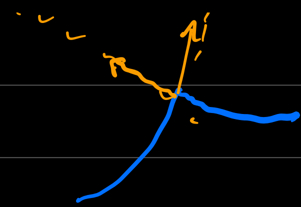
Causality and Spacetimes - Lecture 7



Def: Let (M, g) be a spacetime and $x, y \in M$.

We say that:

(i) $x \ll y$ (x chronologically precedes y) if there is a future-directed piecewise smooth timelike curve from x to y.

(ii) $x \leq y$ (x causally precedes y) if $x = y$ or if there is a future-directed piecewise smooth causal curve from x to y.

With this in place, we set:

$$I^+(x) = \{ y \in M \mid x \ll y \} = \text{chronological future of } x$$

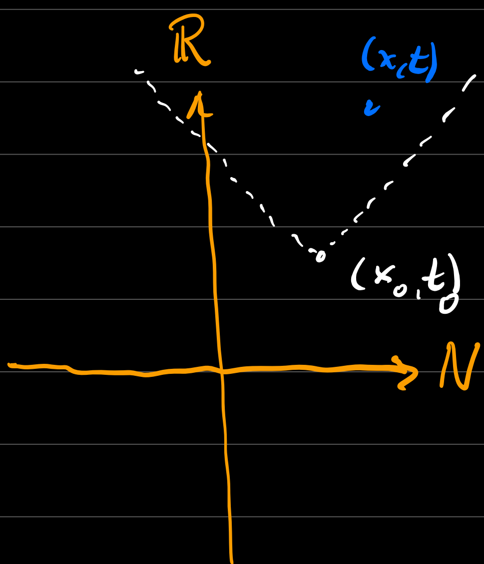
$$J^+(x) = \{ y \in M \mid x \leq y \} = \text{causal future of } x.$$

Can also let, for any $S \subseteq M$, $\left(I^-(x) = \{ y \in M \mid y \ll x \} \right)$

$$I^+[S] = \bigcup_{x \in S} I^+(x) \quad \text{and} \quad J^+[S] = \bigcup_{x \in S} J^+(x).$$

Remark: it doesn't make any difference whether one uses piecewise smooth curves, smooth curves, or piecewise smooth geodesics in those definitions.

Example: Let (N, g^0) be a connected Riemannian manifold, and consider $(M, g) = (\mathbb{R} \times N, -dt^2 \oplus g^0)$. Let's compute, for $(t_0, x_0) \in M$, the future $I^+(t_0, x_0)$.



Claim.

$$I^+((t_0, x_0)) = \{(t, x) \in M \mid t > t_0 \text{ and } d_{g^0}(x, x_0) < t - t_0\}$$

We can assume WLOG that $t_0 = 0$.

Reason: translations in the time factor $\mathbb{R}$ are orthochronous isometries of (M, g)

First inclusion. Let $(\ell, x) \in M$ be such that $(0, x_0) \ll (\ell, x)$. There is a piecewise smooth future-directed timelike curve γ from $(0, x_0)$ to (ℓ, x). $\rightarrow \ell > 0$

Reparametrizing γ if needed, we may assume that

$\gamma : [0, \ell] \to M$ has the form

$\gamma(t) = (t, \alpha(t))$ and $\| \dot\alpha(t) \| < 1$

$\left(\dot\gamma(t) = \frac{\partial}{\partial t} + \dot\alpha(t) \Rightarrow \underbrace{g(\dot\gamma, \dot\gamma) = -1 + \|\dot\alpha(t)\|^2}_{<0} \right)$

$$d_{g^0}(x, x_0) \le L[\alpha] = \int_0^\ell \|\dot\alpha(t)\| dt < \int_0^\ell dt = \ell.$$

Second inclusion: Assume that we take $(\ell, x) \in M$ s.t. $\ell > 0$ and $d_{g^0}(x, x_0) < \ell$.

Let α be a curve in N such that $L[\alpha] < \ell$.

Reparametrizing α if needed, we may assume that α is defined on $[0, \ell]$ and $\|\dot\alpha(t)\| < 1$.

Define $\gamma : [0, \ell] \to M$ by $\gamma(t) = (t, \alpha(t))$.

This is timelike and future-directed by construction.

$\left(g(\dot\gamma, \dot\gamma) = -1 + \|\dot\alpha(t)\|^2 < 0 \right)$.

What happens in the warped product case?

If (N, g^0) is a connected Riem. manifold and $(M, g) = (\mathbb{R} \times N, -dt^2 \oplus \phi^2 g^0)$, for $\phi : \mathbb{R} \to \mathbb{R}_{>0}$?

Define the galactic function of (M, g) by $\Phi : \mathbb{R}^2 \to \mathbb{R}$,

$$\Phi(a,b) = \int_a^b \frac{dt}{\phi(t)}.$$ <u>FLRW</u>

If $\gamma : [a,b] \to M$ has the form $\gamma(t) = (t, \alpha(t))$ and γ is lightlike, then

$$0 = -1 + \phi(t)^2 \|\dot{\alpha}(t)\|^2$$

$$\Rightarrow \|\dot{\alpha}(t)\|^2 = \frac{1}{\phi(t)^2}$$

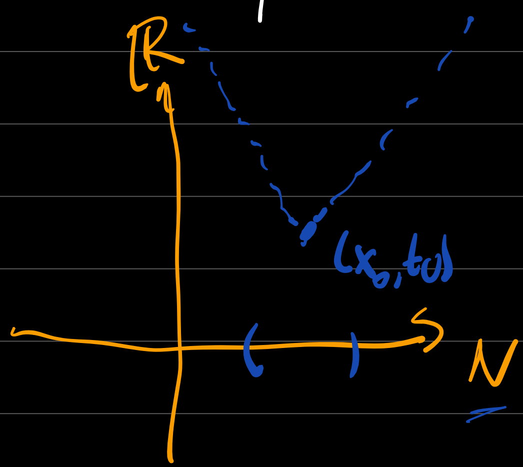
$$\Rightarrow \|\dot{\alpha}(t)\| = \frac{1}{\phi(t)} \Rightarrow L[\alpha] = \Phi(a,b).$$

<u>Fact</u>. Let $(t_0, x_0) \in M$. Then:

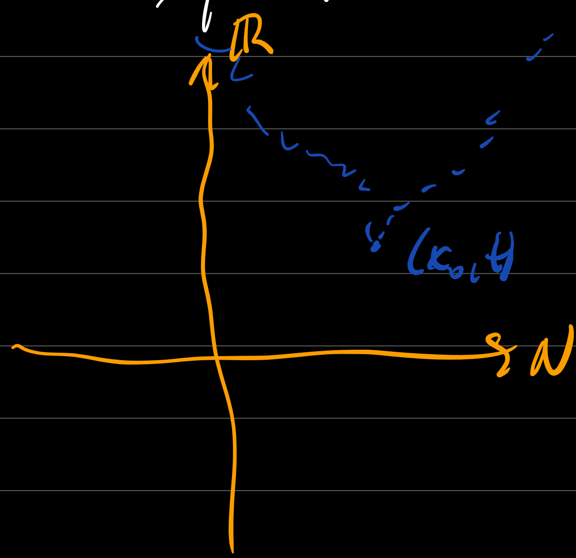
$$I^+((t_0,x_0)) = \{(t,x) \in M \mid t > t_0 \text{ and } d_{g_0}(x,x_0) < \Phi(t_0,t)\}$$

Geometrically:

Direct product



Warped product



Particular case: (V, g) Lorentz vector space, $w \in V$ a timelike vector. So $V = \mathbb{R}w \oplus w^{\perp}$

unit.

For every $x \in V$, we may write

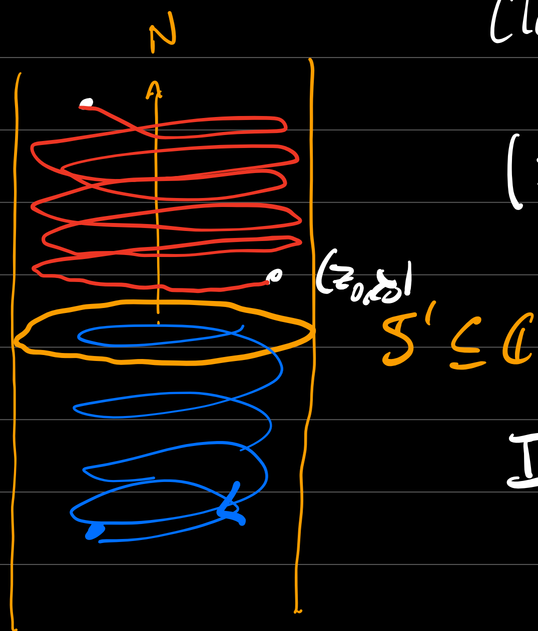
$$x = -g(x,w)\, w + x_0 \quad , \quad \text{with } x_0 \in w^{\perp}$$

$$I^+(x) = \{ y \in V \mid g(x-y, w) > 0 \text{ and } \|x_0 - y_0\| < g(x-y,w) \}.$$

$x \ll y$
$\Rightarrow$ $y - x$ is timelike and f.d. $g(y-x, w) < 0$

Example: Let (N, g^0) be a connected Riemannian manifold, and consider $(M,g) = (S^1 \times N, -d\theta^2 \oplus g^0)$.

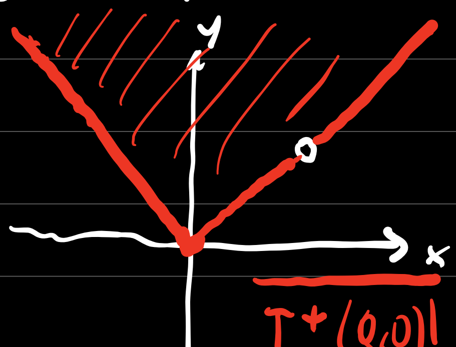
In this case, we have that:

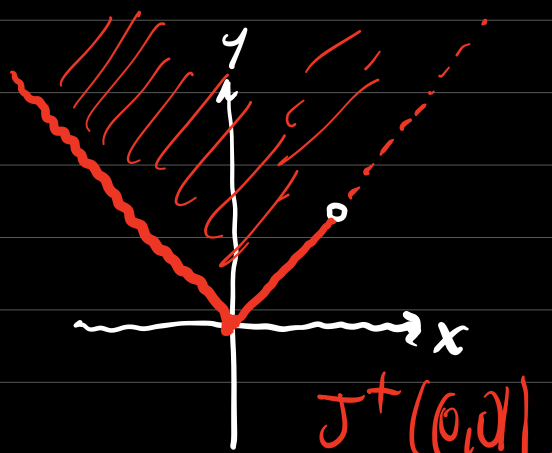


Claim: for every $(z_0, x_0) \in M$, we have that $I^+(z_0,x_0) = M$

We say that (M,g) is totally vicious.

Ex: $\mathbb{L}^2 \setminus \{(1,1)\}$.





Proposition: (M,g) spacetime, $S \subseteq M$. Then:

(i) I^+ and J^+ are inclusion-preserving.

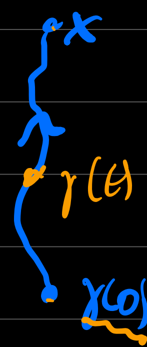
(ii) $I^+[S] \subseteq J^+[S]$

(iii) $I^+[I^+[S]] = I^+[S]$ $\qquad \gamma(0) \ll \gamma(t) \ll x.$

(iv) $J^+[J^+[S]] = J^+[S]$.

Proof: (i) clear. $\left(S_1 \subseteq S_2, \quad I^+[S_1] \subseteq I^+[S_2] \right)$

(ii) clear.



(iii) $I^+[I^+[S]] \subseteq I^+[S]$.

$I^+[S] \subseteq I^+[I^+[S]]$

Let $x \in I^+[S]$ and take $\gamma : [0,1] \to M$ p.w.-smooth, f.d., timelike w/ $\gamma(0) \in S$ and $\gamma(1) = x$.

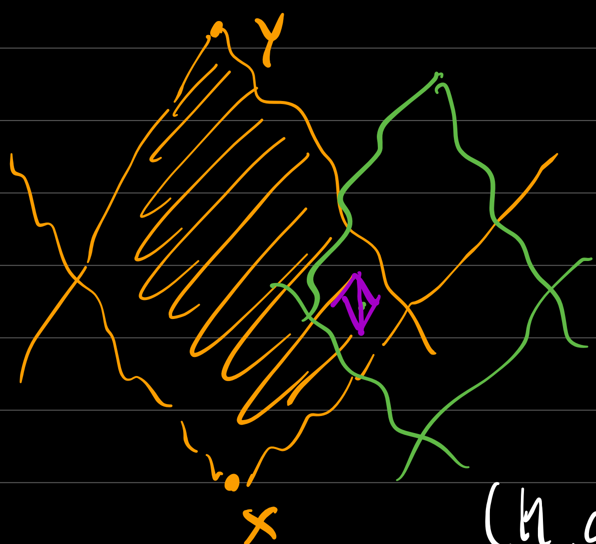
Know: $\gamma(t) \in I^+[S]$

Def.: (M,g) spacetime, $x,y\in M$.

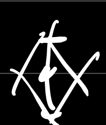
$$I(x,y) := I^+(x)\cap I^-(y) = \{z\in M \mid x \ll z \ll y\}$$

$$J(x,y) := J^+(x)\cap J^-(y) = \{z\in M \mid x \leq z \leq y\}.$$

They are called the chronological and causal diamonds spanned by x and y.



Fact: $\{I(x,y) : x,y\in M\}$ is a basis for a topology in M, called the Alexandrov topology of (M,g).



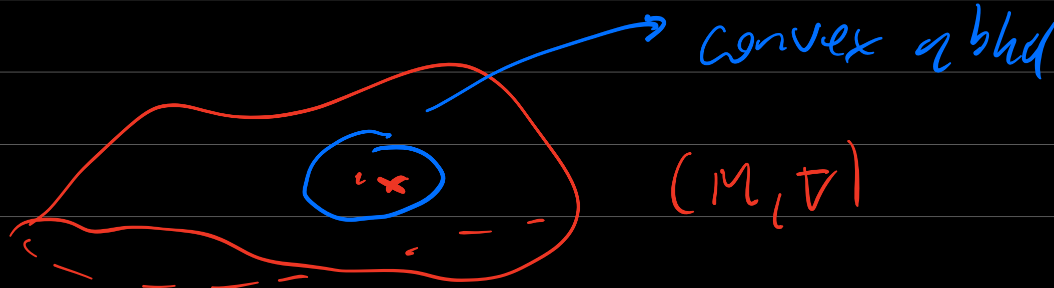
— // —

Causality and Spacetimes - Lecture 8

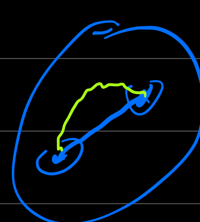
•) General facts about <u>local causality</u>

For a causal curve $\gamma : [a,b] \to M$, we let

$$\tau[\gamma] = \int_a^b \sqrt{-g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\, dt$$ be the proper time of γ.



$$L[\gamma] = \int_a^b \|\dot\gamma(t)\| dt$$

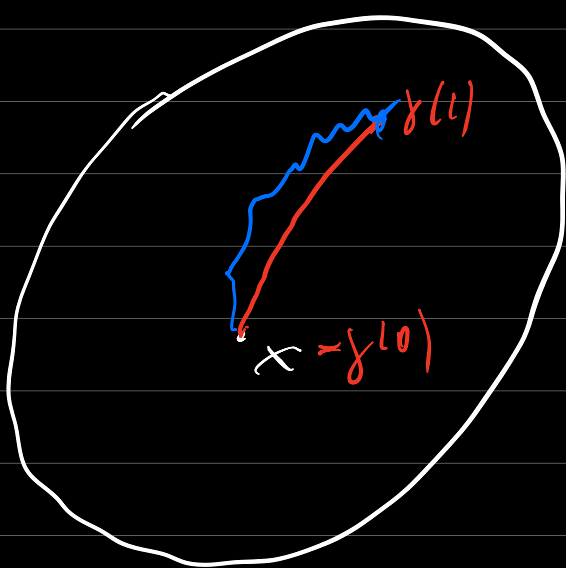


$$= \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))}\, dt$$

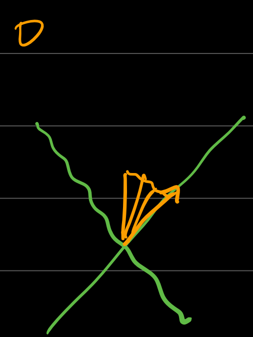
Riemannian case: geodesics "locally" minimize arclength

<u>Prop</u>. (M,g) Lorentz manifold, $U \subseteq M$ normal neighborhood of x, $\gamma : [0,1] \to U \subseteq M$ timelike geodesic with $\gamma(0) = x$. If $\alpha : [0,1] \to U \subseteq M$ is any piecewise smooth timelike curve with $\alpha(0) = x$ and $\alpha(1) = \gamma(1)$, then

$\tau[\gamma] \geq \tau[\alpha]$, with equality if and only if α is a positive reparametrization of γ.



$\tau[\gamma] \geq \tau[\alpha]$



Timelike geodesics locally maximize proper time!

Proposition: (M,g) Lorentz manifold, $x \in M$, $U \subseteq M$ convex neighborhood of x, and $y \in U$ be such that the radial geodesic ℓ_{xy} is lightlike. Then, up to reparametrization, ℓ_{xy} is the unique causal curve in U from x to y.

"Principle of General Covariance"

Express a law of physics in $\mathbb{R}^n$, with Minkowski metric $\eta_{\mu\nu}$ and $T_{\mu\nu,\lambda} = \partial T_{\mu\nu}/\partial x^\lambda$

Replace $\eta_{\mu\nu}$ w/ $g_{\mu\nu}$ and $,$ w/ $;$.

Prop. (M,g) spacetime, $x \in M$. Then $I^+(x)$ and $I^-(x)$ are non-empty open subsets of M.

Note: $J^+(x)$ and $J^-(x)$ are not closed in general! (or equal to the closures of $I^+(x)$ and $I^-(x)$.

$F = m \cdot a$ $\ln \mathbb{R}^n$, $a(t) = \ddot{\gamma}(t)$

$$= \frac{d^2}{dt^2}\gamma(t)$$

$$a(t) \sim (\gamma^1{}_{,tt}, \gamma^2{}_{,tt}, \dots, \gamma^n{}_{,tt})$$

$$\rightsquigarrow (\gamma^1{}_{;tt}, \dots, \gamma^n{}_{;tt})$$

$$\gamma^k{}_{;tt} = \ddot{\gamma}^k + \sum \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j$$

$$X = X^i \partial_i$$

$$X^i{}_{;j} = \partial_j X^i + \Gamma^i_{jk}\, dx^k$$

$(x \ll y \text{ and } y \leq z \Longrightarrow x \leq z)$

<u>Push-up Lemma:</u> (M,g) spacetime, $x,y,z \in M$. Then

$x \leq y$ and $y \ll z \Rightarrow x \ll z$. (And similarly, $x \ll y$ and $y \leq z \Rightarrow x \ll z$).

<u>Proof sketch:</u> Assume that $x \ll y$ and $y \leq z$.

Take $\gamma : [0,1] \to M$ future-directed, piecewise smooth, and causal, with $\gamma(0) = y$ and $\gamma(1) = z$.

Consider the set

$$I = \{ t \in [0,1] \mid x \ll \gamma(s), \text{ for all } s \in [0,t] \}$$

•) $I \neq \emptyset$ because $0 \in I$, as $x \ll y$.

•) I is an interval.

•) I is open in $[0,1]$ (because $I^+(x)$ is open).

If we show that I is closed, then $I = [0,1]$

and $1 \in I \Rightarrow x \ll \gamma(1) = z$.

Let $b = \sup I$. Have to show that $b \in I$.

Consider an open convex nbhd of $\gamma(b)$, U.

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $\gamma(t) \in U$ for $b - \varepsilon < t < b$.

Fix $t_0 \in (b-\varepsilon, b)$, and choose a (f.d.) timelike curve $\alpha : [0,1] \to M$ s.t. $\alpha(0) = x$ and $\alpha(1) = \gamma(t_0)$.

Know that

$\alpha(t_1) \ll \gamma(t_0)$ and

$\gamma(t_0) \leq \gamma(b)$

$\Rightarrow \alpha(t_1) \leq \gamma(b)$

Radial geodesic from $\alpha(t_1)$ to $\gamma(b)$ is causal and future-directed.

Claim: this segment is actually timelike.

(By previous prop; lightline $\Rightarrow$ hopot.)

So $\alpha(t_1) \ll \gamma(b)$ and $x \ll \alpha(t_1)$

$\Rightarrow x \ll \gamma(b) \Rightarrow b \in I$. ↯

((M, ∇) affine manifold $\Rightarrow$ any two points can be joined by a broken geodesic)

Fix $z \in M$

$A = \{ x \in M \mid \exists$ broken geod. $z \to x \}$.

Future sets and past sets

Def: (M,g) spacetime, $F \subseteq M$. We say that F is a future set if $I^+[F] = F$. (Dual definition for past sets).

(Why "F" twice? $F = I^+[S]$
$\Rightarrow \quad I^+[F] = I^+[I^+[S]] = I^+[S] = F$)

Note: future and past sets are automatically open, and $U \subseteq I^+[U]$ for every open set U. For the reverse inclusion, there's a general equivalence:

Prop: (M,g) spacetime, $S \subseteq M$. TFAE:

(i) $I^+[S] \subseteq S$

(ii) $I^-[M\setminus S] \subseteq M\setminus S$

(iii) $I^+[S] \cap I^-[M\setminus S] = \emptyset$

(iv) $\mathring{S} = I^+[S]$

(v) $\partial S = (M\setminus I^+[S]) \cap (M \setminus I^-[M\setminus S])$.

Proof: easy but long, with lots of point-set stuff!

More properties: (M,g) spacetime, $S\subseteq M$

a) $I^+[\overline{S}] = I^+[S]$.

b) $I^+[J^+[S]] = J^+[I^+[S]] = I^+[S]$

c) $J^+[S] \subseteq \overline{I^+[S]}$ and $\overline{J^+[S]} = \overline{I^+[S]}$

d) $J^+[S]^\circ = \overline{J^+[S]}^\circ = I^+[S]$.

Proof of (a): $S\subseteq \overline{S} \Rightarrow I^+[S] \subseteq I^+[\overline{S}]$.

If $x\in I^+[\overline{S}]$, there is $y\in\overline{S}$ s.t. $x\in I^+(y)$.

$\Rightarrow \underset{\text{open}}{y \in I^-(x)} \Rightarrow \exists\ z \in I^-(x) \cap S.$

$\Rightarrow x \in I^+(z),\ z \in S$

$\Rightarrow x \in I^+[S].$ $\blacksquare$

<u>Proposition</u>: Let (M,g) be a spacetime and $F \subseteq M$ be a future set. Then:

i) $\overline{F} = \{x \in M \mid I^+(x) \subseteq F\}$

ii) $\partial F = \{x \in M \mid x \notin F \text{ and } I^+(x) \subseteq F\}.$

<u>Proof</u>. Let $x \in \overline{F}$. $\exists\ (x_n)_{n \geq 0} \subseteq F$, $x_n \to x$.

Let $y \in I^+(x)$. Goal: $y \in F$.

$\Rightarrow \underset{\text{open}}{x \in I^-(y)} \Rightarrow \exists\, n$ large enough s.t.

$x_n \in I^-(y)$

$\Rightarrow$ for this n, $y \in I^+(x_n)$

But $x_n \in F \Rightarrow y \in I^+[F] = F.$

For the reverse inclusion, let $x \in M$ be such that $I^+(x) \subseteq F$.

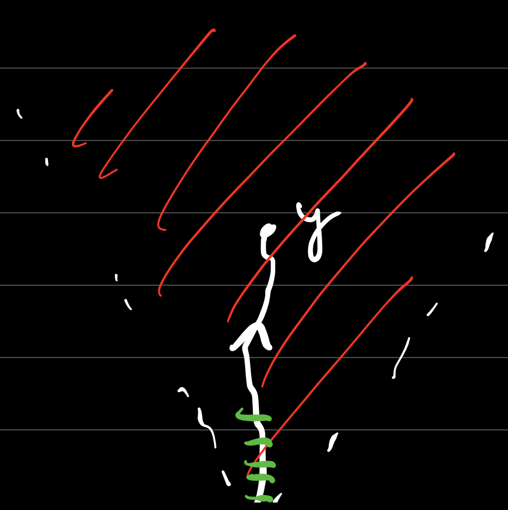
Goal: $x \in \overline{F}$.

Take $y \in I^+(x)$, and a future-directed p.w. smooth timelike curve $\gamma : [0,1] \to M$, $\gamma(0) = x$ and $\gamma(1) = y$. Note that

$$x \ll \gamma(t) \quad \forall t \in (0,1].$$

Choose $(t_n)_{n\geq 0} \subseteq (0,1]$, s.t. $t_n \downarrow 0$, set $x_n = \gamma(t_n) \in I^+(x) \subseteq F$

Have $(x_n)_{n\geq 0} \subseteq F$, and $x_n \to x \in \overline{F}$.



Def. Let (M,g) be a spacetime. A subset $B \subseteq M$ is called an:

(i) achronal set if no two points in B are chronologically related

(ii) achronal boundary if $B = \partial F$ for some future set $F \subseteq M$.

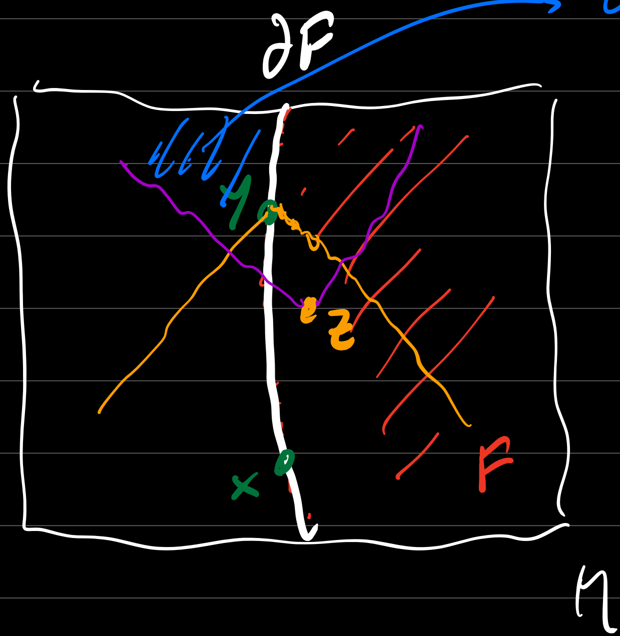
"achronal"

Fact: Achronal boundaries are achronal.

Proof. Let's show that for any $F \subseteq M$, if ∂F is not achronal, then F is not a future set.

Take $x, y \in \partial F$ s.t. $x \ll y$.

$\Rightarrow \exists\, z \in I^-(y) \cap F. \Rightarrow I^+(z) \cap (M \setminus F) \neq \emptyset$



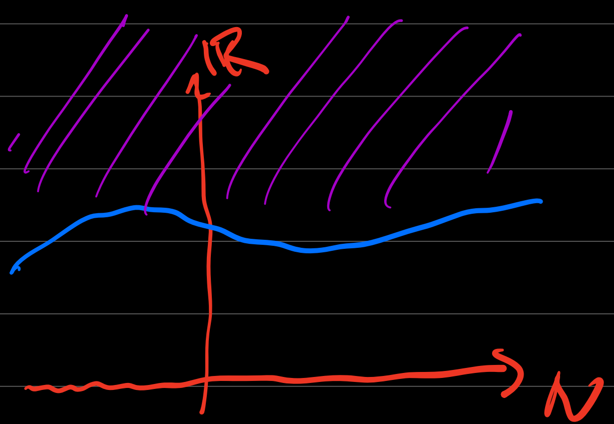
any point
should that

$I^+[F] \subseteq F$.

Example: Spacelike hypersurfaces in warped product spacetimes. More precisely, let $M = \mathbb{R} \times_\phi N$ as usual, and let $f : N \to \mathbb{R}$ be such that $|df_x(v)| < \phi(f(x))\|v\|$ for all $x \in N$ and $v \in T_x N \setminus \{0\}$. Then

$B = \{(f(x), x) : x \in N\}$ is an achronal boundary.

$(M, g) = (\mathbb{R} \times N, -dt^2 \oplus \phi^2 g^0)$



$|df_x v| \leq \phi(f(x))\|v\|$ ensures that B is a spacelike hypersurface.

$B = \partial F, \quad F = \{(t, x) \in M \mid t > f(x)\}.$

Theorem: Let (M,g) be a spacetime. Then:

(i) Every achronal set is contained in a maximal achronal set.

(ii) Every maximal achronal set is an achronal boundary. Moreover, we may write $A_{max} = \partial F$ for a future set $F \subseteq M$ such that $I^+[F] = I^+[\partial F]$.

Proof:

$I^+[A] \cap A = \phi$ (no two points on A are chronologically related)

(i) Zorn!

(ii) Let A_{max} be a maximal achronal set.

Let $F = I^+[A_{max}]$

Since A_{max} is maximal, it suffices to show that $A_{max} \subseteq \partial F$

Know that $A_{max} \subseteq \overline{F}$, and also

$A_{max} \cap F = A_{max} \cap I^+[A_{max}] = \emptyset$

$$\Rightarrow A_{max} \subseteq \partial F.$$

Finally, $I^+[\partial F] = I^+[A_{max}]$
$$= F$$
$$= I^+[F].$$

Causality and Spacetimes - Lecture 9 (Final)

Theorem: Let (M, g) be a spacetime and $B \subseteq M$ be a non-empty achronal boundary. Then B is a topological hypersurface of M.

Idea: If $B = \partial F$ for a future set $F \subseteq M$, since F is open, then the codimension of B must be 1 whenever it is a "submanifold" of M.

Proof: Write $B = \partial F$, for $F \subseteq M$ - future set, and take $x \in B$. We know that $I^+(x) \subseteq F$.

Similarly, we have that $I^-(x) \subseteq M \setminus \bar{F}$.

Conclusion: every timelike curve starting at $I^-(x)$ and ending at $I^+(x)$ must intersect ∂F (and hence B).

[If (X, d) is a metric space, $A \subseteq M$ is any subset, and $C \subseteq M$ is connected and intersects both A and $M \setminus A$, then C intersects ∂A.

Consider $f: C \to \mathbb{R}$ given by

$$f(x) = d(x, A) - d(x, M \setminus A).$$

$\hookrightarrow$ Continuous.

$x \in C \cap \partial A \Leftrightarrow f(x) = 0.$

Apply IVT to f, noting that one among $d(x, A)$ and $d(x, M \setminus A)$ must be zero, ☺

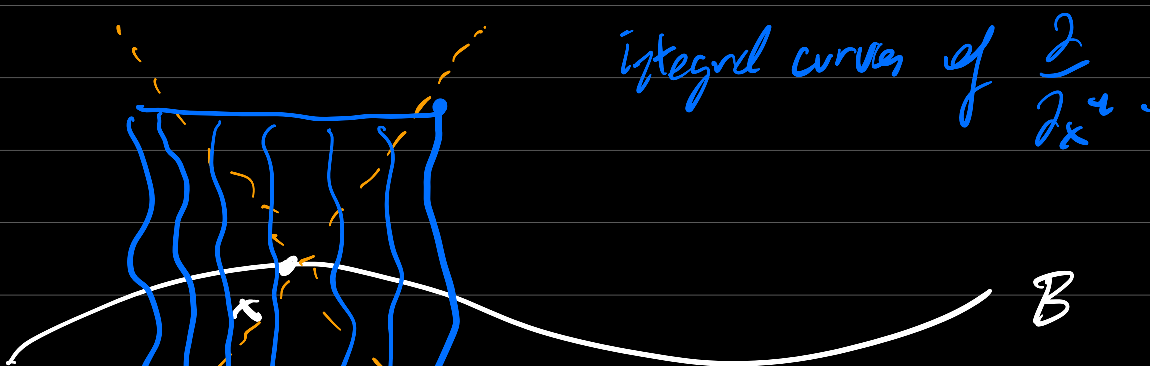
$d(x, A) = d(x, M \setminus A) \Rightarrow$ both are zero.

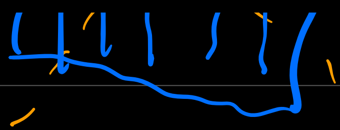
Now, let $U \subseteq M$ be an open neighborhood of x s.t. every timelike curve from $I^-(x) \cap U$ to $I^+(x) \cap U$ meets B exactly once.

Reducing U if needed, we may also assume that:

- U is a normal nbhd of x, carrying normal coordinates $\varphi = (x^1, \ldots, x^n)$.
- $\partial/\partial x^n$ is timelike and future-directed
- $\varphi[U] = U_0 \times (-2\varepsilon, 2\varepsilon) \subseteq \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n$, with $\varepsilon > 0$, $U_0 \subseteq \mathbb{R}^{n-1}$ open, and the points of

coordinates in $U_0 \times \{\pm \varepsilon\}$ are in $I^{\pm}(x) \cap U$.





Every integral curve of $\frac{\partial}{\partial x^n}$, defined on the interval $(-\varepsilon, \varepsilon)$ intersects $U \cap B$ in a single point.

Write the coordinates of this point as

$(x^1, \ldots, x^{n-1}, h(x^1, \ldots, x^{n-1}))$, for some

function $h: U_0 \to (-\varepsilon, \varepsilon)$.

If we show that h is continuous, then

$(x^1|_{U\cap B}, \ldots, x^{n-1}|_{U\cap B})$ will be coordinates

for B around x.

Assume that h is not continuous, and take a sequence $(u_k)_{k\geq 0} \subseteq U_0$ s.t. $u_k \to u \in U_0$, but

that $h(u_k) \not\to h(u)$.

Passing to a subsequence, assume that $h(u_k) \to r$, but $r \neq h(u)$.

Writing $y = \varphi^{-1}(u, h(u)) \in B$, we have that

$\varphi^{-1}(u,r)$ is in $I^+(y) \cap U$ or $I^-(y) \cap U$,

according to whether $r > h(u)$ or $r < h(u)$.

Let's say that $r > h(u)$. So $\varphi^{-1}(u,r) \in I^+(y) \cap U$.

Since $\varphi^{-1}(u_k, h(u_k)) \rightsquigarrow \varphi^{-1}(u,r)$. So, there is

k large enough so that

$$\varphi^{-1}(u_k, h(u_k)) \in I^+(y) \cap U \cap B$$

$\rightsquigarrow$ contradicts that B is achronal. $\boxtimes$

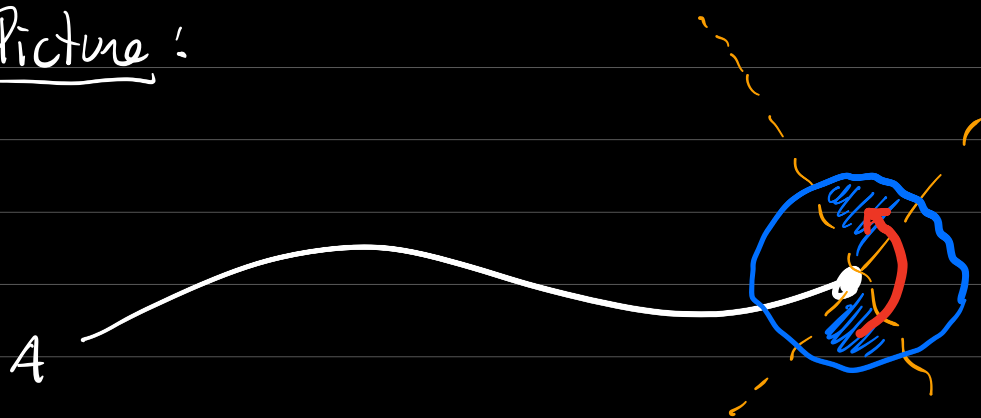
Q: What could make an achronal set to fail to be a topological hypersurface of (M,g)?

A: "Boundary" points!

Def'n: Let (M,g) be a spacetime and $A \subseteq M$ be an achronal set. We say that a point $x \in \overline{A}$ is an *edge point* of A if for every open neighborhood $U \subseteq M$ of x, there is a timelike curve from $I^-(x) \cap U$ to $I^+(x) \cap U$ never intersecting A.

$$\emptyset \subseteq I^+[S] \cap S \subseteq I^+[A] \cap A = \emptyset$$

Picture:



(if $S \subseteq A$, A achronal $\Rightarrow$ S is achronal)

We write $\mathrm{edge}(A)$ for the set of edge points of A, and say that A is *edgeless* if $\mathrm{edge}(A) = \emptyset$.

Relation w/ "boundary": $\mathrm{edge}(A) \subseteq \partial A$.

If A_{max} is any maximal achronal set w/ $A \subseteq A_{max}$,

$\Rightarrow$ $\mathrm{edge}(A)$ is the boundary of A relative to A_{max}.

Prop.: Let (M, g) be a spacetime. Then:

(i) If $A_1 \subseteq A_2 \subseteq M$ are achronal sets, then

$$\mathrm{edge}(A_2) \cap \overline{A_1} \subseteq \mathrm{edge}(A_1).$$

(ii) achronal boundaries are edgeless.

Proof: (i) Note that if $x \in M$ is any point, any continuous curve from $I^-(x) \cap U$ to $I^+(x) \cap U$ not meeting A_2, does not meet A_1 either.

(ii) Done.

— " —

The Achronal Identity: Let (M,g) be a spacetime, and $A \subseteq M$ be an achronal set. Then $\overline{A} \setminus A \subseteq \mathrm{edge}(A)$ and, in particular, every edgeless achronal set is closed.

Note: A achronal $\Rightarrow$ $\overline{A}$ achronal, and $\mathrm{edge}(\overline{A}) \subseteq \mathrm{edge}(A)$, but the inclusion is generally strict.

If $x \in I^+[\overline{A}] \cap \overline{A} = I^+[A] \cap \overline{A}$

$\Rightarrow$ $I^+[A]$ is an open nbhd of x

$x \in \overline{A}$ $\qquad \Rightarrow I^+[A] \cap A \neq \emptyset$

↯

Proof: Let's show that $\overline{A} \setminus \mathrm{edge}(A) \subseteq A$.

Let $x \in \overline{A}$ and assume that x is not an edge point of A.

$\Rightarrow \exists \underset{\text{open}}{U} \ni x$ s.t. every timelike curve from

$I^-(x) \cap U$ to $I^+(x) \cap U$ must meet A.

If $x \notin A$, such a curve γ would have to meet A in a point different than x.

Contradicts that $\bar{A}$ is achronal!

$\square$

Theorem: Let (M,g) be a spacetime, and $A \subseteq M$ be any achronal set. Then $A \setminus \mathrm{edge}(A)$, whenever non-empty, is a topological hypersurface in (M,g).

Where to go from here?

- Chronological and causal spacetimes
- distinguishing and reflecting spacetimes
- strongly causal spacetimes and the Alexandrov topology.
- time functions and stably causal spacetimes
- Causally simple and causally continuous spacetimes
- Lorentzian separation, Cauchy hypersurfaces and globally hyperbolic spacetimes.
- Singularity theorems (Lorentzian version of the Morse Index Theorem, trapped and marginally trapped surfaces, Bartnik's splitting conjecture, etc.)

Def. If (M,g) is a spacetime, the set of chronology violation of (M,g) is

$$\mathcal{C} = \{x \in M \mid x \ll x\}.$$

(M,g) is chronological if $\mathcal{C} \neq \emptyset$.

Define, on M, an equivalence relation by

$$x \sim y \iff x \ll y \ll x$$

$\iff \exists$ closed timelike curve passing through x and y

Carter's equivalence relation

Equivalence classes are chronological diamonds.

$$\mathcal{C} = \bigcup I(x,x).$$

$x \in \mathfrak{C}$

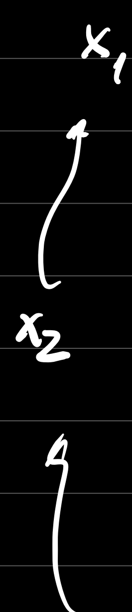
Theorem. If (M,g) is a compact spacetime, then $\mathfrak{C} \neq \emptyset$.

Proof. Consider the open cover of M,

$$\{ I^+(x) \}_{x \in M}.$$

$\rightsquigarrow$ extract finite subcover

$$\{ I^+(x_1), \dots, I^+(x_k) \}.$$



$\Rightarrow \exists\, i \in \{1, \dots, k\}$ s.t.

$$x_i \in I^+(x_i)$$

$\Rightarrow x_i \in \mathfrak{C}.$

Lorentzian separation

(N, g^0) conn. Riem.

$d_{g^0}(x,y) =$

$= \inf \{ L[\gamma] : \gamma_x \text{ to } y \}$

$\rightsquigarrow (N, d_{g^0})$ is a metric space.

(M,g) spacetime

$$\tau_g(x,y) = \begin{cases} \sup \tau[\gamma], & \gamma \text{ causal from } x \text{ to } y \text{ if } y \in J^+(x) \\ 0 & \text{if } y \notin J^+(x) \end{cases}$$

$\downarrow$

NOT a distance function.

(M,g) is totally vicious $\Leftrightarrow \tau_g = +\infty$

$\tau_g(x,y) > 0 \Leftrightarrow y \in I^+(x)$

$\tau_g(x, \cdot) = 0$ on $M \setminus I^+(x)$

If $0 < \tau_g(x,y) < +\infty \Rightarrow \tau_g(y,x) = 0$.

"timelike diameter" $\operatorname{diam}_{\tau_g}(S) =$

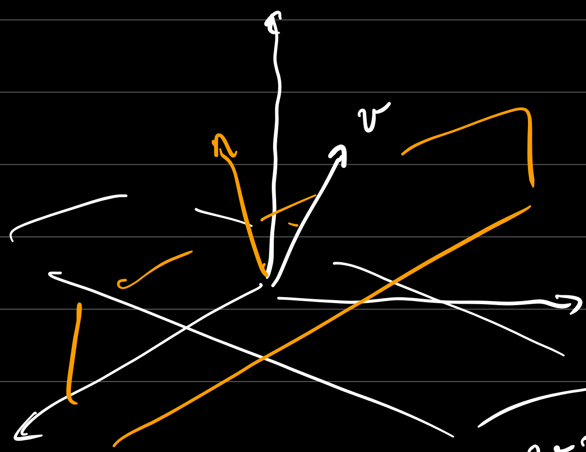
$= \sup\{\tau_g(x,y) : x,y \in S\}.$

A is achronal $\Leftrightarrow$ $\operatorname{diam}_{\tau_g}(A) = 0$.

$(\mathbb{R}\times N, -dt^2 \oplus g^0)$

$$\tau_g((t_0,x_0),(t_1,x_1)) = \sqrt{(t_1-t_0)^2 - d_{g^0}(x_0,x_1)^2}.$$

<u>Simultaneity is not absolute.</u>



$\mathbb{L}^3$

v timelike

$v^\perp =$ "restspace of v"