A Collection of 30 *Sangaku* Problems

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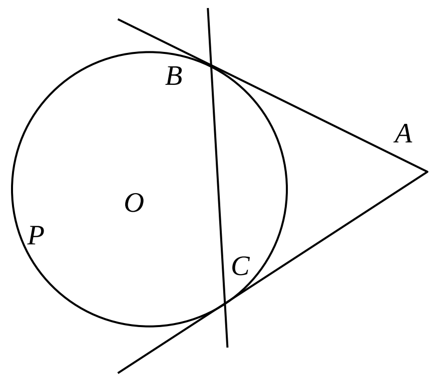
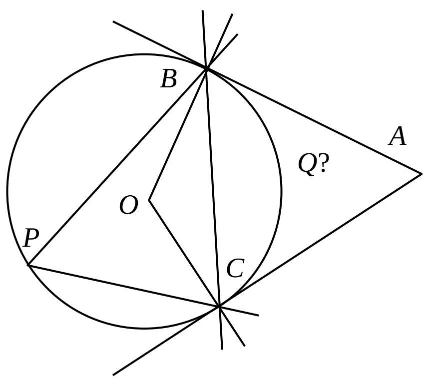
PROBLEM 1: Given two lines tangent to circle (O) at B and C from a common point A, show that the circle passes through the incenter of triangle ABC.[1]



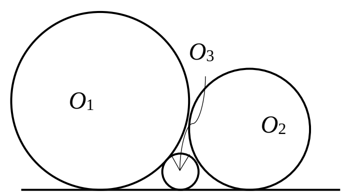


SOLUTION 1 (JMU): Since AB and AC are tangents, each of the base angles of the isosceles triangle ABC measures half of BOC, so the sum of half these angles is also half of BOC. Therefore wherever the intersection of the bisectors, Q, may be, BQC is 180° – $BOC/2$.

Pick any point P on the arc exterior to the triangle; $BPC = BOC/2$. Since BPC and BQC are supplementary, $BQCP$ must be a cyclic quadrilateral. Therefore Q lies on (O). □

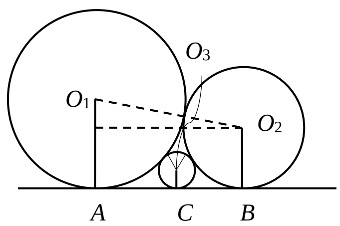
(One can alternatively prove that the midpoint of the arc interior to *ABC* is the incenter.)

PROBLEM 2: What is the relationship between the radii of three circles of different size all tangent to the same line and each externally tangent to the other two?[2]



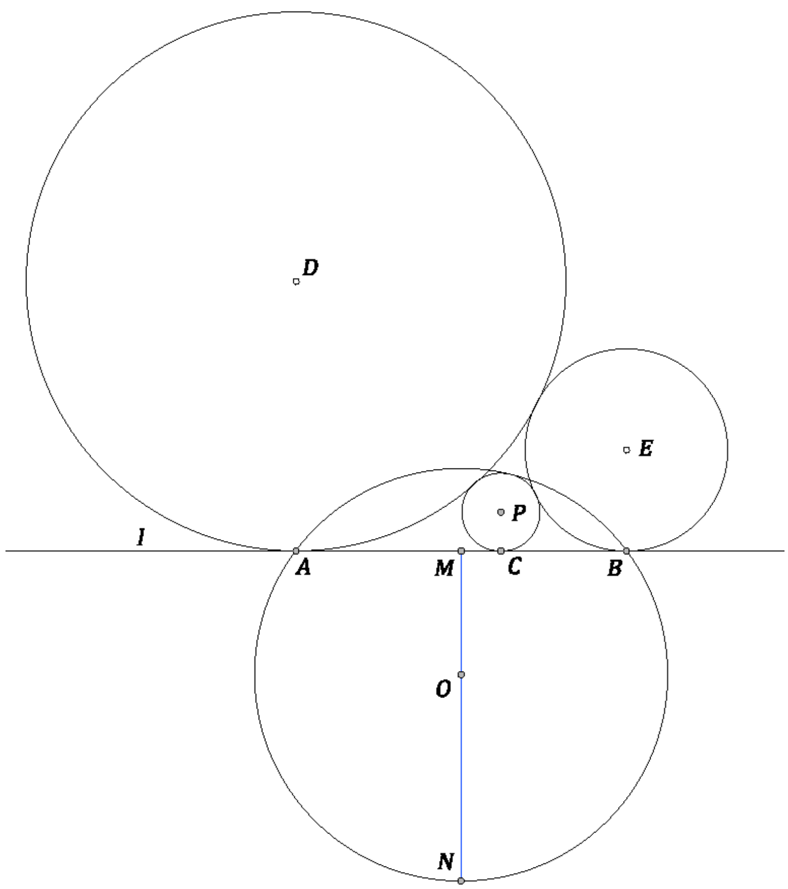
[1] Fukagawa & Pedoe 1989, 1.1.4; lost tablet from Ibaragi, 1896; no solution given.
[2] Fukagawa & Pedoe 1989, 1.1.1; well-known; tablet from Gunma, 1824.

SOLUTION 2 (F&P): The hypotenuse of the right triangle is $r_1 + r_2$. Its short leg is $r_1 - r_2$, so the other leg is the square root of $(r_1 + r_2)^2 - (r_1 - r_2)^2$. I.e., AB is $2\sqrt{r_1 r_2}$ (twice the geometric mean of the radii).



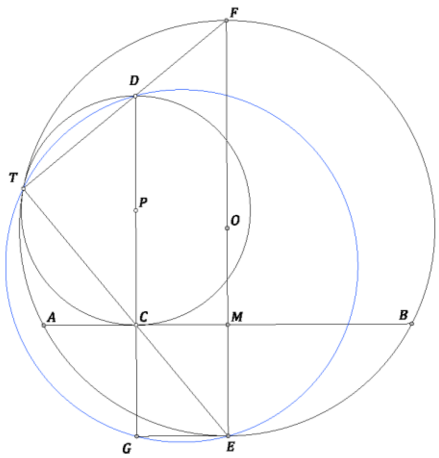
Likewise, $AC = 2\sqrt{r_1 r_3}$ and $BC = 2\sqrt{r_2 r_3}$. Adding and dividing through by $2\sqrt{r_1 r_2 r_3}$, we obtain $\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_2}} + \frac{1}{\sqrt{r_1}}$. □

PROBLEM 3: Suppose circles (D) and (E) with diameters d and e touch one another externally and line l at A and B, respectively. Let (P) with diameter p be the circle that touches (D), (E), and l. Show that the circle (O) that passes through A and B and touches (P) internally is the same as for all positive values of d and e.[3]



[3] Fukagawa & Pedoe 1989, 1.1.2; lost tablet from Miyagi, n.d.; the hint $OA = 5AB/8$ is given, but no solution.

SOLUTION 3 (JMU): A quick proof is possible using a lemma about the figure below (Casey 1888, III: 6, p. 31): *Let circle* (P) *with diameter* p, *inscribed in a segment of circle* (O) *with chord* AB, *touch* AB *at* C *and* (O) *at* T. *Let* M *be the midpoint of* AB *and* $v = ME$ *be the sagitta of the opposite segment. Then* $pv = AC \cdot CB$.

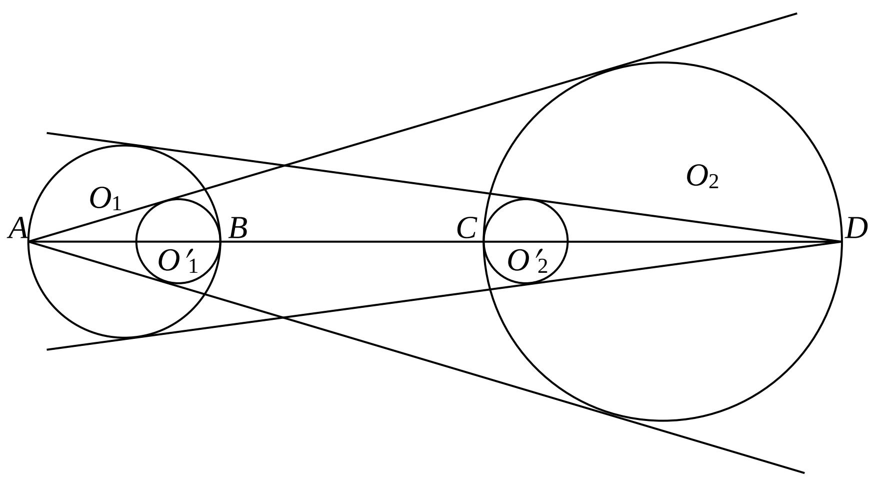


Proof: Because (P) touches AB at C, if $CD \perp AB$, then CD is a diameter of (P). (O) and (P) are homothetic with respect to T, so, if TC and TD produced cut (O) in E and F, respectively, EF is a diameter of (O) and parallel to CD; therefore, it passes through M. Let CD meet the parallel to AB through E in G. Because $\angle ETF$ and $\angle DGE$ are both right angles, the circle (blue) with diameter DE passes through T and G. In that circle, by the Crossed Chords theorem, $TC \cdot CE = DC \cdot CG$. In (O), $TC \cdot CE = AC \cdot CB$. Thus, since $CD = p$ and $CG = ME = v$, we have $AC \cdot CB = pv$. □

In the problem, the position of C on l and length of p depend on d and e, but as explained in Solution 2, we always have $AC = \sqrt{dp}$ and $CB = \sqrt{ep}$. Hence $AC \cdot CB = p\sqrt{de}$. Let M be the midpoint of AB and say that MO cuts $\widehat{AB}$ remote from P in N. Applying the lemma, $AC \cdot CB = p \cdot MN$. Immediately, $MN = \sqrt{de} = AB$. Since A and B are fixed, so are M and N. As only one circle passes through three points (here A, B, N), the size and position of (O) is independent of d and e. □

The fact that $OA = OB = 5AB/8$ (see note 3), which is to say that $\triangle OMA$ and $\triangle OMB$ are 3:4:5 right triangles, follows from the case of $d = e$, in which $AC = CB$. The *wasanka* almost certainly solved the problem by proving this fact first, as Fukagawa and Pedoe imply and is illustrated here.

PROBLEM 4:

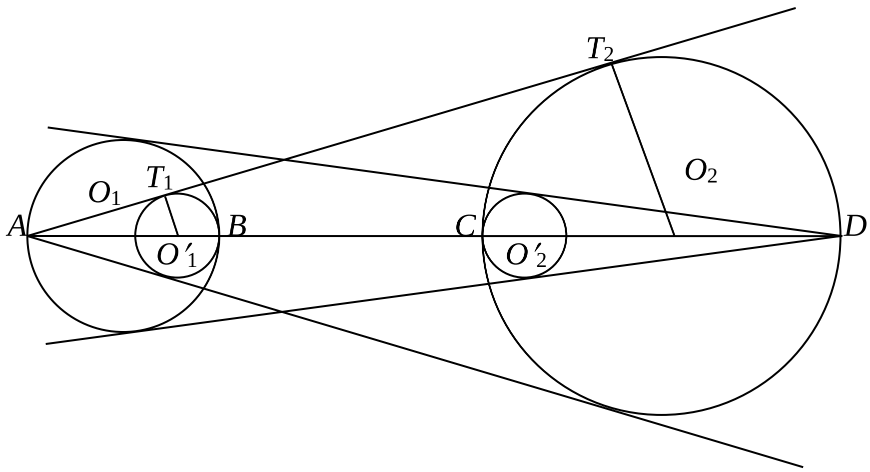


Given two unequal circles with concurrent diameters AB and CD as shown, tangents from A (resp. D) to (O_2) (resp.

(O_1)), and circles tangent to B (resp. C) and the two tangents from A (resp. D), prove that the radii of these two circles are equal.[4]

SOLUTION 4 (F&P):

From $AT_1O_1 \sim AT_2O_2$, it follows that



$$\frac{r_1'}{AB - r_1'} = \frac{r_2}{AO_2}$$

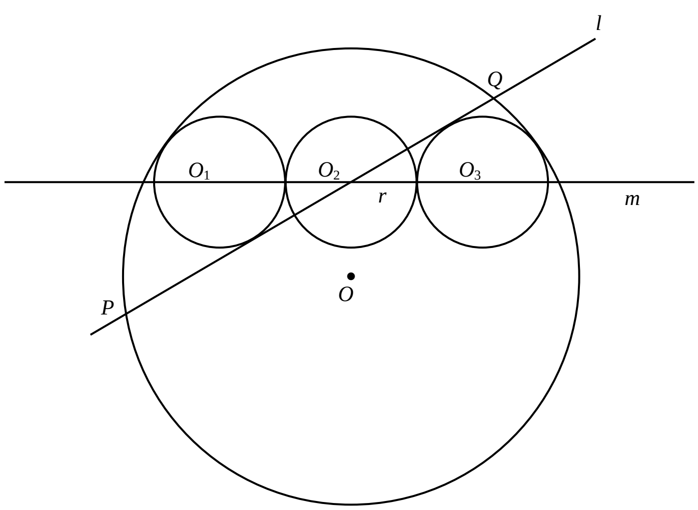
and so

$r'_1(AB + BC + CO_2) = r_2(AB - r'_1)$. Hence

$2r'_1r_1 + r'_1BC + r'_1r_2 = 2r_1r_2 - r_2r'_1$
$2r'_1r_1 + r'_1BC + 2r'_1r_2 = 2r_1r_2$
$r'_1(2r_1 + BC + 2r_2) = 2r_1r_2$

$$r_1' = \frac{2r_1r_2}{2r_1 + BC + 2r_2}.$$

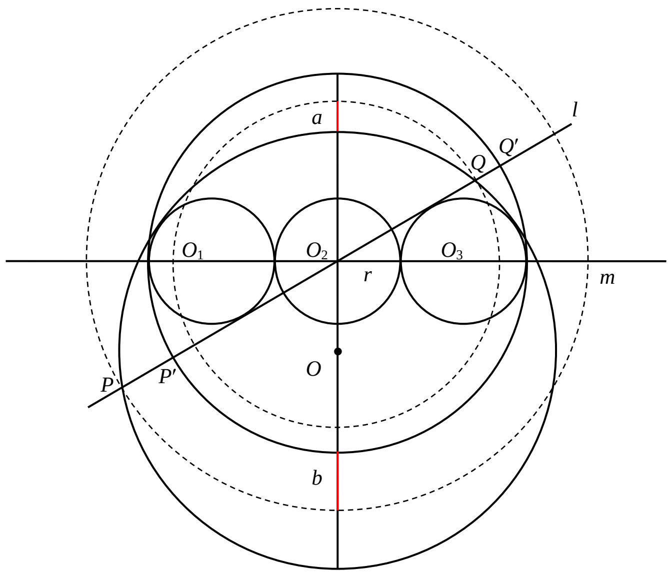
This is algebraically symmetrical: we would have arrived at the same right-side expression for r'_2 if we had started at the other end of the figure. Thus $r'_1 = r'_2$. □



PROBLEM 5: (O_1), (O_2), and (O_3) all have radius r, centers in line m, and form a chain as shown. Line l passes through O_2 and is tangent with O_1 and O_3 on opposite sides of m. Circle (O)r' is internally tangent to (O_1) and (O_3), and is cut by l in P and Q. Prove that $PQ = r' + 3r$.[5]

[4] Fukagawa & Pedoe 1989, 1.3; lost tablet from Aichi, 1842; solution given.
[5] Fukagawa & Pedoe 1989, 1.3.3; tablet from Ibaragi, 1871; no solution given.

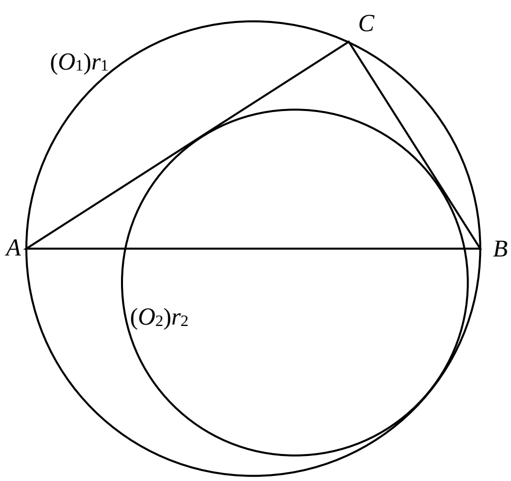
SOLUTION 5 (JMU): The trick is to superimpose the simplest case on the general case. Start with O coincident with O_2: $P'Q' = 6r = r' + 3r$.



Now move O off O_2 along the perpendicular to m. The net change in r' is $r' - 3r$. Measured along l, this is $PP' - QQ'$. Since $P'Q'$ is a diameter of (O_2), $P'Q' = 6r$. Therefore $PQ = (r' - 3r) + 6r = r' + 3r$. □

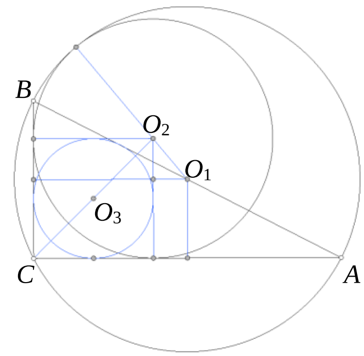
Notice that, measured along the perpendicular to m, the net change in r' is $b - a$ where a and b (red segments) are half the distance between the circumferences of (O_2) and (O). The dashed circles help one see that $PP' = b$ and $QQ' = a$.

A trigonometric solution and a generalization are posted elsewhere on the web.



PROBLEM 6: Given right triangle ACB and its circumcircle $(O_1)r_1$, construct circle $(O_2)r_2$ tangent externally to legs a and b and internally to (O_1). Prove that $r_2 = a + b - c$.[6]

[6] Fukagawa & Pedoe 1989, 2.2.7; lost tablet from Hyōgo, n.d.; no solution given.



SOLUTION 6: Let $C = (0, 0)$. Then $O_2 = (r_2, r_2)$ and $O_1 = (b/2, a/2)$. Since O_1O_2 extended cuts both (O_2) and (O_1) where they touch, $O_1O_2 = r_1 - r_2 = c/2 - r_2$. But as the hypotenuse of the small right triangle, $(O_1O_2)^2 = (b/2 - r_2)^2 + (r_2 - a/2)^2$.

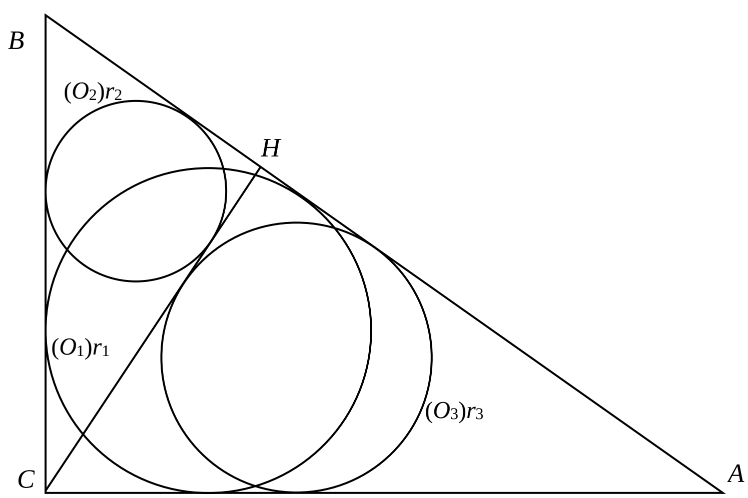
Therefore,

$$c^2 - 4cr_2 + 4r_2^2 = a^2 + b^2 - 4ar_2 - 4br_2 + 8r_2^2.$$

Since we can subtract $c^2 = a^2 + b^2$, this equation reduces quickly to $r_2 = a + b - c$.[7] □

COROLLARY: In any triangle with semiperimeter $s = (a + b + c)/2$, the distance from C to the point where the incircle touches a or b is $s - c$. So in a right triangle such as ACB with incircle $(O_3)r_3$, $r_3 = s - c = (a + b - c)/2$. Therefore $r_2 = 2r_3$.[8]

PROBLEM 7: Right triangle ACB is partitioned into two triangles by the altitude CH as shown. Prove that this altitude is the sum of the radii of the three incircles.[9]



SOLUTION 7 (JMU): All three triangles are right. We use the corollary just stated to calculate $2r_1 = a + b - c$, $2r_2 = BH + CH - a$, and $2r_3 = AH + CH - b$. Adding these equations, we get $2r_1 + 2r_2 + 2r_3 = AH + BH + 2CH - c = 2CH$.

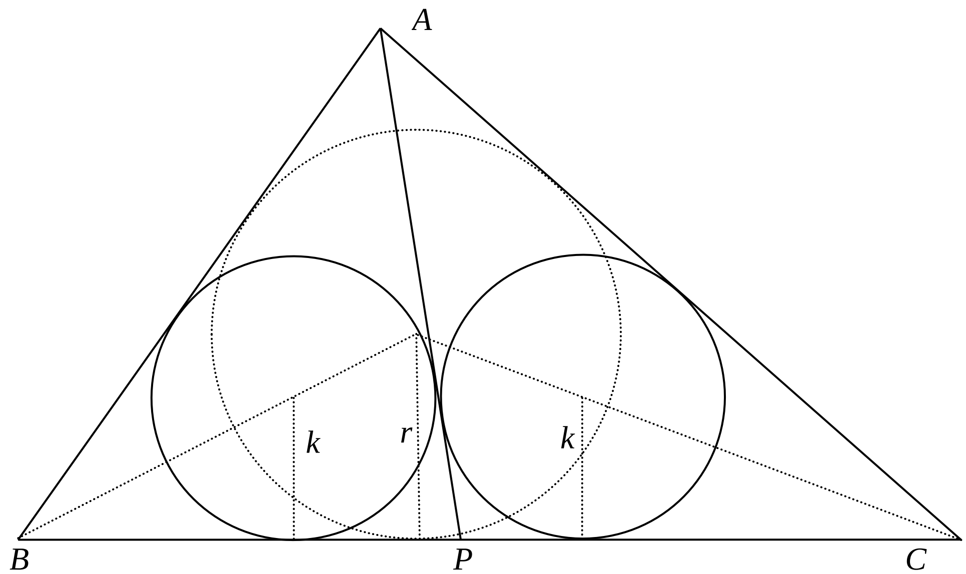
So $r_1 + r_2 + r_3 = CH$. □

[7] See also Okumura & Watanabe 2001 for a theorem that handles this problem as well as Problems 18 and 19 below.

[8] Protasov (Exercise 5) points out that the same relation holds for the radius of the excircle on side AB, which is s, and the radius of the circle tangent to the legs extended and to the excircle externally.

[9] Fukagawa & Pedoe 1989, 2.3.2; tablet from Iwate, n.d.; no solution given.

PROBLEM 8: Given two circles of equal radius inscribed as shown below, prove $AP = \sqrt{s(s-a)}$.[10]



SOLUTION 8 (JMU): In the figure above, *r* is the inradius of *ABC*, *s* is its semiperimeter; *ABP* and *ACP* have semiperimeters s_1 and s_2, respectively, but the same inradius *k*. Using *x* for *AP*, observe that $s_1 + s_2 = s + x$. Consequently, $rs = ks_1 + ks_2 = k(s+x)$ and $k = rs/(s+x)$. Now, by similar triangles,

$$\frac{s-b}{s_1-x} = \frac{r}{k} = \frac{s-c}{s_2-x}.$$

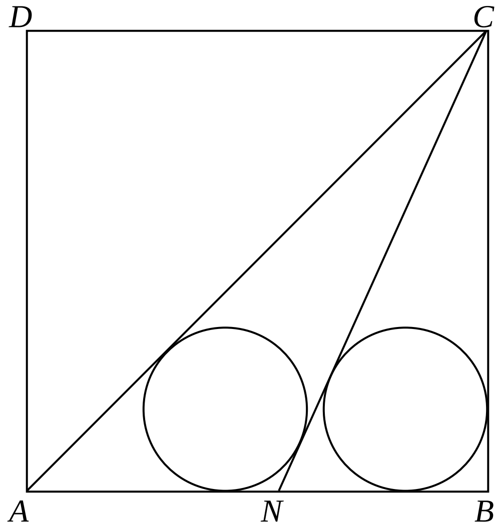
Thus $k(s-b) = r(s_1-x)$ and $k(s-c) = r(s_2-x)$. Adding, $ka = r(s-x)$. Substituting the foregoing $rs/(s+x)$ for k, $ars/(s+x) = r(s-x)$ or $as = s^2 - x^2$. Therefore $x = \sqrt{s(s-a)}$.

COROLLARY: If, in *ABC*, vertex *A* is a right angle, then $k = \dfrac{ab}{\sqrt{2ab}+a+b+c}$.[11]

[10] Fukagawa & Pedoe 1989, 2.2.5; surviving tablet from Chiba, 1897; no solution given.
[11] This is Fukagawa & Pedoe 1989, 2.2.3; lost tablet from Miyagi, 1847; equation given, no solution provided.

PROOF: We have $\Delta = k(b + BP + AP)/2 + k(c + CP + AP)/2 = k(a + b + c + 2AP)/2$, so $k = \frac{ab}{a+b+c+2AP}$. In a right triangle, $s - a = r$, so $AP^2 = s(s - a) = rs = \Delta = ab/2$. That is, $4AP^2 = 2ab$ or $2AP = \sqrt{2ab}$. □

PROBLEM 9: *ABCD* is a square with side a and diagonal AC. The incircles of ACN and BCN are congruent. What is their radius r in terms of a?[12]



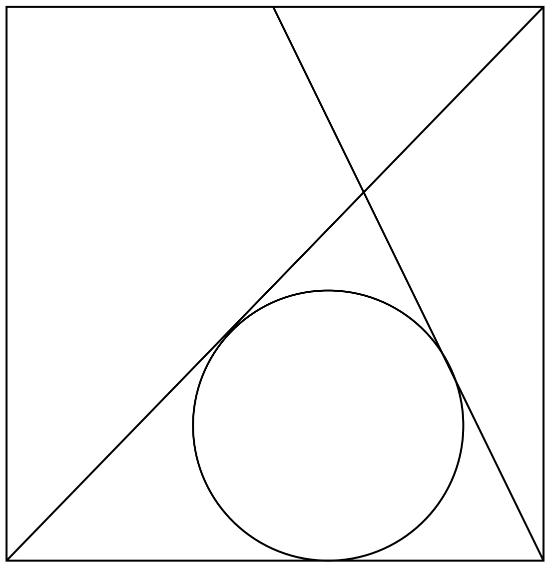
SOLUTION 9 (JMU): Because BCN is a right triangle, $r = (BC + BN - CN)/2$ (see problem 6). The congruence of the two incircles implies $CN^2 = s(s - AB)$, where s is the semiperimeter of ABC (proven in problem 8).

We know $AC = a\sqrt{2}$, so $s = \frac{a\sqrt{2}}{2} + a$.
Hence $CN^2 = \left(\frac{a\sqrt{2}}{2} + a\right)\frac{a\sqrt{2}}{2} = \frac{a^2(\sqrt{2}+1)}{2}$
and $CN = a\sqrt{\frac{\sqrt{2}+1}{2}}$.

Now, since $BN^2 = \frac{a^2(\sqrt{2}+1)}{2} - a^2 = \frac{a^2(\sqrt{2}-1)}{2}$, we also have $BN = a\sqrt{\frac{\sqrt{2}-1}{2}}$.

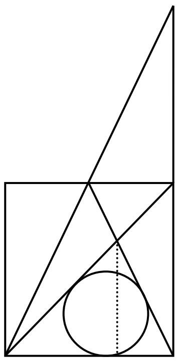
So $r = \frac{1}{2}\left(a + a\sqrt{\frac{\sqrt{2}-1}{2}} - a\sqrt{\frac{\sqrt{2}+1}{2}}\right) = \frac{a}{2} - \frac{a}{2}\sqrt{\sqrt{2} - 1}$. □

[12] Fukagawa & Pedoe 1989, 3.1.7; surviving tablet from Hyōgo, 1893; the solution is given in the form $r = \frac{1}{2}\left(1 - \sqrt{\sqrt{2} - 1}\right)a$.



PROBLEM 10: A square with one diagonal is cut by a line from a third vertex to the midpoint of an opposite side. A circle is inscribed in the resulting triangle opposite the midpoint. What is its radius?[13]

SOLUTION 10 (a posted solution): Imagine completing the figure as shown below.



By congruent triangles, it is easy to see that the top of the square bisects the sides of the large right triangle. Hence the two crossing lines within the square are medians of the large right triangle. The apex of the small triangle containing the incircle is its centroid, and divides the two lines within the square in the ratio 1 : 2. For the same reason, if the side of the square is a, the altitude of the small triangle is $\frac{2}{3}a$ (imagine a line parallel to the top and bottom of the square through the apex of the triangle).

Now the diagonal of the square is $a\sqrt{2}$ and line crossing it is $\dfrac{a\sqrt{5}}{2}$. The sides of the small triangle are $\frac{2}{3}$ of these lengths, respectively. But in any triangle with altitude h on base a, perimeter p, and inradius r, $2\Delta = pr = ha$. Consequently,

$$\left(a+\frac{2}{3}a\sqrt{2}+\frac{2}{3}\frac{a\sqrt{5}}{2}\right)r=\frac{2a}{3}a$$

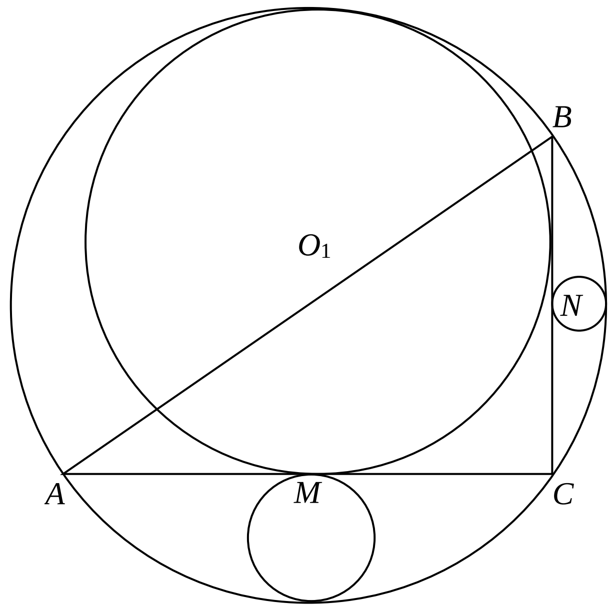
$$(3a+2a\sqrt{2}+a\sqrt{5})r=2a^2$$

$$r=\frac{2a}{3+2\sqrt{2}+\sqrt{5}}$$

□

[13] Fukagawa & Pedoe 1989, 3.1.3; surviving tablet from Miyagi, 1877; solution given in the form $r=\dfrac{2a}{3+\sqrt{5}+\sqrt{8}}$.

PROBLEM 11: A right triangle has three circles tangent to its legs and internally tangent to its circumcircle: (O_1) is tangent to both legs; (O_2) and (O_3) are tangent to legs AC and BC at their midpoints M and N, respectively. Show that $r_1^2 = 32r_2r_3$.[14]



SOLUTION 11 (JMU): The diameters of (O_2) and (O_3) are the sagittae of chords AC and BC: $v_b = 2r_2$ and $v_a = 2r_3$.

Lemma: In any right triangle, the inradius $r = \sqrt{2v_av_b}$. Proof:

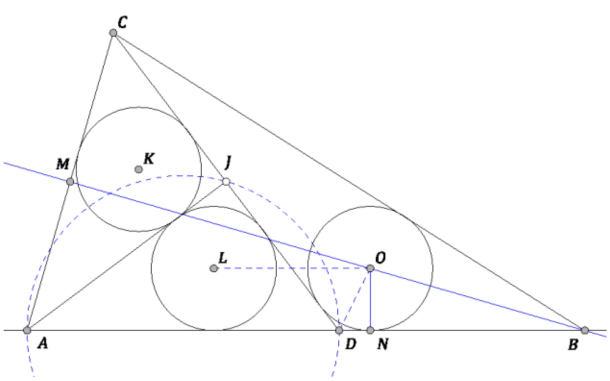
$v_a = R - b/2$ $\qquad$ $v_b = R - a/2$

$2v_a = c - b$ $\qquad$ $2v_b = c - a$

$$4v_av_b = ab - c(a + b - c)$$
$$4v_av_b = ab - 2cr$$
$$2v_av_b = ab/2 - cr$$
$$2v_av_b = rs - cr = r(s - c) = r^2. \square$$

But $r_1 = 2r$ (problem 6), so $r^2 = r_1^2/4 = 8r_2r_3$. Thus $r_1^2 = 32r_2r_3$. □

PROBLEM 12: In ΔABC, $AB = BC$. If one chooses D on AB and J on CD such that $AJ \perp CD$ and the incircles of ΔACJ, ΔADJ, and ΔBCD all have radius r, then $r = AJ/4$.[15]



SOLUTION 12 (JMU):

Given AD, it is easy to construct ΔCAD, $(K)r$, $(L)r$, and a third circle $(O)r$ that touches CD and AD extended. The second tangent to (O) through C meets AD in B. As one moves J along the semicircle with diameter AD, OB cuts AC at different points, passing through the midpoint M of AC for just one choice of J. With that in mind,

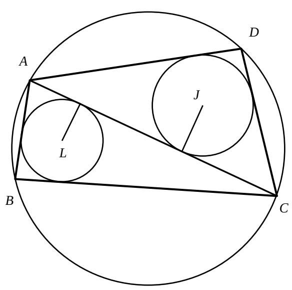
[14] Fukagawa & Pedoe 1989, 2.4.6; surviving tablet from Iwate, 1850; no solution given.
[15] Fukagawa & Rothman 2008:194–96, 212–16; slightly edited version available at http://www.cut-the-knot.org/pythagoras/Ch6Pr3Sangaku.shtml.

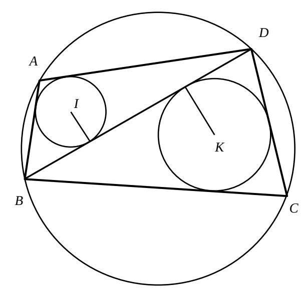
that, I showed that if $r = AJ/4$, then AJC and AJD are 3:4:5 right triangles and $r = 2DN$ (https://www.cut-the-knot.org/pythagoras/Ch6Pr3Unger.shtml, 2009). Then I showed that, if $(K), (L)$, and (O) have radius r and $r = 2DN$, then AJC and AJD are 3:4:5 right triangles. This only indirectly proves the problem theorem; N. Dergiades posted a simpler and more concise direct proof in 2017 (http://www.cut-the-knot.org/pythagoras/Ch6Pr3Dergiades.shtml), using Stewart's theorem, which avoids references to segment DN.

Another proof, by M. Cabart (2010), uses trigonometry (http://www.cut-the-knot.org/pythagoras/Ch6Pr3Cabart.shtml). (The reader should note that "$\angle DAJ$" and "$\angle ADJ$" in the beginning should be $\angle DAJ/2$ and $\angle ADJ/2$.)

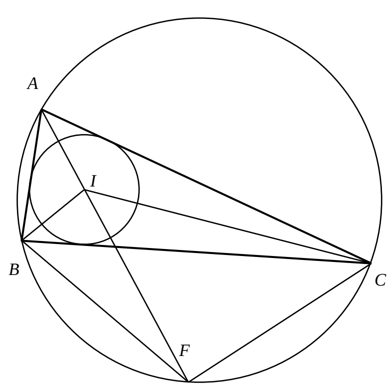
The following is perhaps the sangaku *result most celebrated outside Japan.*

PROBLEM 13: Prove that the sums of the radii of the incircles in both triangulations of a (convex) cyclic quadrilateral are equal.[16]





SOLUTION 13 (JMU): There are many ways to prove this theorem. I have put together the following sequence of results on the basis of hints from several different sources.[17]



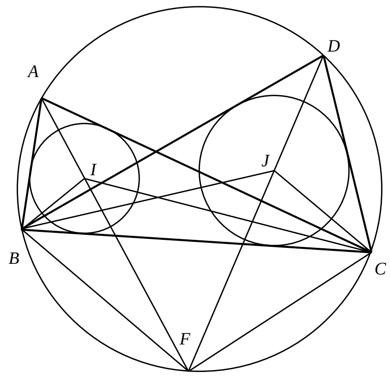
Lemma 1: The bisector from one vertex of a triangle, extended, cuts the circumcircle at the midpoint of the arc subtended by the opposite side of the triangle, which is the center of the circle defined by the other two vertices and the incenter.

Proof: $\angle BIF = \angle BAI + \angle ABI$, that is, half the sum of the vertex angles at A and B. $\angle IBF = \angle CBI + \angle CBF = \angle CBI + \angle CAF$, the same sum. So $\angle BIF = \angle IBF$ and ΔBFI is isosceles. By similar reasoning, so is ΔCFI. Hence $BF = IF = CF$. Moreover, since the $\angle BAF$ and

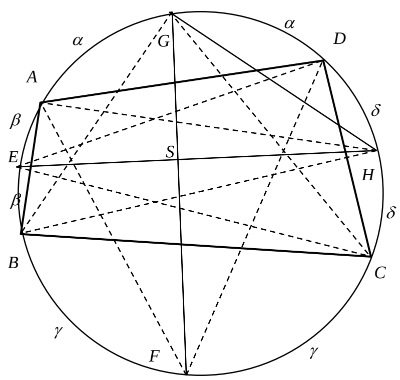
[16] Fukagawa & Pedoe 1989, 3.5(1); lost tablet from Yamagata, 1800.
[17] Most helpful is Ahuja, Uegaki, and Matsushita 2004.

$\angle CAF$, which subtend arcs BF and CF, respectively, are equal, F turns out to be the midpoint of arc BC. □

If we add another point D on the circumcircle as shown, it immediately follows that DJ and AI, extended, concur at F and that all four line segments BF, IF, JF, and CF are equal.



Complete the quadrilateral $ABCD$ and construct the eight bisectors that meet at E, F, G, and H, the midpoints of arcs AB, BC, CD, and DA, respectively. (The diagonals of the quadrilateral have been omitted.) It is easy to prove that EH and FG are perpendicular:



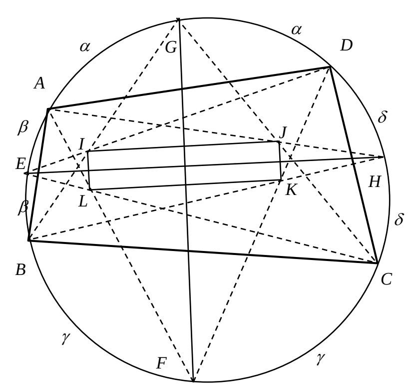
Lemma 2: If a circle is partitioned into four sectors, the lines joining the midpoints of the opposing pairs of arcs are perpendicular.

Proof: By hypothesis, $2\pi = 2\alpha + 2\beta + 2\gamma + 2\delta$. Add auxiliary line GH. $\angle GHE = ½(\alpha + \beta)$. $\angle FGH = ½(\gamma + \delta)$. So $\angle GSH = \pi - ½(\alpha + \beta + \gamma + \delta) = \pi - \pi/2 = \pi/2$. □

This leads to the last lemma, which is an impressive theorem in its own right:

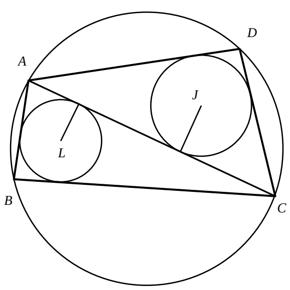
Lemma 3: The incenters of the four triangles formed by the sides of a convex cyclic quadrilateral and its diagonals are the vertices of a rectangle with sides parallel to the lines joining the midpoints of the arcs subtended by the sides of the quadrilateral.

Proof: In the figure, $\angle DEH$ and $\angle HEC$ subtend equal arcs, so EH bisects $\angle DEC$. Lemma 1 assures that $EI = EL$. Thus ΔEIL is isosceles with base IL perpendicular to EH. Applying the same reasoning at H, we conclude that JK is perpendicular to EH, and therefore parallel to JL. Likewise, IJ and LK are parallel and perpendicular to FG. Since EH and FG are themselves perpendicular (Lemma 2), $IJKL$ is a rectangle. □

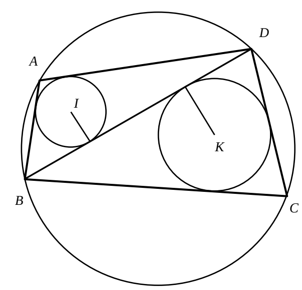


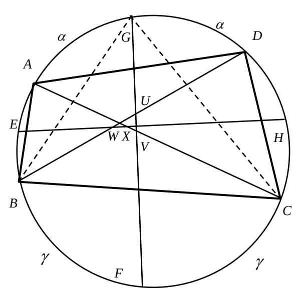
We are now ready to prove the original theorem, which states:

The sums of the radii of the incircles in both triangulations of a (convex) cyclic quadrilateral are the same.

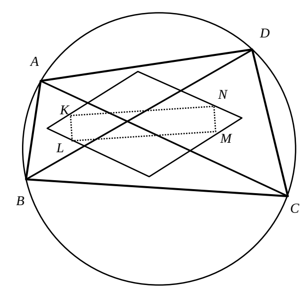


PROOF: If we draw lines through L and J parallel to AC (left) and through I and K parallel to BD (right), the perpendicular distances between each pair of lines will be the sum of the radii of the corresponding pairs of incircles. To prove these sums are equal, it suffices to show that the parallelogram produced by superimposing the two sets of parallel lines is a rhombus, because the two altitudes of a rhombus are equal.

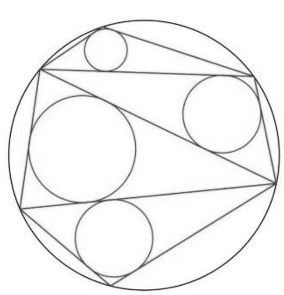




To that end, observe that $\angle ACG = \angle DBG$ because they subtend equal arcs. $\angle BGF = \angle CGF$ for the same reason. Hence, $\Delta BUG \sim \Delta CVG$ with $\angle BUG = \angle CVG$. That is, AC and BD cut GF at the same angle in opposite directions.

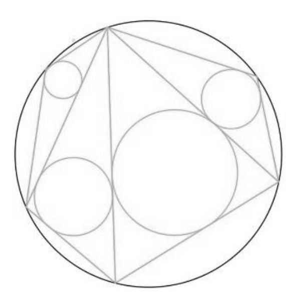


Since EH and FG are perpendicular (Lemma 2), AC and BD likewise cut EH at W and X at the same angle in opposite directions. Hence all lines parallel to the diagonals of the quadrilateral cut the axes of rectangle $KLMN$ (Lemma 3) at the same angles. So the four triangles based on the sides of the rectangle that, together with it, make up the parallelogram, are all isosceles, and we have a rhombus (four sides equal). (Another necessary and sufficient condition for a parallelogram to be a rhombus is that its diagonals be perpendicular: the diagonals of this rhombus lie on EH and FG.) □



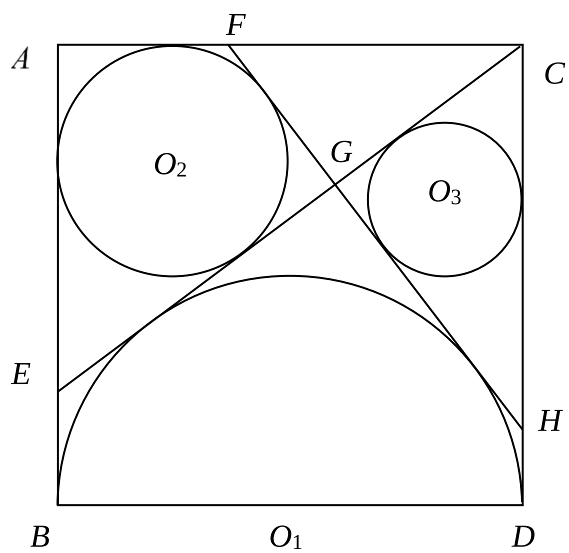
COROLLARY: The sums of the inradii in any of triangulation of a (convex) cyclic polygon are all the same.

For example, here are two of triangulations of the same cyclic hexagon. There are many others. Yet the sum of the radii of the incircles is the same for all of them.



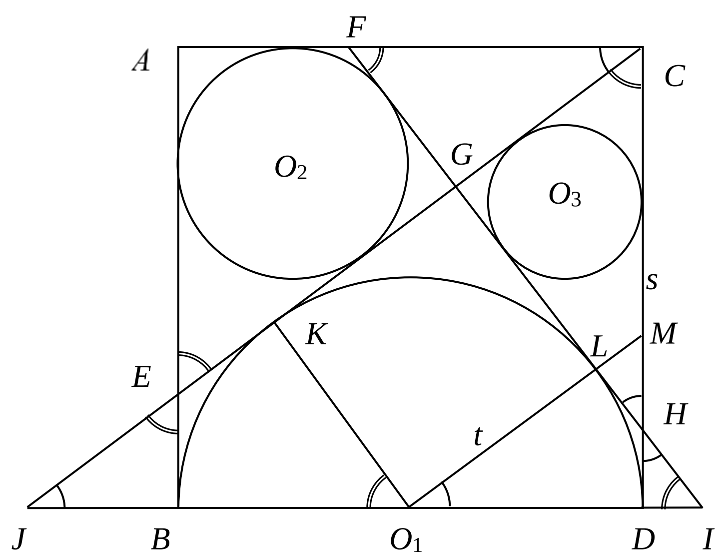
PROOF: The previous theorem establishes this theorem for cyclic quadrilaterals. Assume it holds for cyclic n-gons. Every cyclic polygon of $n + 1$ sides can be analyzed as a cyclic n-gon plus a triangle by selecting three adjacent vertices of the starting polygon for the triangle and regarding all the vertices other than the middle one of these three as a cyclic n-gon. Since the same triangle is added to every triangulation of the cyclic n-gon, the theorem holds for the larger polygon too. □

This corollary is frequently described as a theorem by itself.

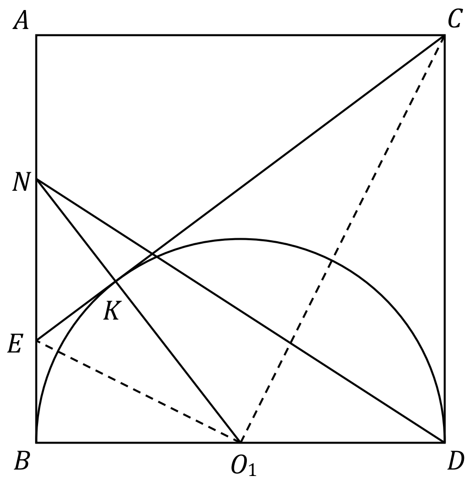


PROBLEM 14: In square $ABCD$, CE is tangent to semicircle BO_1D. (O_2) is the incircle of ACE. The tangent to (O_1) and (O_2) meets the sides of the square in F and H and intersects CE in G. (O_3) is the incircle of CGH. Prove that $r_2/r_3 = 3/2$.[18]

SOLUTION 14 (JMU): First, we prove $CE \perp FH$. Extend BD, CE, and FH and draw the normals KO_1 and LO_1 as shown below. Mark equal angles noting where parallels are cut by transversals, complementary acute angles in known right triangles, vertical angles, and equal angles in similar triangles. There are two kinds of acute angles in each right triangle. Both kinds are found at O_1; since they are complementary, KO_1L must be a right angle. All the right triangles containing both kinds of acute angle are similar, and, by the lemma proved presently, have sides in the ratio 3:4:5.



[18] Fukagawa & Pedoe 1989, 3.2.5, lost tablet of 1838 from Iwate prefecture; no solution given.



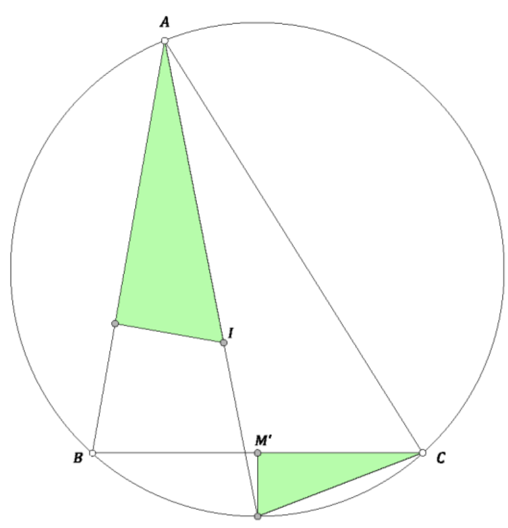
Let s be the side of square $ABCD$ and $t = s/2$ be the side of square GLO_1K. Note pairs of tangents from the same points to the same circles: $BE = EK$, $CD = s = CK$, and $DH = HL$. Because of this last pair, if we extend LO_1 to meet CD in M, $HLM \cong HDI$. For later convenience, say that a, b, and c are the lengths of $DI = LM$, $DH = HL$, and $HI = HM$, respectively, noting that a:b:c :: 3:4:5.

We now prove the key lemma. In the auxiliary figure below, we extend KO_1 to meet AB in N, and add lines EO_1 and CO_1. EO_1 and CO_1, which form congruent triangles with radii of and equal tangents to circle O_1, bisect supplementary angles, so $\angle CO_1E = 90°$ and KO_1 is the altitude to the hypotenuse of right ΔCO_1E. Hence $KO_1^2 = CK{\cdot}EK$, or $t^2 = s{\cdot}EK = 2t{\cdot}EK$. Therefore $EK = t/2 = BE$. Observe that this implies $AE = ¾\, AC$, so ACE is a 3:4:5 right triangle.

Now, returning to the figure above, in ΔACE, $AE + AC - CE = 2r_2 = (s - BE) + s - (s + EK) = s - 2BE = t$ (by the lemma). Thus $r_2 = t/2$. In ΔCGH, $CG + GH - CH = 2r_3 = (s - t) + (t + HL) - (s - DH) = 2b$. Thus $r_3 = b$.

But $c/b = 5/4$, so $b + c = 9b/4$. In ΔDO_1M, $t/(b + c) = 4/3$. Thus $3t = 9b$, or $r_3 = t/3$, $r_2/r_3 = 3/2$. □

PROBLEM 15: In circumscribed triangle ABC, let M' and M be the midpoints of, respectively, chord and arc BC. Then $v_a = M'M$ is the SAGITTA of the chord a. Prove that the square of the distance from a vertex of a triangle to its incenter is four times the product of the sagittae to the adjacent sides.[19]



SOLUTION 15 (F&P modified):

We write a' for $s - a$, etc. for convenience.

Square Heron's Formula and divide by s: $r^2s = a'b'c'$ or $a'b'c' = r^2(a' + b' + c')$.

[19] Fukagawa & Pedoe 1989, 2.2; lost tablet of 1825 from Musashi; much lengthier traditional solution provided.

AM bisects A because angle bisectors in circumscribed triangles pass through the midpoints of the arcs they subtend. $\angle MAB = \angle BCM$ (both subtend arc BM) $= \angle CAM$, so $a'/r = CM'/M'M = (½CB)/v_a = (b' + c')/2v_a$. That is, $2v_a a' = (b' + c')r$. Likewise, $2v_b b' = (a' + c')r$ and $2v_c c' = (a' + b')r$.

Multiply these last two equations together and use the foregoing relationship to simplify:

$4v_b v_c b'c' = (a' + c')(a' + b')r^2 = b'c'r^2 + a'(a' + b' + c')r^2 = b'c'r^2 + a'^2 b'c'$.

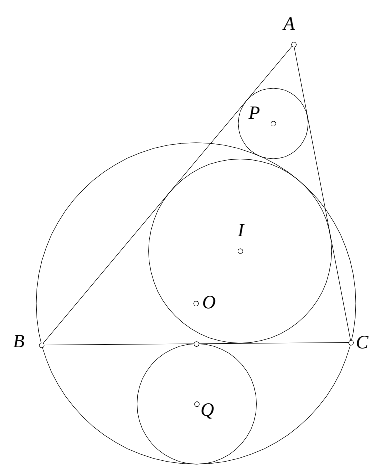
Now divide by $b'c'$: $4v_b v_c = r^2 + a'^2$. From the definition of a', $a'^2 + r^2 = AI^2$. □

COROLLARY 1: since $4v_b v_c = AI^2$, $4v_a v_c = BI^2$, and $4v_a v_b = CI^2$, $4^3(v_a v_b v_c)^2 = (AI \cdot BI \cdot CI)^2$, or $8v_a v_b v_c = AI \cdot BI \cdot CI$.

COROLLARY 2: if ACB is a right angle, $CI^2 = 2r^2$, so $(AI \cdot BI)^2 = 16 v_a v_b v_c^2 = 8r^2 v_c^2$. But v_c is the radius of the circumcircle when ACB is a right angle, so $4v_c^2 = AB^2$. Therefore, in a right triangle, $AI \cdot BI = r \cdot AB\sqrt{2}$.[20]

PROBLEM 16: Triangle ABC has incircle $(I)r$, to which (O), passing through B and C, is internally tangent. Circle $(P)p$ is tangent to AB and AC and externally tangent to (O). Circle $(Q)q$ is internally tangent to (O) and tangent to BC at its midpoint M. Show that $r^2 = 4pq$.[21]

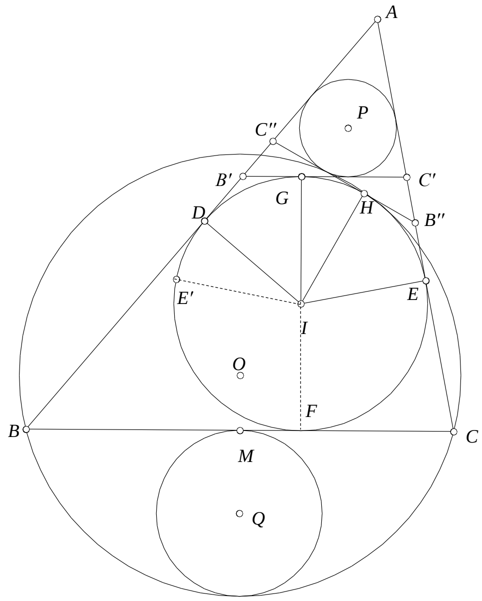
SOLUTION 16 (JMU)[22]:



Construct the two common internal tangents of (I) and (P), and label them as shown. $AB'C'$ is the reflection in AI of $AB''C''$; both triangles share incircle (P) and excircle (I).

In (I) as excircle, $DIG = EIH = B' = B''$ and $DIH = EIG = C' = C''$. Hence $GIH = C' - B'$. But in (I) as incircle, $FI \perp BC$, so $DIF = \pi - B$ and $EIF = \pi - C$. That is, $DIF - EIF = C - B$.

[20] This is Fukagawa & Pedoe 1989 problem 2.2.1 (Fukushima, n.d.); no solution given.
[21] Fukagawa & Pedoe 1989, 2.4.2.
[22] This solution supersedes the one offered in Unger 2010.

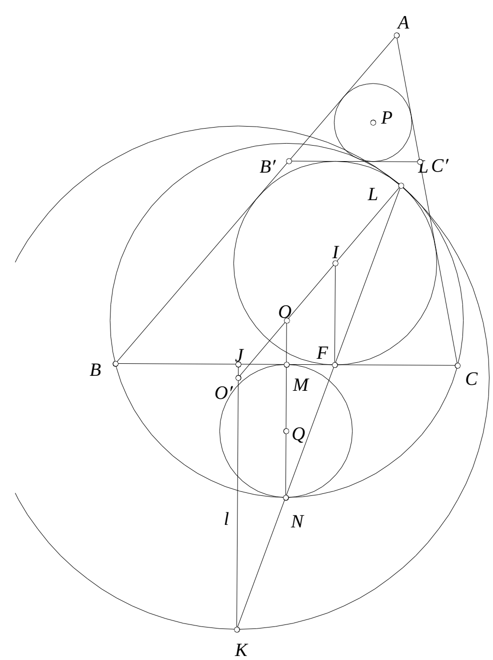


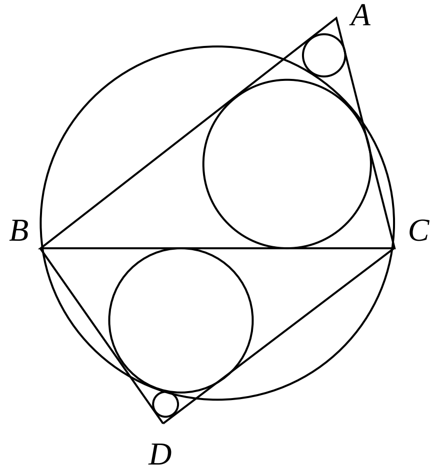
Now construct E', the reflection of E in IG. On the one hand, $DIF - EIF = DIF - E'IF = E'ID$. On the other, IG bisects $E'IE$ and $DIG = EIH$, so $E'ID = GIH$. Thus $C - B = C' - B'$. But $C + B = \pi - A = C' + B'$. Adding and subtracting equations, $C = C'$ and $B = B'$. Therefore $BC \parallel B'C'$ and $ABC \sim AB'C'$.

Consequently, if s is the semiperimeter of ABC, then the semiperimeter of $AB'C'$ is $s - a$. Hence $p/r = (s - a)/s$, or $4qr(s - a)/s = 4pq$. So $4pq = r^2$ is equivalent to $4q(s - a) = rs$. But rs is the area of ABC, which equals $4q(s - a)$ if and only if the radius of the excircle to ABC on side BC is $4q$. We now show that it is.

Note that $CF = s - c$. Place J on BC so that $BJ = CF$, and draw $l \perp BC$ through J; line l passes through the center of the relevant excircle (K) (in fact, $K = l \cap AI$.) Notice too that $JM = FM$ since M is the midpoint of BC.

Let N be the point diametrically opposite M in (Q), and say that L is the point of contact of (I) and (O). These circles are homothetic with respect to L, so $LIF \sim LON$. LO is the locus of centers of other circles homothetic to (I) and (O), and LN is the locus of the points where those circles intersect lines parallel to IF and ON. Thus, for $O' = l \cap LO$, (O') with radius $O'K$ is homothetic to (I) and (O), and L, F, N, and K are collinear. Hence FJK is a right triangle (FNK is a straight line) with median MN. Therefore $JK = 2MN = 4q$. □





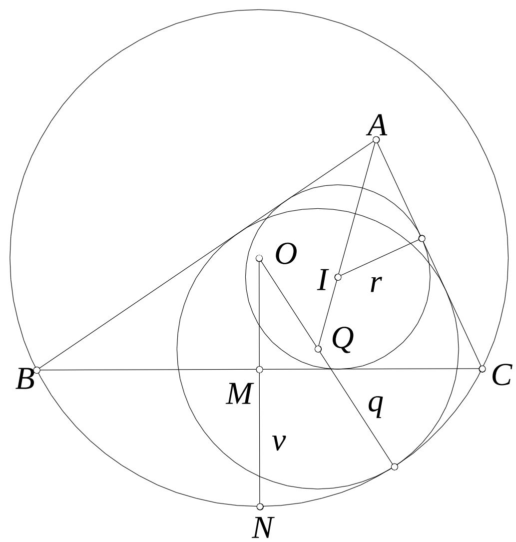
COROLLARY[23]: In the case of two triangles, ABC and BCD, if the radii of the two circles tangent to BC are r_1 and r_2 and the radii of the two small circles at A and D are r_1' and r_2', then $r_1r_2 = (r_1'r_2'/2BC)^2$. For if the diameter perpendicular to BC measures d_1 above BC and d_2 below, $2d_1r_1 = r_1'^2$ and $2d_2r_2 = r_2'^2$. Multiply these equations together, noting that $d_1d_2 = (BC/2)^2$. □

PROBLEM 17:

Let the semiperimeter of triangle ABC with inradius r (below left) be s, and the sagitta to side BC be v. Circle *(O)* passes through B and C. Let circle *(Q)*q be tangent to AB, AC, and *(O)* internally. Prove that

$$q = r + \frac{2v(s-b)(s-c)}{as}.$$ [24]

N.B. A can be anywhere on the plane, but the problem is presented with A inside *(O)*.



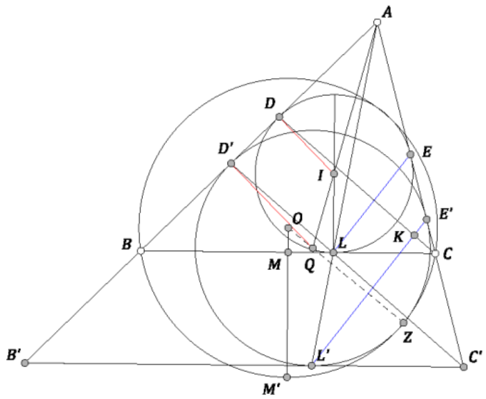
SOLUTION (JMU):

In Solution 16, we showed that, if *(I)* touches *(O)* internally, then $r = 2v(s - a)/s$. We now add *(Q)*q touching AB, AC, and arc BC. *(Q)* and *(I)* are homothetic with respect to A, and $AID \sim AQD'$ (see the figure below, where $ID = r$ and $QD' = q$ are marked in red). By similar triangles,

[23] Fukagawa & Pedoe 1989, 2.5.5.

[24] Fukagawa & Pedoe 1989, 2.2.8 (1781, n.pl.), "a hard but important problem." This is an edited version of the solution in Unger 2010.

$$\frac{q-r}{r} = \frac{AD' - AD}{AD} = \frac{DD'}{AD}.$$



But $AD = s - a$. Therefore,

$$q - r = r \cdot \frac{DD'}{s-a} = \frac{2v(s-a)}{s} \cdot \frac{DD'}{s-a}$$
$$= \frac{2v \cdot DD'}{s},$$

and $\frac{2v \cdot DD'}{s} = \frac{2v(s-b)(s-c)}{as}$ provided that

$$\frac{DD'}{s-b} = \frac{s-c}{a}.$$

Since $BD = s - b$ and $CL = s - c$, this last proportion is true if $BLD' \sim BCD$. To prove that, it suffices to show that $LD' \parallel CD$ because D' lies on BD and L lies on BC.

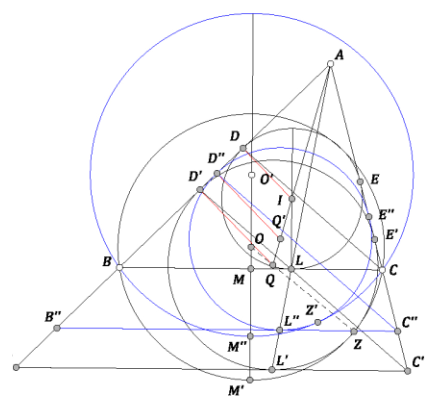
Extend AL to cut *(Q)* in L', and let $B'C'$, as shown, be the tangent to *(Q)* through L'. Since *(Q)* and *(I)* are homothetic, $BC \parallel B'C'$ and $ABC \sim AB'C'$. *(Q)* is the incircle of $AB'C'$, so the corresponding sides of the intouch triangles DEL and $D'E'L'$ are parallel. In particular, $EL \parallel E'L'$.

Since the line joining any two intouch points of a triangle is perpendicular to the line joining the third with the opposite vertex, $C'D' \parallel CD$. But $EL \parallel E'L'$ also implies that right triangle $C'KE'$ is similar to $C'LE$ (their hypotenuses coincide). Hence the point where $C'D'$ and EL meet (there can only be one) is L. Since L lies on $C'D'$, $LD' \parallel CD$.

To get the general case, we imagine moving O along OM to a new location O' (blue lines in the adjoining figure). Notice that we still have $AID \sim AQ'D''$, $LE \parallel L''E''$, and $C''D'' \parallel CD$, so

$$\frac{DD''}{s-b} = \frac{s-c}{a}$$

as before, which means that



$$q - r = \frac{2v(s-b)(s-c)}{as}$$

as claimed. □

Remarks:

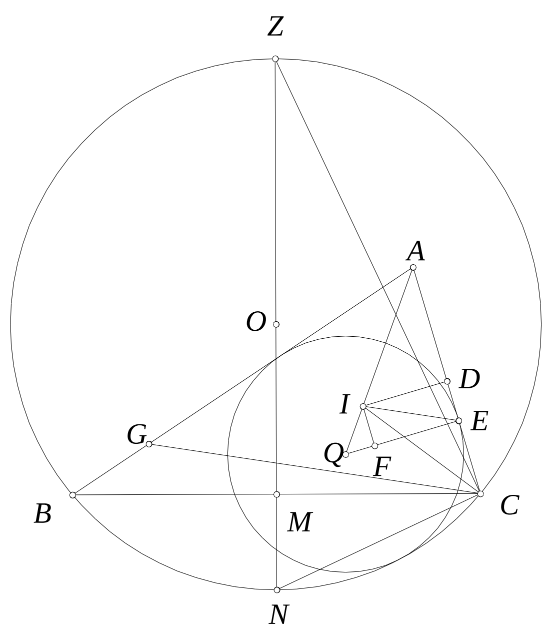
1. Notice that the foregoing proof does not require the Sawayama Lemma.[25]

2. For a different analysis, see Fukagawa & Rigby 2002: 32, 97. They attribute Problem 17 and a related one, with *A* outside *(O)* and *(Q)* externally tangent to *(O)*, to Ajima Naonobu (1732–1798) but do not give the name of their source. The only explicit Japanese proof of which I am aware is by Aida Yasuaki (1747–1817).

3. Fukagawa and Rigby sketch what they say is the traditional solution of Problem 17, ending up with the quadratic equation

$$2avs(s-a)q^2+\left[\frac{a^2\Delta}{s-a}-\frac{4v^2\Delta}{s}-2av(b+c)\right]\Delta q+\left[2av-\frac{2\Delta^2 av}{s^2(s-a)^2}-\frac{a^2\Delta}{s(s-a)}+\frac{4v^2\Delta}{s(s-a)}\right]\Delta^2=0$$

where Δ is the area of *ABC*). They assert that this leads to $q=r+\dfrac{2v(s-b)(s-c)}{as}$ and that the solution of the related problem in remark 5 below "is similar."



4. Fukagawa and Rigby also observe that, in the adjoining figure,

$$\tan\frac{\alpha}{2}=\frac{r}{s-a}$$

and, if *δ* is the angle *CBN* = *BCN* = *CZN*, then

$$\tan\delta=\frac{v}{a/2}.$$

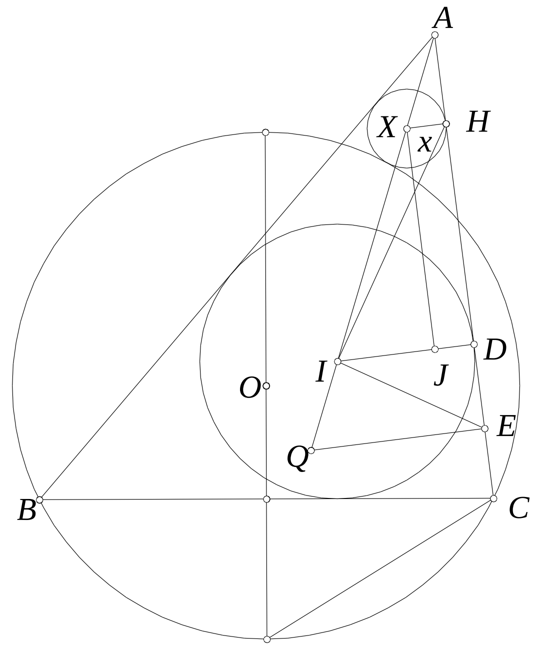
Therefore, if we use Heron's formula in the form $r^2s=(s-a)(s-b)(s-c)$, we obtain $\dfrac{2v(s-b)(s-c)}{as}=\dfrac{2v}{a}\cdot\dfrac{r^2}{s-a}$. This amounts to saying that the problem is equivalent to proving that $q-r=r\tan\dfrac{\alpha}{2}\tan\delta$.

This is not hard to do provided that one can prove *IEQ* = *δ*, or, equivalently, only if *CG*, isogonal to *CZ* with respect to angle *ACB*, is parallel to *IE*. Let *E* be the point

[25] Ayme (2003). Y. Sawayama, an instructor at the Central Military School in Tōkyō published the lemma in 1905 coincidental to solving another problem. The algebraic solution by "yetti" posted on MathLinks, 1 January 2005, does require the Sawayama Lemma.

where *(Q)* touches *AC*, and *F* be the foot of the perpendicular from *I* to *EQ*. Then *DI* || *EQ* and *IEQ* = *DIE*. We immediately have *EQ* – *EF* = *q* – *r* = *FQ* = *IF* tan (α/2) = (*EF* tan *IEF*) tan (α/2) = *r* tan δ tan (α/2). *IEQ* = δ can be proven independently, but the proof is no simple matter (see Protasov 1992, 1999). In any event, the Japanese did not use trigonometric functions, and arrived at the equivalent of Solution 17 algebraically.

5. The related problem mentioned in remark 3 above is the following:

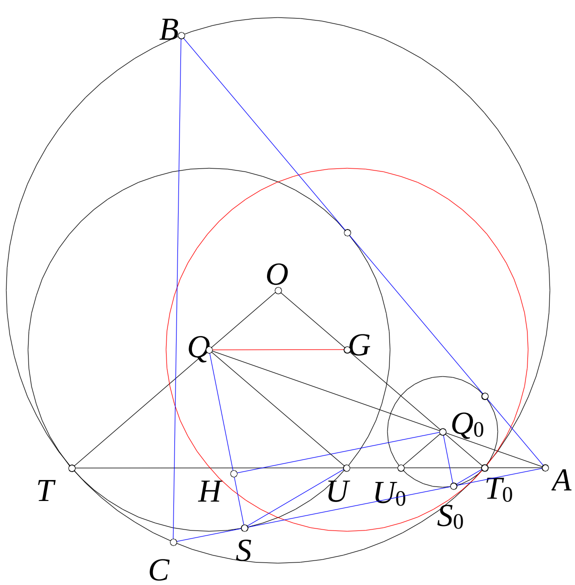


If *A* is outside *(O)*, then, $x = r - \dfrac{a(s-b)(s-c)}{2vs}$.[26]

Note that $\dfrac{a(s-b)(s-c)}{2vs} = r\tan\dfrac{\alpha}{2}\cot\delta$ for the reasons previously stated. Thus this problem is equivalent to proving that $r - x = r\tan\dfrac{\alpha}{2}\cot\delta$.

6. When *A* is outside (*O*), there is another variation that Fukagawa and Rigby do not mention, but which can be solved using more or less traditional *sangaku* methods.

A second circle (Q_0)q_0 that, like (*Q*)*q*, is internally tangent to (*O*)*R* and tangent to *AB* and *AC*. The Japanese knew that $\dfrac{k^2R^2}{x^2} = (R-q)(R-q_0)$, where $k = HQ_0 = SS_0$, and $x = TT_0$. A laborious proof this using just algebra appears in Nakayama 2008. It can be simplified with an easy use of the Law of Cosines as follows:



Place *G* on OT_0 such that (*G*)*q* touches (*O*) at T_0, and let $t = QG = UT_0$.

[26] Fukagawa & Rigby 2002 (p. 32) incorrectly write $x = r - \dfrac{2v(s-b)(s-c)}{as}$.

In $\triangle QGQ_0$, we have $QQ_0^2 = t^2 + (q - q_0)^2 - 2t(q - q_0) \cos(\pi/2 + GOQ/2)$. Hence

$$QQ_0^2 = t^2 + (q - q_0)^2 + 2t(q - q_0) \sin(GOQ/2).$$

But in right $\triangle QHQ_0$, we have $QQ_0^2 = k^2 + (q - q_0)^2$. Therefore,

$$k^2 = t^2 + 2t(q - q_0) \sin(GOQ/2).$$

Since $x/2 = R \sin(GOQ/2)$, this is equivalent to $k^2 = t^2 + xt(q - q_0)/R$, or

$$k^2 = t[t + x(q - q_0)/R].$$

And since $\frac{t}{x} = \frac{R-q}{R}$, we can replace t to get $k^2 = \frac{x(R-q)}{R}\left[\frac{x(R-q)}{R} + \frac{x(q-q_0)}{R}\right]$, or

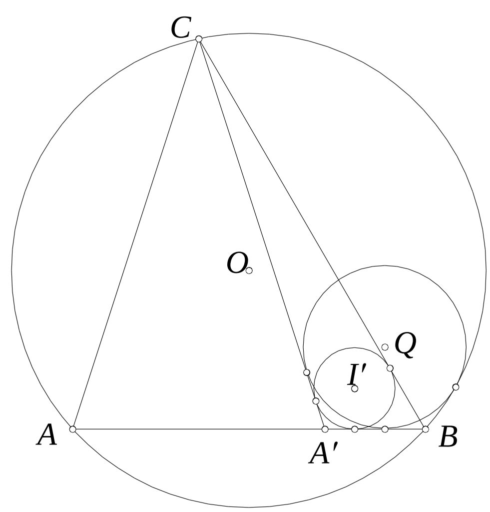
$$\frac{k^2R^2}{x^2} = (R-q)[(R-q)+(q-q_0)].\text{ Thus } q_0 = R - \frac{k^2R^2}{x^2(R-q)}. \square$$

An even quicker way to a formula for q_0 follows from the theorem of Menelaus:

$\frac{AT_0}{AT}\cdot\frac{QT}{OQ}\cdot\frac{OQ_0}{Q_0T_0} = 1$, or $\frac{AT_0}{AT}\cdot\frac{q}{R-q}\cdot\frac{R-q_0}{q_0} = 1$. Since $\frac{AT_0}{AU} = \frac{q_0}{q}$, $\frac{AT_0}{AT}\cdot\frac{R-q_0}{R-q} = \frac{AT_0}{AU}$ or $q_0 = R - \frac{AT}{AU}(R-q)$. □

Here are four problems that can be solved with the aid of the theorem inferred from Solution 17.

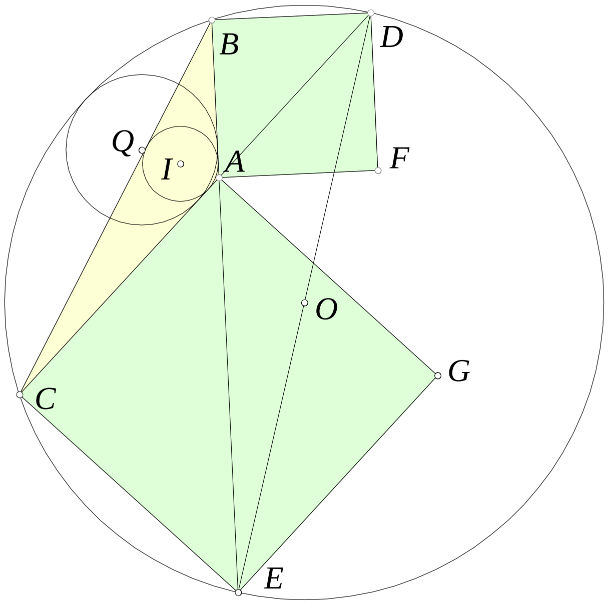
PROBLEM 18:[27] Given ABC inscribed in (O), $AC = A'C$, $(I')r'$ the incircle of $A'BC$, and mixtilinear circle $(Q)q$ in $A'BC$ as shown, prove that $2r' = q$.



SOLUTION 18 (JMU): We know that $q = r' + r'(\tan BA'C/2)(\tan BOC/4)$.

But $BOC/4 = BAC/2 = AA'C/2$ (since it is given that ACA' is isosceles) $= \pi/2 - BA'C/2$ (since $AA'C$ and $BA'C$ are supplementary), so we have $\tan BOC/4 = \cot BA'C/2$. Hence $(\tan BA'C/2)(\tan BOC/4) = 1$. □

[27] Fukagawa & Pedoe 1989, 2.3.4 (1857, Miyagi); no solution given.

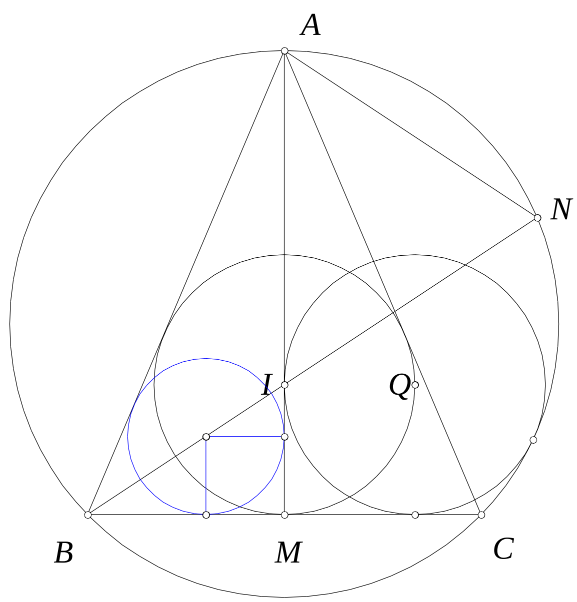


PROBLEM 19:[28] Suppose that, for a point A inside (O), there are chords BD and CE such that $ABDF$ and $ACEG$ are squares. Let $(I)r$ be the incircle of ABC and $(Q)q$ the mixtilinear circle shown. Prove that $2r = q$.

SOLUTION 19 (JMU): Note that, given one square, say $ABDF$, the other is uniquely determined: $C = AD \cap (O)$ and $E = (O) \cap AB$. DE is a diameter of (O); both BE and CD are straight lines. Since $BEC = BDC = \pi/4$. $BOC = \pi/2$ and $BAC = 3\pi/4$.

Therefore $(\tan BAC/2)(\tan BOC/4) = (\tan 3\pi/8)(\tan \pi/8)$. But $3\pi/8$ and $\pi/8$ are complementary, so $\tan 3\pi/8 = \cot \pi/8$. Thus $(\tan BAC/2)(\tan BOC/4) = 1$, and, once again, $q = 2r$. □

PROBLEM 20:[29] ABC, an isosceles triangle with sides $b = c$ and base a, has incircle $(I)r$, circumcircle (O). M is the midpoint of BC. $(Q)q$ touches AM, MC, and (O) as shown. Prove that $r = q$.



SOLUTION 20 (JMU): Let N be midpoint of the arc AC remote from B, r' be inradius of AMB or AMC, and $k = a/2$ for convenience. We know that

$q = r'(1 + \tan AMC/2 \tan CAN)$.

Since AMC is a right angle, this is $q = r'(1 + \tan CAN)$.

Since (O) circumscribes ABC, BIN is a straight line and $CBN = CAN$. But $\tan CBN = r/k = r'/(k - r')$.

[28] Fukagawa & Pedoe 1989, 3.2 (1799, Musashi); longer solution given. F&P imply that DE being a diameter of (O) is a necessary condition, but Okumura & Watanabe 2001 show that it is not.

[29] Fukagawa & Pedoe 1989, 2.3.5 (1901, Fukushima); no solution given.

Thus $q = r'[1 + r'/(k - r')] = kr'/(k - r') = r$. □

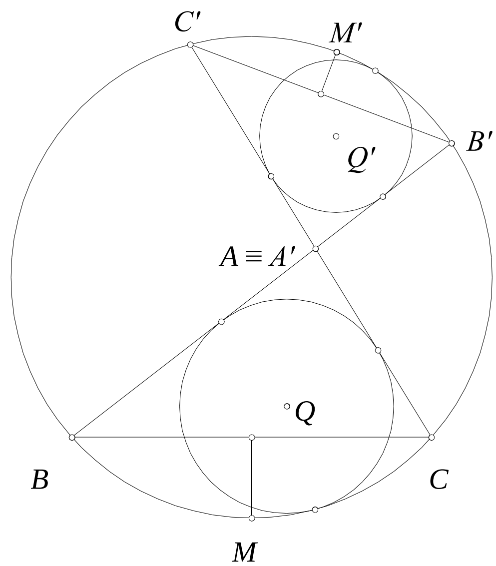
PROBLEM 21: Given two similar triangles ABC and $A'B'C'$ in $(O)R$ formed by the diagonals and opposite sides a and a' of a cyclic quadrilateral, prove that $q/q' = va'/v'a$, where v and v' are the sagittae of the triangles.

SOLUTION 21:[30]

Let BCM (= CBM) = δ and $B'C'M'$ (= $C'B'M'$) = δ'. Note that

$BC'C$ (= $BB'C$) = 2δ,
$B'CC'$ (= $B'BC'$) = $2\delta'$, and
$2\delta + 2\delta' = A$ (= A'),

That is, $\delta + \delta' = A/2$.



To facilitate the calculation, we want to express q and q' as products of like factors as nearly as possible. To that end, note first that

$\cos\delta' = \cos A/2 \cos\delta + \sin A/2 \sin\delta$,
$\cos\delta = \cos A/2 \cos\delta' + \sin A/2 \sin\delta'$.

Therefore,

$(\cos\delta')/(\cos A/2 \cos\delta) = 1 + \tan A/2 \tan\delta$,
$(\cos\delta)/(\cos A/2 \cos\delta') = 1 + \tan A/2 \tan\delta'$.

Next, $a = r(\cot B/2 + \cot C/2)$. Using $\cot x = \cos x/\sin x$, and noting that

$\cos B/2 \sin C/2 + \cos C/2 \sin B/2 = \sin(B/2 + C/2) = \cos A/2$,

this becomes $a = r(\cos A/2)/(\sin B/2 \sin C/2)$. Therefore

$r = a(\sin B/2 \sin C/2)/\cos A/2$, and so too
$r' = a'(\sin B/2 \sin C/2)/\cos A/2$.

Finally, $a = 2R \sin 2\delta = 4R \sin\delta \cos\delta$. Likewise, $a' = 4R \sin\delta' \cos\delta'$. Therefore

$r = (4R \sin\delta \cos\delta \sin B/2 \sin C/2)/\cos A/2$, and
$r' = (4R \sin\delta' \cos\delta' \sin B/2 \sin C/2)/\cos A/2$.

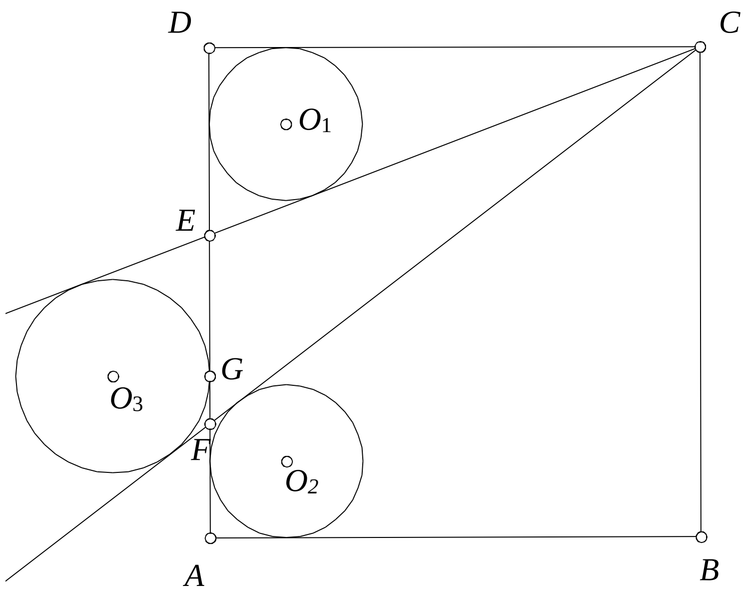
Hence the equations $q = r(1 + \tan A/2 \tan\delta)$ and $q' = r'(1 + \tan A/2 \tan\delta')$ become

[30] Solution sketched in Fukagawa & Rigby 2002 (p. 97).

$q = (4R \sin\delta \sin B/2 \sin C/2 \cos\delta')/(\cos^2 A/2)$,
$q' = (4R \sin\delta' \sin B/2 \sin C/2 \cos\delta)/(\cos^2 A/2)$.

Thus $q/q' = \tan\delta / \tan\delta'$. Since $\tan\delta = 2v/a$ and $\tan\delta' = 2v'/a'$, we have $q/q' = va'/v'a$. □

Remark: This is virtually the same problem as one described as "exceedingly difficult"[31] that asks for a proof, in the same figure, that $1/q + 1/r' = 1/q' + 1/r$. This is equivalent to $1/r' - 1/q' = 1/r - 1/q$ or $\frac{q'-r'+}{q'r'} = \frac{q-r}{qr}$. That is, $\frac{\tan A/2 \tan\delta' +}{q'} = \frac{\tan A/2 \tan\delta}{q}$, which is what was just proved.



PROBLEM 22:[32] Circles $(O_1)r$ and $(O_2)r$ are inscribed in corners A and D of square $ABCD$, which has side a. CE is tangent to (O_1); CF is tangent to (O_2). $(O_3)r'$ is tangent to CE, CF, and AD at G. Prove that, if $r' = r$, $r = a/6$.

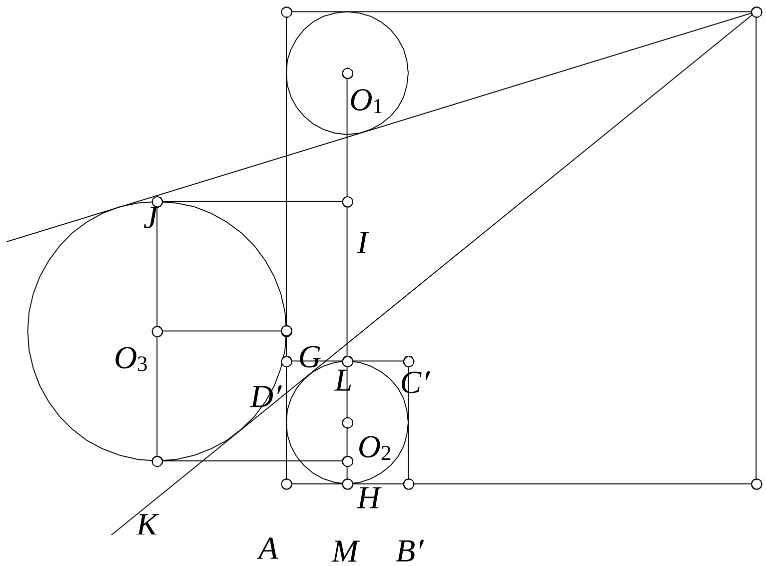
SOLUTION 22 (JMU):

(O_3) is easily constructed: it is the excircle on side EF of triangle CEF.

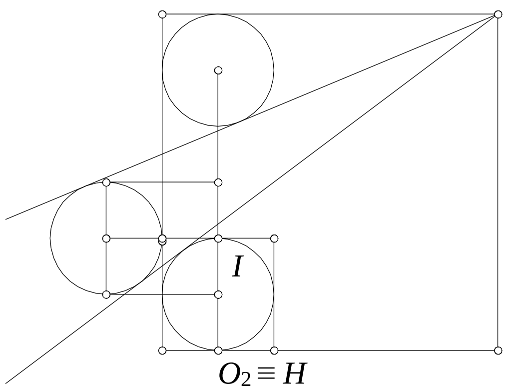
Consider the square $AB'C'D'$ and rectangle $HIJK$ in the general case (next figure). Note that $KO_3 = GO_3$ and $LO_2 = D'L$.

Siuce $HK = IJ = r + r'$ and $JK = HI = 2r'$, in the special case of $r' = r$, $HIJK$ will be a square. Hence $GO_3 = D'L$, and so $LO_2 = KO_3$. But O_3 is the midpoint of JK, so L must now be the midpoint of HI. That is, if and only if $r' = r$ do we have $G \equiv D'$, O_3GLC' a straight line, and $H \equiv O_2$.

[31] Fukugawa & Pedoe 1989 1.4.7, (1844, Aichi).
[32] Fukagawa & Pedoe 1989, 3.2.2 (1893, Fukushima). They stipulate $r < a/4$, but one finds empirically that E and F coincide for $r < a/4.37$.

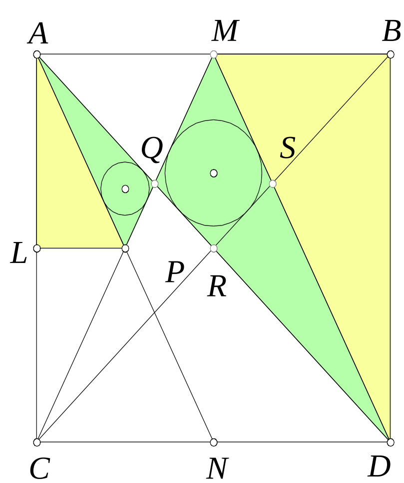


Therefore, when $r' = r$, $IO_2 = HI = 2r$ and $IM = 3r$. If we repeat the whole construction using vertex B rather than C at the start, we obtain the same I because the resulting figure is a reflection of the one above in the horizontal axis of square $ABCD$.



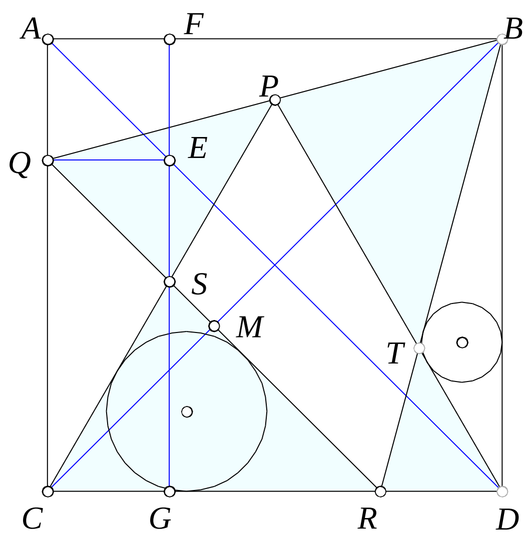
Thus $IM = a/2$, so $r = a/6$. □

PROBLEM 23:[33] Given square $ABCD$, with M, N the midpoints of AB, CD, inscribe a circle in kite $QRSM$ and in triangle APQ. Prove that the radius of the larger circle is twice that of the smaller.



SOLUTION 23: Note that the incircle of the kite is also the incircle of DMQ. Let L be the midpoint of AC; then the yellow right triangles are similar, and DBM is a dilation of ALP by a factor of 2. The green triangles are similar because $AN \parallel DM$. Since $DM = 2AP$, DMQ is a dilation by 2 of AQP. So their inradii have the same ratio. □

[33] Fukagawa & Pedoe 1989, 3.1.5 (1835, Miyagi); no proof given.



PROBLEM 24:[34] In square *ABCD*, *P* is the apex of equilateral triangle *CPD*. *BP* meets *AC* in *Q*. Show how to find *R* on *CD* such that *BQR* is also equilateral. Then prove that the inradius of *CSR* is twice the inradius of *BDT*.

SOLUTION 24 (JMU): As for part 1, any two lines isogonal to angle *ABD* cutting *AC* in *Q* and *CD* in *R* define an isosceles triangle *BQR* with *CQR* = *CRQ* = 45°. Let *M* be the midpoint of *QR*; *BC* is its perpendicular bisector. If we select *CBQ* = *CBR* = 30°, we therefore have *BQ* = 2*MQ* = 2*MR* = *BR* = *MQ* + *MR*. That is, *BQR* is equilateral. □

Knowing the location of some 45° and 60° angles, we can calculate all the rest and find that the four blue triangles are similar, with angles of 45°, 60°, and 75° at the corresponding vertices. Moreover, we can prove that *CSR* is a dilation of *RDT* by a factor of 2.

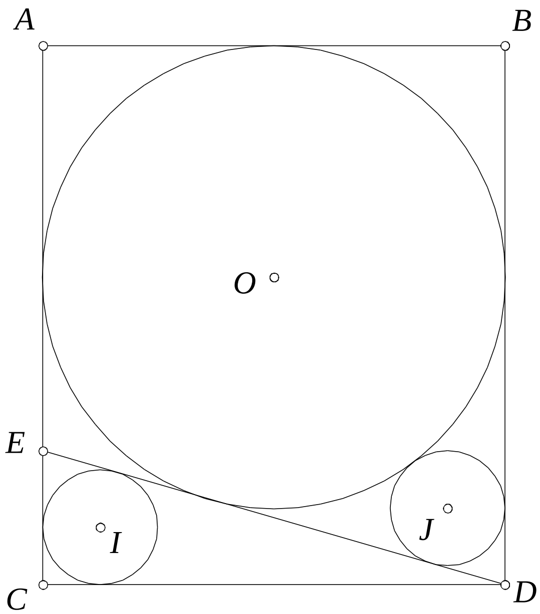
Let *a* be the side of *ABCD*, *b* be the side of *BQR* and *c* be *AQ* = *DR*. Say the extension of altitude *GS* of *CSR* cuts *AD* in *E* and *AB* in *F*. Since *AFE* is isosceles with 45° base angles, $AF\sqrt{2} = EF\sqrt{2} = AE$. Select *Q′* on *AC* such that *Q′E* ⊥ *FG* or, which is the same thing, *Q′E* || *AF*. Since *CAE* = *FAE* = 45° = *AEQ′*, *AQ′E* is isosceles and hence congruent to *AFE* . I.e., *Q* ≡ *Q′*, *AFEQ* is a square of side *c*, and $c\sqrt{2} = AE$. Thus *CG* = *c*, and, because *CSG* = 30°, *CS* = 2*c*.

Using this fact and setting *c* = 1, we can compute the length of other segments in the figure. We can then use the fact that the area of a triangle is the product of its inradius and semiperimeter to calculate (with some effort) the ratio of the inradii of *CSR* and *BDT*.

A quicker method is based on Problem 8 above, the solution of which shows that the incircles of *DRT* and *BDT* are equal if and only if $DT = \sqrt{s(s-b)}$, where *s* is the semiperimeter of *BDR*. To make use of this theorem, it suffices to note that *DR* = 1, *a* = *BD* = $2 + \sqrt{3}$, *b* = *BR* = $\sqrt{2} + \sqrt{6}$, and *DT* = $(1 + \sqrt{3})/2$. A little arithemetic then shows that $s(s - b) = (2 + \sqrt{3})/2$, which is DT^2, and we are done. Moreover, since *BDR* is a right triangle, we could use the corollary to Problem 8 to compute the length of inradius if we wish. □

[34] Fukagawa & Pedoe 1989, 3.1.6 (1881, Yamagata); no proof given.

Yet another solution has been posted elsewhere on the web.



PROBLEM 25: In rectangle $ABCD$, $(I)r$ is the incircle of triangle CDE, $(O)u$ is tangent to AB, BC, BD, and DE.and (J) is tangent to BD and DE. If (J) also has radius r, prove that $AB = (6/7)AC$.[35]

SOLUTION: Because (I) and (J) have the same radius, DCE and DFE are congruent, and $CDEF$ is a rectangle.

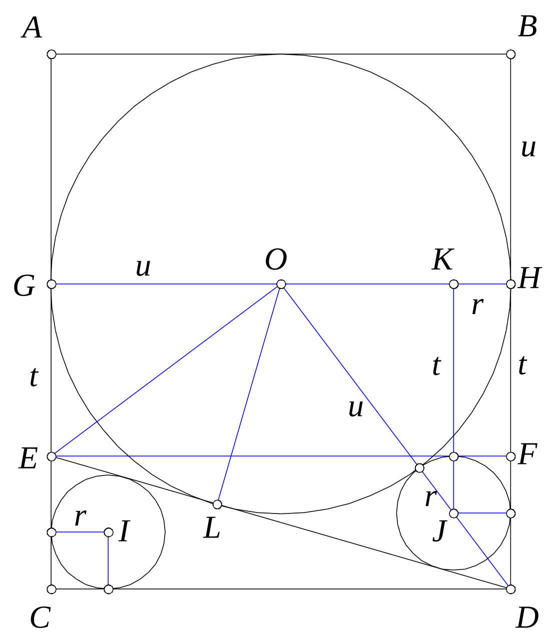
Let $EG = FH = t$.

Since DE is tangent to (O) at L, $OL \perp DE$, $EGO \cong ELO$, $DHO \cong DLO$, and DOE is a right triangle. All these right triangles and JKO are similar.

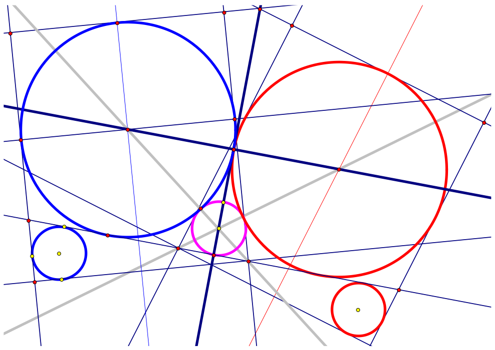
Comparing the legs of $EGO \sim JKO$, we have $t/u = (u - r)/(t + r)$. This is obviously satisfied by $u = t + r$. (That is, EGO and JKO are not just similar but congruent.)

Substituting u for $t + r$ in $(t + r)^2 = (u + r)^2 - (u - r)^2 = 4ur$ (in JKO), we have $u^2 = 4ur$ or $u = 4r$. And substituting $t + r$ for u in this equation, we get $t = 3r$. That is, JKO is a 3:4:5 right triangle.

Since DHO is similar, $DH = 4x$ and $HO = 3x$ for some unit x. Hence $BD = 7x$ and $GH = 6x$. Thus $AB/AC = 6/7$. □

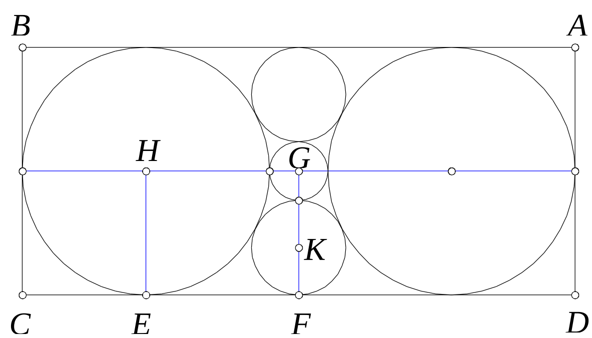


[35] Fukagawa & Pedoe 1989, 3.4.4; posed differently in Fukagawa & Rothman 2008, pp. 256, 278–80.



Here's an elegant geometric shortcut. Reflect the original figure in the axis through J as shown and apply the result of Problem 2. Immediately we get $4r = u$. As before, $EGO \sim OKJ$ together with this implies $t = 3r$. The rest follows as before.

The solution in Fukagawa & Rothman, based on Japanese sources, involves solving a cubic equation and discarding two roots. Given the shortcut, that is a particularly striking piece of evidence of the Japanese preference for algebra at the expense of geometric reasoning.

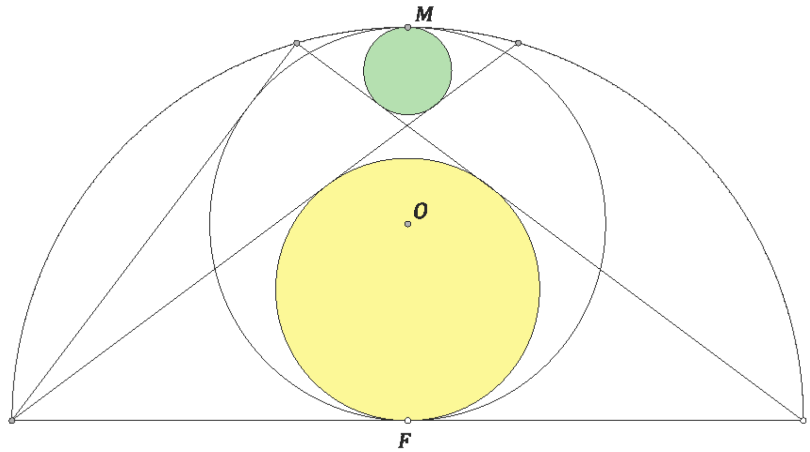


PROBLEM 26: Rectangle $ABCD$ contains two large circles of radius r, two smaller circles of radius s, and one yet smaller circle of radius t situated as shown. Prove that $AB = BC\sqrt{5}$.[36]

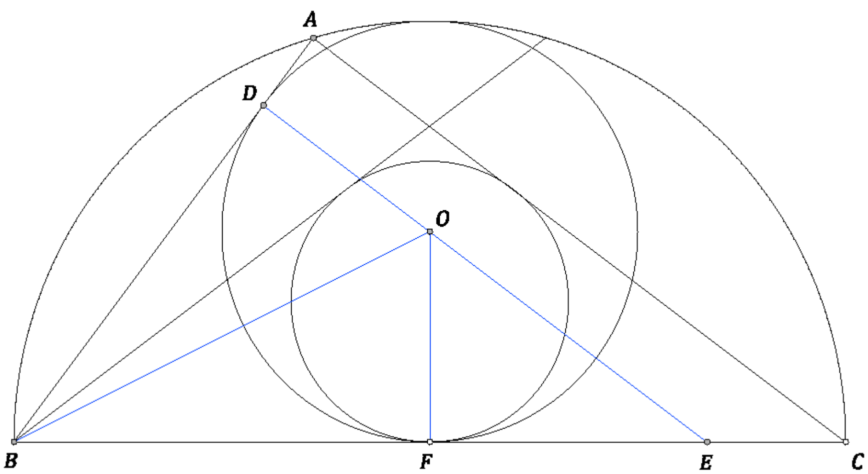
SOLUTION (JMU): $AB = 2r + 2d$ where $d = EF$. But $EF = r + t$ and $FG = r = 2s + t$. Eliminating t from these equations, $d = 2r - 2s$. Squaring, $d^2 = 4r^2 - 8rs + 4s^2$. Since (H) and (K) touch each other and CD, $d^2 = (r + s)^2 - (r - s)^2 = 4rs$ (see Solution 2). Eliminating d^2, $0 = r^2 - 3rs + s^2$, which leads to $r = s(3 + \sqrt{5})/2$ and $s = r(3 - \sqrt{5})/2$. Replacing s in $d = 2r - 2s$, $d = 2r - r(3 - \sqrt{5}) = r(-1 + \sqrt{5})$. Thus $AB = 2r + 2r(-1 + \sqrt{5}) = 2r\sqrt{5} = BC\sqrt{5}$. □

[36] Fukagawa & Pedoe 1989, 3.4.5 (1820, Iwate); no proof given.

PROBLEM 27: Circle (O) touches the chord and the arc of a semicircular segment at their respective midpoints, F and M. The second tangents to (O) through the endpoints of the chord cut the arc in two points, which are joined to the opposite endpoints. What are the diameters of the circles (yellow and green) that touch both of these joining lines and (O)? Assume that the diameter d of (O) is known.[37]



SOLUTION 27: Start with the yellow circle. Notice (figure below) that, if D is the point where (O) touches AB, then $DO \parallel AC$.



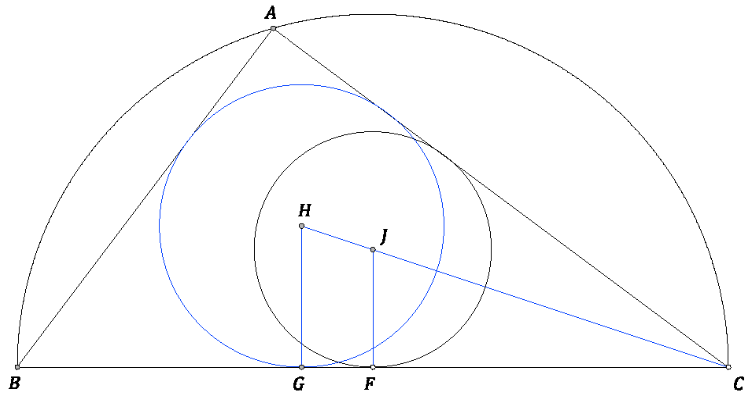
Extend DO to cut BC in E, and note that $BD = BF = FC = d$. Since $\triangle BDE \sim \triangle OFE$, $DE = 2FE$. Therefore, from $\triangle DBE = \triangle BDO + \triangle BOE$, we obtain

$$\frac{1}{2} \cdot d \cdot 2 \cdot FE = \frac{1}{2} \cdot \frac{d}{2} \cdot d + \frac{1}{2}(d + FE)\frac{d}{2}.$$

[37] Kotera 2013: 134–35. A tablet dated 1857 in a shrine straddling the border between Nagano and Gunma prefectures shows this problem on the Nagano side (where the shrine is called Kumano kōdai). Another tablet, dated 1872, on the Gunma side (where the shrine is called just Kumano) shows the same problem with corrections. Kotera gives the solution for the yellow circle only.

Thus $FE = 2d/3$. This means that $\triangle BDE$ is a 3:4:5 right triangle, and hence that $\triangle BAC$ is too.

Now, label the yellow circle $(J)F$. Since it is homothetic to $(G)H$, the incircle of $\triangle BAC$, with respect to C, $\triangle HGC \sim \triangle JFC$, and so



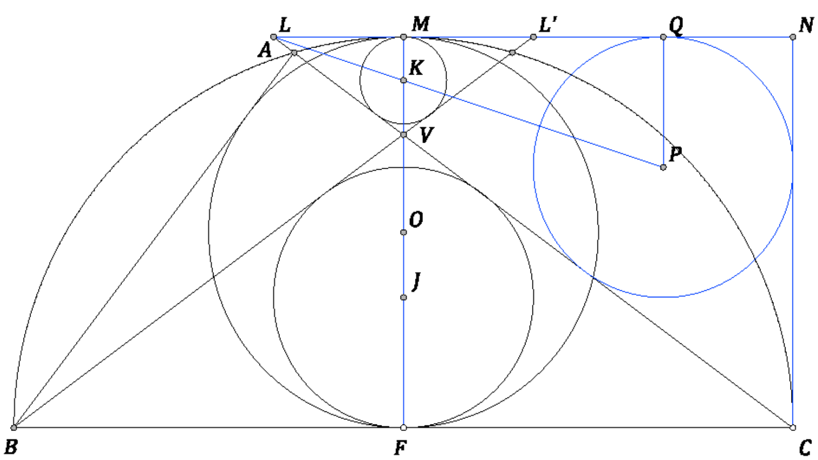
$$\frac{FJ}{FC} = \frac{GH}{GC}.$$

But $FC = d$ and

$$\frac{GH}{GC} = \frac{(3+4-5)/2}{(3+4+5)/2-3} = \frac{1}{3}$$

(recall the corollary to Solution 6 above). Therefore the diameter of (J) is $2d/3$. □

Next, label the green circle (K) and construct the common tangent to $(K), (O), (F)$ through M as shown below. Extend CA, BV to meet this line in L, L'. Since (K) is the incircle of $\triangle LVL'$, K is the point where the bisectors of $\angle VLL'$ and $\angle LVL'$ intersect.



Furthermore, if the perpendicular to BC through C meets LL' extended in N, then, since $CN \perp LN$ and $LN \parallel BC$, $\triangle CNL \sim \triangle BAC$, and $\triangle CNL$ is another 3:4:5 right triangle. Since the sides of square $CNMF$ are d, it follows that $LN = 4d/3$ and $LC = 5d/3$.

Now the center of the incircle $(P)Q$ of $\triangle CNL$ lies on LK extended, so $\triangle LMK \sim \triangle LNC$, and

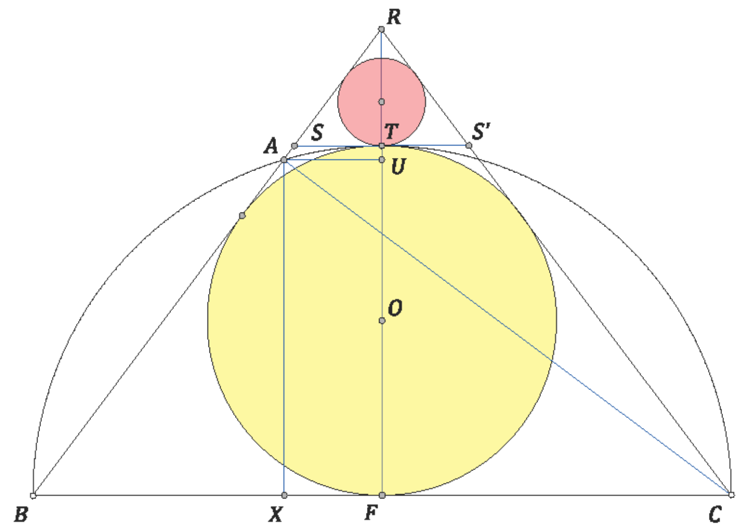
$$\frac{MK}{LM} = \frac{PQ}{LQ}.$$

We can easily compute the lengths of all the segments other than MK in this proportion:

$$PQ = QN = \frac{CN + LN - LC}{2} = \frac{d}{3}, \qquad LQ = LN - QN = d, \qquad LM = LN - MN = \frac{d}{3}.$$

Thus $\frac{MK}{d/3} = \frac{d/3}{d}$ or $MK = d/9$. Therefore, the diameter of (K) is $2d/9$. □

PROBLEM 28:[38] Suppose in the foregoing configuration we have the incircle (red) of $\triangle SRS'$. What is its diameter in terms of d?

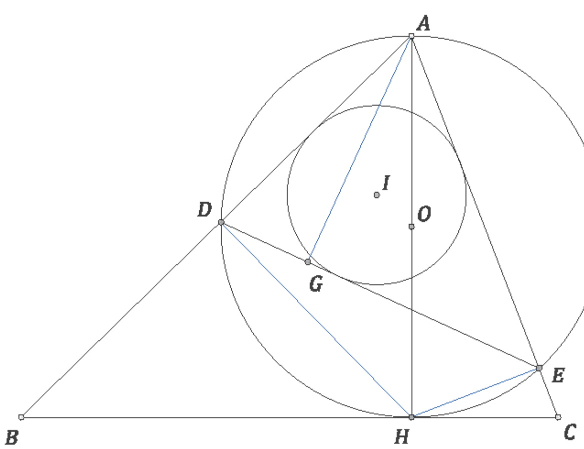


SOLUTION 28 (JMU): $BF = FC = TF = d$, so $RT = RF - d$ and, taking note of similar 3:4:5 right triangles,

$AB = 6d/5 = 30d/25$
$BX = 18d/25$
$AX = 24d/25.$

Thus $RF = AX \cdot BF/BX = 4d/3$ and $RT = d/3 = 4d/12$. Hence $ST = 3d/12$ and $AR = 5d/12$. Since $ST = TS'$, the area of $\triangle SRS'$ is $d^2/12$ and its semiperimeter is $8d/12$. Therefore, the inradius of $\triangle SRS'$ is $d/8$, which means the diameter of the incircle is $d/4$.

PROBLEM 29.[39] Triangle ABC has altitude $h = AH$. Its midpoint is O. $(O)H$ cuts AB in D and AC in E. $(I)r$ is the incircle of ADE. Express r in terms of a, b, c, and h.



SOLUTION 29 (JMU): Let $AD = e, AE = d$ and $DE = f$. Using the inradius and circumradius formulae for the area of ADE, we have

$$r\frac{d+e+f}{2} = \frac{def}{4OH}$$

or

$$r = \frac{def}{2OH(d+e+f)}.$$

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[38] Fukagawa & Pedoe 1989, 2.3.1 (1891, Fukushima); no solution given, or any mention of the clearly related Problem 27.

[39] Fukagawa & Pedoe 1989, 2.2.6 (1805, Toyama); answer given, but without a proof.

But $h = 2OH$, so $r = \frac{def}{h(d+e+f)}$. Our goal is to rewrite this with a, b, c instead of d, e, f.

Since AH is a diameter of (O), ADH and AEH are right angles. Since BC touches (O) at H, $AHC \sim AEH$, so $\angle ACB = \angle AHE$. But $\angle AHE = \angle ADE$ because they subtend the same arc, so $\angle ACB = \angle ADE$. Likewise, $\angle AED = \angle ABC$. Hence, DE is antiparallel to BC, and $ADE \sim ACB$.

Given altitude $g = AG$ in ADE, this similarity implies $\frac{b}{h} = \frac{e}{g}, \frac{c}{h} = \frac{d}{g}$, and $\frac{a}{h} = \frac{f}{g}$. Therefore,

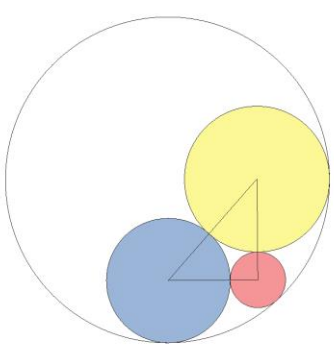
$$\frac{def}{h(d+e+f)} = \frac{def}{g(a+b+c)}.$$

Now because $\angle AHE = \angle ADE$, $AGD \sim AEH$. Hence, $\frac{d}{g} = \frac{h}{e}$, or $de = gh$. But $\frac{bc}{h^2} = \frac{de}{g^2}$, so

$$bcde = \frac{h^2 de}{g^2} gh = \frac{deh^3}{g} = h^4.$$

Rearrange $bcde = h^4$ as $\frac{de}{h} = \frac{h^3}{bc}$, and multiply its left and rights sides by $\frac{fh}{g}$ and a, respectively. (Recall that $\frac{a}{h} = \frac{f}{g}$.) The result is $\frac{def}{g} = \frac{ah^3}{bc}$. Plugging this into $r = \frac{def}{g(a+b+c)}$, we obtain

$$r = \frac{ah^3}{bc(a+b+c)}. \square$$



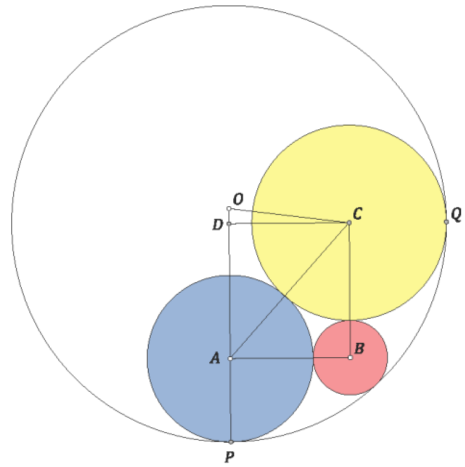
Problem 30:[40] Suppose that the centers of three circles, each touching the other two externally, lie at the vertices of a right triangle, and that a fourth circle touches all three internally. Prove that the largest diameter is the sum of the other three.

Solution 30 (JMU): Let the diameters of $(A), (B), (C), (O)$ be a, b, c, d, respectively. Let D be the fourth vertex of the rectangle determined by A, B, C.

Here is the proof in Nakamura 2008, slightly elaborated. Say that $(A), (C)$ touch (O) at P, Q, respectively. PA and CQ meet at O. If $\angle POQ$ is a

[40] Fukagawa & Pedoe 2.4.5 (Tochigi, 1853; tablet lost but problem mentioned in *Sanpō jojutsu*). No solution given.

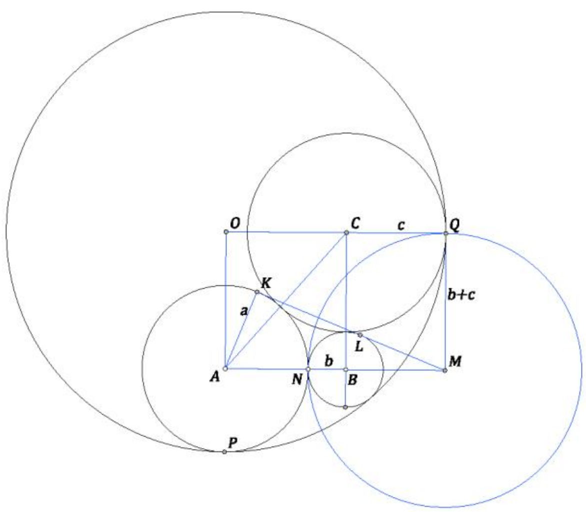
right angle, i.e. if $O \equiv D$ in the figure, then by inspection $\frac{d}{2} = \frac{a}{2} + \frac{b}{2} + \frac{c}{2}$. If $O \not\equiv D$, then either O doesn't lie on DA or it doesn't lie on DC. Without loss of generality, assume the latter. Then in right triangle ODC,



$$\left(\frac{d}{2} - \frac{c}{2}\right)^2 = \left(\frac{a}{2} + \frac{b}{2}\right)^2 + \left(\frac{d}{2} - \frac{a}{2} - \frac{b}{2} - \frac{c}{2}\right)^2.$$

Simplifying, $(a+b)(a+b+c-d) = 0$. Since $a + b \neq 0$, it must be that $d = a + b + c$.

This solution comes close to begging the question, and it sheds no light on why (O) should touch (B) internally, or, more generally, how to construct the figure. It is worth knowing that $(A), (B)$ touching at N determine C, O uniquely, apart from which side of AB we choose to put them on.



Construct the external bitangent KL, which meets AB at the homothetic excenter M. Draw the perpendicular to AM through M, and say that $(M)N$ cuts it in Q. Then C is the fourth vertex of the rectangle determined by B, M, Q, and O is the fourth vertex of the rectangle determined by A, M, Q.

Proof: Since $\triangle AKM \sim \triangle BLM$, $\frac{a}{b} = \frac{KM}{LM}$ or $\frac{a-b}{b} = \frac{KL}{LM} = \frac{2\sqrt{ab}}{LM}$; therefore $LM = \frac{2b\sqrt{ab}}{a-b}$ and $KM = \frac{2a\sqrt{ab}}{a-b}$.

Algebraically, their geometric mean is $\frac{2ab}{a-b}$; constructively, it is MN ($\triangle KNM \sim \triangle NLM$). But $MN = MQ = b + c$. Thus $c = MN - b = \frac{2ab - b(a-b)}{a-b} = \frac{ab+b^2}{a-b}$. On the other hand, since ABC is a right triangle, $(a+c)^2 = (a+b)^2 + (b+c)^2$; solving for c, again $c = \frac{ab+b^2}{a-b}$, so (C) touches both (A) and (B) externally. Furthermore, $AM = a + b + c = OQ$. Since O, C, Q are collinear, $(O)Q$ touches (C) internally. Since $OA = QM = b + c$ and (A) has radius a, $(O)Q$ touches (A) internally (at P). Likewise, since $OB = CA = c + a$ by construction and (B) has radius b, $(O)Q$ touches (B) internally. □

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