Causality and Spacetime - Lecture 4

Last time: (M,g) spacetime, Γ is a discrete group of orthochronous isometries acting freely and properly disc. on M, then M/Γ becomes a spacetime.

Ex 1. Every Lorentz vector space (V,g) is a spacetime, once a timelike vector has been chosen. Assume that $\Gamma \subseteq V$ is a lattice (discrete additive subgroup of V which generates V). Γ acts on V by translations, and translations are orthochronous.

$\therefore$ V/Γ is a spacetime. $(n = \dim V)$

Γ is a lattice $\Rightarrow \exists$ isomorphism $V \to \mathbb{R}^n$ that maps Γ into $\mathbb{Z}^n$.

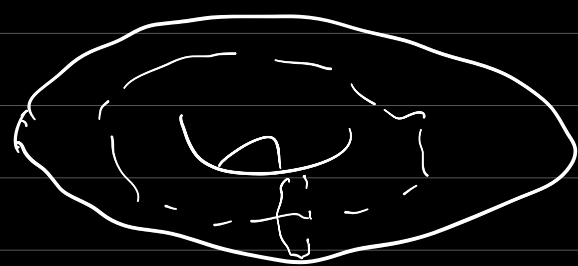
So $V/\Gamma \cong \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n = S^1 \times \cdots \times S^1$.

If $V = \mathbb{L}^n$, $g = (dx^1)^2 + \cdots + (dx^{n-1})^2 - (dx^n)^2$,

then the metric on $\mathbb{T}^n$ is $(d\theta^1)^2 + \cdots + (d\theta^{n-1})^2 - (d\theta^n)^2$,

where $d\theta^i$ is the non-exact angle form on the

i-th factor S^1.



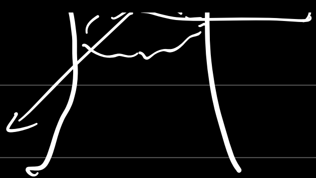
$\mathbb{R}^{n+1}_1$
$\|$

Ex 2. In $\mathbb{L}^{n+1}$, consider

$$S^n_1 = \{ x \in \mathbb{L}^{n+1} \mid \langle x, x \rangle_L = 1 \}$$

$$= \{ (x_1, \ldots, x_n, t) : x_1^2 + \cdots + x_n^2 - t^2 = 1 \}.$$

We have a diffeomorphism



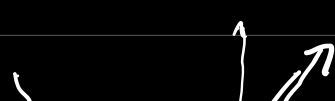
$$S^n_1 \ni (x_0, t) \longmapsto \left((1+t^2)^{-1/2} x_0, t\right) \in S^{n-1} \times \mathbb{R}.$$

$(x_0, t) = x$

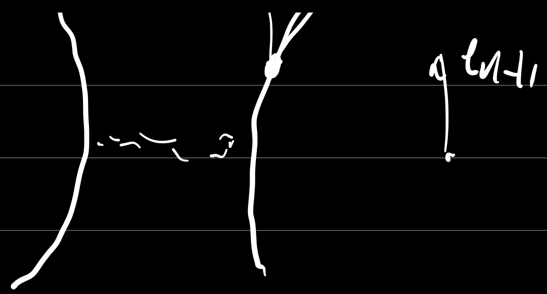
$\Rightarrow$ S^n_1 is connected (and simply connected if $n \geq 3$).

$\forall x \in S^n_1$, $T_x S^n_1 = x^\perp$.

Sine x is spacelike, $x^\perp$ is a timelike hyperplane in $\mathbb{L}^{n+1}$. We conclude that $\langle \cdot, \cdot \rangle_L$ makes a Lorentzian metric on S^n_1. Moreover, projecting the timelike field $e_{n+1} = (\vec{0}, 1)$ onto the tangent spaces to S^n_1 defines a timelike field on S^n_1.



$$\mathbb{L}^{n+1} = x^\perp \oplus \mathbb{R}x$$



S^n_1 is called the <u>de Sitter space</u> in dim n.

(Fact: S^n_1 is indeed a Lorentzian analogue of S^n, and its sectional curvature is 1).

<u>Next.</u> When we have S^n, $\mathbb{Z}_2 \curvearrowright S^n$, via 1_{S^n} and $-Id_{S^n}$. We obtain $\mathbb{RP}^n = S^n/\mathbb{Z}_2$.

The same happens here: consider $\mathbb{RP}^n_1 := S^n_1/\mathbb{Z}_2$.

$\mathbb{RP}^n_1$ has a natural Lorentzian metric.

<u>Q</u>: Does the time orientation survive in $\mathbb{RP}^n_1$?

Fix $p=(1,0,\dots,0,0) \in S^n_1$.

Recall that a basis $(v_2,\dots,v_{n+1})$ for $T_p S^n_1$ is positive $\Leftrightarrow$ $(p, v_2, \dots, v_{n+1})$ is a positive basis for $\mathbb{E}^{n+1}$!

$-d(\mathrm{Id}_{S^n_1})_p : T_p S^n_1 \longrightarrow T_{-p} S^n_1$.

$(-1)^{n+1}$

maps $(p, v_2, \dots, v_{n+1}) \mapsto (-p, -v_2, \dots, -v_{n+1})$

$\Rightarrow \mathbb{RP}^n_1$ is orientable $\Leftrightarrow$ n is odd.

$-d(\mathrm{Id}_{S^n_1})_p (e_{n+1}) = -e_{n+1}$

$\Rightarrow \mathbb{RP}^n_1$ is never time-orientable.

Summary so far:

$\mathbb{T}^n$ w/ natural metric = orientable and time-orient.
$(S^n_1, \mathbb{E}^{n+1})$

$\mathbb{RP}^n_1$ = orientable or non-orientable, never time-orientable.

What is missing? non-orientable but time-orientable.

Here's one example: consider $\mathbb{L}^2 = (\mathbb{R}^2, dx^2 - dy^2)$

and $\Phi_1, \Phi_2 : \mathbb{L}^2 \to \mathbb{L}^2$ given by

$\Phi_1(x,y) = (x+1, y)$ and $\Phi_2(x,y) = (-x, y+1)$.

They are both isometries, and $\Phi_2 \Phi_1 \Phi_2^{-1} = \Phi_1^{-1}$.

The group $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle$ acts on $\mathbb{L}^2$ by

$(a = \Phi_1, b = \Phi_2)$ isometries.

By the chain rule, for every $\Phi \in \Gamma$ (generated

by Φ_1 and Φ_2), we have $\frac{\partial \Phi}{\partial y} = 1$. $(0,1) \mapsto (*, \text{pos.})$

$\Rightarrow$ every $\Phi \in \Gamma$ is an orthochronous isometry.

$\mathbb{L}^2/\Gamma$ = Klein bottle.

2nd component of $\frac{\partial \Phi}{\partial y}$ equals 1

$$d\Phi_{(x,y)} = \begin{pmatrix} * & + \\ + & * \end{pmatrix} \in O^{\uparrow}_{2,1}$$

$$d\Phi_{(x,y)}(0,1) = (x, 1).$$

One more example: Stick w/ $\mathbb{L}^2$. Consider $\Phi : \mathbb{L}^2 \circlearrowleft$ given by $\Phi(x,y) = (x+1, -y)$.

Let $\mathbb{Z} \curvearrowright \mathbb{L}^2$ by $(n, (x,y)) \longmapsto \Phi^n(x,y)$
$$\Phi^n(x,y) = (x+n, (-1)^n y).$$

$$d\Phi_{(x,y)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ = time-reversing Lorentz transformation.

$$\left(d\Phi_{(x,y)} \in O^{-\downarrow}_{2,1} \right)$$

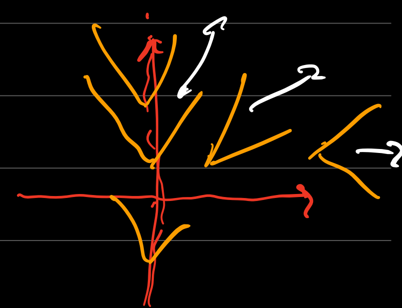
$\mathbb{L}^2/\mathbb{Z}$ is not orientable nor time-orientable.

↳ Möbius strip.

Extra: On $\mathbb{R}^2$, consider the metric

$$g = \cos(2\pi x)\,(dx^2 - dy^2) - 2\sin(2\pi x)\,dx\,dy.$$

$$= \begin{pmatrix} \cos(2\pi x) & -\sin(2\pi x) \\ -\sin(2\pi x) & -\cos(2\pi x) \end{pmatrix}$$



$\hookrightarrow \det = -1.$

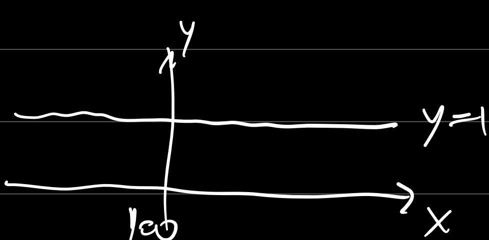
[So g is Lorentzian by Sylvester's Criterion from linear algebra].

a) $\Gamma = \mathbb{Z}^2$ acting by translation

b) $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle$

c) $\Gamma = \mathbb{Z} = \langle (x,y) \mapsto (-x, y+1) \rangle$.

	orient.	time-orien.
a)	Y	N
b)	N	N
c)	N	Y



$\hookrightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$X_{(x,y)} = (\cos(\pi x), -\sin(\pi x))$? Spacelike field

$\cos^2(\pi x) - \sin^2(\pi x) = \cos(2\pi x)$.

$$g(X_{(x,y)}, X_{(x,y)}) = \cos(2\pi x)(\cos^2(\pi x) - \sin^2 \pi x) - 2\sin(2\pi x)\cos(\pi x)\sin(\pi x)$$

$$= \cos^2(2\pi x) + \sin^2(2\pi x) = 1$$

Probably: $(\sin(\pi x), -\cos(\pi x))$ should be timelike.

$$g = \cos(2\pi x)(dx^2 - dy^2) - 2\sin(2\pi x)\,dx\,dy$$

$\Phi(x,y) = (-x, y+1)$.

$$\Phi^* g = \cos(2\pi(-x))(d(-x)^2 - d(y+1)^2) - 2\sin(2\pi(-x))\,d(-x)\,dy$$

$$= \cos(2\pi x)(dx^2 - dy^2) - 2\sin(2\pi x)\,dx\,dy$$

$= g.$

Thm. Let (M, g) be a Lorentz manifold. Set

$$M^\tau = \{(x, C) : x \in M \text{ and } C \text{ is a time-orientation for } (T_xM, g_x)\}.$$

Define $\pi^\tau : M^\tau \to M$ by $\pi^\tau(x, C) = x$.

There is a smooth manifold structure on M^τ for which π^τ becomes a smooth two-fold covering of M, and a unique Lorentzian metric g^τ on M^τ for which π^τ is a local isometry and (M^τ, g^τ) is time-orientable. Moreover, if M is connected, then: M^τ is disconnected

$$\Updownarrow$$

(M, g) is already time-orientable.

Lecture 5

Proof.

$M^{\tau} = \{ (x, C) : x \in M$ and C is a time-orientation on $T_x M \}$

$\pi^{\tau} : M^{\tau} \to M$, $\pi^{\tau}(x, C) = x$.

Fix a countable family $\{(U_\alpha, X_\alpha)\}_{\alpha \in A}$, where $U_\alpha \subseteq M$ is open and X_α is a timelike field on U_α. ($\{U_\alpha\}_{\alpha \in A}$ is an open cover for M).

Define set-theoretic sections $\psi_\alpha^{\pm} : U_\alpha \to M$ by

$$\psi_\alpha^{+}(x) = (x, C^{\uparrow}(X_\alpha|_x)) \text{ and}$$

$$\psi_\alpha^{-}(x) = (x, C^{\downarrow}(X_\alpha|_x)).$$

Equip M^{τ} with the final topology induced by $\{\psi_\alpha^\pm\}_{\alpha\in A}$. That is, $W \subseteq M^\tau$ is open

$\Leftrightarrow \forall \alpha\in A, (\psi_\alpha^\pm)^{-1}[W]$ is open in M.

The characteristic property of this topology makes

all $\psi_\alpha^\pm$ continuous (in fact, homeomorphisms

between U_α and $\psi_\alpha^\pm[U_\alpha] = U_\alpha^\pm$).

Other consequences:

•) $\pi^\tau : M^\tau \to M$ becomes a continuous map
(b/c all inclusions $U_\alpha \hookrightarrow M$ are continuous).

•) $\forall \alpha \in A$, $U_\alpha^\pm$ is open in M^τ. For example,

$(\psi_\beta^+)^{-1}[U_\alpha^+]$ = union of the connected components of $U_\alpha \cap U_\beta$ on which X_α and X_β determine the same time-orientation

•) M^τ is Hausdorff. If $(x, C), (x', C') \in M^\tau$

which are distinct, then:

- $x \neq x'$. If this happens, we have $U, V \subseteq M$ open s.t. $U \cap V = \emptyset$, $x \in U$, $x' \in V$.

  So $\left.\begin{array}{l}(x, C) \in (\pi^{\tilde{}})^{-1}[U] \\ (x', C') \in (\pi^{\tilde{}})^{-1}[V]\end{array}\right\}$ disjoint.

- $x = x'$ but $C \neq C'$. Take $\alpha \in A$ s.t. $x \in U_\alpha$, and note that $(x, C) \in U_\alpha^+$ and $(x', C') \in U_\alpha^-$. ($U_\alpha^+ \cap U_\alpha^- = \emptyset$).

i) $M^{\tilde{}}$ is second-countable b/c $\{(U_\alpha, X_\alpha)\}_{\alpha \in A}$ is countable.

c) $M^{\tilde{}}$ is locally Euclidean: reducing the U_α's if needed, we may assume that they carry charts $\varphi_\alpha : U_\alpha \to \varphi_\alpha[U_\alpha] \subseteq \mathbb{R}^2$.

$_\alpha$ $_\alpha$

open

$$\begin{tikzcd}
U_\alpha^{\pm} \arrow[d, "\pi^\tau"'] \arrow[dr, "\varphi_\alpha^{\pm}"] \arrow[d, phantom, shift left=6ex, "\circlearrowleft"] & \\
U_\alpha \arrow[r, "\varphi_\alpha"'] & \varphi_\alpha[U_\alpha] \subset \mathbb{R}^n .
\end{tikzcd}$$

$\Rightarrow \{(U_\alpha^{\pm}, \varphi_\alpha^{\pm})\}_{\alpha \in A}$ is an atlas for M^τ.

Transitions are $\varphi_\alpha^a \circ (\varphi_\beta^b)^{-1} = \varphi_\alpha \circ \varphi_\beta^{-1}$

$\forall\, a, b \in \{+, -\}$.

So far: M^τ is a topological manifold of dimension $n = \dim M$, and $\pi^\tau : M^\tau \to M$ is a continuous double-cover of M.

Next step: smooth structure in M^τ.

$\Rightarrow M^{\tau}$ is smooth.

Claim: $\pi^{\tau}: M^{\tau} \to M$ is actually a smooth double-cover of M (in particular, it is a local diffeomorphism).

Reason: $\pi^{\tau} \circ (\varphi_{\alpha}^{\pm})^{-1} = \varphi_{\alpha}^{-1}$

Claim: all the sections $\varphi_{\alpha}^{\pm}$ are smooth.

Reason:

$$\varphi_{\beta}^{b} \circ \psi_{\alpha}^{a} = \varphi_{\beta} \quad , \; a, b \in \{+, -\}$$

Metric. $\pi^{\tau}: M^{\tau} \to M$ local diffeomorphism, so the only choice of metric in M^{τ} which makes π^{τ} a local isometry is $g^{\tau} := (\pi^{\tau})^{*} g$.

We'll show next that (M^{τ}, g^{τ}) is always time-orientable.

Consider $\{(U_\alpha^{\pm}, \pm X_\alpha^{\pm})\}_{\alpha \in A}$, where

$$X_\alpha^{\pm} = \underbrace{(\psi_\alpha^{\pm})}_{}{}_{*} X_\alpha$$

↳ are also local isometries.

$U_\alpha^{\pm} = \psi_\alpha^{\pm}[U_\alpha]$

$$\left((\psi_\alpha^{\pm})^* g^{\tau} = (\psi_\alpha^{\pm})^* (\pi^{\tau})^* g \right.$$

$$\left. = (\pi^{\tau} \circ \psi_\alpha^{\pm})^* g = \mathrm{Id}_M^* g = g \right)$$

For example: if $\alpha, \beta \in A$ are such that $U_\alpha \cap U_\beta \neq \emptyset$

and $(x, C) \in U_\alpha^{+} \cap U_\beta^{+}$, then $x \in U_\alpha \cap U_\beta$,

and $C = C^{\uparrow}(X_\alpha|_x) = C^{\uparrow}(X_\beta|_x)$.

Then:

$$g^{\tau}_{(x,C)}\left(X_\alpha^{+}|_{(x,C)}, X_\beta^{+}|_{(x,C)} \right) =$$

$$= g\left(X_\alpha|_x, X_\beta|_x \right) < 0$$

g_x ...

Another situation: if $(x,C) \in U_\alpha^+ \cap U_\beta^-$,

$C = C^\uparrow(X_\alpha|_x) = C^\downarrow(X_\beta|_x)$

$$g^\tau_{(x,C)}\left(X_\alpha^+|_{(x,C)}, X_\beta^-|_{(x,C)}\right) =$$

$$= -\underbrace{g_x(X_\alpha|_x, X_\beta|_x)}_{>0} < 0.$$

(If $(x,C) \in U_\alpha^- \cap U_\beta^-$, do the same thing as for the case $(x,C) \in U_\alpha^+ \cap U_\beta^+$).

Finally, assume that M is connected.

Note that $\pi^\tau : M^\tau \to M$ is a principal $\mathbb{Z}_2$-bundle.

(principal G-bundles are trivial $\Leftrightarrow$ they admit a global section) and that global sections of π^{τ} are precisely time-orientations for (M, g).

(if there's a time-orientation for (M, g), then $M^{\tau} \cong M \times \mathbb{Z}_2$ is disconnected).

(if M^{τ} is disconnected, π^{τ} restricted to any of the exactly two connected components of M^{τ} is a diffeomorphism onto M).
(isometry)

$\boxtimes$

Remark: M connected.

$\exists$ two deck transformations of $\pi^{\tau}: M^{\tau} \to M$.

(the identity $(x, C) \mapsto (x, C)$, and $F: M^{\tau} \to M^{\tau}$ which switches the sheets, i.e. $(x, C) \mapsto (x, -C)$)

F is automatically an isometry for (M^τ, g^τ).

$$\Big[F^* g^\tau = F^* (\pi^\tau)^* g = (\pi^\tau \circ F)^* g$$
$$= (\pi^\tau)^* g = g^\tau.\Big]$$

So $\{ Id_{M^\tau}, F \} \cong \mathbb{Z}_2$ acts freely and properly discontinuously on M^τ, so

$$\tilde{\pi}^\tau : M^\tau / \mathbb{Z}_2 \to M \text{ is an isometry.}$$

Conclusion: (M,g) is time-orientable

$\iff F : M^\tau \to M^\tau$ is an orthochronous isometry.

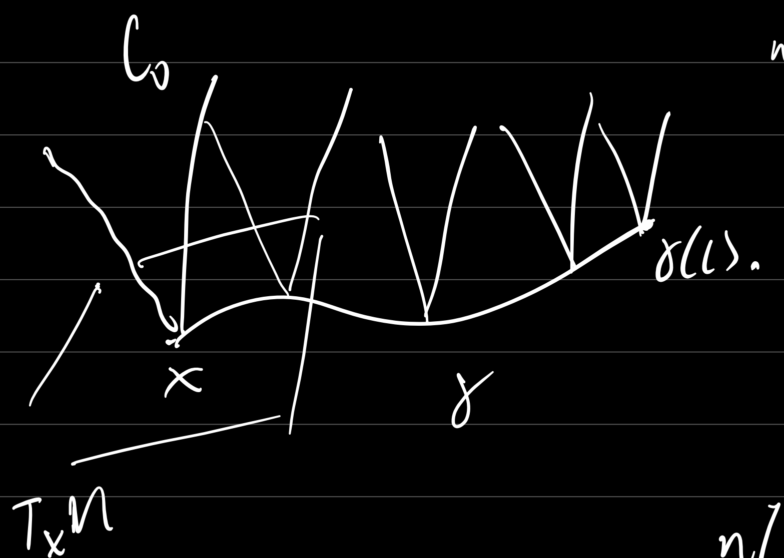
$$\begin{tikzcd} \tilde{M} \arrow[r] & M^\tau \arrow[l, bend right] \end{tikzcd}$$

Corollary. The universal cover of any (connected) Lorentz manifold is time-orientable. In particular, any simply connected Lorentz manifold is time-orientable.

What about uniqueness of time-orientable double covers?

Idea: Fix $x \in M$, and $C_0 \subseteq T_x M$ a time-orientation.

For any curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$,



we may continuously extend C_0 to a curve of time-orientation C_t on $T_{\gamma(t)} M$, w/ $C_{(0)} = C_0$.

If $\gamma(1) = \gamma(0) = x$ (i.e., γ is closed), we say that γ is : • orthochronous if $C_1 = C_0$

• anti-orthochronous if $C_1 = -C_0$.

Note that being o.c. or not depends only on the homotopy class of γ.

Also, local diffeomorphisms send orthochronous curves to orthochronous curves.

We have a homomorphism $\Theta_x : \pi_1(M,x) \to \mathbb{Z}_2$.

Corollary : Let (M,g) be a Lorentz manifold, (M^τ, g^τ) be a time-orientable double cover of M w/ projection $\pi^\tau : M^\tau \to M$, and we fix $x \in M$ and $x^\tau \in M^\tau$ s.t. $\pi^\tau(x^\tau) = x$.

then the image of $(\pi^\tau)_\# : \pi_1(M^\tau, x^\tau) \to \pi_1(M,x)$

is precisely $\operatorname{Ker}\Theta_x$.

<u>Proof</u>. $\Theta_x \circ (\pi^\tau)_\# = Id. \Rightarrow \operatorname{Im}(\pi^\tau)_\# \subseteq \operatorname{Ker}\Theta_x$.

So, since $\forall$ group G, if $H_1, H_2 \leq G$ with the same finite index and $H_1 \subseteq H_2$, then $H_1 = H_2$.

$\Big([G : \cancel{H_1}] = [G : \cancel{H_2}] \cdot [H_2 : H_1]$

1

$\Rightarrow [H_2 : H_1] = 1 \Rightarrow H_2 = H_1 \Big)$.

The proof ends by noting that both $\operatorname{Im}(\pi^\tau)_\#$ and $\operatorname{Ker}\Theta_x$ have index 2 in $\pi_1(M,x)$. $\blacksquare$

<u>Consequence</u>: time-orientable double-covers are unique up to isomorphism (of covering spaces), and such isomorphisms are in fact isometries.

isomorphism classes of covering maps over M $\rightleftarrows$ conjugacy classes of subgroups of $\pi_1(M)$.

(Galois connection.)

Corollary. If (M,g) is a Lorentz manifold s.t. $\pi_1(M)$ does not have subgroups of index 2, then (M,g) is automatically time-orientable.

Lecture 6.

Theorem: Let M be a smooth manifold. then, the following conditions are equivalent.

(i) M admits a Lorentzian metric.

(ii) M admits a time-orientable Lorentzian metric

(iii) There is a nowhere-vanishing vector field $X \in \mathfrak{X}(M)$

(iv) Either M is non-compact, or $\chi(M) = 0$.

Proof:

$$\begin{tikzcd}
 & \text{(i)} \arrow[dl] & \\
\text{(iv)} \arrow[dr, leftrightarrow] & & \text{(ii)} \arrow[ul] \arrow[dl, shift right] \\
 & \text{(iii)} \arrow[ur, shift right] &
\end{tikzcd}$$

(Poincaré-Hopf thm).

Take any Riemannian metric g^0 on M, and let

$$g_x(v,w) := g^0_x(v,w) - 2\,\frac{\underbrace{g_x(v,X_x)\,g_x(w,X_x)}}{g_x(X_x,X_x)}$$

$\Rightarrow$ Check that g is Lorentzian.

Note: $g(X,X) = -g^0(X,X) < 0$

Assume that M is compact, and consider a time-orientable double cover (M^τ, g^τ) for

M. Now, $M^{\mathbb{C}}$ is compact.

Moreover, there is a nowhere-vanishing vector field on $M^{\mathbb{C}}$. So, $\chi(M^{\mathbb{C}}) = 0$.

But $\chi(M^{\mathbb{C}}) = 2\chi(M) \Longrightarrow \chi(M) = 0$.

$\boxtimes$

Ex. •) the only compact orientable surface admitting a Lorentzian metric is $\mathbb{T}^2$.

If M has genus g, then $\chi(M) = 2 - 2g$.

And $2 - 2g = 0 \Leftrightarrow g = 1 \Longleftrightarrow M = \mathbb{T}^2$.

•) non-orientable case: only the Klein bottle. Know $M = \underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{k \text{ times}}$,

and $\chi(M) = 2 - k$, and so $\chi(M) = 0$ $\Leftrightarrow k = 2$, but $\mathbb{RP}^2 \# \mathbb{RP}^2 \cong K$.

•) any compact odd-dimensional manifold has a Lorentzian metric (b/c $\chi(M)=0$ by Poincaré Duality)

$(\chi(S^n) = 1+(-1)^n)$

Examples. Let (N, g^0) be a connected Riemannian manifold, $I \subseteq \mathbb{R}$ be an open interval, and $\phi : I \to \mathbb{R}_{>0}$ a smooth function.

$I \times_\phi N$

Consider $(M, g) = (I \times N, -dt^2 \oplus \phi^2 g^0)$

$\left(\begin{array}{l} \pi_1 : M \to I \\ \pi_2 : M \to N \end{array} \right. , \quad g = \pi_1^*(-dt^2) + (\phi \circ \pi_1)^2 \pi_2^* g^0.$

This is a spacetime w/ canonical time-orientation given by the coordinate field $\frac{\partial}{\partial t}$.

This is called a warped product spacetime.

When (N, g^0) is a simply connected, complete space w/ constant sectional curvature, then

$(N, g^0) \cong \mathbb{R}^n, S^n, \mathbb{H}^n$

For those choices, $I \times_\phi N$ is called a FLRW space (Friedmann-Lemaître-Robertson-Walker)

Example: Let $m > 0$ and $q \in \mathbb{R}$ be constants (mass, electric charge). Define $h : (0, \infty) \to \mathbb{R}$, by

$$h(r) = \underbrace{1 - \frac{2m}{r} + \frac{q^2}{r^2}}_{\text{horizon function}}.$$

Consider $P = \{(t,r) : t \in \mathbb{R} \text{ and } r^2 - 2mr + q^2 > 0\}$,

and the metric $-h(r)\,dt^2 + h(r)^{-1} dr^2$.

$$\det \begin{pmatrix} -h(r) & 0 \\ 0 & h(r)^{-1} \end{pmatrix} = -1 < 0$$

Now, we can consider $P \times_r S^2$,

$(M = P \times S^2,\ (-h(r)\,dt^2 + h(r)^{-1} dr^2) \oplus r^2 g^0,$

where g^0 is the round metric on S^2).

$$(-h(r)\,dt^2 + h(r)^{-1} dr^2) \oplus r^2 g^0 \to -dt^2 + dr^2 + r^2 g^0$$

as $r \to +\infty$

? Expression of the Minkowski metric in spherical coordinates in the $\mathbb{R}^3$ factor of $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$

This is a model of a spacetime in the proximities of a black hole w/ mass m and electric charge q.

(t,r,p)

$(t,r)\in P$

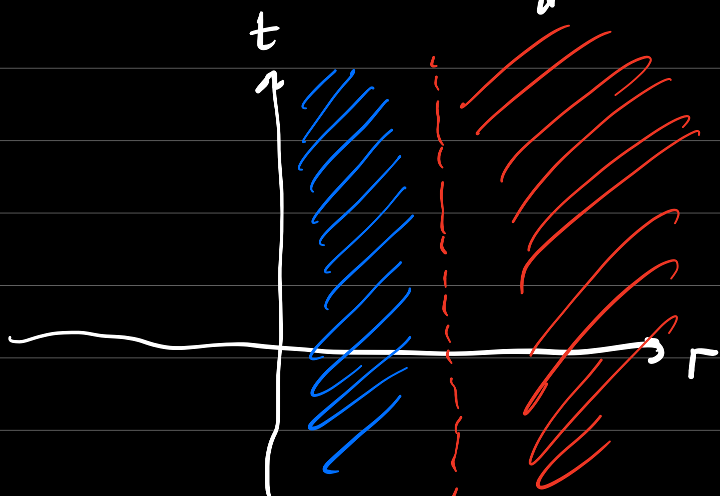
$p\in S^2$

This is called a Reissner-Nordström spacetime.

When $q=0$, it is called a Schwarzschild spacetime.

In the Schwarzschild case, $h(r)=1-\frac{2m}{r}$,

so $P=\{(t,r)\in\mathbb{R}^2 \mid r>2m\}$.



$r=2m$

$$\mathrm{Ric} \sim \frac{q^2}{r^4}\begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

$\hookrightarrow$ has the same "metric signature" of the electromagnetic energy momentum tensor.

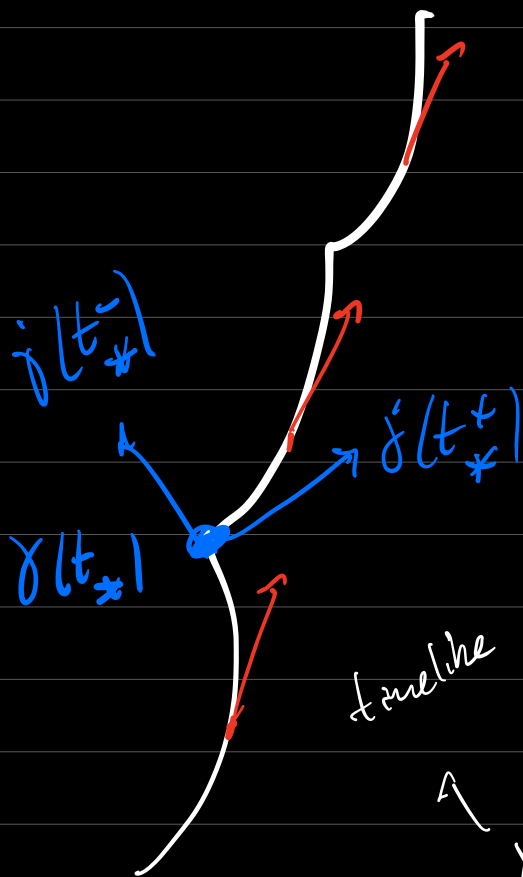
Birkhoff's theorem: every spherically symmetric, asymptotically Minkowski, Ricci flat spacetime is locally Schwarzschild.

Chronology relations.

From here on, (M,g) is a spacetime.

Let's say that a piecewise smooth curve

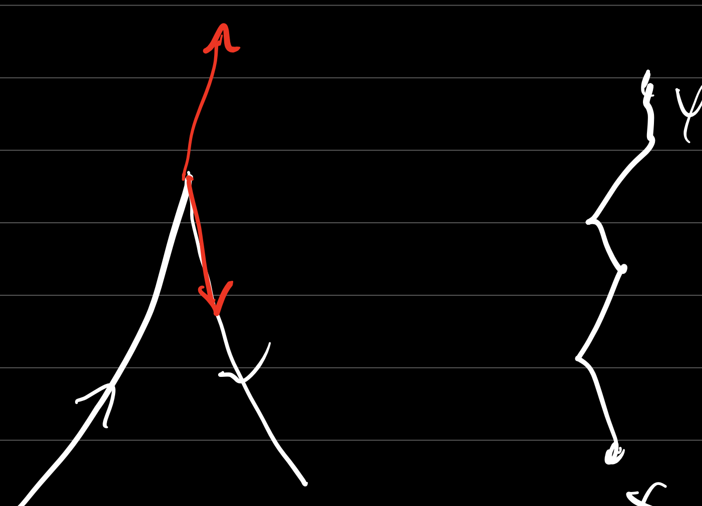
$\gamma : [a,b] \to M$ is timelike if:



1) $\dot{\gamma}(t)$ is timelike for every t in the interior of an interval on which γ is smooth

2) if $t_* \in I$ is a discontinuity of $\dot{\gamma}$ and we write $\dot{\gamma}(t_*^+)$ and $\dot{\gamma}(t_*^-)$ for the one-sided derivatives, then $g_{\gamma(t)}(\dot{\gamma}(t_*^+), \dot{\gamma}(t_*^-)) < 0$.

Forbidden:



Def: If $x, y \in M$, we say that

- x chronologically precedes y if there is a piecewise smooth timelike curve future-directed $\gamma : [0,1] \to M$ w/ $\gamma(0) = x$ and $\gamma(1) = y$. We write $x \ll y$.
- x causally precedes y if there's a future-directed piecewise smooth causal curve $\gamma : [0,1] \to M$ w/ $\gamma(0) = x$ and $\gamma(1) = y$, or $x = y$. We write $x \leq y$.

Remark. Let $x, y, z \in M$.

a) $x \ll y$ and $y \ll z \Rightarrow x \ll z$.

b) $x \ll y$ and $y \leq z \Rightarrow \underline{x \leq z}$ $x \ll z$

$x \leq y$ and $y \ll z \Rightarrow \underline{x \leq z}$. $x \ll z$

Remarkable fact: in item (b), the conclusion with $\leq$ may be replaced w/ $\ll$.

Terminology:

- the chronological future of x is

$$I^+(x) = \{ y \in M \mid x \ll y \}.$$

- the causal future is

$$J^+(x) = \{ y \in M \mid x \leq y \}.$$

Dual past-versions:

$$I^-(x) = \{ y \in M \mid y \ll x \}$$

$$J^-(x) = \{ y \in M \mid y \leq x \}.$$