \mid Y\rangle = X(Yf) - (\nabla_X Y)f.$$

[Hint: To understand any tensor field of type $\binom{k}{\ell}$, it suffices to understand how it acts on tuples $(X_{1p},\dots,X_{kp},\omega_{1p},\dots,\omega_{\ell p}) \in T_pM^{\times k} \times T_p^*M^{\times \ell}$. One can compute

$$\nabla^2 f(X,Y) = \nabla_X(\nabla_Y f) = \nabla_X(\nabla f(Y)).$$

The point is that by compatibility with the metric,

$$X\langle \operatorname{grad} f, Y\rangle = \langle \nabla_X \operatorname{grad} f \mid Y\rangle + \langle \operatorname{grad} f \mid \nabla_X Y\rangle$$

and $\langle \operatorname{grad} f \mid X\rangle = f_*(X) = X(f)$ by definition of $\operatorname{grad} f$.]

Lemma 42

The naive way to define higher order partial derivatives for a smooth real-valued function on M is not, in general, a tensorial construction.

Proof. Suppose we have overlapping charts x and y. The naive way to define partial derivatives of f in coordinates is as

$$\frac{\partial^2 f}{\partial x^i \partial x^j} \overset{\text{def}}{=} \frac{\partial^2 (f\circ x^{-1})}{\partial x^i \partial x^j}$$

which denotes the usual mixed partial.

To see how this transforms under a change coordinates, note that $\frac{\partial f}{\partial x^i} = \sum \frac{\partial f}{\partial y^j}\frac{\partial y^j}{\partial x^i}$ (with the usual summation convention) by the chain rule and so in particular

$$\frac{\partial}{\partial x^i} = \sum \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial y^j}.$$

This means the first derivative construction transforms like a tensor.

To compute the change of coordinate for our mixed partial derivative, we follow the same strategy and we compute

$$\frac{\partial([f\circ x^{-1}])}{\partial x^i \partial x^j} = \frac{\partial([f\circ y^{-1}][y\circ x^{-1}])}{\partial x^i \partial x^j}$$

according to the chain rule. By the Faa di Bruno formula, this is (being pedantic about what symbols mean)

$$\sum_k \left(\frac{\partial(f\circ y^{-1})}{\partial y^k}\frac{\partial^2 (y\circ x^{-1})^k}{\partial x^i \partial x^j}\right) + \sum_{\alpha,\beta}\left(\frac{\partial^2(f\circ y^{-1})}{\partial y^\alpha \partial y^\beta}\frac{\partial(y\circ x^{-1})^\alpha}{\partial x^i}\frac{\partial(y\circ x^{-1})^\beta}{\partial x^j}\right)$$

which we might more typically write as

$$\sum_k \left(\frac{\partial f}{\partial y^k}\frac{\partial^2 y^k}{\partial x^i \partial x^j}\right) + \sum_{\alpha,\beta}\left(\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\right).$$

Since $f \in C^\infty(M)$ was arbitrary, this, in particular, means our symbol takes the form

$$\frac{\partial^2}{\partial x^i \partial x^j} = \sum_k \frac{\partial^2 y^k}{\partial x^i \partial x^j}\frac{\partial}{\partial y^k} + \sum_{\alpha,\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2}{\partial y^\alpha \partial y^\beta}.$$

This almost looks right, except the first term has the wrong type! If the construction $\frac{\partial^2}{\partial x^i \partial x^j}$ is coordinate independent, it can be easily seen that the first term must vanish.

□

Exercise 44

Fix $f\colon M \to \mathbf{R}$ smooth with critical point p.

(a) Show that $\frac{\partial^2 f}{\partial x^i \partial x^j}$ as defined above *is* intrinsically defined at a critical point p (i.e., gives a well-defined element of the tangent space $T_{f(p)}\mathbf{R}$).

(b) Show that the naive coordinate definition of $\mathrm{Hess}_p(f)$ agrees with definition given above at p. Then show that at a critical point p, $\mathrm{Hess}_p(f)$ may be defined by sending $(X, Y) \in T_pM \times T_pM$ to $X(Y(f))$ and that this agrees with the other formulations.

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