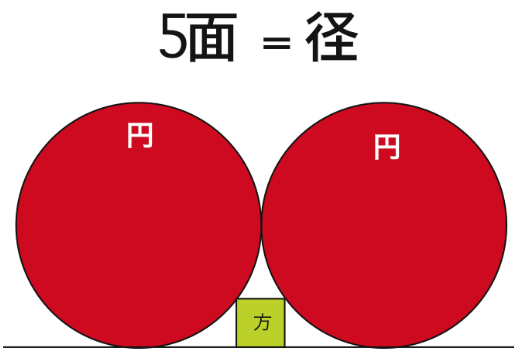
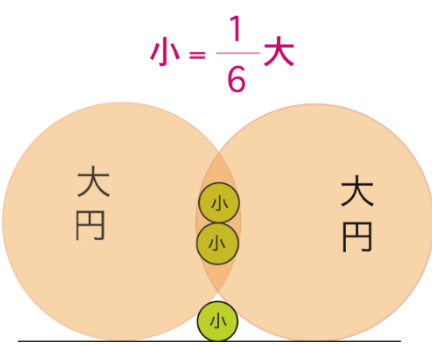
The following is a selection of problems posted in Facebook group *Wasan nau!* (Traditional Japanese Mathematics Now!) arranged very roughly in order of difficulty. I have provided English translations and solutions. —JMU



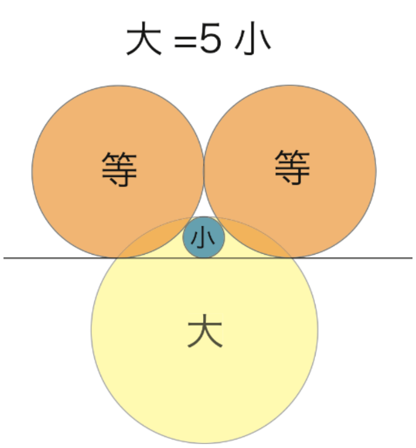
Prove that 5 times the side of the square is the diameter of the congruent red circles.

Solution



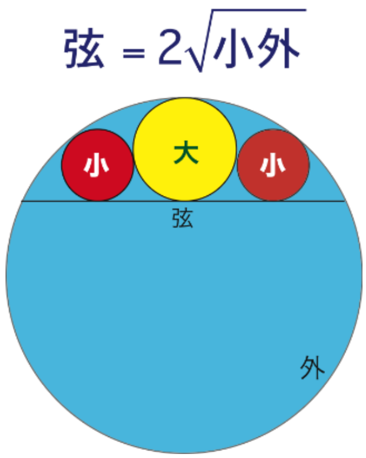
Prove that the diameters of the equal green circles are $\frac{1}{6}$ the diameters of the large orange circles.

Solution



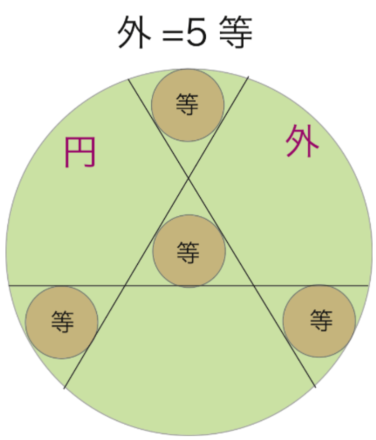
Prove that the diameter of the large yellow circle is 5 times the diameter of the small blue circle. The two brown circles have equal diameters.

Solution



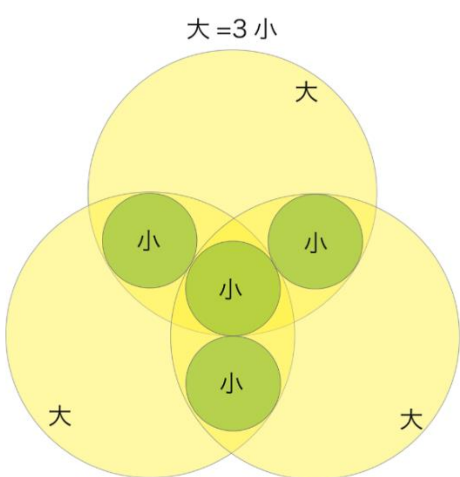
Prove that the chord is twice the geometric mean of the yellow and red diameters.

Solution



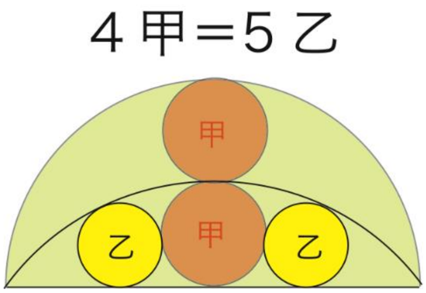
Prove that the diameter of the large green circle is 5 times the diameter of the small brown congruent circles.

Solution



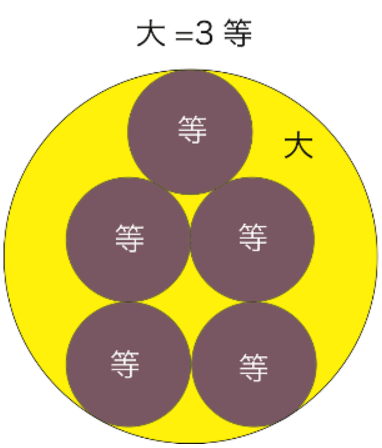
Prove that the diameter of the three yellow congruent circles are 3 times the diameter of the four green congruent circles.

Solution



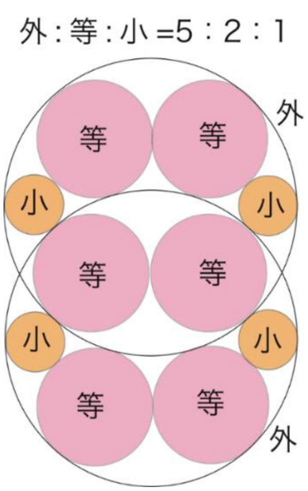
Prove that the ratio of the diameter of congruent brown circles to the diameter of the congruent yellow circles is 4 : 5.

Solution



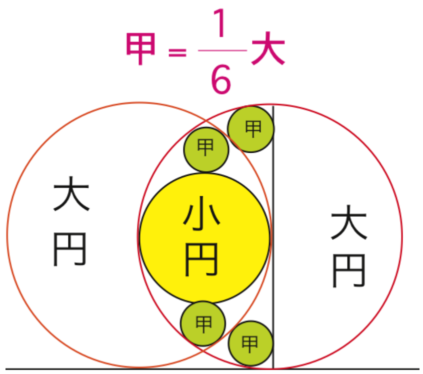
Prove that the diameter of the yellow circle is three times the diameter of any of the five congruent circles

Solution



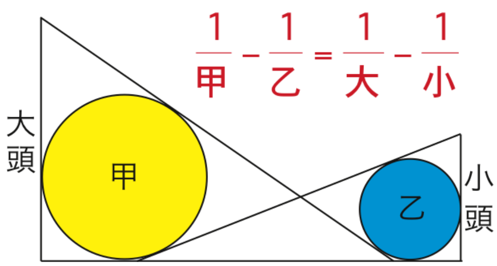
Prove that the ratios of the diameters of the two uncolored congruent circles, the six pink congruent circles, and the four orange congruent circles are 5 : 2 : 1.

Solution



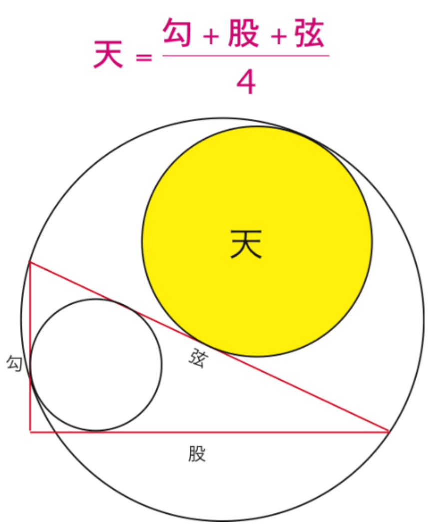
Prove that the diameters (radii) of all four green circles are the same and equal to $\frac{1}{6}$ the diameters of the uncolored circles.

Solution



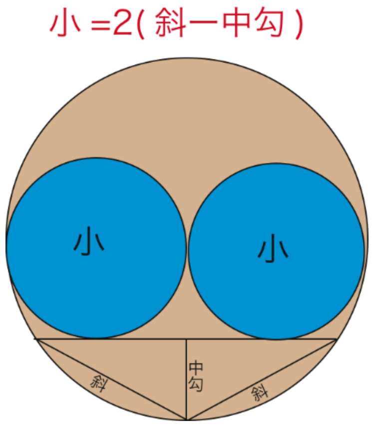
Two right triangles are positioned so that the hypotenuse of each touches the incircle of the other and two legs lie in the same line. Prove that difference of the reciprocals of the incircle diameters equals the difference of the reciprocals of the triangle legs not on the shared line.

Solution



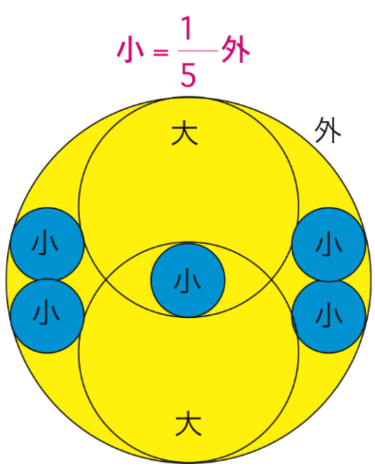
Prove that the diameter of the yellow circle, which touches the midpoint of the hypotenuse of the right triangle, is one quarter of its perimeter.

Solution



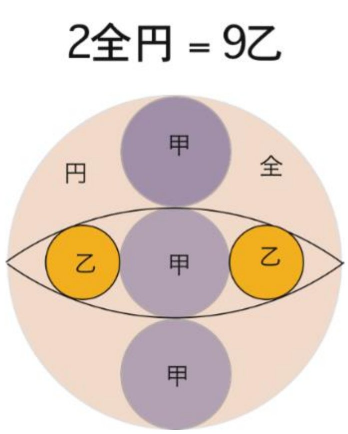
Prove that, if the blue circles have diameter d and the congruent right triangles have sides $a < b < c$, then $d = 2(c - a)$.

Solution



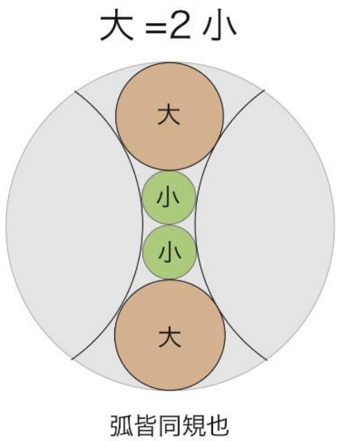
Prove that the diameters (radii) of the congruent blue circles are $\frac{1}{5}$ the diameter (radius) of the outermost yellow circle.

Solution



Prove that the ratio of the diameter of the large pink circle to the diameter of the congruent orange circles is 2 : 9.

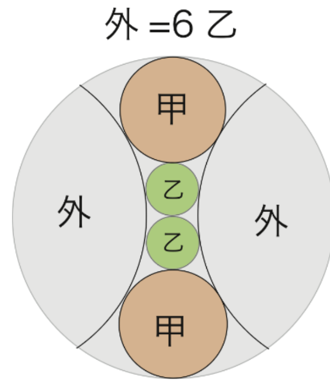
Solution

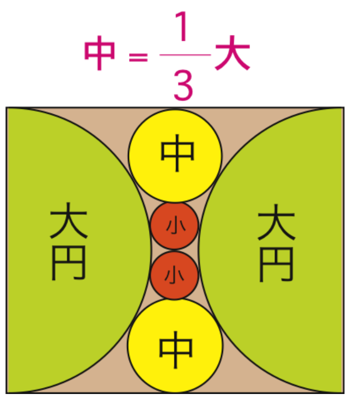


Two problems combined:

The diameters of the gray circle and the impinging arcs are the same. Prove the brown circles are twice as large as the green circles, and that gray circle is 6 times as large.

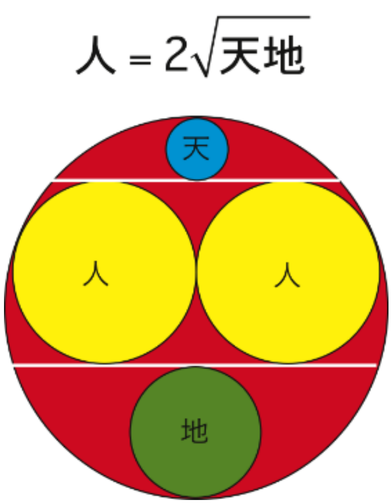
Solution





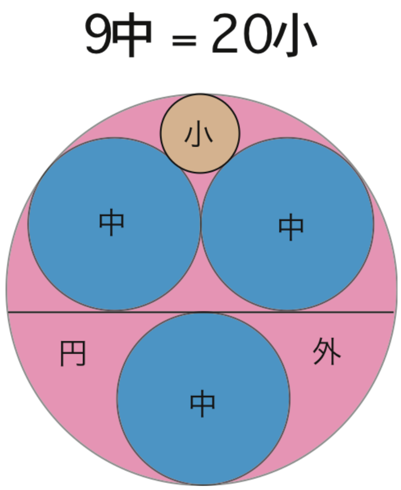
Prove that the diameter of the yellow circles is $\frac{1}{3}$ the diameters of the green circles.

Solution



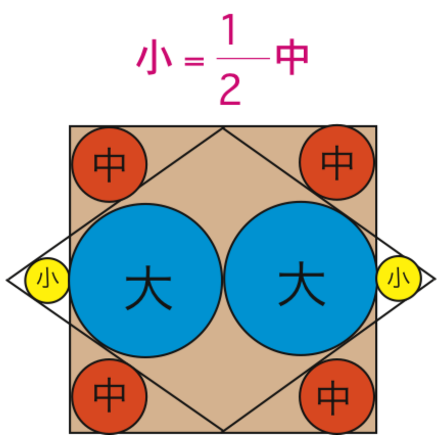
Prove that the diameter of the congruent yellow circles is twice the geometric mean of the diameters of the blue and green circles.

Solution



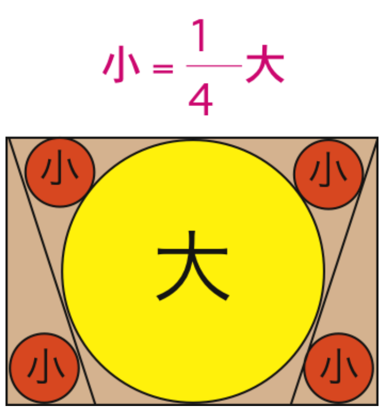
Prove that the ratio of the diameters of the congruent blue circles to the diameter of the brown circle is 9 : 20.

Solution



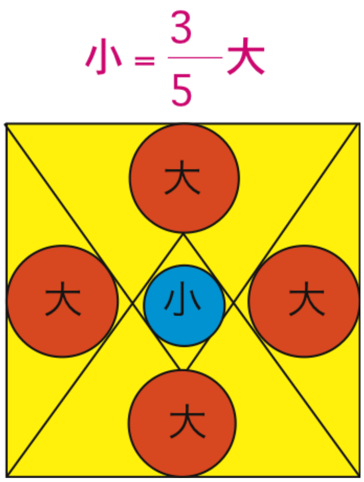
The brown quadrilateral is a square. Prove that the diameters of the yellow circles are half the diameters of the red circles.

Solution



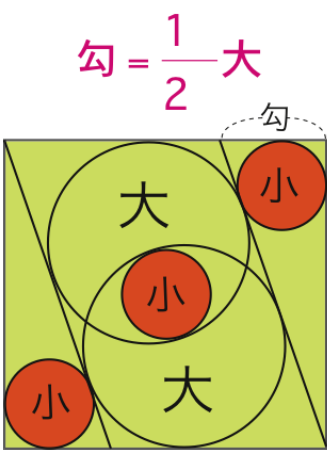
The four orange circles have equal diameters. The diagonal lines touch the yellow circle and two orange circles each. Prove that an orange diameter is a quarter of the yellow diameter.

Solution



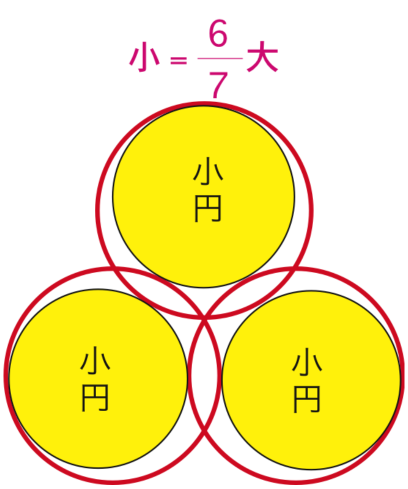
The yellow quadrilateral is a square. Prove that the diameters of the blue circle is 3/5 of the diameters of the red circles.

Solution



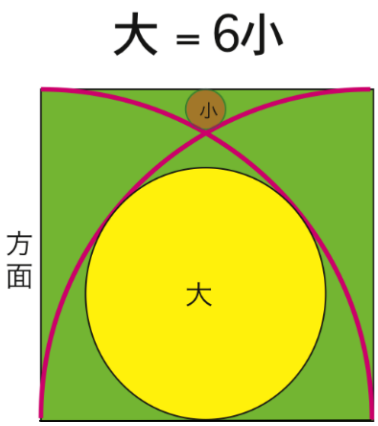
The green quadrilateral is a square and the red circles have equal diameters. Prove the short legs of the right triangles equal the radii of the large circles.

Solution



The three yellow circles are centered on the vertices of equilateral triangle. The three red circles are internally tangent to them and concur at the center of the triangle. Prove that $r = \frac{6}{7}R$.

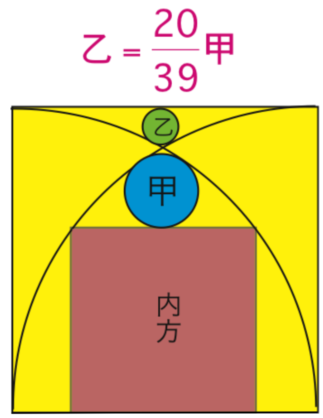
Solution

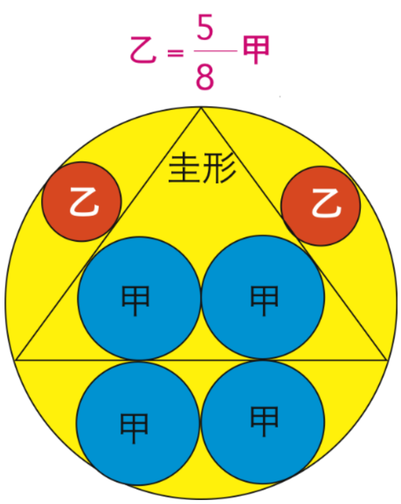


Two problems combined:

Prove that the yellow circle in the green square has a diameter 6 times that of the brown circle. Prove that the ratio of the diameters of the green and blue circles in the yellow square is 20 : 39.

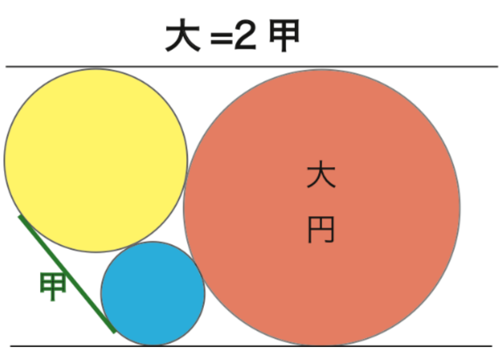
Solution





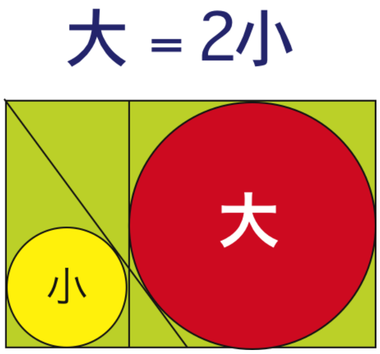
Two equal red circles touch the midpoints of the sides of an isosceles triangle and its circumcircle. Two equal blue circles touch its base and sides, and two more touch its base and circumcircle. Prove the that diameters of the red circles are $\frac{5}{8}$ the diameters of the blue circles.

Solution



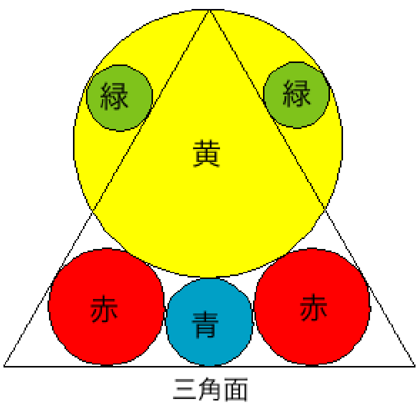
Prove that twice the length, from touch point to touch point, of the common external tangent of the yellow and blue circles is the diameter of the orange circle.

Solution



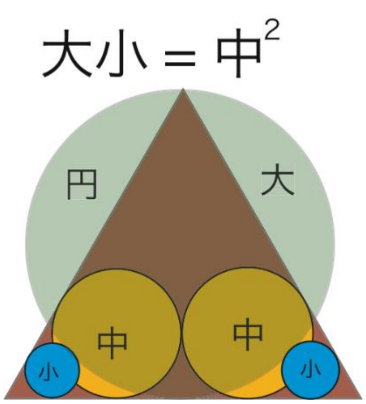
A yellow circle of radius y is the incircle of the a right triangle. A red circle of radius r touches its hypotenuse and one of its legs extended, as shown; the right triangle thus has legs $2r > m$. Prove that $2y = r$.

Solution



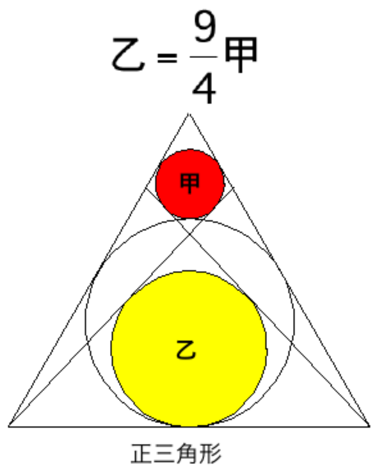
Yellow, blue, and equal green and red circles are arranged relative to an equilateral triangle as shown. Express the diameters of the green, blue, and red circles and the linear dimensions of the triangle in terms of the diameter of the yellow circle.

Solution



Prove that the product of the diameters of the large gray and small blue congruent circles is the square of the diameter of the congruent yellow circles.

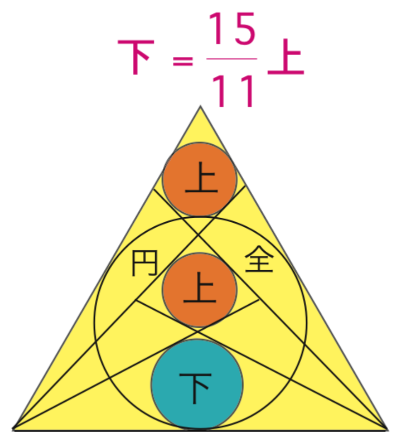
Solution

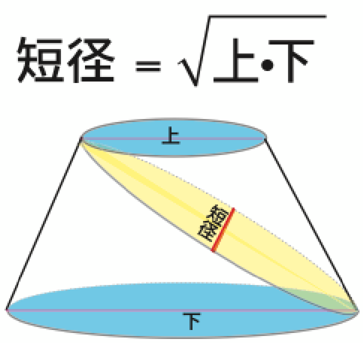


Two problems combined:

Both triangles are equilateral, and the uppermost circle touches the incircle. Prove that the diameter of the yellow circle is 9/4 the diameter of the red circle, and that the diameter of the blue circle is 15/11 the diameters of the orange circles, which are equal.

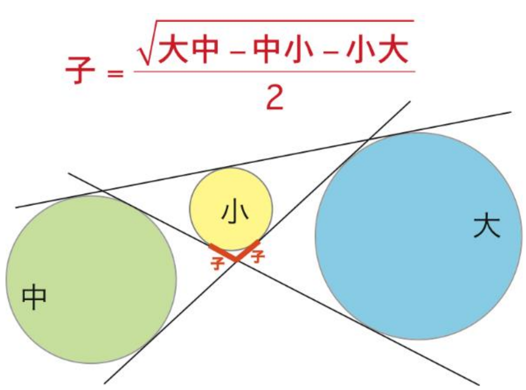
Solution





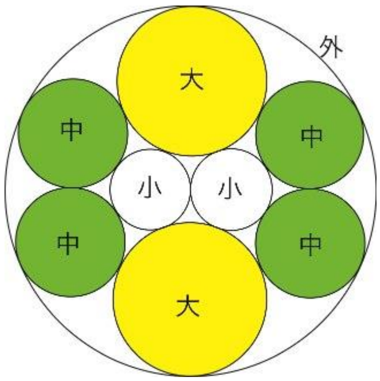
A right circular cone is intercepted by a plane making an elliptical section. Two planes orthogonal to the axis of the cone pass through the vertices of the ellipse, forming a frustum. Prove that the axis of the ellipse marked in red is the geometric mean of the diameters of the bases of the frustum.

Solution



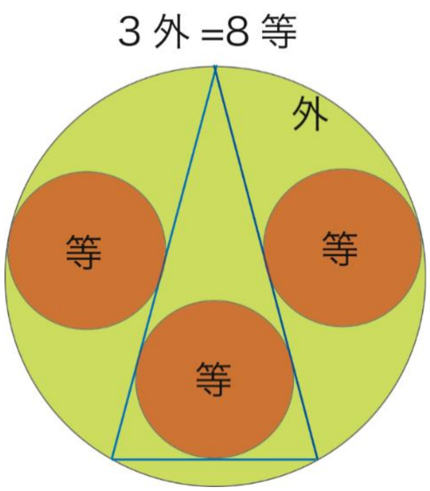
Let the diameters of the green, blue, and yellow circles be p, q, r, respectively. Prove the the red line segments are $\frac{1}{2}\sqrt{pq - pr - pq}$.

Solution



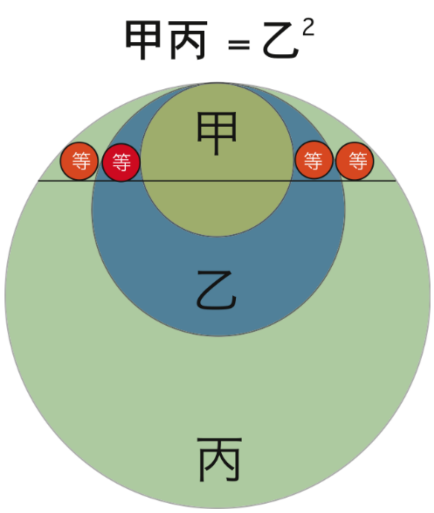
Prove that the diameter of the large uncolored circle is the sum of the diameter of the one of the yellow circles and two of the green circles.

Solution



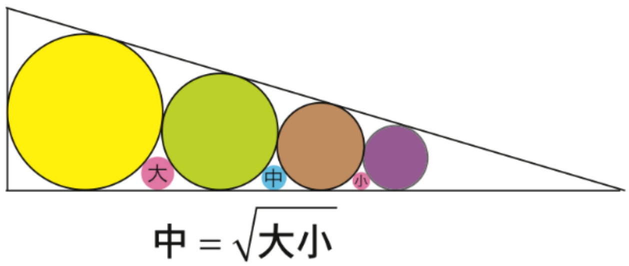
Prove that the ratio of the diameter of the green circle to the diameters of the three congruent brown circles is 3 : 8.

Solution



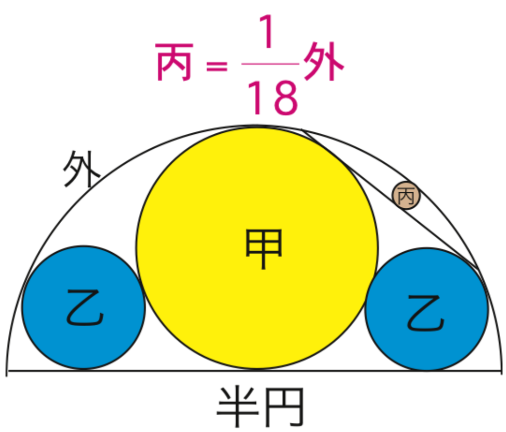
Given this figure with red circles of equal diameter, prove that the product of the diameters of the large and small green circles is the square of the diameter of the blue circle.

Solution



Prove that the diameter of the light blue circle is the geometric mean of the diameters of the two pink circles.

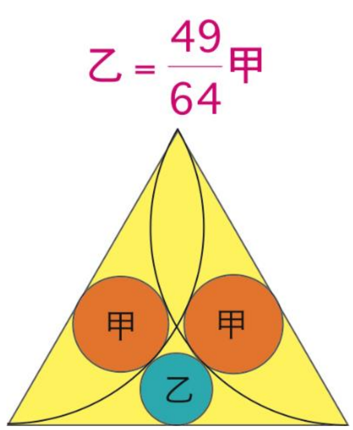
Solution



The circle inscribed in a circular segment touching the midpoint of its chord is its SAGITTAL CIRCLE. A circle touching the chord, arc, and sagittal circle of a circular segment is a SATELLITE CIRCLE of the segment.

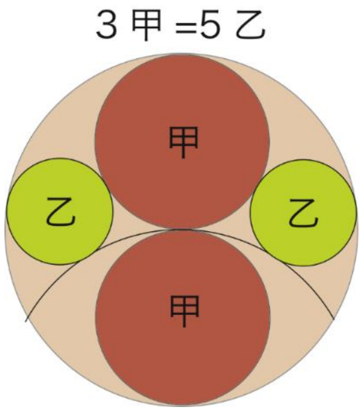
Given the segment defined by the external bitangent to the sagittal and one satellite circle of a semicircular segment, prove that the diameter of its sagittal circle is $\frac{1}{18}$ the diameter of the semicircle.

Solution



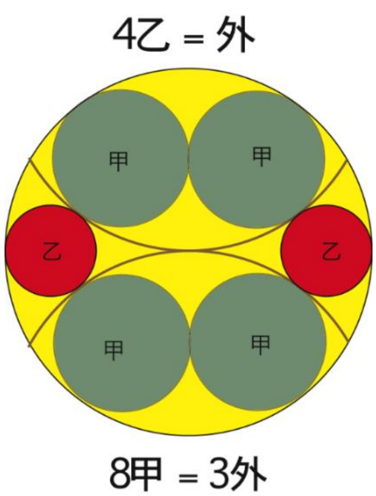
Prove that the ratio of the diameters (so too, the radii) of the orange and blue circles is $\frac{49}{64}$.

Solution



Prove that the ratio of the diameters of the brown circles to the diameters of the green circles is 3 : 5. N.B. It is understood that the radius of the arc is the same as the diameter of one of the brown circles.

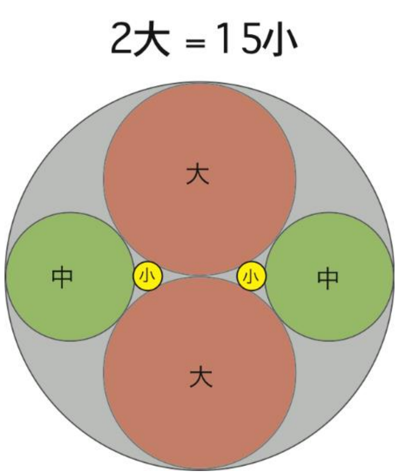
Solution



Two problems combined:

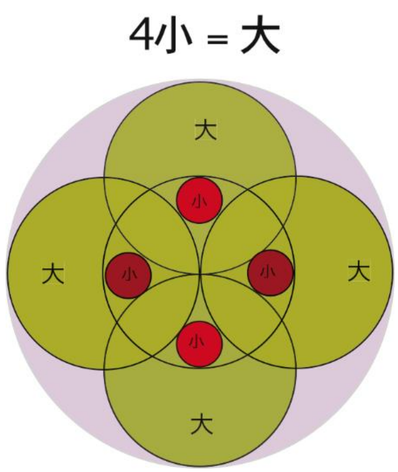
Prove that the ratios of the diameters of the red, green, and yellow circles are 12 : 8 : 3.

Solution



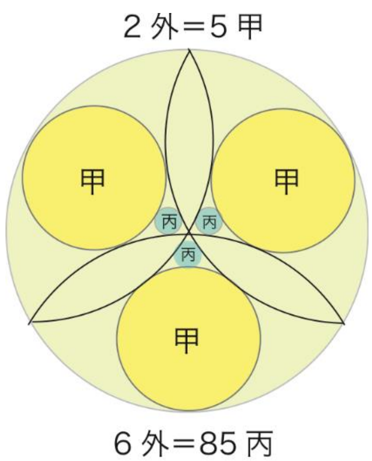
Prove that the ratio of the diameters of the brown and yellow circles is 2 : 15. (One proves *en passant* that the ratio of the diameters of the brown and green circles is 2 : 3.)

Solution



Prove that the ratio of the diameters of the red circles to those of the green is 4 : 1.

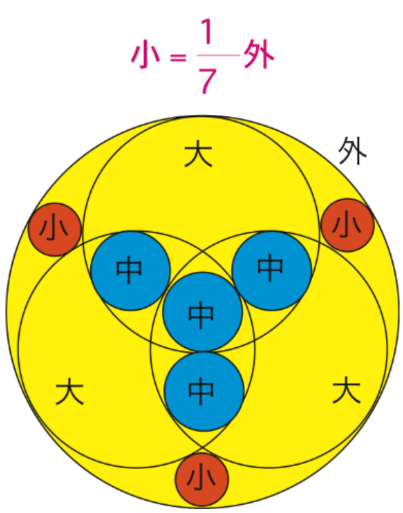
[Solution]



Two problems combined:

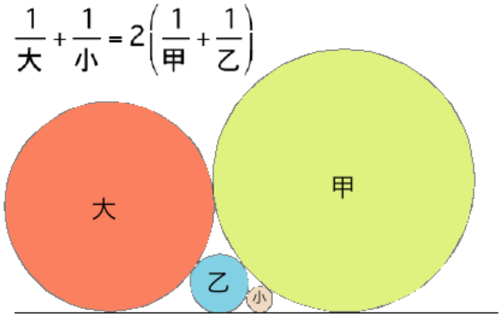
Prove that the ratio of the diameter of the outermost circle to those of the yellow circles is 2 : 5, and to those of the small blue circles is 6 : 85. (N.B. the black arcs subtend sides of an equilateral triangle.)

[Solution]



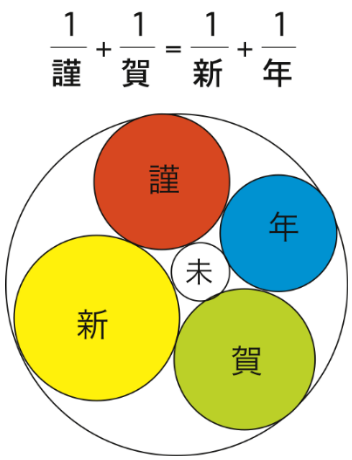
Given three congruent triplets of circles—blue, red, and uncolored—arranged around a fourth blue circle that shares the center of a circumscribing yellow circle, as shown, prove that the diameters of the red circles are $\frac{1}{7}$ the diameter of the yellow circle.

[Solution]



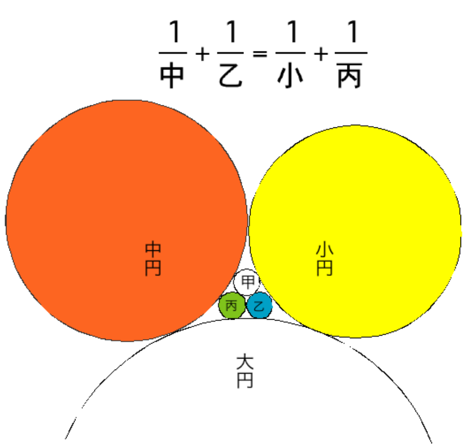
Given four circles all touching line *l* and each other as shown, prove that the sum of the reciprocals of the diameters (resp. radii) of the red and tan circles is twice the sum of the reciprocals of the diameters (resp. radii) of the green and blue circles.

Solution



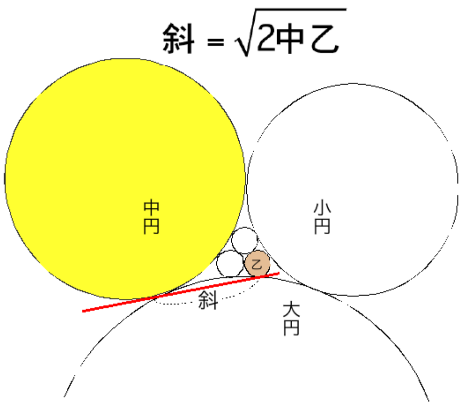
Prove that the sum of the reciprocals of the diameters of the orange and green circles equals the sum of the reciprocals of the diameters of the yellow and blue circles.

Solution



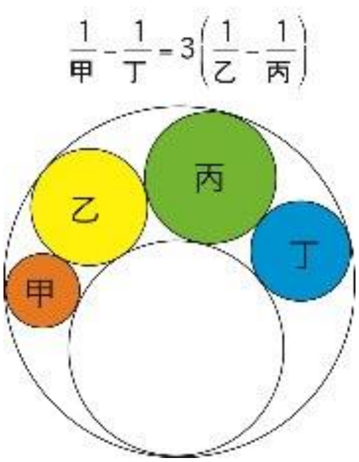
Prove that the sum of the reciprocals of the diameters of the orange and blue circles equals the sum of the reciprocals of the diameters of the yellow and green circles. (If this is true, then, by symmetry, the sum of the diameters of the uncolored circles is the same.)

Solution



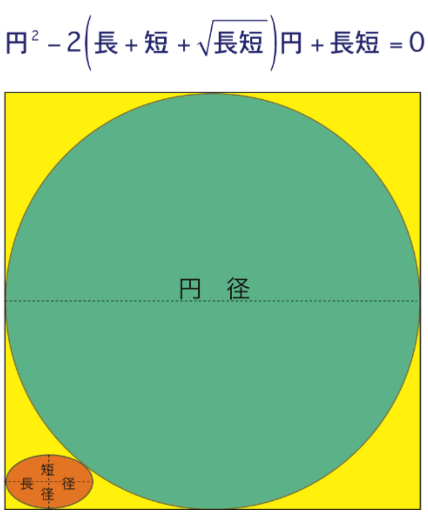
Prove that the length of the red tangent between the touch points of the yellow and brown circles is the square root of twice their product.

Solution



Prove that difference of the reciprocals of the diameters of the orange and clue circles is 3 times the difference of the reciprocals of the diameters of the yellow and green circles.

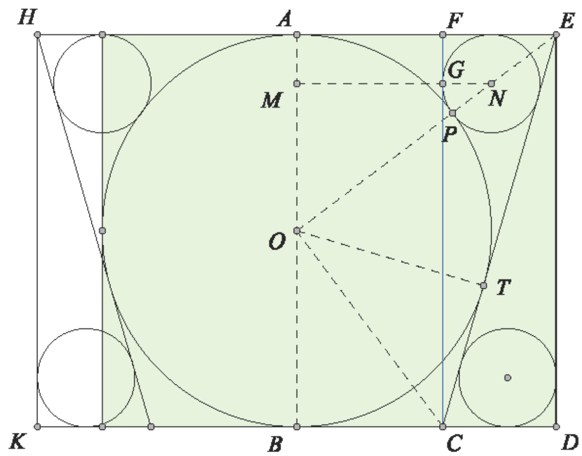
Solution



If the diameter of the green circle is d and the major, minor axes of the orange ellipse are u, v, respectively, prove that $d^2 - 2(u + v + \sqrt{uv})d + uv = 0$.

Solution

It is easy to see that $OBC \cong OTC$, $OAE \cong OTE$, and that these four right triangles and OMN (not to mention COE) are all similar. Denote $OA\ (= OB = OT = OP)$ as R, $AM\ (= FG = GN = NP)$ as r, and $MG\ (= AF = BC = CT)$ as t. $OBC \sim OMN$, so $\frac{BC}{OB} = \frac{OM}{MN} \Leftrightarrow \frac{t}{R} = \frac{R-r}{t+r}$.



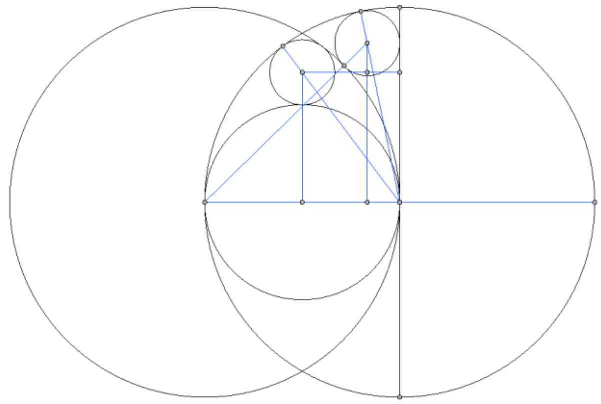
Adding 1 to each side and simplifying, $R = t + r$ so, in fact, $OBC \cong OMN$.

In OMN, $MN^2 = ON^2 - OM^2$ or $(t+r)^2 = (R+r)^2 - (R-r)^2$, so $R^2 = 4Rr$ or $R = 4r$.

Going a step further, since $R = t + r$, $t = 3r$, so the similar right triangles are all $3 : 4 : 5$. Therefore, if $R = 3x$, $AE = 4x$ and the ratio of the sides of the shaded rectangle are $6 : 7$, which is the claim of a less symmetrical but slightly more challenging version of this problem.

Back to problem

Say the radii of the circles depicted are $a > b > c$. Looking at the yellow circle and one of the green circles tangent to it (and internally to the two uncolored circles), we have $(a-c)^2 = b^2 + (b+c)^2$, or $a^2 - 2ac = 2b^2 + 2bc = 2b(b+c)$. But $2b = a$. Therefore $a^2 - 2ac = a\left(\frac{a}{2} + c\right)$, or $2a - 4c = a + 2c$. Hence $a = 6c$.



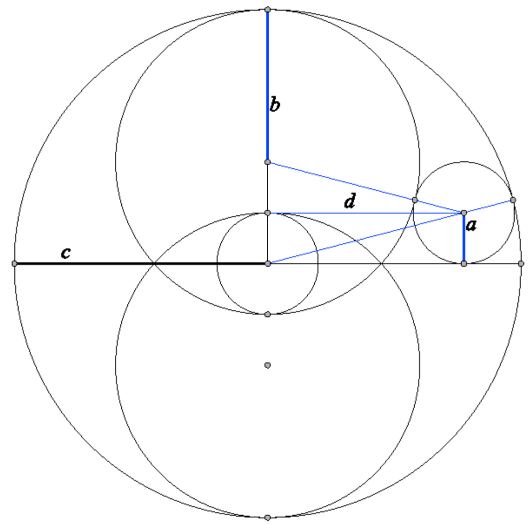
Now look at one of the green circles touching the diameter of the uncolored circle that it touches internally and the other uncolored circle externally: $(a+c)^2 - (a-c)^2 = (a-c)^2 - c^2$, so $a^2 + 2ac = 2(a-c)^2 = 2a^2 - 4ac + 2c^2$ or $0 = a^2 - 6ac$, and again $a = 6c$. Therefore all the green circles have the same radius, viz. $c = \frac{1}{6}a$.

[Back to problem]

We see at once that $c = 2b - a$.

We only a bit more effort, we see that $(c - a)^2 - a^2$ and $(b + a)^2 - (b - 2a)^2$ are both equal to d^2.

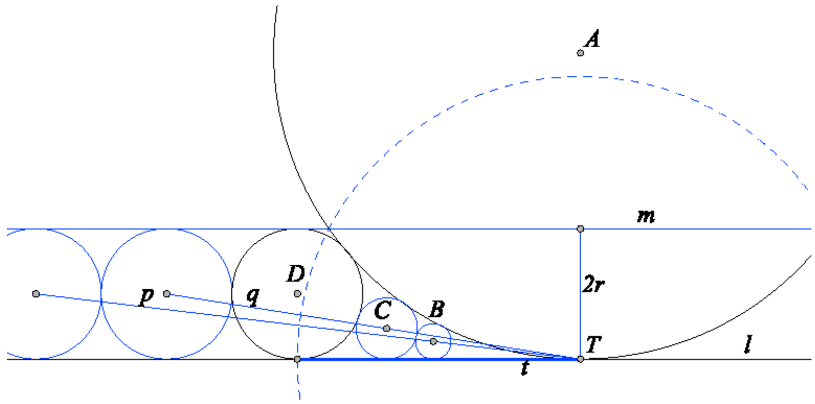
Substituing $\frac{a+c}{2}$ for b in the second expression, equating it with the first, and simplifying, we get $5ac = c^2$ or $a = \frac{1}{5}c$.



Back to problem

Add the tangent m to circle $(D)r$ parallel to l and circle $(T)t$ orthogonal to (D), where T is the touch point of $(A)a$ on l.

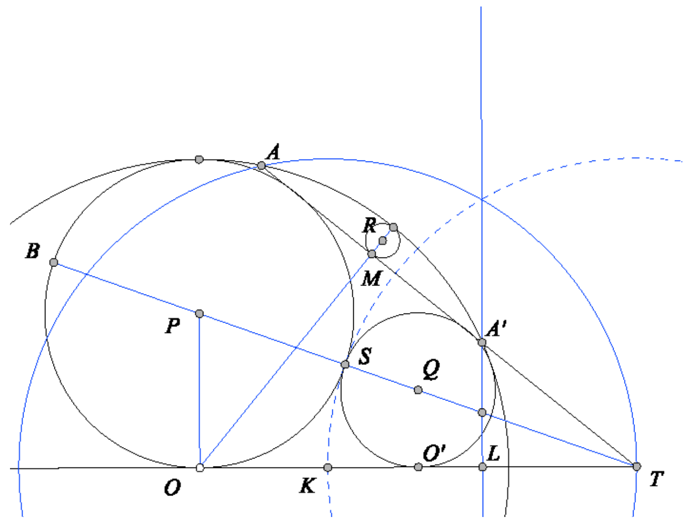
With respect to (T), (A) is the inverse of m, (D) is its own inverse, and the inverses of $(B)b$, $(C)c$ are circles congruent with (D) forming a contact chain between l and m as shown.



Now $p^2 = (t+4r)^2 + r^2$ and $q^2 = (t+2r)^2 + r^2$. Since (B) and (C) are the images of circles of radius r with centers at distances p and q away from T, $b = \frac{rt^2}{p^2-r^2}$ and $c = \frac{rt^2}{q^2-r^2}$. Hence $\frac{1}{b} + \frac{1}{r} = \frac{2(8r^2+4rt+t^2)}{rt^2}$ and $2\left(\frac{1}{a}+\frac{1}{c}\right) = \frac{2(4ar^2+4art+at^2+rt^2)}{art^2}$. Substituting $2\sqrt{ar}$ for t, we get $\frac{2(a+2r+2\sqrt{ar})}{ar}$ for both expressions, so they are equal.

Back to problem

Label the figure as shown. Define $2OT = a, 2PO = b, 2QO' = c, 2RM = d$.



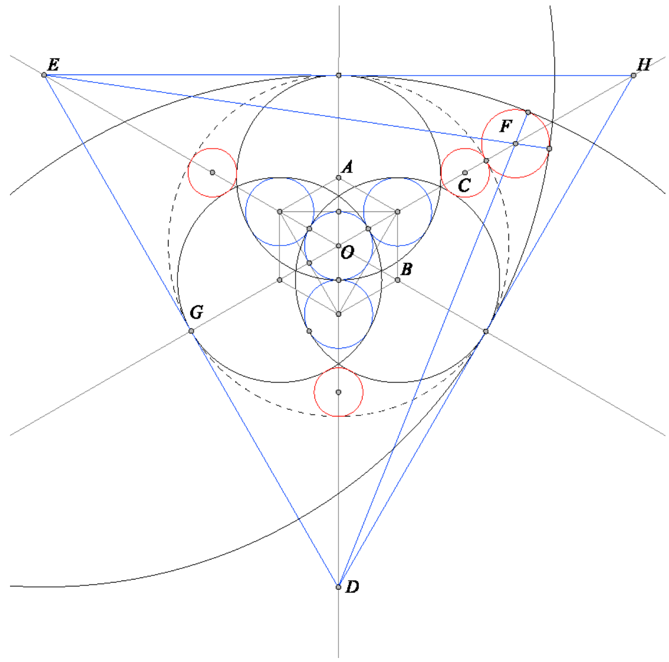
A well-known *wasan* theorem states that for any chord x in a circle of diameter y and a satellite circle with diameter z, $x = 2\sqrt{yz}$.* From this, we immediately know that $\frac{a}{2} = b = 2c$. $(T)S$ is orthogonal to $(O)T$, and (P), (Q) are inverses of one another in it. (This is why O' is so labeled.) From the orthogonality, $AT \cdot A'T = k^2$; from $BT \cdot B'T = k^2$, $(k + 2c)(k - c) = k^2$, we have $k = 2c$. Since $OO' = \sqrt{bc} = \frac{a}{2\sqrt{2}}$, $OT = \frac{a}{\sqrt{2}}$.

Now the inverse of line $A'L \perp OT$ is a circle through A and the two points in which $(T)k$ meets $A'L$. Its center H must therefore lie on OT, and its radius must be k; i.e. $HL = LT = \frac{a}{4}$. Since $\tan \angle PTO = \frac{a/2}{a/\sqrt{2}} = \frac{1}{2\sqrt{2}}$ and PT bisects $\angle ATO$, we find that $\tan \angle ATO = \frac{4\sqrt{2}}{7}$ and so $A'L = \frac{a\sqrt{2}}{7}$. Hence $A'T = \frac{9a}{28}$. Because $A'LT \sim OMT$, $\frac{OM}{a/\sqrt{2}} = \frac{a\sqrt{2}/7}{9a/28}$, so $OM = \frac{4a}{9}$. From this, $d = \frac{a}{2} - OM$ or $d = \frac{1}{18}a$.

* This is presented in this collection as an elementary problem in its own right.

Back to problem

Adding auxiliary lines (gray) that highlight the hexagonal symmetry of the figure, it is easy to see that, if the blue circles have radius r, then the centers of the uncolored circles are $2r$ from the center of symmetry, and have radius $3r$.* Hence, the yellow circle has radius $5r$.

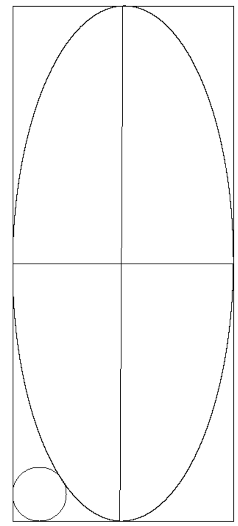


Using the yellow circle $(O)G$, invert (A), (B), and (C) into (D), (E), and (F), respectively. Because (A), (B) touch (C) externally, (D), (E) touch (F) internally. From OA, OB, OG, we find that the radii of (D), (E) are $\frac{25r^2}{9r^2-4r^2} \cdot 3r = 15r$.

Say the radius of (F) is x. Then in EGF, we have $EG^2 = (15r - x)^2 - (10r + x)^2 = 125r^2 - 50rx$. Now ED is the side of the equilateral triangle EDH with incircle $(O)G$; therefore $\frac{ED\sqrt{3}}{6} = 5r$ or $ED = \frac{30r}{\sqrt{3}}$, so $EG^2 = \frac{225r^2}{3} = 75r^2$. Substituting, $75r^2 = 125r^2 - 50rx$, or $x = r$. Therefore, the radius of (C) is $\frac{25r^2}{36r^2 - r^2} \cdot r = \frac{5}{7}r$. Since $OG = 5r$, this proves the proposition.

* This is presented in this collection as an elementary problem in its own right.

Back to problem



Scale the figure horizontally by v/u so that the square becomes a rectangle, the ellipse becomes a circle of diameter v, and the circle becomes an ellipse. We then apply the *wasan* theorem

$$mn + \sqrt{m^2n^2 - p^2q^2} - 2\left(m + n + \sqrt{mn - \sqrt{m^2n^2 - p^2q^2}}\right)d + d^2 = 0,$$

where $m > n$ are the sides of a rectangle in which an ellipse is inscribed and d is the diameter of the circle inscribed in the corner niche. The axes $p > q$ of the ellipse need not be parallel to the sides of rectangle. (A proof is available here.)

If they are parallel, as in the present case, then $p = m, q = n$, and

$$pq - 2(p + q + \sqrt{pq})d + d^2 = 0.$$

The lengths in the adjacent figure, to which the formula applies, correspond to lengths in the problem figure as follows: $d \leftrightarrow v, p \leftrightarrow d, q \leftrightarrow dv/u$. Hence, in the problem figure,

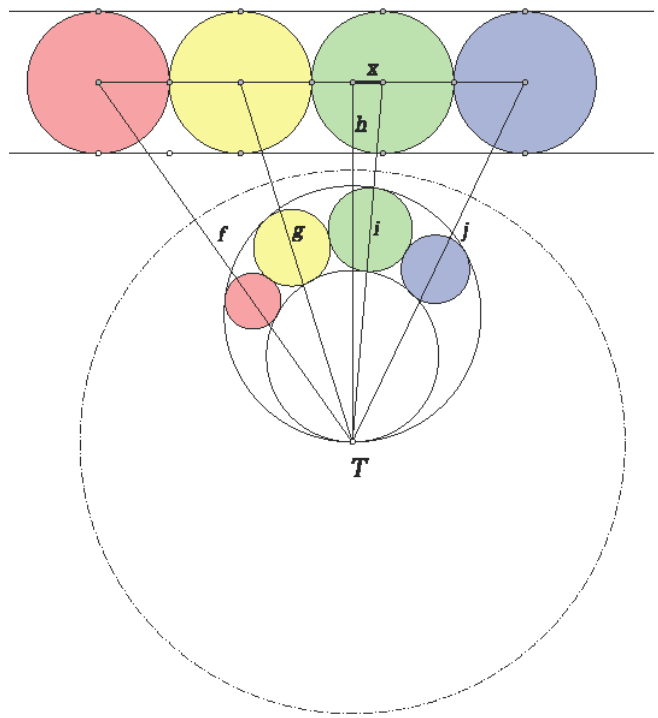
$$\frac{d^2v}{u} - 2\left(d + \frac{dv}{u} + d\sqrt{\frac{v}{u}}\right)v + v^2 = 0$$

Multiply through by u/v, note that $u\sqrt{\frac{v}{u}} = \sqrt{uv}$, and factor out d in the second term:

$$d^2 - 2(u + v + \sqrt{uv})d \ + uv = 0.$$

Back to problem

Consider the chain of four colored circles sandwiched between the two circles that touch at T in the figure. Inverting these in a circle (T), the circles touching at T become parallel lines and, since inversion preserves tangencies, the image circles have equal radii. Setting $a' = s' = m' = b'$ to 1, and dropping a perpendicular, h, from T to the line of centers of the images, the hypotenuses f, g, i, j of the right triangles with leg h are



$$f^2 = h^2 + (4 - x)^2,$$
$$g^2 = h^2 + (2 - x)^2,$$
$$i^2 = h^2 + x^2,$$
$$j^2 = h^2 + (2 + x)^2.$$

By a general theorem about the ratio of pre-image and image radii, we know that

$$\frac{a}{a'} = \frac{k^2}{|f^2 - a'^2|}, \qquad \frac{s}{s'} = \frac{k^2}{|g^2 - s'^2|}, \qquad \frac{m}{m'} = \frac{k^2}{|i^2 - m'^2|}, \qquad \frac{b}{b'} = \frac{k^2}{|j^2 - b'^2|}.$$

Therefore,

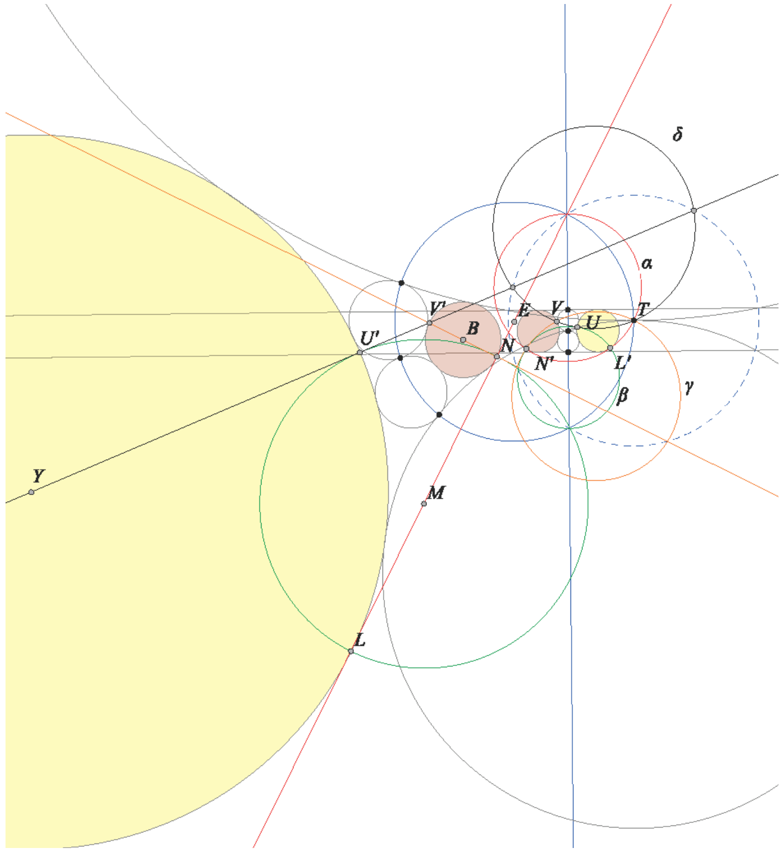
$$a = \frac{k^2}{h^2 + 15 - 8x + x^2}, s = \frac{k^2}{h^2 + 3 - 4x + x^2}, m = \frac{k^2}{h^2 - 1 + x^2}, b = \frac{k^2}{h^2 + 3 + 4x + x^2}.$$

Consequently, $\frac{1}{a} - \frac{1}{b} = \frac{12-12x}{k^2}$ and $\frac{1}{m} - \frac{1}{s} = \frac{4-4x}{k^2}$, so $\frac{1}{a} - \frac{1}{b} = 3\left(\frac{1}{m} - \frac{1}{s}\right)$.

Back to problem

If M is the midpoint of the bitangent segment LN, $(M)N$ is orthogonal to both (Y) and (B). Therefore, if $(M)N$ meets (Y) a second time in U', MU' is tangent to (Y) and YU' is tangent to $(M)N$.

Let BN produced meet (B) in V'. Provided that YU' passes through V', $MU'V'N$ is a cyclic quadrilateral and, since $MU' = MN$, a kite; therefore $(V')N$, with a radius equal to the diameter of (B), touches (Y) at U', which establishes $LN = 2\sqrt{2BN \cdot YL}$. The problem is proving that V' lies on YU' produced.



Placing the center of inversion T at the contact point of the two large uncolored circles, we invert them into two parallel lines. The other two uncolored circles invert into equal, externally tangent circles with centers on a perpendicular, each touching one parallel. The two colored circles invert into equal circles (Y') and (B'), each touching both parallels and the two smaller circles.

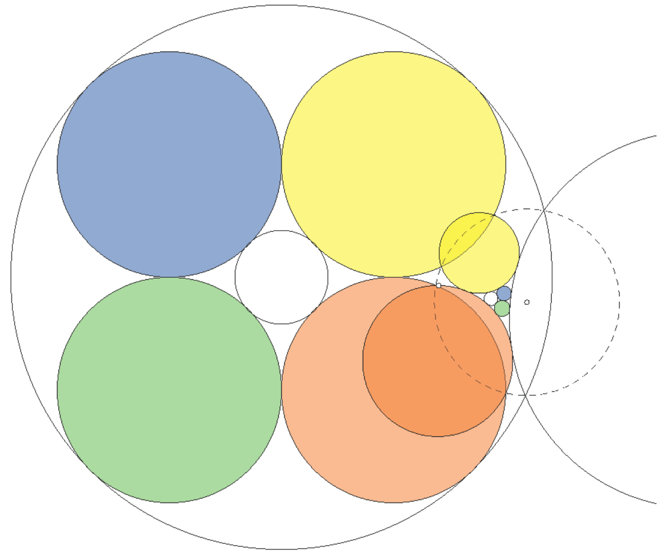
The three contact points of the uncolored circles other than T have inverses on the perpendicular to the parallel lines through the centers and contact points of the two uncolored image circles, and so are concyclic on some circle (E). E is the homothetic excenter of circles (Y) and (B) because the perpendicular is equidistant from Y' and B', the centers of the colored image circles. For that reason, E is collinear with L, M, N. It is this key geometric fact that makes this particular configuration of (Y), (B), and LN (up to scaling) in the problem special.

Line LN (red) and circle $(M)N$ (green) are orthongal to each other and (Y) and (B), so their images α, β are orthogonal to each other and to (Y') and (B'), which they meet in L' and N', respectively. Since LN is a line, α passes through T. Likewise, the image γ of line BN (orange) is a circle through T and N', orthogonal to α, and internaly tangent at N' to β, to which it is homothetic. The points U and V in which β and γ meet (Y') and (B') a second time are the inverses of U' and V', respectively. Circle $(T, U, V) = \delta$ touches β externally because it is orthogonal to (Y'), and its touches it at U because δ is the image of YU', which is tangent to $(M)N$ at U'.

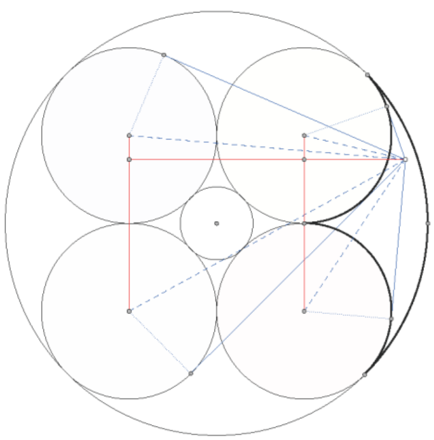
Now since V lies on δ, V' lies on YU', which is all that remained to be proven.

Back to problem

Since one problem figure can be transformed into the other (both can be inverted into the same annulus and four inscribed congruent circles), a separate proof isn't strictly necessary, but to illustrate the theorem that $\frac{r}{r'} = \frac{k^2}{L^2}$, where L is the length of a tangent from T to an image circle if the pre-image circle does not pass through T, here is another proof.



Let the radii of the orange, blue, yellow, and green pre-image circles be A, a, B, b, respectively. Letting $r' = 1$ for each colored image circle, $\frac{L^2}{k^2} = \frac{1}{r}$ for each pre-image, and if d is the distance from T to the center of an image circle, then $L^2 + 1 = d^2$ in each case. That is, $\frac{1}{r} = \frac{d^2-1}{k^2}$.

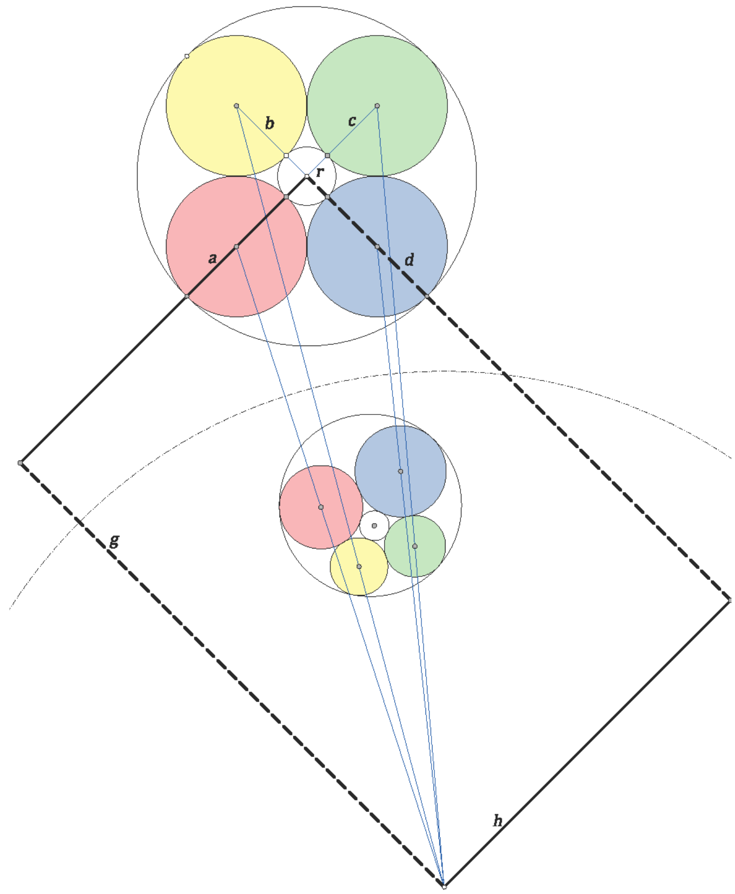


Therefore, $\frac{1}{A} + \frac{1}{a} = \frac{d_A^2-1}{k^2} + \frac{d_a^2-1}{k^2} = \frac{x_A^2+y_A^2+x_a^2+y_a^2-2}{k^2}$, where the x's and y's are the legs of the right triangles in red in the figure to the left. Since $x_A - x_a = 2 = y_A + y_a$, we have $x_A^2 + y_A^2 + x_a^2 + y_a^2 = 8 + 2x_A x_a - 2y_A y_a$, so $\frac{1}{A} + \frac{1}{a} = \frac{6+2x_A x_a - 2y_A y_a}{k^2}$. But also $x_A = x_b, x_a = x_B, y_A = y_B$, and $y_a = y_b$; therefore, $\frac{1}{B} + \frac{1}{b} = \frac{6+2x_B x_b - 2y_B y_b}{k^2} = \frac{6+2x_A x_a - 2y_A y_a}{k^2} = \frac{1}{A} + \frac{1}{a}$.

Back to problem

The problem solved here is actually the same as this one except for the location of the center T of inversion, which, for this case, is a point inside the largest circle but not on or within any of the smaller circles.

The problem figure can be inverted into an annulus with four inscribed congruent circles of radius 1 as shown.



In terms of right triangles, the distances from the center of inversion to the centers of the image circles are, respectively,

$$m_a^2 = g^2 + (h - r - 1)^2$$
$$m_c^2 = g^2 + (h + r + 1)^2$$
$$m_b^2 = h^2 + (g + r + 1)^2$$
$$m_d^2 = h^2 + (g - r - 1)^2.$$

Since in general $\frac{r'}{r} = \frac{k^2}{|m^2 - r^2|}$,

$$a' = k^2/[g^2 + (h - r - 1)^2]$$
$$c' = k^2/[g^2 + (h + r + 1)^2]$$
$$b' = k^2/[h^2 + (g + r + 1)^2]$$
$$d' = k^2/[h^2 + (g - r - 1)^2].$$

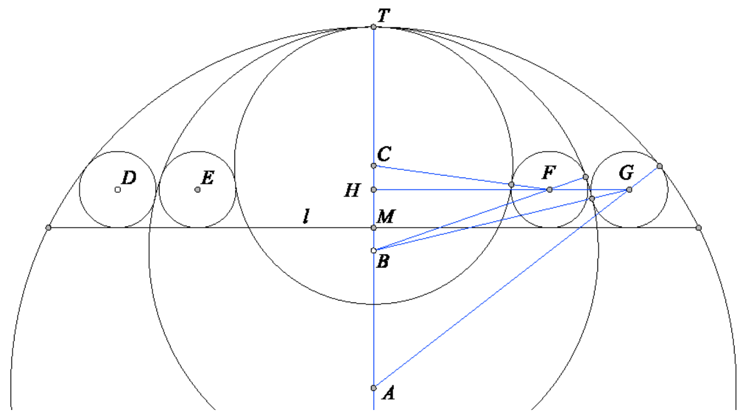
$$a' = k^2/[g^2 + h^2 + r^2 - 2hr - 2(h + r)]$$
$$c' = k^2/[g^2 + h^2 + r^2 + 2hr + 2(h + r)]$$
$$b' = k^2/[h^2 + g^2 + r^2 + 2gr + 2(g + r)]$$
$$d' = k^2/[h^2 + g^2 + r^2 - 2gr - 2(g + r)].$$

Hence

$$\frac{1}{a'} + \frac{1}{c'} = \frac{2(g^2 + h^2 + r^2)}{k^2} = \frac{1}{b'} + \frac{1}{d'},$$

which proves the proposition.

Back to problem



The problem can be solved without the use of inversion, as shown below. But to construct the problem figure exactly without the aid of a circle of inversion centered at T is not too easy.

Say that the diameters of $(A), (B), (C), (D)$ are a, b, c, d. It is given that the diameters of $(E). (F), (G)$ are also d.

Let $TM = p$.

Then in right triangles CHF and BHF, we have $CF^2 - BF^2 = CH^2 - BH^2$, or

$$\left(\frac{c}{2}+\frac{d}{2}\right)^2-\left(\frac{b}{2}-\frac{d}{2}\right)^2=\left(p-\frac{d}{2}-\frac{c}{2}\right)^2-\left(\frac{b}{2}-p+\frac{d}{2}\right)^2,$$

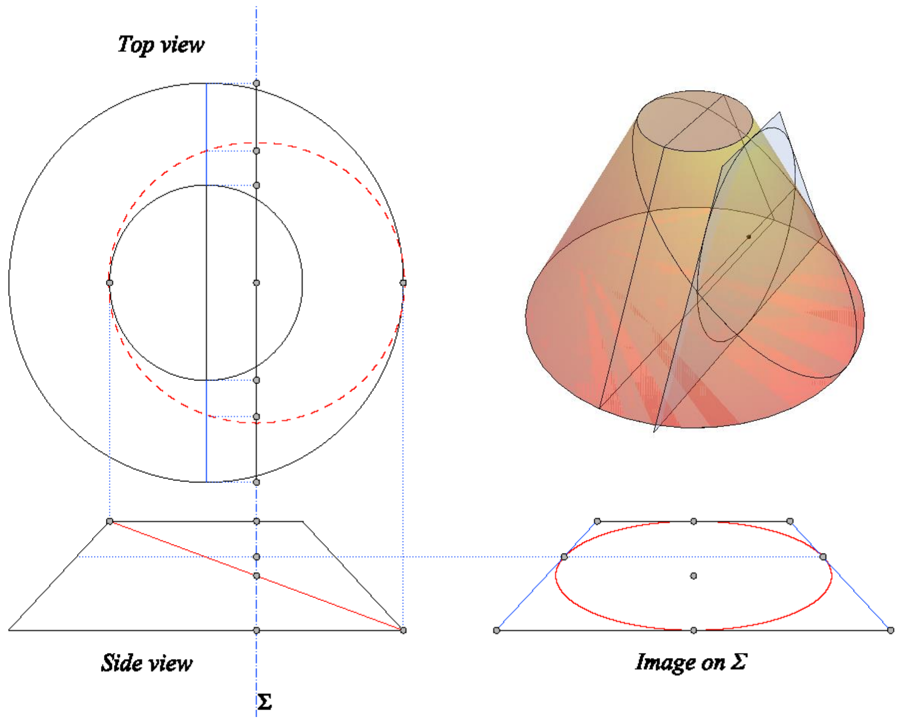
which simplifies to $\frac{b}{2}=\frac{p}{p-d}\cdot\frac{c}{2}$. Likewise, in right triangles, BHG and AHG, we have $AG^2 - BG^2 = AH^2 - BH^2$, or

$$\left(\frac{a}{2}+\frac{d}{2}\right)^2-\left(\frac{b}{2}-\frac{d}{2}\right)^2=\left(\frac{a}{2}-p-\frac{d}{2}\right)^2-\left(\frac{b}{2}-p+\frac{d}{2}\right)^2$$

or $\frac{a}{2}=\frac{p}{p-d}\cdot\frac{b}{2}$. Thus $\frac{p}{p-d}=\frac{b}{c}=\frac{a}{b}$.

Back to problem

LEMMA: *if the major axis of an ellipse inscribed in an isosceles trapezium is parallel to its bases, then it is the geometric mean of the bases.*

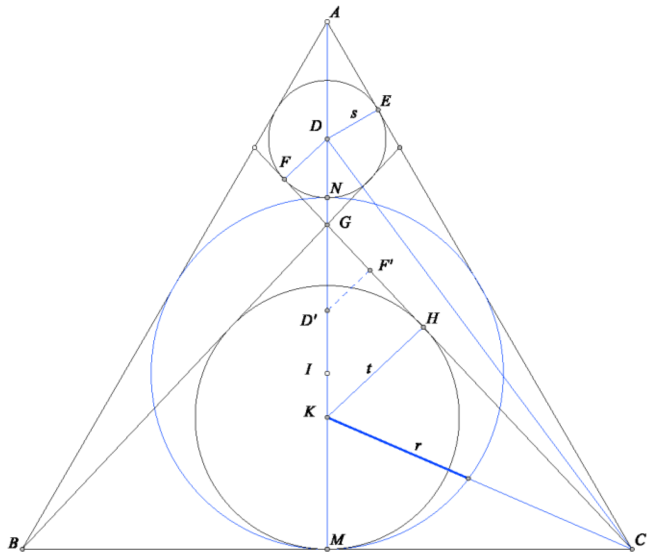


Let the plane of the ellipse be Π and consider the plane Σ, orthogonal to the bases of the frustum, in which the minor axis of the ellipse lies. Onto Σ, orthogonally project the ellipse and the trapezium bases. The images of bases are the diameters parallel to Σ, and the segments joining their endpoints form an isosceles trapezium circumscribing the image of the ellipse. The image is also an ellipse, and one of its axes (in the orthographic drawing, the major axis) is the minor axis of the pre-image. The claim in the problem follows from applying the lemma to the whole image.

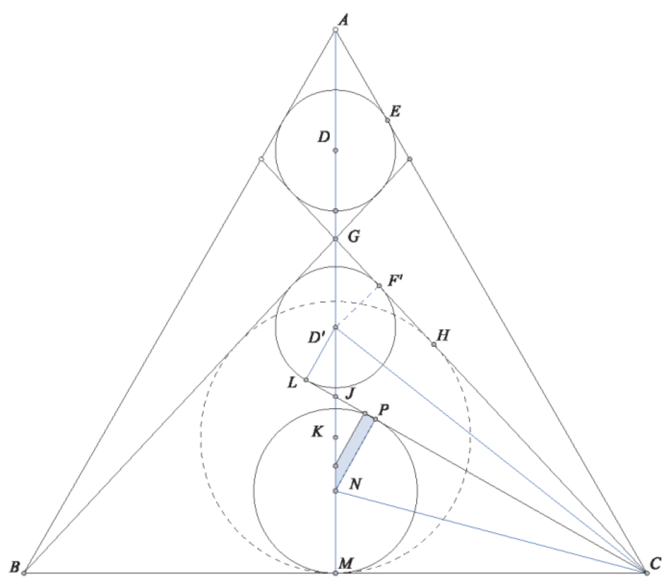
To help with visualization, a 3D figure is included in the uppr right corner of the orthographic drawing .

Back to problem

Observe that $3s = r$. Letting $BC = 2$, $FC = EC = 2 - s\sqrt{3}$. Therefore $FH = 1 - s\sqrt{3}$, and, since $DFG \sim KHF$, $FG = 1 - s\sqrt{3} - \frac{t}{s}FG$ and $GH = 1 - s\sqrt{3} - \frac{s}{t}GH$. Hence $FG = \frac{1-s\sqrt{3}}{1+t/s}$ and $GH = \frac{1-s\sqrt{3}}{1+s/t}$, and $KHG - DFG = KHF'D' = \frac{t}{2}\left(\frac{1-s\sqrt{3}}{1+s/t}\right) - \frac{s}{2}\left(\frac{1-s\sqrt{3}}{1+t/s}\right) = \frac{(1-s\sqrt{3})(t-s)}{2}$. Adding areas, $AMC = ADC + DFC - DFG + GCK + KMC = ADC + DFC + KHF'D' + 2KMC$. I.e. $\frac{\sqrt{3}}{2} = s + s\left(\frac{2-s\sqrt{3}}{2}\right) + \frac{(1-s\sqrt{3})(t-s)}{2} + t$.



Substituting $\frac{\sqrt{3}}{9}$ for s, this simplifies to $4t = \sqrt{3}$. Therefore $\frac{t}{s} = \frac{9}{4}$.



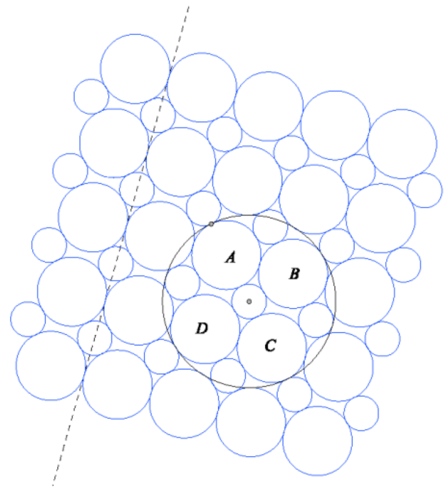
Given the foregoing result, we can express GF' and GH in terms of s, and so too GC, GM, LC, and LP. LJ and JP correspond to FG and GH in the previous proof, so, letting $NM = n$, the shaded trapezium is $\frac{LP(n-s)}{2}$. Adding areas as before,

$$\frac{GM}{2} = s\frac{GC}{2} + s\frac{LC}{2} + \frac{LP(n-s)}{2} + n.$$

Needless to say, it is easier to set s to $\frac{\sqrt{3}}{9}$ before replacing the lengths and doing the algebra. One obtains $n = \frac{5\sqrt{3}}{33}$ and so $\frac{n}{s} = \frac{15}{11}$.

Back to problem

Construct an indefinite number of circles of radius 2 with collinear centers, each touching two others. Circles of radius 3 with the same centers intersect at the centers—also collinear—of circles of radius 1, each touching two of the larger circles.

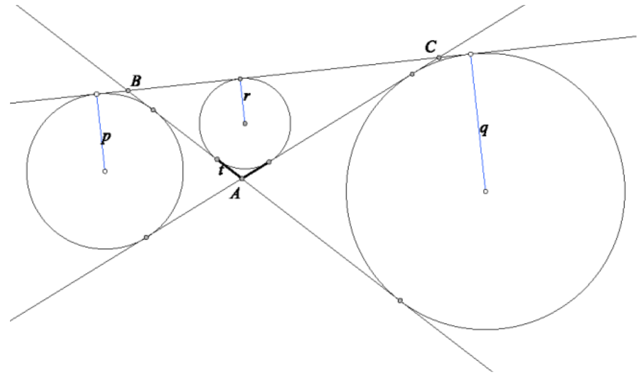


By reflecting circles across the lines of centers, we obtain a symmetric pattern as large as we please. The tangents to the large circles where they meet are parallel lines, such as the dashed line shown, separated by a distance of 4. The centers of all small circles lie on these lines.

Let O be the center of an arbitrary small circle and label the four large circles touching it A, B, C, D as shown. Then $(O)5$ is externally tangent to the four small circles touching A, B, C, D, and internally tangent to all of them and the two small circles touching A, D and B, C.

The problem figure consists of $(O)5$ plus the circles it touches internally and the similar image centered on one of the small circles adjacent to the central circle untouched by (O).

Back to problem



Given two circles of diameter $2p, 2q$ and triangle ABC formed by their internal bitangents and one of their external bitangents, say that the diameter of the incircle of ABC is $2r$. Then the segments t from the A to the touch points of the incircle on the internal bitangents have length $\frac{1}{2}\sqrt{4pq-4pr-4q}$. I.e., $t^2 = pq - pr - qr$.

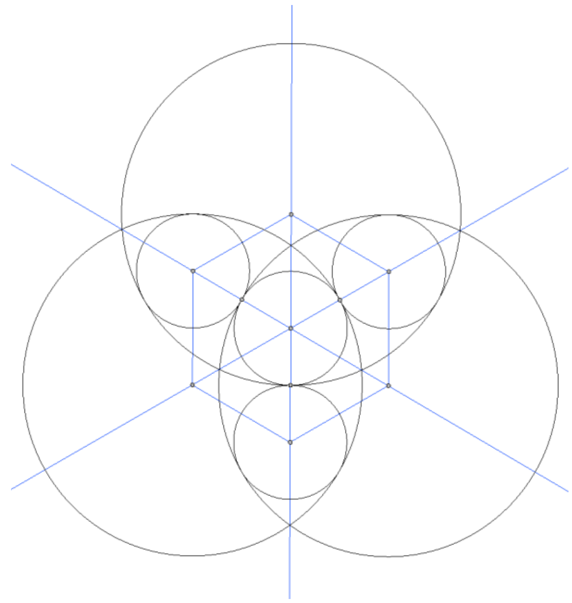
We use the formulae for the area K of ABC that make use of the inradius and exradii. Let $s = \frac{1}{2}(a+b+c)$. Then $t = s - a$ and $K = rs = p(s-b) = q(s-c) = \sqrt{st(s-b)(s-c)}$.

From $Kr = pr(s-b) = qr(s-c)$, we have $pr + qr = Kr\left(\frac{1}{s-b} + \frac{1}{s-c}\right) = Kr\left(\frac{2s-b-c}{(s-b)(s-c)}\right) = Kr\left(\frac{ast}{K^2}\right) = \frac{arst}{K} = at$. Therefore $pq - pr - qr = pq - at$.

From $Kq = pq(s-b)$ and $Kp = pq(s-c)$, we have $K(p+q) = pq(2s-b-c) = apq$. But in addition $K(p+q) = K\left(\frac{K}{s-b} + \frac{K}{s-c}\right) = K^2\left(\frac{ast}{K^2}\right) = ast$. Therefore $pq = st$ and $pq - at = (s-a)t = t^2$, which proves the claim.

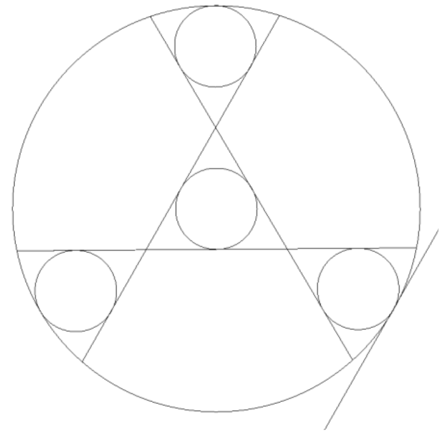
Back to problem

By synmmetry, the center of three green circles must be alternate vertices of a regular hexagon of side s. In a regular hexagon, the sides and the rays from the center to the vertices are equal; therefore, all four green circles are $\frac{s}{2}$. Furthermore, circles centered on the remaining vertices with radius $\frac{3s}{2}$ touch three of the four green circles internally and one externally.



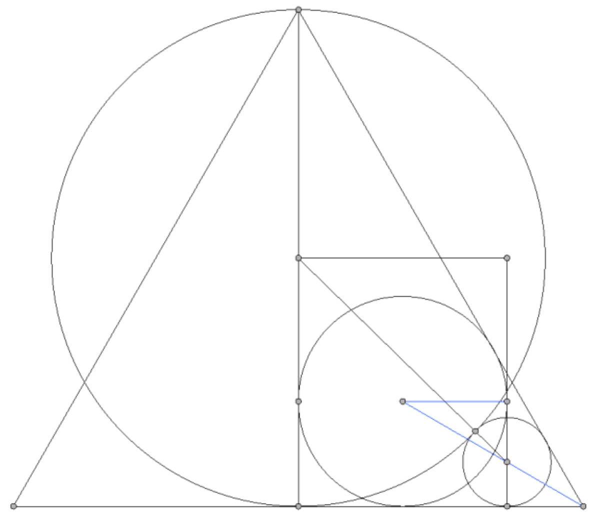
[Back to problem](#)

The median of an equilateral triangle of side s is $\frac{s\sqrt{3}}{2}$. The incenter trisects this segment; therefore, the inradius is $\frac{s\sqrt{3}}{6}$. Draw a tangent to the large circle where one of the small circles touches it. Its distance to the parallel chord is clearly $s\sqrt{3}$, so the large circle has radius $s\sqrt{3} - \frac{s\sqrt{3}}{6} = \frac{5s\sqrt{3}}{6}$, which is five times the radius of the smaller circles.



Back to problem

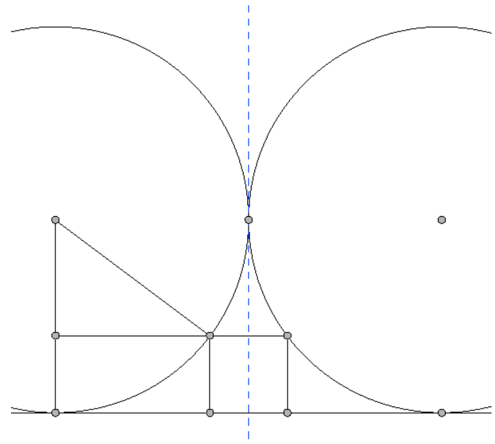
If the side the equilateral triangle is s, then the diameter of the large circle is is $a = \frac{s}{2}\sqrt{3}$, the length of the medians. The medium-sized circles are incircles of right triangles with sides $a, \frac{s}{2}$, and s, so they have diameter $b = a - \frac{s}{2}$. The small circles of radius c touch the large circle and two sides of the triangle.



The key observation is that the centers of the medium and small circle one each side of the median lie on the same vertex bisector. Since the also touch the same sides of the triangle, they are homothetic. Therefore, by similar right triangles, the small circle touches each side at distance b from the midpoint of the side. Hence $\left(\frac{a}{2} + \frac{c}{2}\right)^2 = b^2 + \left(\frac{a}{2} - \frac{c}{2}\right)^2$, which immediately gives $ac = b^2$.

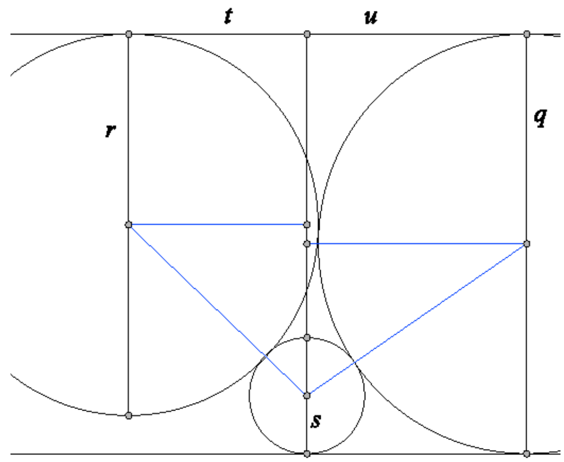
Back to problem

Let the radius of the red circles be r and the side of the square be s. In the right triangle shown, we have $(r-s)^2 + \left(r - \frac{s}{2}\right)^2 = r^2$. This has solutions $2r/5$ and $2r$, of which the first is germane to the figure.



Back to problem

LEMMA: *The distance between the touch points on a common tangent to two externally touching circles is the geometric mean of their diameters.* Hence $t + u = 2\sqrt{qr}$ or $t = 2\sqrt{qr} - u$. Squaring, $t^2 = 4qr - 4u\sqrt{qr} + u^2$.



On the other hand, $t^2 = (r+s)^2 - (2q - r - s)^2$ or $t^2 = 4q(r+s) - 4q^2$. Likewise, $u^2 = 4q(q+s) - 4q^2$.

Substitute these expressions for t^2 and u^2 in the first equation and simplify:

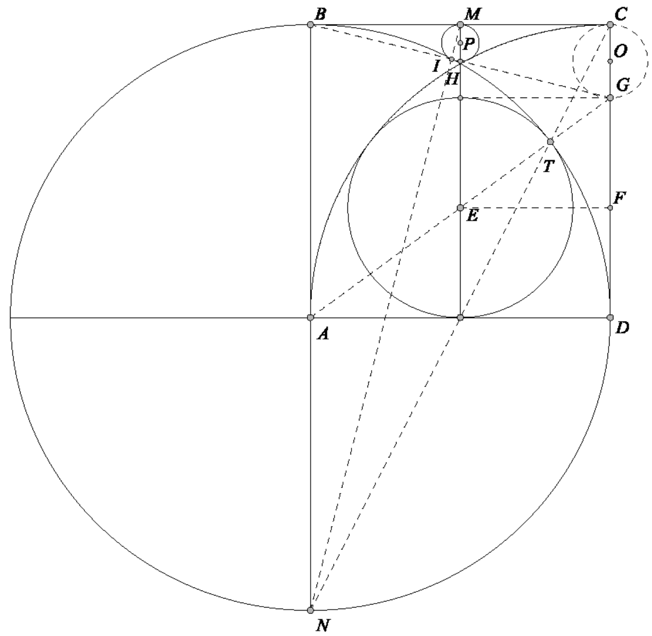
$$4q(r+s) - 4q^2 = 4qr - 4u\sqrt{qr} + 4q(q+s) - 4q^2$$
$$-4q^2 = -4u\sqrt{qr}$$
$$q^3 = ru^2$$

But, as shown earlier, $ru^2 = 4qr(q+s) - 4q^2r$, so $q^3 = 4qr(q+s) - 4q^2r$ or $q^2 = 4rs$, which proves the assertion in the problem figure by the same lemma we started with.

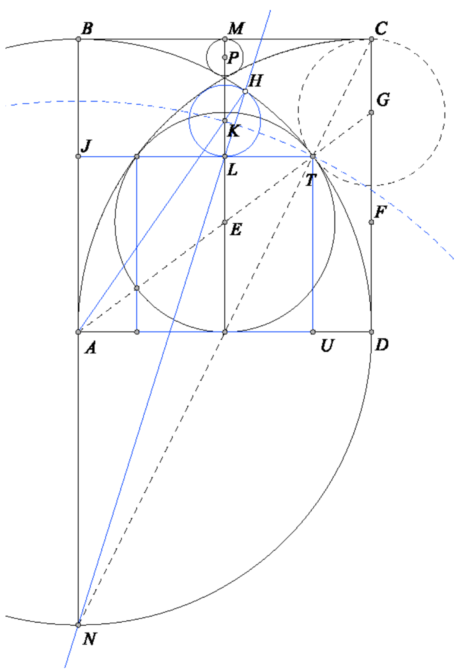
(This proof was given by Aida Yasuaki.)

Back to problem

Let the side of square $ABCD$ be $1, r = ET$, and $s = TG$. Since $NAT \sim CGT$, $CG = s$, so $1 - s = 2r$. On the other hand, in EFG, $(r+s)^2 = [1-(r+s)]^2 + \frac{1}{4}$; hence $r + s = \frac{5}{8}$. Adding equations and solving, $r = \frac{3}{8}$, so $s = \frac{2}{8} = \frac{1}{4}$. Thus, if O is the midpoint of CG, $(O)C$ has radius $\frac{1}{8}$.



Now if I is the touch point of $(P)M$ on $(A)B$, BIN is a right angle. If H is the point diametrically opposite from M in $(P)M$, then MIH is also a right angle. Since $AB \parallel ME$, MN and BH are straight lines, and BH extended passes through G because G is diametrically opposite from the touch point of $(O)C$ on BC. By similar triangles, $PH = \frac{1}{2}OG = \frac{1}{16}$, which is $\frac{r}{6}$.



We already know that $ET = \frac{3}{8}$, $PM = \frac{1}{16}$, and $CG = \frac{1}{4}$. Denote the side of the small square TU as q. $TU : AT :: GD : AG$ or $q : 1 :: \frac{3}{4} : \frac{5}{4}$, so $q = \frac{3}{5}$.

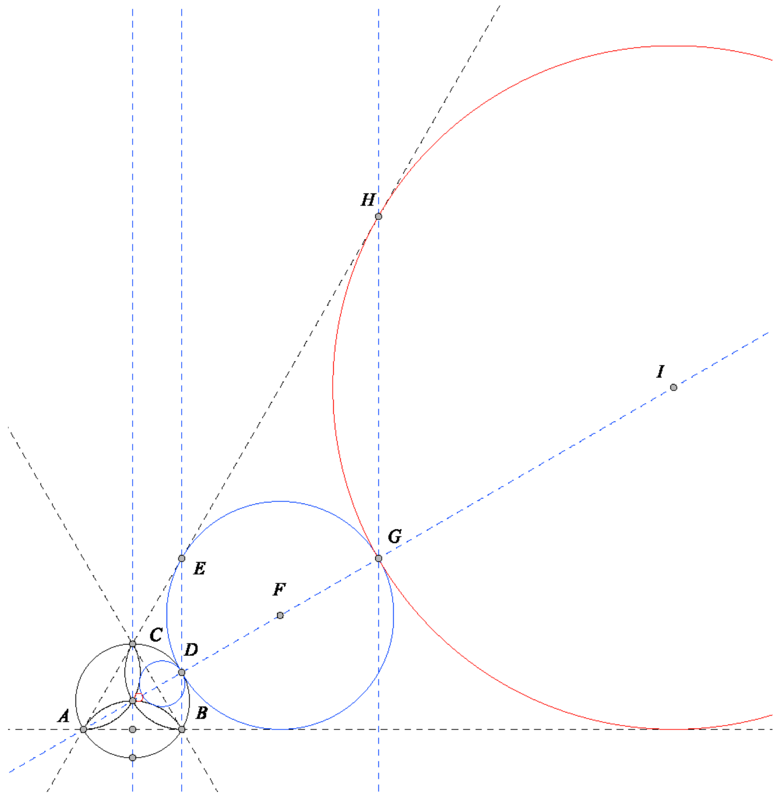
In circle $(N)T$, L and H are inverses of one another; i.e. $NL \cdot NH = NT^2$. Since $NAH \sim LKH$ and $AN = 1$, $LK = \frac{NH - NL}{NH} = 1 - \frac{NL}{NH} = 1 - \frac{NL^2}{NT^2}$. We therefore compute $NJ = \frac{8}{5}, JT = \frac{4}{5}, NT^2 = \left(\frac{8}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{16}{5}$; and $NL^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{8}{5}\right)^2 = \frac{281}{100}$. Since $\frac{NL^2}{NT^2} = \frac{281}{320}$, we have $LK = \frac{39}{320}$, and thus

$$\frac{PM}{LK} = \frac{1}{16} \cdot \frac{320}{39} = \frac{20}{39}.$$

[Back to problem]

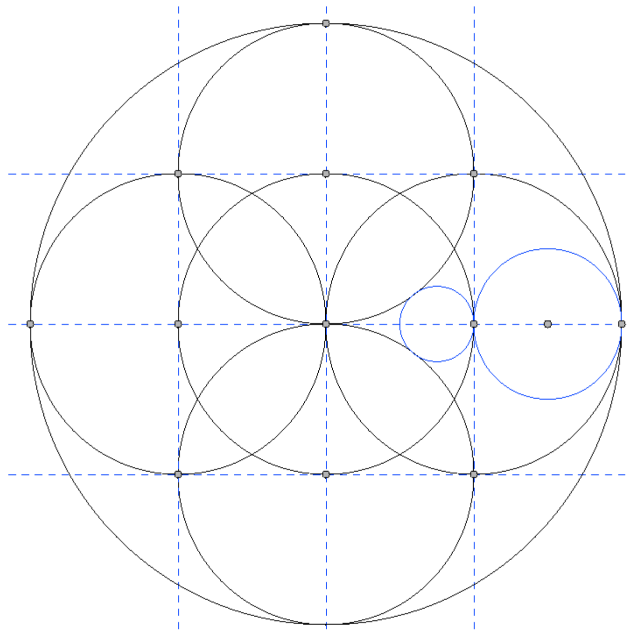
All the black curves have the same radius r. $AD = DE = DF = 2r$, so the image of the blue circle has radius $\frac{r^2}{|(3r)^2-(2r)^2|} \cdot 2r = \frac{2r}{5}$. Likewise, $AG = GH = GI = 6r$, and the image of the red circle has radius $\frac{r^2}{|(11r)^2-(6r)^2|} \cdot 6r = \frac{6r}{85}$.

Back to problem



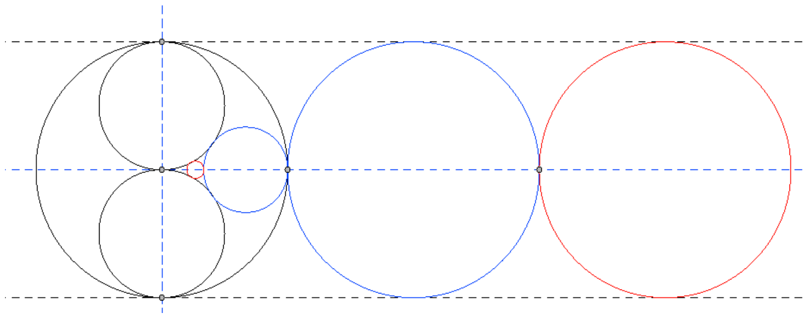
The five large circles with radius r are easily constructed. The small circles are the inversions in the central circle of circles that touch the central circle, the outermost circle, and one of the circles that pass through the center of the figure. They have radius $\frac{r}{2}$, so their radii are

$$\frac{r^2}{\left|\left(\frac{3r}{2}\right)^2 - \left(\frac{r}{2}\right)^2\right|} \cdot \frac{r}{2} = \frac{4}{8} \cdot \frac{r}{2} = \frac{r}{4}.$$



Back to problem

Say that the radii of the large black, blue, and red circles are r. The radii of the smaller black circles are $\frac{r}{2}$; the radius of the image of the blue circle is $\frac{r^2}{|(2r)^2-r^2|} \cdot r = \frac{r}{3}$; and the radius of the image of red circle is $\frac{r^2}{|(4r)^2-r^2|} \cdot r = \frac{r}{15}$.

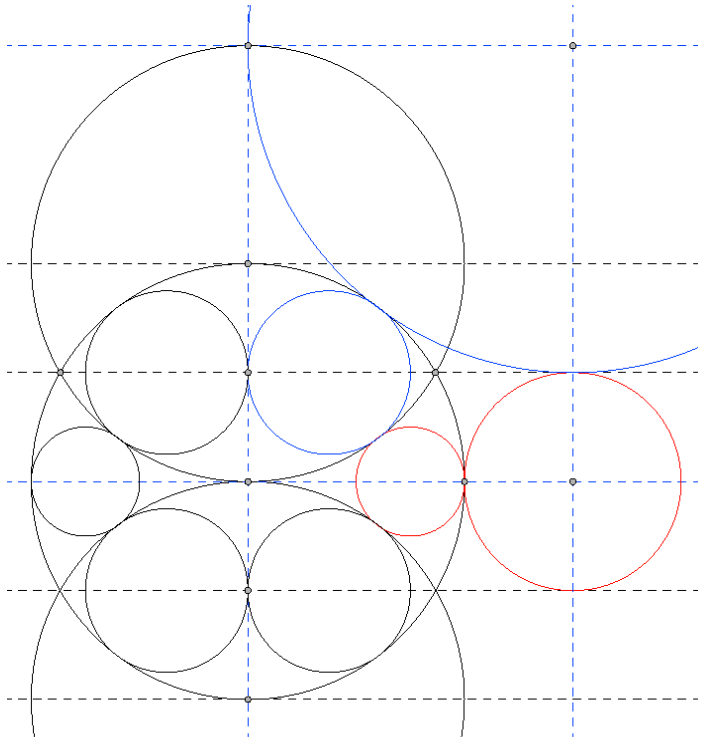


Back to problem

Let the radius of the larger red circle be r. The radius of the large blue circle is thus $3r$, and the radii of the large black circles is $2r$.

The image of the red circle therefore has radius $\frac{(2r)^2}{|(3r)^2-r^2|}\cdot r=\frac{1}{2}r$ while the image of the blue circle has radius $\frac{r^2}{|(4r^2+3r^2)-(3r)^2|}\cdot 3r=\frac{3}{4}r$.

Back to the problem

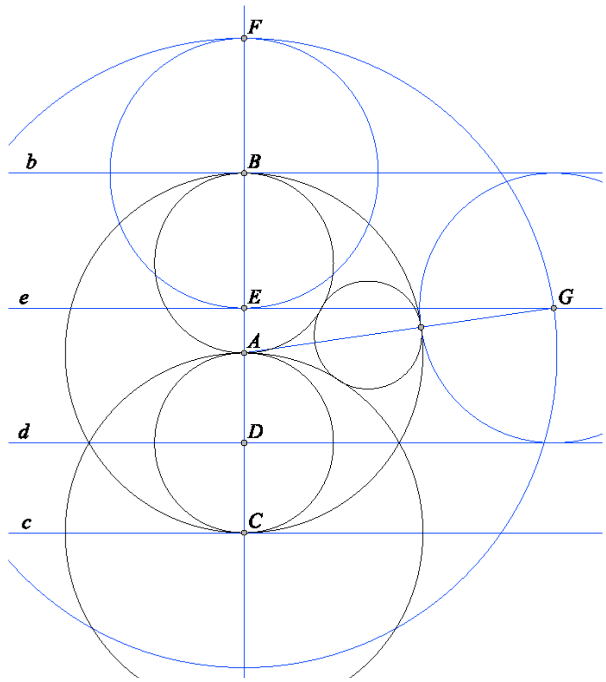


Draw $(A)B$ with radius r; $b \perp AB$ at B; $c \perp AC$ at C; $d \perp AC$ at D, the midpiont of AC; $e \perp AB$ at E, the midpiont of BD; $(B)E$ meeting AB produced in F; $(A)F$ meeting e in G; and finally (G), which by construction touches $(A)B$, b, and d. (G) has radius $\frac{3}{4}r$.

Now invert (G), b, c, and d in $(A)B$ to get the problem figure.

By a well-known theorem of [inversive geometry](), the radius of the image of (G) is then

$$\frac{r^2}{\left|\left(r+\frac{3r}{4}\right)^2-\left(\frac{3r}{4}\right)^2\right|}\cdot\frac{3r}{4}=\frac{r^2}{\left|r^2+\frac{3r^2}{2}\right|}\cdot\frac{3r}{4}$$
$$=\frac{2}{5}\cdot\frac{3r}{4}=\frac{3r}{5}.$$

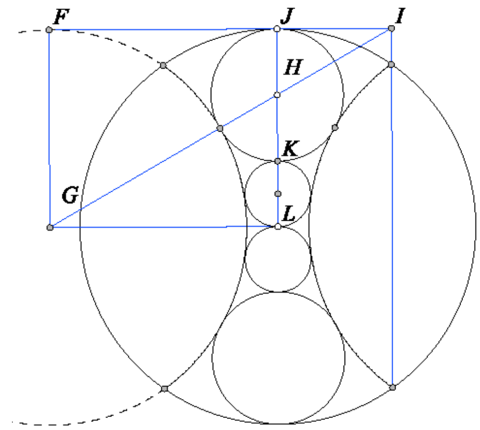


To solve this problem without inversion would probably be difficult.

[Back to problem]()

It is given that (G) and (L) have equal radii; therefore FI touches (H) at J and $FI \perp JL$. Hence GH, which passes through the point where (G) touches (H), meets FJ in I, the center of similitude of (G) and (H).

By construction, $FJ = 2JI$, so $FI = 3JI$. By similar triangles, $FG = 3JH$. But $FG = JK + KL$ and $JK = 2JH$; therefore, $KL = JH$, so $FG = 3KL$ and $JK = 2KL$.

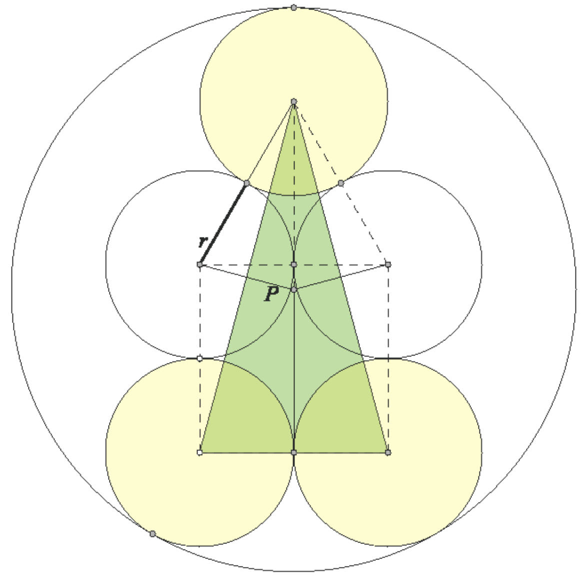


[Back to problem]

The centers of the colored circles are vertices of an isosceles triangle with circumcenter P. The perpendicular bisectors of its sides concur in P.

The distance from P to the apex of the isosceles triangle is its circumradius, and is therefore easily seen to be $2r$.

As the large circle touches the three colored circles, its center must also be P and its radius $R = 2r + r = 3r$. Thus $R : r :: 3 : 1$.



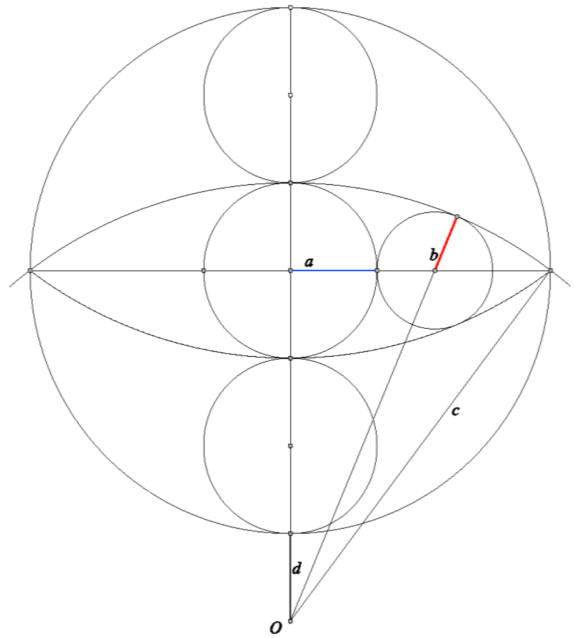
Back to problem

It is easier to work with radii geometrically. Let O be the center of one of the arcs. O, the center of either small circle, and its farther point of contact with the arc are collinear.

From $c^2 = (4a + d)^2 = 9a^2 + (3a + d)^2$, we have $2ad = 2a^2$ or $d = a$, so

$$(a + b)^2 + 16a^2 = (5a - b)^2,$$

from which we obtain $12ab = 8a^2$ or $b = 2a/3$. Thus $18b = 12a$ or $9h = 2g$.

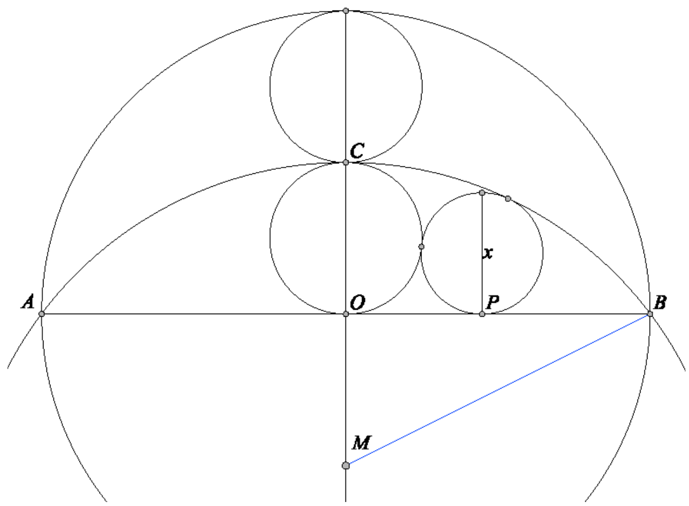


Back to problem

Denote $MC = MB$ as r. From $r^2 = (r - d)^2 + 4d^2$ (Pythagorean theorem) and $d(2r - d) = 4d^2$ (Crossed Chords theorem), we find that $r = 5d/2$. Since $OB = 2d$, MOB is a 3 : 4 : 5 right triangle, and $OM = 3d/2$.

We know $OP = \sqrt{dx}$, and a lemma of Casey (Prop. 6) tells us that $AP \cdot PB = x \cdot (OM + r)$, Thus

$$(2d - OP)(2d + OP) = x\left(\frac{3d}{2} + \frac{5d}{2}\right)$$
$$4d^2 - dx = 4dx$$
$$4d = 5x.$$



Back to problem

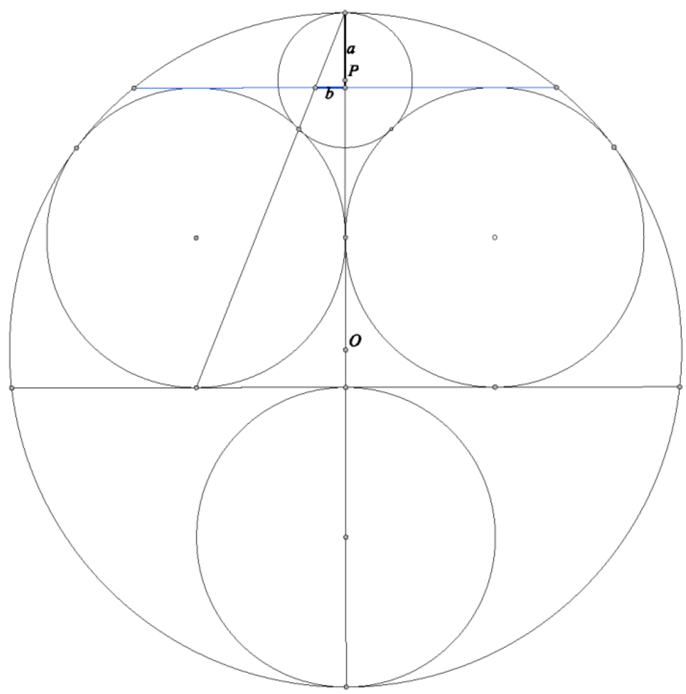
Add the second common tangent of the kissing circles and let a denote the distance of its midpoint from (O). Applying the result of this problem, $m = 2\sqrt{am}$ or $m = 4a$ and $d = 9a$.

By similar triangles, $\frac{b}{a} = \frac{2a}{5a}$ or $5b = 2a$, and, since (O) and (P) are homothetic, $\frac{a-\frac{p}{2}}{b} = \frac{\frac{9a}{2}-4a}{2a}$ or $b = 4a - 2p$.

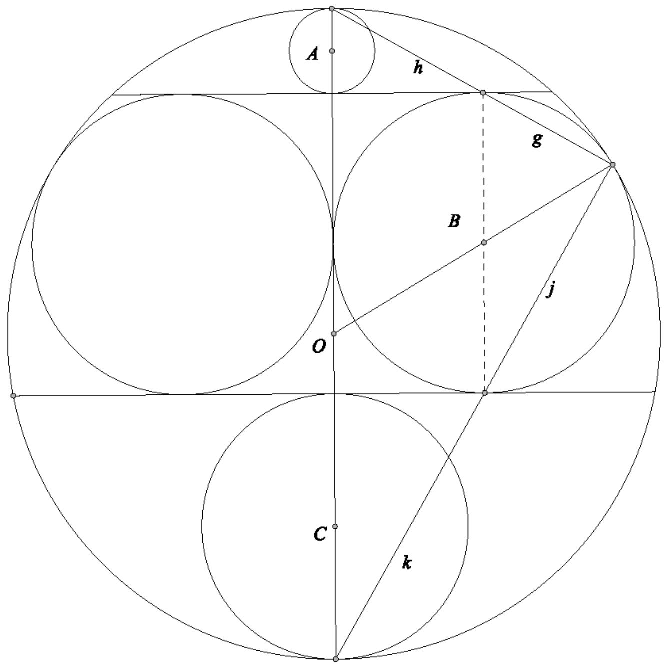
Multiplying by 5, $20a - 10p = 5b = 2a$, so, eliminating b, $18a = 10p$.

Multiplying by 2 to make the coefficient of a divisible by 4, $36a = 20p$. Thus $9m = 20p$.

Back to problem



The diameter of (O) through A and C has length $d = a + b + c$. (B) and (O) are homothetic. By similar triangles,



$$\frac{b/2}{g} = \frac{d/2}{g+h} \text{ and } \frac{b/2}{j} = \frac{d/2}{j+k}.$$

That is, $(a + b + c)g = b(g + h)$ and $(a + b + c)j = b(j + k)$, or $(a + c)g = bh$ and $(a + c)j = bk$. On the other hand, $h^2 = a^2 + b^2/4$ and $k^2 = c^2 + b^2/4$. Therefore

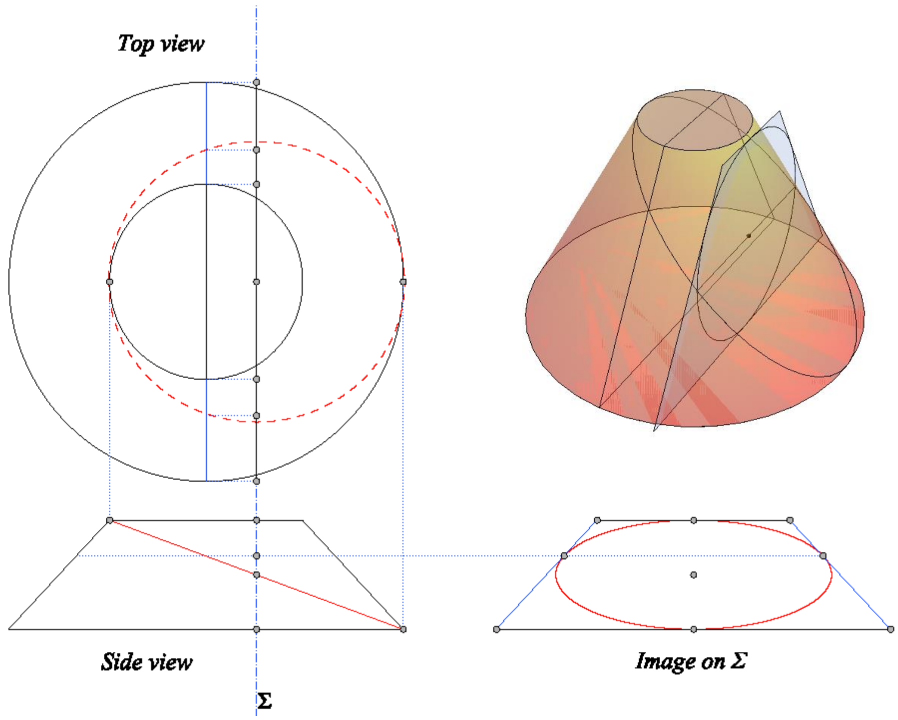
$$(a + c)^2 g^2 = b^2(a^2 + b^2/4) \text{ and } (a + c)^2 j^2 = b^2(c^2 + b^2/4).$$

Adding and applying the Pythagorean theorem once again, we get

$$\begin{aligned} (a + c)^2(g^2 + j^2) &= b^2(a^2 + c^2 + b^2/2) \\ (a + c)^2 &= a^2 + c^2 + b^2/2 \\ 2ac &= b^2/2 \\ 2\sqrt{ac} &= b. \end{aligned}$$

Back to problem

LEMMA: *if the major axis of an ellipse inscribed in an isosceles trapezium is parallel to its bases, then it is the geometric mean of the bases.*



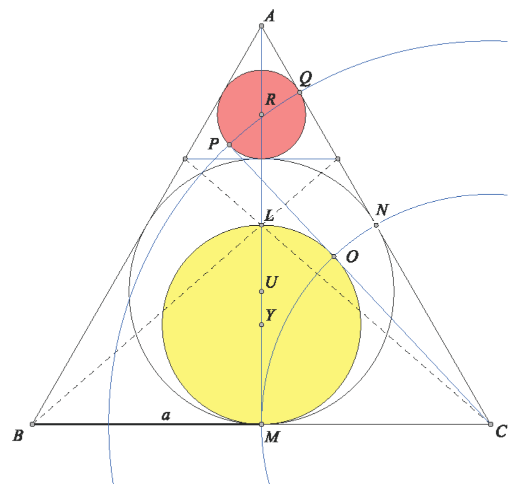
Let the plane of the ellipse be Π and consider the plane Σ, orthogonal to the bases of the frustum, in which the minor axis of the ellipse lies. Onto Σ, orthogonally project the ellipse and the trapezium bases. The images of bases are the diameters parallel to Σ, and the segments joining their endpoints form an isosceles trapezium circumscribing the image of the ellipse. The image is also an ellipse, and one of its axes (in the orthographic drawing, the major axis) is the minor axis of the pre-image. The claim in the problem follows from applying the lemma to the whole image.

To help with visualization, a 3D figure is included in the uppr right corner of the orthographic drawing .

Back to problem

For convenience, we shall use radii instead of diameters.

Assume $(Y)y$ is absent from the figure for the time being; we will construct it later.



Draw the common tangent of the red and uncolored circles, $(R)r$ and $(U)u$. The altitude of the smaller equilateral triangle is $\frac{1}{3}$ that of the larger because, in equialteral triangles, the medians and angle bisectors are identical. Therefore, $r = \frac{1}{3}u$, and the sides of the smaller triangle are $\frac{2a}{3}$.

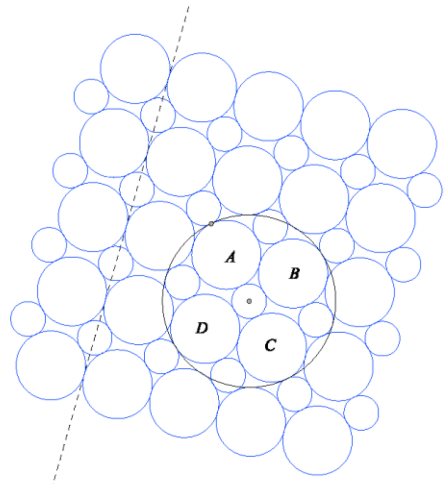
The diagonals of the trapezium with incircle (U) concur at a point L on AM. The height of the trapezium is $2u$, so, by a well-known formula, $LM = \frac{2a \cdot 2u}{2a+2a/3} = \frac{2u}{4/3} = \frac{3}{2}u$.

Choose an endpoint of the base, say C, and let $C(R)P$ and $C(R)Q$ be the two tangents to (R) through it. $(C)P$ passes through Q, the contact point of (R) and AC. Likewise $(C)M$ passes through N, the contact point of (U) and AC. Say that $(C)M$ meets CP in N and construct circle (L, N, M).

The center of this circle is the midpoint of LM; call it Y. Let $(C)M$ meet CP in O. Because $(C)M$ is orthogonal to both (Y) and (U), (Y) touches CP at O just as (U) touches AC at N. L is the point on (Y) diametrically opposite M, so $LM = 2y$. Hence $y = \frac{3}{4}u$, and $\frac{y}{r} = \frac{y}{u} \cdot \frac{u}{r} = \frac{3}{4} \cdot 3 = \frac{9}{4}$, as claimed.

Back to problem

Construct an indefinite number of circles of radius 2 with collinear centers, each touching two others. Circles of radius 3 with the same centers intersect at the centers—also collinear—of circles of radius 1, each touching two of the larger circles.



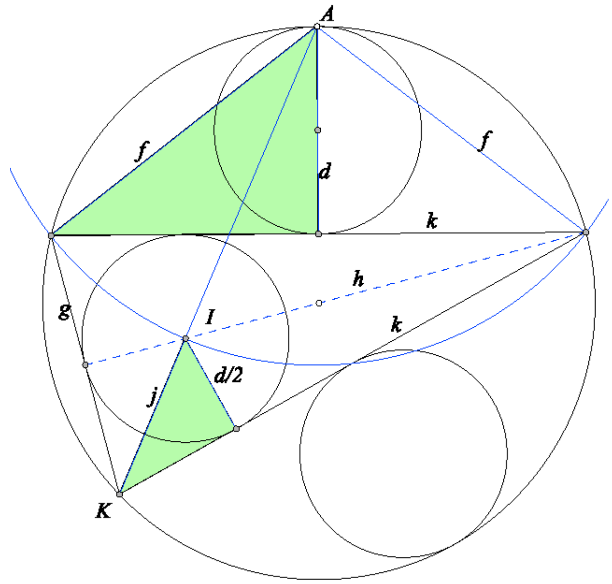
By reflecting circles across the lines of centers, we obtain a symmetric pattern as large as we please. The tangents to the large circles where they meet are parallel lines, such as the dashed line shown, separated by a distance of 4. The centers of all small circles lie on these lines.

Let O be the center of an arbtirary small circle and label the four large circles touching it A, B, C, D as shown. Then $(O)5$ is externally tangent to the four small circles touching A, B, C, D, and internally tangent to all of them and the two small circles touching A, D and B, C.

The problem figure consists of $(O)5$ plus the circles it touches internally and the similar image centered on one of the small circles adjacent to the central circle untouched by (O).

Back to problem

Denote large and small diameters as D and d. In *Sanpō kantsū jutsu*, Aida shows that, for four different diameters D, d, e, f, one has $-e^2D - 8def + 16dfD - 16df(d+f) = 0$. If $d = e = f$, this becomes $-d^2D - 8d^3 + 16d^2D - 32d^3 = 0$, or $15dD - 40d^2 = 5d(3D - 8d) = 0$, which proves the assertion. But there is a shortcut available in this case.



Denote the base of the isosceles triangle as g, its legs as k, and the segments f, j as shown in the figure. Note the following lemma: *The bisector of an angle of a triangle cuts its circumcircle at the midpoint of the arc subtended by the opposite side, which is the center of a circle through the endpoints of that side and the incenter of the triangle.* Hence A, I, K are collinear and, by similar triangles, $j = f/2$. Applying the Ptolemy's theorem, $fg + fk = \frac{3fk}{2}$, so $2g = k$.

Equating the inradius-semiperimeter and base-altitude expressions for the area of the isosceles triangle, and using the Pythagorean and Crossed Chords theorems for k^2, we have

$$\frac{1}{4}d(g+2k) = \frac{gh}{2}, \qquad k^2 - \frac{g^2}{4} = h^2, \qquad \frac{k^2}{4} = d(D-d),$$

or, simplifying,

$$2gh = d(g+2k), \quad g^2 + 4h^2 = 4k^2, \qquad 4d^2 + k^2 = 4Dd.$$

Applying $2g = k$ to eliminate k,

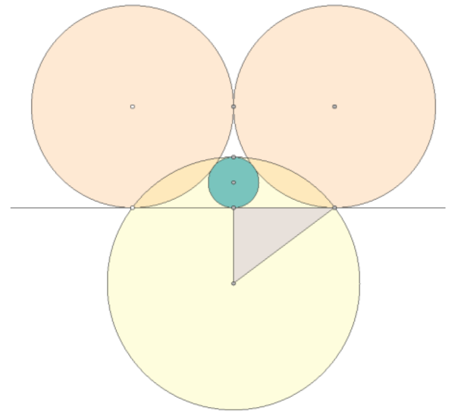
$$2h = 5d, \qquad 4h^2 = 15g^2, \qquad d^2 + g^2 = Dd.$$

Eliminating h, $25d^2 = 15g^2$ and $d^2 + g^2 = Dd$. Eliminating g^2, $8d^2 = 3Dd$, or $8d = 3D$.

Back to problem

Let the diameters of the large, medium, and small circles be a, b, c, respectively. Then $4c = b$ because, measuring along the bitangent, $\frac{b}{2} = \sqrt{bc}$. By the Crossed Chords theorem, $\left(\frac{b}{2}\right)^2 = c(a - c)$, so $4c^2 = ac - c^2$, or $5c^2 = ac$, from which $5c = a$.

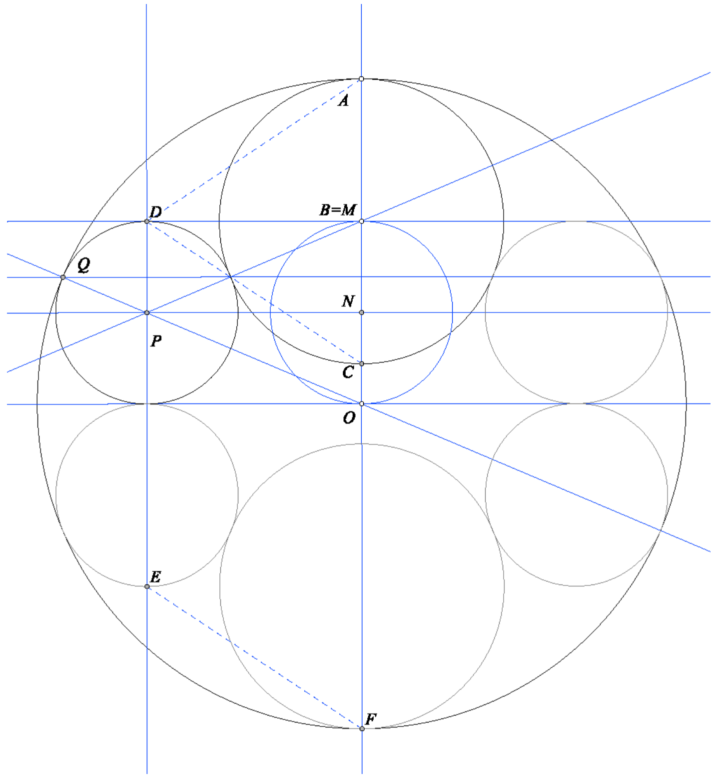
Another way to state the assertion is to say that the shaded triangle is a 3: 4: 5 right triangle.



Back to problem

Lemma: If (B) and (P) are internally tangent to (O), and (P) touches the diameter of (O) perpendicular to OB, then (B) touches (P) externally if and only if (P) touches the diameter of (B) perpendicular to OB.

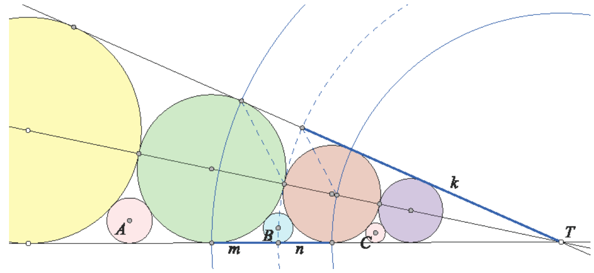
Add auxiliary circle $(N)O$ and note that the polar of A with respect to it meets (O) in Q and that OQ meets the parallel line through N in P. (This is the fact that allows us to construct (P) exactly.) If (P) touches (B), PB and the polar must be concurrent at the contact point of the circles because their common tangent passes through the second point where the polar meets (P). But this concurrency occurs if and only if B and M coincide.



In the problem, the other four colored circles are just reflections in horizontal and vertical diameters of (O) of the ones corresponding to (B) and (P) in the lemma. The lemma shows that the horizontal diameter of the yellow circle touches the green circle regardless of the ratio $AC:MO$. It also guarantees that M is the midpoint of AC, i.e. that ADN is an isosceles triangle; therefore, $CDEF$ is always a parallelogram. The small uncolored circles in the problem figure are extraneous: $AF = DE + AC$ whether or not the small uncolored circles each touch the yellow circles.

Back to problem

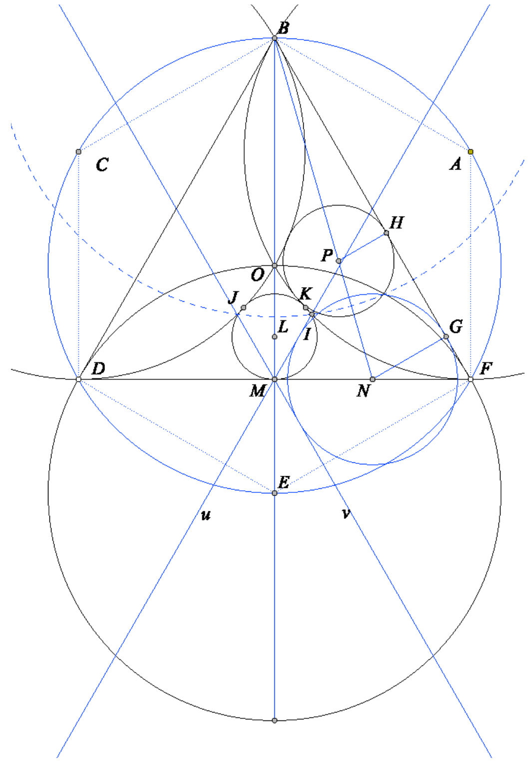
It is obvious that circles $(T)(k+m)$ and $(T)(k-n)$ are orthogonal to the green and brown circles, respectively, and easy to prove that $(T)k$ is orthogonal to the light blue circle. The yellow and green circles are the inverses of the brown and purple circles in $(T)k$, and (A) and (C) are inverses of (B) in $(T)(k+m)$ and $(T)(k-n)$. (N.B. m, n are not the radii of the green and brown circles.)



$TB^2 = b^2 + k^2$, so by the usual formula for image radii, $a = \frac{(k+m)^2}{k^2} b$ and $c = \frac{(k-n)^2}{k^2} b$. Therefore $\sqrt{ac} = \frac{(k+m)(k-n)}{k^2} b$. But the coefficient of b in this equation is 1 because, by similar triangles, $\frac{k+m}{k} = \frac{k}{k-n}$. Thus $\sqrt{ac} = b$.

Back to problem

Let the sides and circumradius of regular hexagon $ABCDEF$ with center O be 1. Then the sides of equilateral triangle BDF are $\sqrt{3}$, and $\widehat{AOD}$ and $\widehat{AOF}$ have radius 1. (L) touches (A), (C) at J, K, and DF at its midpoint M. $MF = \frac{\sqrt{3}}{2} = 2\sqrt{LM \cdot 1}$, so $LM = \frac{3}{16}$.



Say that the parallel u to BD through M meets (A) in I, and let v be the reflection of u in DF. Then, for N the midpoint of MF, there is a circle (N) touching u, v, and BF at, say, G, $GN \perp BF$. $NF = \frac{\sqrt{3}}{4}$, and since $\angle GNF = 30°$, $GF = \frac{\sqrt{3}}{8}$. Thus $GN = \sqrt{\frac{3}{16} - \frac{3}{64}} = \frac{3}{8} = 2LM$.

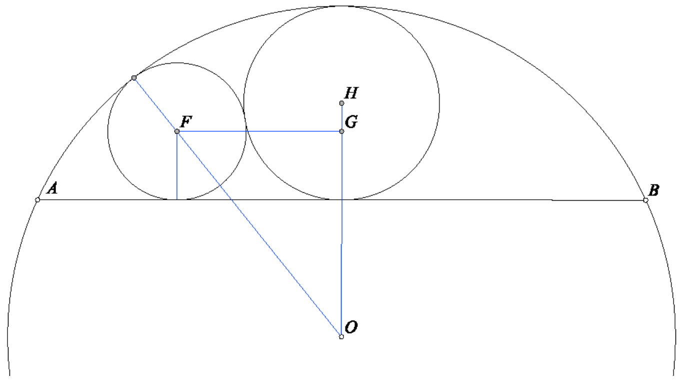
Moreover, with respect to a circle (B) of suitable radius, u is the inverse of (C), v is the inverse of (A), and BF is its own inverse. In fact, the radius is BI, but the important point is that the inversion of $(N)G$ in (B) is the circle $(P)H$ tangent to BF, (A), and (C).

Because B, P, N are collinear and $BH = 2\sqrt{PH \cdot 1}$, we have, by similar triangles, $\frac{2LM}{7\sqrt{3}/8} = \frac{PH}{2\sqrt{PH}} = \frac{\sqrt{PH}}{2}$, or $4LM^2 = \frac{3 \cdot 49}{64} \cdot \frac{PH}{4}$. But $16LM = 3$, so this is $LM = \frac{49}{64}PH$, as claimed.

Back to problem

Let the radii of (F), (G), and (O) be f, g, and r. In FGO, $GO^2 + FG^2 = FO^2$, or

$$(r - 2g + f)^2 + 4fg = (r - f)^2.$$



As a quadratic in g, this is

$$0 = g^2 - gr + fr,$$

so $g = \frac{1}{2}\left[r - \sqrt{r(r - 4f)}\,\right]$.

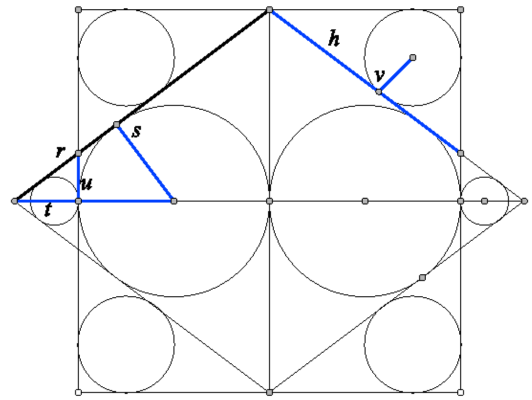
Denoting AB as c,

$$\left(\frac{c}{2}\right)^2 = 2g(2r - 2g).$$

Hence $\frac{c^2}{4} = \left[r - \sqrt{r^2 - 4fr}\,\right]\left[r + \sqrt{r^2 - 4fr}\,\right] = r^2 - (r^2 - 4fr) = 4fr$ or $c = 2\sqrt{fr}$.

Back to problem

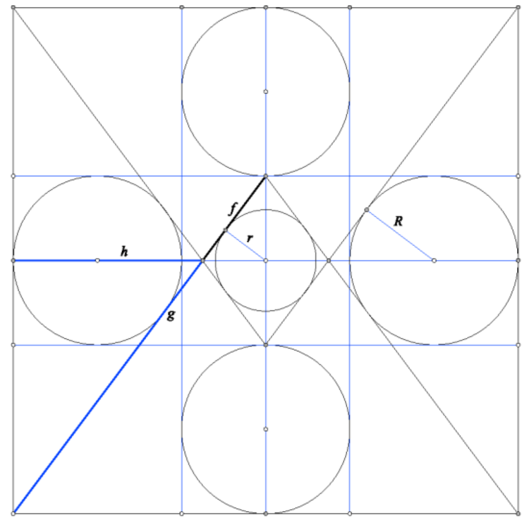
Say the square has side $4s$ and the yellow circles have radius y. Comparing the right triangles with hypotenuses r and t, $\frac{2s}{r} = \frac{s}{t}$, so $r = 2t$. Because $(t+s)^2 + 4s^2 = r^2 = 4t^2$, $t = \frac{5s}{3}$. Hence, in the four triangles with hypotenuse r, $s : r - h : t :: 3 : 4 : 5$. All the other right triangles shown are similar.



In the smallest, if $u = 3x$, then $t - s = 4x, r - h = 5x$. Since $t = \frac{5s}{3}, \frac{s}{6} = x$. The areas of the isosceles triangles with yellow incircles are $tu = y(u + h)$ or $\frac{2s}{3} \cdot \frac{s}{2} = y \cdot \frac{8s}{6}$, so $y = \frac{s}{4} = \frac{u}{2}$. In the right triangles with red incircles, $2v = 3u + 4u - 5u = 2u$. Thus $\frac{y}{v} = \frac{1}{2}$.

Back to solution

Since the red circles are all equal and touch the square at the midpoints of its sides, the symmetry of the figure requires that the side of the square be $6R$, as the auxiliary lines show. The right triangles with hypotenuse $f+g$ are evidently $3:4:5$, so $f+g=5R$. Since $\frac{f}{f+g}=\frac{1}{4}$, so $f=\frac{5R}{4}$, $g=\frac{15R}{4}$. Hence $h^2=\frac{225R^2}{16}-9R^2=\frac{81}{16}R^2$. The short diagonal of the rhombus is therefore $6R-\frac{9}{2}R=\frac{3}{2}R$. Its long diagonal is $2R$, so its area is $\frac{3}{2}R^2$, which is also $2fr=\frac{5}{2}Rr$. Therefore $3R=5r$.

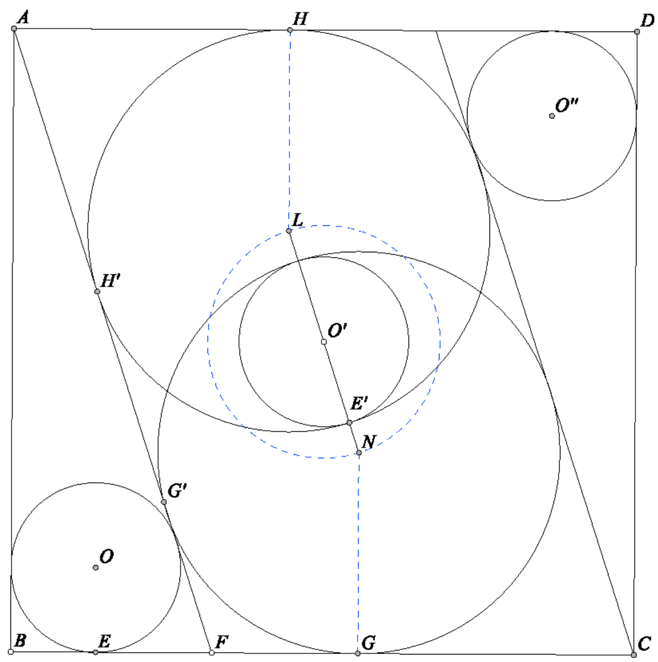


[Back to solution](#)

Let the smaller radius be r and larger R. Say the side of the square is $2a$. It is claimed that $BF = R$.

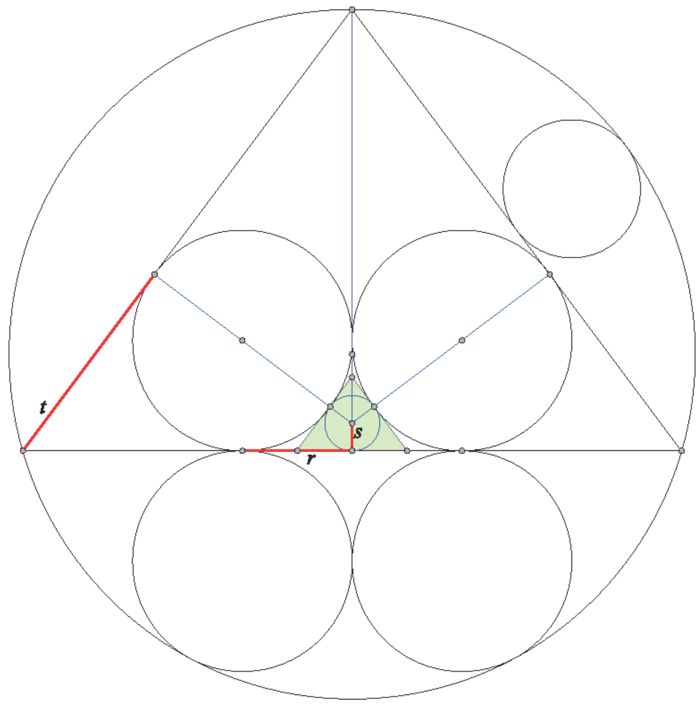
Denote $BF, FG = FG', GC = AH = AH'$ as x, y, z. By construction, $G'H' = LN = 2R - 2r$, and, because $2r = 2a + x - \sqrt{4a^2 + x^2}$, $x = \frac{r(2a-r)}{a-r}$. Since $z = 2a - x - y$ and $LN + y + z = \sqrt{4a^2 + x^2}$, $r(2a + R) = r^2 + aR$, so $a = \frac{r(R-r)}{R-2r}$. Substituting $\frac{r(R-r)}{R-2r}$ for a in the equation for x, $x = R$.

It is not easy to construct the problem figure exactly. $\angle AFB$ is slightly greater than 72°. E, O, H' are collinear, but this is not easy to prove.



Back to solution

Let the equal sides of the isosceles triangle, its base, its altitude, the diameters of the red and blue circles, and the radius of the yellow circle be, respectively, $a, 2b, h, c, d = 2r$, and R.



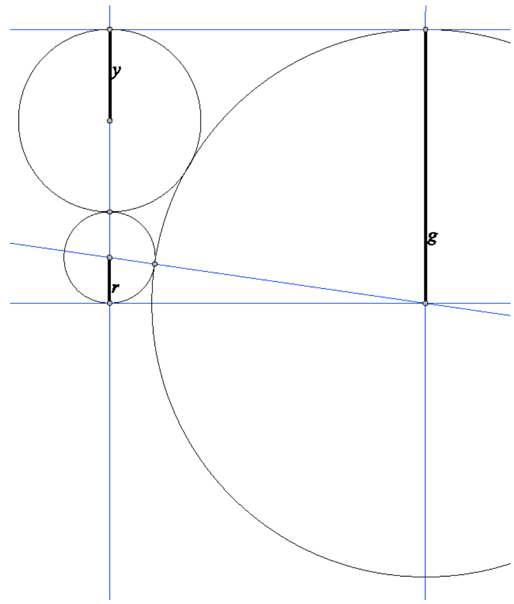
Construct the similar triangle (green) with sides touching the blue circles at the points diametrically opposite their touch points on the sides of the large triangle. Say its inradius is s.

Since $r = 2\sqrt{rs}, r = 4s$. Furthermore, $(h-s)^2 - (9s)^2 = (a-t)^2 = (a-b+4s)^2$. Using $2r = 8s = b + h - a$ and $a^2 = b^2 + h^2$ to eliminate s and simplify, we obtain $a = \frac{5h}{4}$ and $b = \frac{3h}{4}$; i.e., $b, h, a = 3u, 4u, 5u$ for some unit u.

Setting $u = 1$, the area of the isosceles triangle is $12 = \frac{6 \cdot 5^2}{4R}$, so $R = \frac{75}{24} = \frac{25}{8}$. By the Crossed Chords theorem, $c\left(\frac{25}{4} - c\right) = \frac{25}{4}$. We reject the root $c = 5$ because $c < a$, and so obtain $c = \frac{5}{4}$. As $2r = d = b + h - a = 2, c = \frac{5}{8}d$.

Back to problem

Because of the symmetry of the figure, we need consider just one set of green, yellow, and red circles. Say that their radii are g, r, y, respectively.



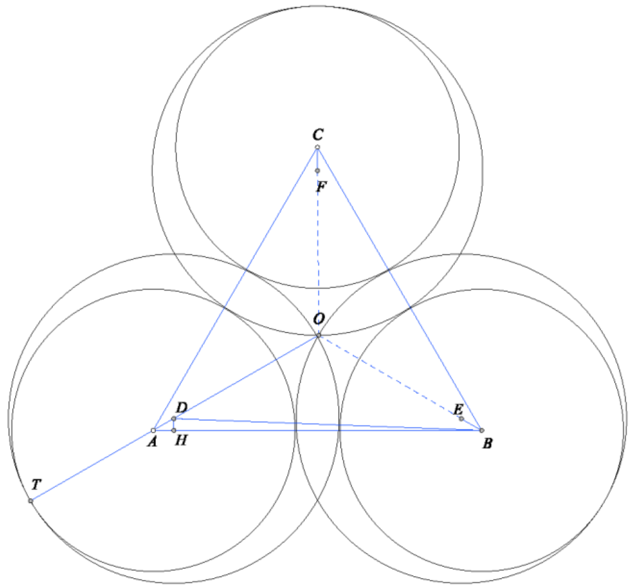
$2\sqrt{gy} = \sqrt{(r+g)^2 - r^2}$ or $4gy = 2gr + g^2$.

That is, $4y - g = 2r$.

But $2r = g - 2y$. Hence $6y = 2g$ or $y = \frac{1}{3}g$.

[Back to problem](#)

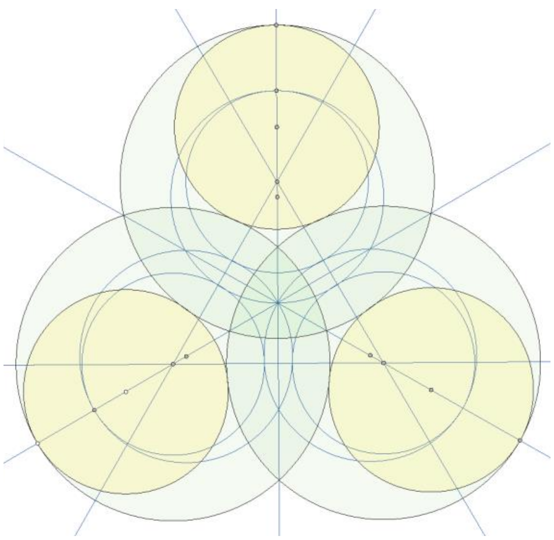
Label the figure as shown.



$AD = R - r$; $2HD = AD$ because $\angle OAB = 30°$. $HB^2 = (R+r)^2 - \left(\frac{R-r}{2}\right)^2$ and $AH^2 = (R - r)^2 - \left(\frac{R-r}{2}\right)^2$.

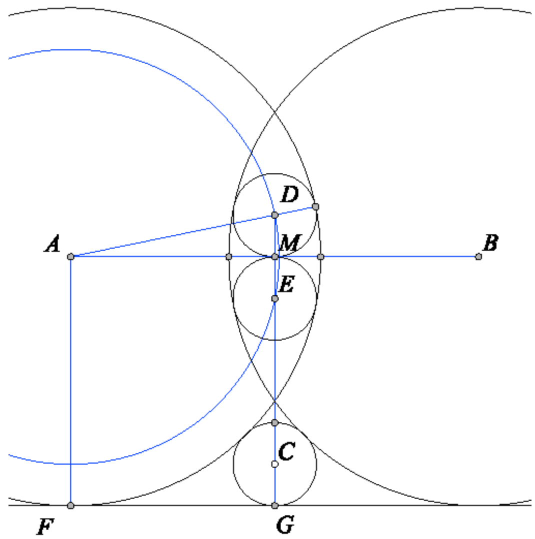
Adding roots, $AB = \frac{1}{2}\left(\sqrt{3}(R-r) + \sqrt{3r^2 + 10rR + 3R^2}\right)$. Call this s.
Then $OA = \frac{s\sqrt{3}}{3} = 2R - r$. That is, $\frac{\sqrt{3}(R-r)+\sqrt{3R^2+10Rr+3r^2}}{2\sqrt{3}} = 2R - r$, or $3r + \sqrt{9r^2 + 30rR + 9R^2} = 9R$, which has the solution $r = \frac{6}{7}R$.

Back to problem



Keeping A, B, C fixed, imagine sliding T along OA to change the length of r and sliding D to compensate, with similar changes in the other two pairs of circles. All the tangencies are maintained, but the ratio $r : R$ obviously changes. This shows that the concurrency of $(D), (E), (F)$ in O is a necessary condition for $r = \frac{6}{7}R$.

Given $(A)R$, $(B)R$, and $(C)r$, $(D)r$, $(E)r$ as shown, let M be the midpoint of AB.



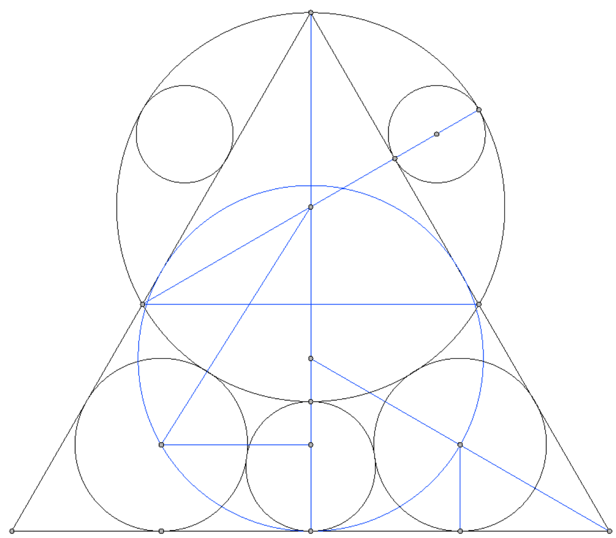
In AMD, $AM^2 + r^2 = (R - r)^2$.

But $AM = FG = 2\sqrt{Rr}$, so $4Rr + r^2 = R^2 - 2Rr + r^2$, or $6r = R$. The ratio of the diameters is, of course, the same.

Back to problem

Note that the radius of the blue circle is $5r$, and is slightly greater than $AM = MB$.

Let the equilateral triangle have side s and altitude h, and let the diameters of the red, blue, green, and yellow circles be a, b, c, d, respectively.



Examining the yellow, blue, and one red circle, it is easy to see that $\left(\frac{a}{2}+\frac{d}{2}\right)^2 - ab = \left(\frac{d}{2}+b-\frac{a}{2}\right)^2$.

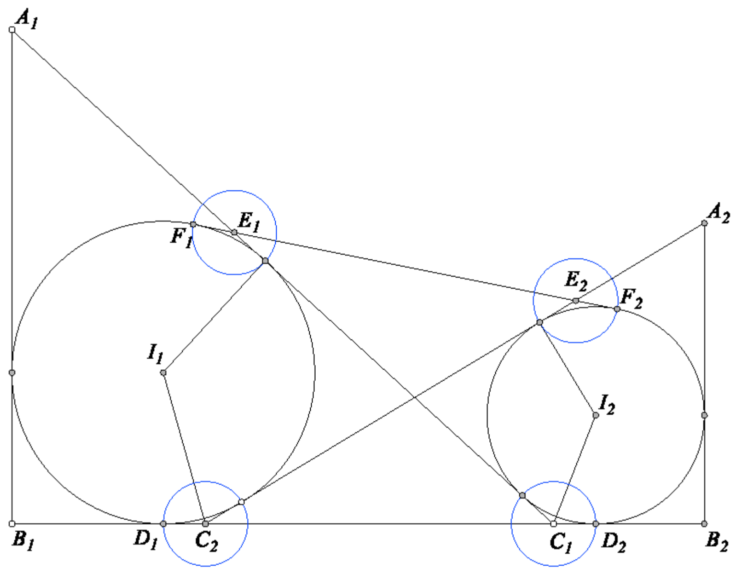
For inradius r, $3r = h = b + d$ and, because $\frac{s}{2} = r\sqrt{3}$, $\frac{a/2}{r} = \frac{r\sqrt{3}-\sqrt{ab}}{r\sqrt{3}}$. Eliminating r and simplifying, we obtain $ad = b(b+d)$ and $3b = d$, from which we find $a = \frac{4d}{9}$ and $b = \frac{d}{3}$. Hence, if $d = 9$, $a = 4$ and $b = 3$, $h = 12$, $r = 4$, and $s = 8\sqrt{3}$.

As for c, consider the equilateral triangle of side t circumscribed by the yellow circle. It has circumradius $R = \frac{\sqrt{3}}{3}t$. For $d = 9$, $R = \frac{9}{2}$, so $\frac{t}{2} = \frac{9\sqrt{3}}{4}$. Applying the Crossed Chords theorem $\left(\frac{t}{2}\right)^2 = c\left(\frac{9}{2}-c\right)$, we find that $c = \frac{9}{4}$, or $4c = d$.

Back to problem

Aida Yasuaki discussed this problem (without the green circles) in his book *Sangaku keiko daizen hyōrin* (A Comprehensive Collection of Critiques of Practice Problems for the Study of Calculation) 算学稽古大全評林 , where he clarifies that d should be the given term.

Circles $(I_1)r_1$ and $(I_2)r_2$ are the incircles of right triangles $A_1B_1C_1$ and $A_2B_2C_2$. $B_1C_2C_1B_2$ are collinear; A_1C_1 touches (I_2) and A_2C_2 touches (I_1). Prove that $\frac{1}{2r_1} - \frac{1}{2r_2} = \frac{1}{c_1} - \frac{1}{c_2}$.



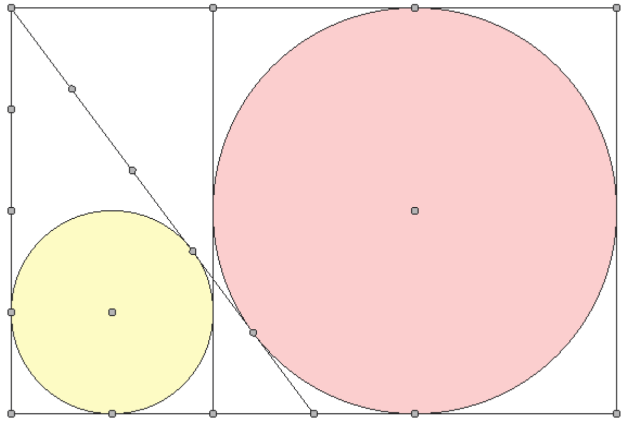
For $n = 1$ or 2, in right triangle $A_nB_nC_n$ we have $a_n + c_n - 2r_n = b_n$ and $\frac{1}{2}a_nc_n = \frac{1}{2}(a_n + b_n + c_n)r_n$. Eliminating b_n and rearranging, $a_n(c_n - 2r_n) = (2c_n - 2r_n)r_n$, so $a_n = \frac{2(c_n - r_n)r_n}{c_n - 2r_n}$.

The Japanese had proven that the internal bitangent segments intercepted by the external bitangents of two circles equal the segments on the external tangents between their touch points; i.e., in the figure above, $D_1D_2 = F_1F_2 = C_1E_1 = C_2E_2$. Hence the radii of the four blue circles are equal. It follows at once that $a_1 - a_2 = r_1 - r_2$.

Thus $\frac{2(c_1 - r_1)r_1}{c_1 - 2r_1} - \frac{2(c_2 - r_1)r_2}{c_2 - 2r_2} = r_1 - r_2 \Leftrightarrow \frac{1}{2r_2} = \frac{c_1c_2 + 2c_1r_1 - 2c_2r_1}{2c_1c_2r_1} = \frac{1}{2r_1} + \frac{1}{c_2} - \frac{1}{c_1}$, or $\frac{1}{2r_1} - \frac{1}{2r_2} = \frac{1}{c_1} - \frac{1}{c_2}$.

Back to problem

Say that the segment on the hypotenuse between the common vertex of the rectangle and right triangle to the the touch point of the yellow circle is j; the segment from there to the touch points of the red circle is k; and the remainder of the hypotenuses is l.



The large rectangle has sides $2y < 2y + 2r$, so, taking note of the equal tangents from the acute vertices of the triangle to the red circle,

$$j + k = 2y + r = m + l.$$

The sum of the legs less the hypotenuse of any right triangle is the diameter of its incircle, so

$$2r + m - (j + k + l) = 2y$$
$$2r - 2y = (j + k + l) - (j + k - l).$$

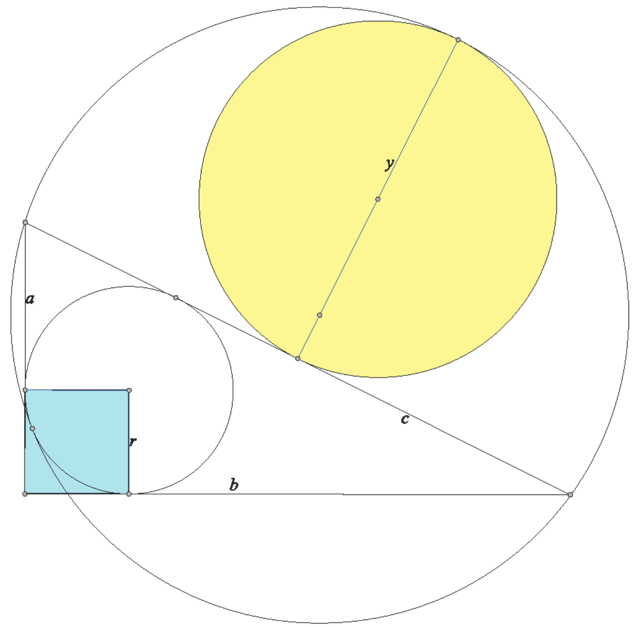
Hence $r - y = l$. Substituting $r - y$ for l in $m + l = 2y + r$, we find that $m = 3y$. The length of the hypotenuse $j + k + l$ is $(2y + r) + (r - y) = y + 2r$, so, applying the Pythagorean theorem to the right triangle with the yellow incircle,

$$9y^2 + 4r^2 = (y + 2r)^2$$
$$9y^2 + 4r^2 = y^2 + 4yr + 4r^2$$
$$8y^2 = 4yr$$
$$2y = r.$$

Corollary: the right triangle is 3 : 4 : 5.

Back to problem

Let the diameter of the yellow circle be y and the side of the right triangle be $a < b < c$.



Notice that the incircle touches the hypotenuse at a point that divides it into segments of lengths $a - r$ and $b - r$ (because tangents to a circle from the same point are equal). By a lemma of Casey (Prop. 6), $(a - r)(b - r) = 2ry$.

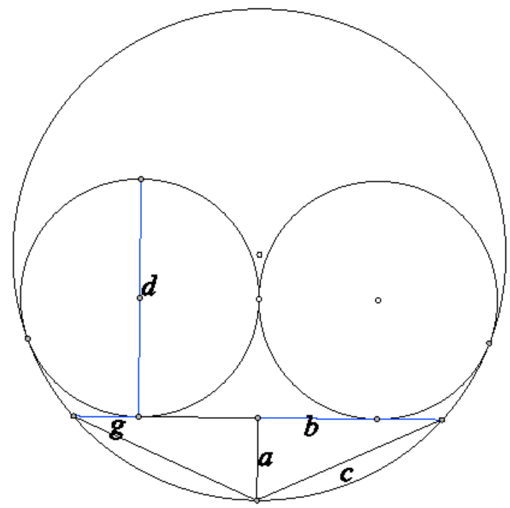
Substituing $\frac{a+b-c}{2}$ for r and simplifying,
$y = -\frac{(a-b-c)(a-b+c)}{4(a+b-c)} =$
$-\frac{a^2}{4(a+b-c)} + \frac{ab}{2(a+b-c)} - \frac{b^2}{4(a+b-c)} + \frac{c^2}{4(a+b-c)}$.

By Pythagoras, this is just $y = \frac{ab}{2(a+b-c)}$.

Using equivalent area formulae, $\frac{ab}{2} = \frac{r(a+b+c)}{2}$, this is easily seen to be $y = \frac{a+b+c}{4}$.

Back to problem

Say that each blue circle touches the chord of length $2b$ at a point that divides it into segments of length g and $2b - g$, with $g \leq \frac{b}{2}$.



By a lemma of Casey (Prop. 6), $g(2b - g) = da$, so $g = b - \sqrt{b^2 - ad}$.

But, by inspection, $b - g = \frac{d}{2}$, so $\frac{d^2}{4} = b^2 - ad$. Hence $d = 2\left(\sqrt{a^2 + b^2} - a\right) = 2(c - a)$ by the Pythagorean theorem.

Back to problem