WestFest

SCIENCE & SUSTAINABILITY

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Packing Oranges to Fix Errors

Wednesday, October 19th, 9:30-10:30 a.m.
Saturday, October 22nd, 12:00-1:00 p.m.
Register at https://go.osu.edu/register4wf

Materials:

Nine 1” round chips	1” graph paper sheet	Thirty 1” balls
Eight 1” half balls	Transparent box	Ruler

Background:

In the 17th century, Johannes Kepler, famous for his work in astronomy, posed the following question: What is the densest way of packing spheres? In other words, what is the best way to arrange a given number of balls so that they occupy the smallest space possible.

While he had a hypothesis, he was not able to prove that it was indeed the most efficient way. Since then, mathematicians have tried to solve the problem. It was not until 1998, that American mathematician Thomas Hale finally proved Kepler’s hypothesis.

However, that did not end the discussion. Throughout the years, mathematicians have not limited their work on this question to 3-dimensional spheres. In fact, Kepler did what mathematicians often do and started by looking at a simpler problem: how to pack 2D versions of spheres, in other words, circles. He was able to answer and prove the result with circles.

However, one can also answer the question for higher dimensional versions of circles and spheres. In 2016, Maryna Viazovska, a Ukranian mathematician, solved the problem in 8 dimensions. A year later, Viazovska and a team of mathematicians also solved the problem for 24 dimensions. This was such an important discovery, that she was awarded the Fields Medal, a very important prize in mathematics that is only awarded every four years.

In this activity, we will explore Kepler's problem in 2 and 3 dimensions and talk about the problem in higher dimensions.

Directions:

Part I. 2D

1. Place the graph paper on a table and arrange the 9 chips on it. The goal is to find an arrangement where the chips are packed as tight as possible. Another way of saying this is that we want the arrangement to cover the smallest possible area of paper.
2. Once you have it, carefully draw a polygon tightly surrounding the chips. Remember that a polygon has straight sides. Use a straight edge if possible.
3. Compute the area of the polygon you drew. To do that, use the ruler to take any necessary measurements. You can also use the grid as reference considering each box is 1 inch per side. Some area formulas are given here for reference.

   Rectangle and romboid: $A = b \times a$ Regular polygon: $A = \frac{P \times a}{2}$
4. Each chip has a diameter of 1 inch. What is the area of a chip?
   Remember that the formula for the area of a circle is $A = \pi \times r^2$
5. What is the total area of 9 chips?
6. To measure how packed or dense an arrangement is, mathematicians use the following formula:

   $$\text{Density} = \frac{\text{Total area of the 9 disks}}{\text{Area of the polygon that surrounds them}}$$

   Use the formula to compute the density of your arrangement.
7. This formula becomes handy when we have several arrangements that we need to compare. Come up with another way of arranging the chips: a second option that you think could compete with the previous. Repeat all the steps to get the density of this new arrangement.
8. Which arrangement is denser?

Part 2. 3D

1. Now we will try to do the same but with the balls.
2. Assemble your box and arrange the balls inside it, trying to fit as many as possible. The sides of this box might bend a little but try not to do that. You should fill the box with balls without letting the sides bulge. The sides of the box should remain straight.
   Hint: You should be able to find a way of nicely fitting 27 balls precisely in the box.
3. The box is a cube with each side measuring 3 inches. Use the formula to compute its volume.
   $$\text{Volume} = \text{width} \times \text{height} \times \text{depth}$$
4. Each ball has a diameter of 1 inch, so its volume is about 0.52 cubic inches. What is the total volume of the 27 balls?
5. CHALLENGE: Use the formula for the volume of a sphere to verify that the approximate volume of these balls is 0.52 cubic inches each.
   $$V = \frac{4}{3}\pi r^3$$
6. Again, calculate the density using the same formula as before (here with volume instead of area):
   $$\text{Density} = \frac{\text{Total volume of the balls}}{\text{Volume of the prism that surrounds them}}$$
7. We can also try to find an arrangement that is similar to the best one we found in 2D. To do this, we should lay the second layer of balls in the dimples between the balls of the first layer. You will notice that by doing this, we actually fit less balls in the box, but we also don’t completely fill the box. Make a mark on a side of the box to the height reached by the balls.
8. Compute the volume of the box up to this mark.

9. Moreover, just as we saw in the 2D optimal solution, in this new arrangement, there are some gaps where we can fit half balls. Use the half balls and fit them as appropriate.
10. What is the density of this new arrangement?
11. CHALLENGE: Find a way of fitting 27 balls in total (counting two half balls as a single ball) without filling the whole box. Hint: It should be 23 whole balls and 8 half balls.

Part 3. Higher Dimensions & Applications

You might be able to see the usefulness of the previous problems of packing disks and spheres. In real life, there might be instances in which we might want to draw circles or arrange balls as tightly as possible. However, you might be wondering why would mathematicians care about studying the problem in an abstract, inexistent world, of more than 3 dimensions.

Sphere packing turns out to have an important application in telecommunications. As we will learn in this activity, the method allows to send information on the internet or in cell networks in a way that makes it easier to correct errors that might arise when sending the information. As a reference, you can watch this Numberphile video:

Spheres and Code Words – https://youtu.be/T46FTuHnbvY