Revamping the Restriction Strategy

by

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Abstract

This study continues the anti-realist's quest for a principled way to avoid Fitch's paradox. It is proposed that the Cartesian restriction on the anti-realist's knowability principle 'φ, therefore $\Diamond K\varphi$' should be formulated as a consistency requirement not on the premise φ of an application of the rule, but rather on the set of assumptions on which the relevant occurrence of φ depends. It is stressed, by reference to illustrative proofs, how important it is to have proofs in normal form before applying the proposed restriction. A similar restriction is proposed for the converse inference, the so-called Rule of Factiveness '$\Diamond K\varphi$ therefore φ'. The proposed restriction appears to block another Fitch-style derivation that uses the KK-thesis in order to get around the Cartesian restriction on applications of the knowability principle.

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Korean saying: Joong-i je meo-ri mot kkak-neun-da.
Translation: A (buddhist) monk cannot shave his own hair.[1]

1 Introduction

In *The Taming of The True* a restriction was proposed on the anti-realist's Knowability Principle, which can be expressed as a rule of inference in natural deduction as follows:

$$\frac{\varphi}{\Diamond K\varphi}$$

It was proposed that this principle be limited, in its applications, to *Cartesian* propositions φ. A proposition φ is Cartesian just in case $K\varphi \nvdash \bot$. So the restricted Knowability Principle would be

$$\frac{\varphi}{\Diamond K\varphi} \quad \text{where } K\varphi \nvdash \bot$$

This way of restricting the Knowability Principle may well be suspected of being overly 'local'. It might be advisable to have a more 'global' restriction. In general, a step of inference from φ to $\Diamond K\varphi$ (with or without an extra condition on φ) takes place within a proof Π which will have some set Δ undischarged assumptions:

$$\begin{array}{c}\Delta\\ \Pi\\ \dfrac{\varphi}{\Diamond K\varphi}\end{array}$$

If we limit ourselves to the 'local' restriction, we ignore the contribution of the set Δ of assumptions, focusing instead on the fact that, via Π, we have just reached the conclusion φ, whatever our starting point might have been:

$$\begin{array}{c}\Delta\\ \Pi\\ \dfrac{\varphi}{\Diamond K\varphi}\end{array} \quad \text{where } K\varphi \nvdash \bot$$

But if our grounds for φ are indeed Δ, then the inferred possibility of knowing that φ surely presupposes the possibility of knowing that Δ. Indeed,

[1] Thanks to Sukjae Lee for the motto.

if it were impossible to know the joint truth of the assumptions in Δ, how could one be confident in inferring from the intermediate conclusion φ to the knowability claim $\Diamond K\varphi$? These considerations lead to the thought that the restriction strategy, instead of looking *down* at φ within Π should rather look *up* at Δ. The proposal, then, is that the restricted Knowability Principle should take the form of the following rule of inference, with a rather more exigent pre-condition for its applicability:

Globally Restricted Knowability Principle

$$\frac{\begin{array}{c}\Delta\\ \Pi\\ \varphi\end{array}}{\Diamond K\varphi}\quad \text{where } K\Delta \nvdash \bot$$

Here $K\Delta$ is defined in the usual Frobenian way as $\{K\delta \mid \delta \in \Delta\}$. When $K\Delta \nvdash \bot$, we shall say that Δ is Cartesian.

In logics whose relation $\vdash$ of deducibility is not effectively decidable, the correctness of applications of the globally restricted rule is accordingly not effectively decidable. This is a modest price to pay, however, when one is concerned to avoid Fitch's paradox.

The main purpose of this study is to explain, investigate and defend this new proposed global restriction on applications of the Knowability Principle. A similar proposal will be made concerning its converse, the Rule of Factiveness for the compound operator $\Diamond K$:

$$\frac{\Diamond K\varphi}{\varphi}$$

As will become clear from details that will emerge below, we need to restrict Factiveness too, in order to avert a different proof of Fitch's paradox, which exploits the KK-thesis but does not fall foul of the global restriction on its application of the Knowability Principle.

1.1 A retraction, for the record

The present author's claim, made in [10], at p. 829 and repeated in [11], at p. 140, to the effect that $\Diamond K\varphi$ is factive, was incautious. While it is valid so long as the sentence φ in question concerns only non-epistemic facts, the inference is not guaranteed always to preserve truth if we allow φ to contain occurrences of K (and adopt the KK-thesis).

A related claim, however, still stands: *to the extent that* $\Diamond K$ *is factive*, $\Diamond$ is not to be analyzed as the familiar alethic modal operator. Its contribution to truth- or assertability-conditions of sentences in which it is prefixed to K will have to be elucidated in terms of possibilities of investigative outcomes, at future times, *within the actual world*. *Those* possibilities will be strongly constrained by relevant contingencies in the actual world. It is this feature of the possibilities adverted to within $\Diamond K$ that make the use of an ordinary alethic $\Diamond$ inappropriate.

1.2 Global restrictions on rules of inference

A 'global' restriction on a rule of natural deduction is one that imposes some pre-condition for applicability by adverting to syntactic features of the proof other than the forms of the sentences standing as the immediate premises, or as the conclusion, of applications of the rule.

As soon as any 'global' restriction is proposed on a rule of natural deduction, the possibility arises that its strictures can be rendered toothless by applying other rules of inference in a roundabout fashion that creates an artificial deductive context that meets the pre-condition in the letter, but not in the spirit, of the proposed restriction. The most obvious way to do this is by constructing proofs that are not in normal form.[2]Thus the most obvious prophylactic against such deductive chicanery is to insist that, when determining whether the pre-condition is met, the proof that is to result from the contemplated application of the restricted rule should be in normal form. With this said by way of foreshadowing, we shall defer detailed illustrative examples to their most natural points of entry below.

[2]Such proofs involve inferring a sentence as the conclusion of an application of an introduction rule, and then treating that sentence as the major premise for an application of the corresponding elimination rule. Such sentence-occurrences within a proof are called *maximal*, and their presence is what prevents the proof from being in normal form. By contrast, a proof in normal form is one that contains no maximal sentence occurrences.

2 The new restriction on Knowability

2.1 How the new restriction works on Fitch's original proof

We recall the proof of the Fitch paradox as given in [9], at pp. 260–1:[3]

$$\Sigma \qquad \dfrac{\dfrac{\dfrac{\varphi \qquad \neg K\varphi}{\varphi\wedge\neg K\varphi}\,(\wedge I)}{\Diamond K(\varphi\wedge\neg K\varphi)}\,(\Diamond K) \qquad \begin{array}{c}\overline{K(\varphi\wedge\neg K\varphi)}^{(1)}\\ \Xi \\ \bot\end{array}}{\bot}\,(\Diamond\bot)^{(1)}$$

where the embedded proof Ξ is

$$\Xi \qquad \dfrac{K(\varphi\wedge\neg K\varphi) \qquad \dfrac{\dfrac{\overline{\varphi\wedge\neg K\varphi}^{(1)}}{\neg K\varphi} \qquad \dfrac{K(\varphi\wedge\neg K\varphi)}{K\varphi}\,(K\wedge)}{\bot}}{\bot}\,(I)^{(1)}$$

The reader will easily verify that the new, global restriction blocks this proof Σ of Fitch's paradox at its application of the rule $(\Diamond K)$. For the premise-set Δ for that step is $\{\varphi, \neg K\varphi\}$. Hence $K\Delta = K\{\varphi, \neg K\varphi\} = \{K\varphi, K\neg K\varphi\}$. And the latter set is not Cartesian:

$$\dfrac{\dfrac{K\neg K\varphi}{\neg K\varphi} \qquad K\varphi}{\bot}$$

We see, then, that the global restriction can do the old work required of the local restriction.

2.2 How the new restriction works on a proof of Salerno

The strengthened restriction also does important new work that the local restriction cannot do. The following is a short proof of a result brought to my attention by Joe Salerno (albeit using a different proof).[4] Consider the

[3] Natural deductions will be set out in tree form below. The reader unfamiliar with this format for proofs is advised that with applications of so-called 'discharge rules' the parenthetically enclosed numeral '(i)' has an occurrence labelling the step at which the indicated assumption-occurrences higher up at 'leaf nodes' of the sub-proof(s) are *discharged* by applying the rule in question. A discharged assumption no longer counts among the assumptions on which the conclusion of the newly created proof depends.

[4] Personal communication. See also [2].

following proof Θ of ψ from the set of assumptions $\{p, \neg Kp\}$. Note that the application of $(\Diamond K)$ uses the old, 'local', restriction.

$$\Theta: \quad \dfrac{\dfrac{\Diamond K\ \dfrac{K(p\wedge(Kp\to K\psi))\nvdash\bot \qquad \dfrac{p \qquad \dfrac{\dfrac{\dfrac{\overset{(2)}{\overline{Kp}} \qquad \neg Kp}{\bot}}{K\psi}}{Kp\to K\psi}{}^{(2)}}{p\wedge(Kp\to K\psi)}}{\Diamond K(p\wedge(Kp\to K\psi))} \qquad \dfrac{\dfrac{\dfrac{\overline{K(p\wedge(Kp\to K\psi))}^{(1)}}{p\wedge(Kp\to K\psi)}}{Kp\to K\psi} \qquad \dfrac{\overline{K(p\wedge(Kp\to K\psi))}^{(1)}}{Kp}}{K\psi}}{\Diamond K\psi}{}^{(1)}}{\psi}$$

Note that the step labelled (1) is an application of the following rule of inference in modal logic:

$$(\Diamond\vdash) \qquad \dfrac{\Diamond\chi \qquad \begin{array}{c}\overline{\chi}^{(i)}\\ \vdots\\ \theta\end{array}}{\Diamond\theta}{}^{(i)} \qquad \text{where } \chi \text{ is the sole assumption of the subordinate proof}$$

Note also that the final step of the proof Θ:

$$\dfrac{\Diamond K\psi}{\psi}$$

is an application of the aforementioned Rule of Factiveness of $\Diamond K$. It is worth stressing that its application here would be correct even if it were subject to the restriction that ψ should not contain any occurrence of K. (We shall return to this rule later.)

Given the pattern of occurrences of ψ within Θ, we could evidently take any contingent atomic proposition q (not involving K) and complete a new proof of Fitch's paradox as follows:

$$\dfrac{\begin{array}{c}\{p,\neg Kp\}\\ \Theta[\psi/q]\\ q\end{array} \qquad \begin{array}{c}\{p,\neg Kp\}\\ \Theta[\psi/\neg q]\\ \neg q\end{array}}{\bot}$$

Remember that these two substitution instances of Θ involve the old, 'local', restriction on $(\Diamond K)$. But those steps of $(\Diamond K)$ will not go through when we impose the new, ‘global’, restriction. For at the point where $(\Diamond K)$ is applied, the subordinate proof of $p \wedge (Kp \to Kq)$ (resp., $p \wedge (Kp \to K\neg q)$) has as its set of assumptions the non-Cartesian set $\{p, \neg Kp\}$.

2.3 A possible objection to the new restriction

A possible objection to the new global restriction is that it is all too easy to comply with. The thought might be that one could still construct a Fitchian reductio by sneaking around the restriction to Cartesian Δ in the Globally Restricted Knowability Principle. The trick would be to successively discharge all the members of a non-empty, non-Cartesian Δ by means of terminal applications of $\to$-introduction within the subordinate proof in the proof-schema below. One would thereby reduce to the empty set the set of assumptions of the resulting subordinate proof for globally restricted $(\Diamond K)$; in which case the restriction in question—that the set of assumptions be Cartesian—would be trivially met:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\underbrace{\overset{(1)}{\delta_1}, \overset{(2)}{\delta_2}, \ldots, \overset{(n-1)}{\delta_{n-1}}, \overset{(n)}{\delta_n}}_{\Pi}}{\varphi}}{\delta_1 \to \varphi}{\scriptstyle (1)}}{\delta_2 \to (\delta_1 \to \varphi)}{\scriptstyle (2)}}{\vdots}}{\delta_{n-1} \to (\ldots(\delta_2 \to (\delta_1 \to \varphi))\ldots)}{\scriptstyle (n-1)}}{\dfrac{\delta_n \to (\delta_{n-1} \to (\ldots(\delta_2 \to (\delta_1 \to \varphi))\ldots))}{\Diamond K(\delta_n \to (\delta_{n-1} \to (\ldots(\delta_2 \to (\delta_1 \to \varphi))\ldots)))}}{\scriptstyle (n)}$$

The objector who takes this line this far will not, however, be able to press it much further. For now we see that any use of

$$\Diamond K(\delta_n \to (\delta_{n-1} \to (\ldots(\delta_2 \to (\delta_1 \to \varphi))\ldots)))$$

as a premise for a rule that involves stripping away the prefix $\Diamond K$ (or, first stripping away $\Diamond$, and thereafter stripping away K) will presumably result in the multiply nested conditional eventually being exposed. Therewith arises a need to assume $\delta_1, \ldots, \delta_n$ in order to winkle out φ for whatever Fitchian mischief is up the objector’s sleeve—mischief which would have been blocked by the new global restriction before the currently contemplated stratagem of multiple →-introductions. Yet there is no guarantee that these forced extra assumptions $\delta_1, \ldots, \delta_n$ will be of the required form (enjoying a dominant occurrence of $\Box$ or of K, say) that might be called for in the application of whatever rule(s) might have been used to strip away both $\Diamond$ and K.

2.4 The importance of normal forms for the new restriction

As foreshadowed earlier, in order for the new global restriction to be effective, we must insist that it be applied only within the context of a proof in normal form. The reason for this is that by resorting to proofs that are not in normal form, one can 'hide from view' the full set Δ that provides the genuine grounds for the possible knowledge-claim φ. Only in the context of a proof in normal form will all those grounds be displayed as undischarged assumptions on which the premise φ (for an application of the Knowability Principle) depends. An example illustrating this point would be the following, which I owe to Salvatore Florio [6]. It exploits the rule called (λ) in [10], at p. 837:

$$(\lambda)\quad \dfrac{\Diamond K\varphi \qquad \begin{array}{c}\mathbf{K}, \overline{K\varphi}^{(i)}\\ \vdots \\ \bot\end{array}}{\bot}{\scriptstyle(i)}$$

The proof using (λ) is as follows.

$$\dfrac{\dfrac{\exists\varphi(\varphi\land\neg K\varphi) \qquad \dfrac{\dfrac{\dfrac{\overline{p\land\neg Kp}^{(2)}}{p} \qquad \dfrac{\dfrac{\dfrac{\overline{Kp}^{(1)} \quad \dfrac{\overline{p\land\neg Kp}^{(2)}}{\neg Kp}}{\bot}}{\neg Kq}}{Kp\to\neg Kq}{\scriptstyle(1)}}{p\land(Kp\to\neg Kq)}}{\exists\varphi(\varphi\land(K\varphi\to\neg Kq))}}{\exists\varphi(\varphi\land(K\varphi\to\neg Kq))}{\scriptstyle(2)} \qquad \dfrac{\dfrac{\overline{p\land(Kp\to\neg Kq)}^{(4)}}{\Diamond K(p\land(Kp\to\neg Kq))} \qquad \dfrac{\dfrac{\dfrac{\overline{K(p\land(Kp\to\neg Kq))}^{(3)}}{Kp} \quad \dfrac{\dfrac{\overline{K(p\land(Kp\to\neg Kq))}^{(3)}}{p\land(Kp\to\neg Kq)}}{Kp\to\neg Kq}}{\neg Kq} \qquad Kq}{\bot}}{\bot}{\scriptstyle(3)}\ (\lambda)}{\bot}{\scriptstyle(4)}$$

This proof, absurdly, reduces an arbitrary proposition of the form Kq to absurdity, on the basis of there being an unknown truth. The proof, however, is not in normal form, for the major premise of the final step of $\exists$-

elimination had been derived two steps earlier as the conclusion of a step of ∃-introduction. By means of the resulting abnormality, this proof is able to harbor an application of the Knowability Principle without, apparently, violating the condition that the ultimate premises for that application should be Cartesian. This can be seen by normalizing the foregoing proof. We do so by applying the appropriate reduction procedure in order to get rid of the maximal occurrence of $\exists\varphi(\varphi \wedge (K\varphi \to \neg Kq))$:

$$\dfrac{\exists\varphi(\varphi \wedge \neg K\varphi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{p \wedge \neg Kp}^{\,(3)}}{p} \qquad \dfrac{\dfrac{\dfrac{\overline{Kp}^{\,(1)} \qquad \dfrac{\overline{p \wedge \neg Kp}^{\,(3)}}{\neg Kp}}{\bot}}{\neg Kq}}{Kp \to \neg Kq}\,(1)}{p \wedge (Kp \to \neg Kq)}}{\Diamond K(p \wedge (Kp \to \neg Kq))}\,(\dagger) \qquad \dfrac{\dfrac{\dfrac{\overline{K(p \wedge (Kp \to \neg Kq))}^{\,(2)}}{Kp} \qquad \dfrac{\dfrac{\overline{K(p \wedge (Kp \to \neg Kq))}^{\,(2)}}{p \wedge (Kp \to \neg Kq)}}{Kp \to \neg Kq}}{\neg Kq} \qquad Kq}{\bot}}{\bot}\,(2)\ (\lambda)}{\bot}\,(3)$$

We find that in the normalized proof the Knowability Principle is, after all, being incorrectly applied at the step marked (†); for its ultimate premise (at this application) is $p \wedge \neg Kp$. And this, as we already know, is not Cartesian. It took conversion into normal form to detect this violation of the global restriction on the Knowability Principle.

3 Restricting the Rule of Factiveness

It was pointed out earlier that the Rule of Factiveness

$$\frac{\Diamond K\psi}{\psi}$$

could be restricted in its applications to sentences ψ that do not contain any occurrences of K. Call the Rule of Factiveness of $\Diamond K$, restricted in this way, $\mathcal{F}$. The restriction in question is very well motivated for the anti-realist who is also an internalist about knowledge. For if, as some internalists do, one subscribes to the so-called KK-thesis:

$$\frac{K\psi}{KK\psi}$$

an *unrestricted* Rule of Factiveness $\mathcal{UF}$ will instate the unwanted Fitch inference, even for Cartesian propositions ψ. For consider the following proof Ψ that ψ implies $K\psi$.[5]

$$\Psi : \qquad \dfrac{\dfrac{(\Diamond K)\dfrac{\psi}{\Diamond K\psi} \qquad \dfrac{\dfrac{}{K\psi}(1)}{KK\psi}(KK)}{\Diamond KK\psi}(1)\ (\Diamond\vdash)}{K\psi}(\mathcal{UF})$$

Given this proof, one will be able, disastrously, to inflate possibility to actuality (for Cartesian propositions ψ),[6] by means of the following proof Π.

$$\Pi : \qquad \dfrac{\dfrac{\Diamond\psi \qquad \begin{matrix}\dfrac{}{\psi}(2)\\ \Psi \\ K\psi\end{matrix}}{\Diamond K\psi}(2)\ (\Diamond\vdash)}{\psi}(\mathcal{F}) \quad \text{, i.e.} \quad \dfrac{\dfrac{\Diamond\psi \qquad \dfrac{\dfrac{(\Diamond K)\dfrac{\dfrac{}{\psi}(2)}{\Diamond K\psi} \qquad \dfrac{\dfrac{}{K\psi}(1)}{KK\psi}(KK)}{\Diamond KK\psi}(1)\ (\Diamond\vdash)}{K\psi}(\mathcal{UF})}{\Diamond K\psi}(2)\ (\Diamond\vdash)}{\psi}(\mathcal{F})$$

The way to avoid this madness, without losing a proper grip on knowability, is to refuse to grant the unrestricted factiveness of $\Diamond K$. It should be possible to hold the KK-thesis without making every truth known (via Ψ), and without inflating the possible truth of a Cartesian proposition to its actual truth (via Π). Suppose one holds that a knower's knowing is always reflectively accessible to the knower, so that if x knows that φ, then x knows that x knows that φ. It follows that if it is known that φ, then it is known that it is known that φ. That is, the KK-thesis holds.

Suppose now that ψ is true. The anti-realist wants to say that it is possible, in principle, for someone to know that ψ. That envisaged in-principle possibility will, if and when actualized (say at time t), bring with it the knowledge (at t) *that* ψ *is known* (at t). So, it is possible also that it be known that ψ is known (i.e., $\Diamond KK\psi$). But does this *intuitively* imply that ψ is actually known, or will ever actually be known (i.e., $K\psi$)? Of course it does not!—for the envisaged possibility *might* never be actualized. There is

[5] Compare Brogaard and Salerno's proof of the KK-knowability paradox in [1].

[6] This was observed by Brogaard and Salerno in [2].

a degree of serendipity in empirical (and even mathematical) inquiry, which even the anti-realist must recognize. So we cannot treat $\Diamond K$ as reliably factive when applied to propositions, such as $K\psi$, that have K dominant. Our formal rules of inference must answer to our pre-theoretic intuitions. We cannot issue the *carte blanche* of the unrestricted rule $\mathcal{F}$.

3.1 Two proposals for restricting Factiveness

How, though, might $\mathcal{F}$ be restricted? Clearly, it is the rot within the proof Ψ that has to be stopped. The pre-theoretic intuitions mulled through above tell us that this much of Ψ is in order:

$$\Psi' : \qquad \dfrac{{\scriptstyle(\Diamond K)}\dfrac{\psi}{\Diamond K\psi} \qquad \dfrac{\overline{K\psi}^{\,(1)}}{KK\psi}{\scriptstyle(KK)}}{\Diamond KK\psi}{\scriptstyle(1)\ (\Diamond\vdash)}$$

It is only the *final* step of Ψ that should give us pause:

$$\dfrac{\Diamond KK\psi}{K\psi}{\scriptstyle(\mathcal{UF})}$$

We have envisaged circumstances in which one would be warranted in asserting the premise $\Diamond KK\psi$, without being warranted in asserting the conclusion $K\psi$. So: what is wrong with *this* application of the rule $\mathcal{UF}$? What might be the general defect exhibited? Could we restrict applications of the rule so as to avoid just those would-be applications that are defective in this way?

3.1.1 The first proposal for restricting Factiveness

A first stab at the problem might be to insist that applications of the rule

$$\dfrac{\Diamond K\varphi}{\varphi}$$

may be made only when the sentence φ is 'about basic, non-epistemic facts'. One could give here a myriad examples from the language of physics, mathematics, chemistry, biology etc., of sentences that are about basic, non-epistemic facts. The proposal would be that sentences like *these* could be substituends for φ in applications of the rule of factiveness. For such sentences, surely, the only reason why it might be possible to know that they are true is that they are indeed true. This is *not* the case, however, with 'epistemic' sentences θ. Here, there can be reason to hold it possible to know that the truth of θ is known, *without* this being a reason for holding that the truth of θ is indeed (or will ever be) known. Our inquiries might, by the heat-death of the universe, never have taken the turns required, even though, had we conducted our investigations otherwise, we *could* have come to know the truth of θ.

Restriction to K-free φ would certainly block the final step of Ψ, since it depends on taking $K\psi$ as a substituend for φ in the statement of the rule. But this seems rather drastic as a proposed logical inoculation against the possibility of incurring Ψ-type rot.

3.1.2 A second proposal for restricting Factiveness

Another way to formulate a restriction that would render the proof Ψ ill-formed (at its final step) would be to observe that the premise for that would-be application of the Rule of Factiveness stands as the conclusion of the application, labeled (1), of the rule $\Diamond \vdash$, in whose subordinate proof there had been an application of the rule KK:

$$\dfrac{\dfrac{(\Diamond K)\dfrac{\psi}{\Diamond K\psi} \qquad \dfrac{\dfrac{}{K\psi}{\scriptstyle(1)}}{KK\psi}{\scriptstyle(KK)}}{\Diamond KK\psi}{\scriptstyle(1)\ (\Diamond\vdash)}}{K\psi}{\scriptstyle(\mathcal{UF})}$$

We can perhaps make more vivid the reason why one might wish to formulate the restriction this way. Let us identify the different occurrences of K by means of italics and boldface. The proof would then look like this:

$$\dfrac{\dfrac{(\Diamond K)\dfrac{\psi}{\Diamond K\psi} \qquad \dfrac{\dfrac{}{K\psi}{\scriptstyle(1)}}{\mathbf{K}K\psi}{\scriptstyle(KK)}}{\Diamond \mathbf{K}K\psi}{\scriptstyle(1)\ (\Diamond\vdash)}}{K\psi}{\scriptstyle(\mathcal{UF})}$$

The idea would be that $\Diamond\mathbf{K}$ is not factive, whereas $\Diamond K$ is. $\Diamond\mathbf{K}$ is not factive because $\mathbf{K}$ was introduced by an application of the rule (KK).

3.1.3 The importance of normal form, again

As with the global restriction on the Knowability Principle, this restriction will work only when we have ensured that the proof is in normal form. In order to illustrate this, let us remind ourselves how proofs *not* in normal form characteristically arise. They come from joining together two proofs, the conclusion of one of them being an undischarged assumption of the other. The occurrences of the sentence serving as such a 'point of accumulation' can thereby be *maximal*: a point of locally increased, and—given the overall context—unnecessary logical complexity. Such local complexity can be eliminated by applying a suitable reduction procedure. The result of applying the appropriate reduction procedure might well be a new proof in which *other* sentences now have maximal occurrences. But these will be of lower complexity than the original one. By repeatedly applying the appropriate reduction procedures, the proof will eventually be transformed into one in normal form.

In the context of epistemic logic, here is a simple proof—call it Φ—that appears to be entirely in order. It does not even sin against the newly proposed restriction on the Rule of Factiveness of $\Diamond K$ (the one that is framed in terms of earlier applications of the rule $\Diamond \vdash$).

$$\Phi: \qquad \dfrac{{\scriptstyle(\Diamond\vdash)}\ \dfrac{\Diamond K(\varphi\wedge\psi) \qquad \dfrac{\dfrac{}{K(\varphi\wedge\psi)}{\scriptstyle(1)}}{K\varphi}}{\Diamond K\varphi}{\scriptstyle(1)}}{\varphi}$$

Below is another proof—call it Ω. The reader ought to look ahead at Ω in order to follow the explanatory comments about to be entered. In Ω, the sentence ψ is indicated as having been proved outright, from the empty set of assumptions. Since we are idealizing the logical abilities of our knowers, we may accordingly infer $K\psi$—for a logical saint knows every logical theorem.

In Ω we also employ the rule

$$\dfrac{K\theta \quad K\chi}{K(\theta\wedge\chi)}$$

This rule can be justified, in its application within Ω, by appeal to the more obvious rule

$$\frac{aK\theta \quad aK\chi}{aK(\theta \wedge \chi)}$$

which employs a logical form making explicit provision for the knower a. (Claims of the form $K\varphi$ are really short for $\exists x(xK\varphi)$.) The latter rule, applied to the materials involved in Ω, would mediate the inference

$$\frac{aKK\varphi \quad aK\psi}{aK(K\varphi \wedge \psi)}$$

—for, since any logical saint knows the truth of the theorem ψ, she will know also the conjunction $\theta \wedge \psi$, for any truth θ that she knows. (Here, the truth θ takes the form $K\varphi$.)

Here at long last is the proof Ω.

$$\Omega : \qquad (\Diamond\vdash)\ \dfrac{\dfrac{\varphi}{\Diamond K\varphi} \qquad \dfrac{\dfrac{\dfrac{}{K\varphi}{\scriptstyle (2)}}{KK\varphi} \qquad \dfrac{\begin{matrix}\emptyset\\ \vdots\\ \psi\end{matrix}}{K\psi}}{K(K\varphi \wedge \psi)}}{\Diamond K(K\varphi \wedge \psi)}{\scriptstyle (2)}$$

Let us substitute $K\varphi$ for φ within the proof Φ, so as to obtain the following proof Φ'.

$$\Phi' : \qquad (\Diamond\vdash)\ \dfrac{\dfrac{\Diamond K(K\varphi \wedge \psi) \qquad \dfrac{\dfrac{}{K(K\varphi \wedge \psi)}{\scriptstyle (1)}}{KK\varphi}}{\Diamond KK\varphi}{\scriptstyle (1)}}{K\varphi}$$

The time has come now to join together the proofs Ω and Φ'. Note that the conclusion of Ω is the undischarged assumption of Φ'. When we make that sentence $\Diamond K(K\varphi \wedge \psi)$ the point of accumulation, we obtain the following proof, which is not in normal form, and which purports to establish the Fitch result $\varphi \vdash K\varphi$:

$$\dfrac{{\scriptstyle(\Diamond\vdash)}\ \dfrac{{\scriptstyle(\Diamond\vdash)}\ \dfrac{\dfrac{\varphi}{\Diamond K\varphi}\qquad \dfrac{\dfrac{\dfrac{}{K\varphi}{\scriptstyle(2)}}{KK\varphi}\qquad \dfrac{\begin{matrix}\emptyset\\ \vdots\\ \psi\end{matrix}}{K\psi}}{K(K\varphi\wedge\psi)}}{\Diamond K(K\varphi\wedge\psi)}{\scriptstyle(2)}\qquad \dfrac{\dfrac{}{K(K\varphi\wedge\psi)}{\scriptstyle(1)}}{KK\varphi}}{\Diamond KK\varphi}{\scriptstyle(1)}}{K\varphi}$$

The advocate of restricting the Rule of Factiveness so that it yields only non-epistemic conclusions will already have objected that the substitution-instance Φ' of Φ sins against this restriction. Moreover, he might (mistakenly, as it happens) complain that the *other* bruited restriction, banning applications of the rule KK within subordinate proofs for applications of the rule $\Diamond\vdash$, does not work: witness the proof just formed, in which the latter restriction does not appear to be violated.

Appearances, however, can be deceptive. Note that the proof in question is not in normal form. It turns out that, if we normalize it, it is transformed into a normal-form proof in which that second proposed restriction *is* violated. To see this, proceed as follows. First, apply the obvious reduction procedure to the proof just constructed, so as to get rid of the maximal occurrence of $\Diamond K(K\varphi\wedge\psi)$. The result is

$$\dfrac{\dfrac{\dfrac{\varphi}{\Diamond K\varphi}\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{K\varphi}{\scriptstyle(1)}}{KK\varphi}\qquad \dfrac{\begin{matrix}\emptyset\\ \vdots\\ \psi\end{matrix}}{K\psi}}{K(K\varphi\wedge\psi)}}{KK\varphi}}{\Diamond KK\varphi}{\scriptstyle(1)}}{K\varphi}$$

in which there is a new maximal sentence occurrence, namely the occurrence of $K(K\varphi\wedge\psi)$. Upon applying the appropriate reduction procedure in order to get rid of this occurrence, we obtain the following proof in normal form:

$$\Psi^{\psi}_{\varphi}: \quad \dfrac{\dfrac{\dfrac{\varphi}{\Diamond K\varphi}{\scriptstyle(\Diamond K)} \qquad \dfrac{\dfrac{}{K\varphi}{\scriptstyle(1)}}{KK\varphi}{\scriptstyle(KK)}}{\Diamond KK\varphi}{\scriptstyle(1)\,(\Diamond\vdash)}}{K\varphi}{\scriptstyle(\mathcal{UF})}$$

—which of course we had already faulted earlier, on the basis of our second proposed restriction on applications of the rule of factiveness.

3.1.4 Blocking an S4-like route to Fitch

Consider the following purported proof Ξ, involving a Cartesian proposition φ:[7]

$$\Xi \qquad \dfrac{\dfrac{\dfrac{\Diamond\varphi \qquad \dfrac{\dfrac{}{\varphi}{\scriptstyle(1)}}{\Diamond K\varphi}}{\Diamond\Diamond K\varphi}{\scriptstyle(1)}\ \diamond\text{-E}}{\Diamond K\varphi}\ \text{S4}}{\varphi}\ \mathcal{F}$$

The topmost step, an application of the Knowability Principle, complies with the global restriction. The purported result is unacceptable: that every Cartesian proposition, if possible, is true. Indeed, so is the inference from $\Diamond\varphi$ to the claim $\Diamond K\varphi$, the penultimate conclusion of the proof Ξ

This 'proof' highlights a point made in §1.1. There are two possibility operators at work, and they need to be distinguished. The one that is introduced by applications of the Knowability Principle adverts not to metaphysical possibilities (which may be contrary to actual fact), but rather to the possibility of an agent coming to know, in accordance with the contingent facts governing his own world, that a certain proposition is true. This epistemic possibility operator should accordingly be distinguished from the metaphysical possibility operator. We shall use $\diamond$ and $\Diamond$ respectively. The 'proof' Ξ accordingly becomes

[7] This proof is due to an anonymous referee.

$$\Xi' \qquad \dfrac{\dfrac{\dfrac{\Diamond\varphi \qquad \dfrac{\overset{\text{---}(1)}{\varphi}}{\diamond K\varphi}}{\Diamond \diamond K\varphi}\,{\scriptstyle (1)}\ \diamond\text{-E}}{\diamond K\varphi}\ \text{S4 (??)}}{\varphi}\ \mathcal{F}$$

in which the allegedly S4-like step is now clearly not valid. While the step

$$\frac{\Diamond\Diamond\theta}{\Diamond\theta}$$

is formally correct and valid for the metaphysical possibility operator $\Diamond$ of S4, and hence also its substitution instance

$$\frac{\Diamond\Diamond K\varphi}{\Diamond K\varphi}$$

is formally correct and valid, matters are very different with the step

$$\frac{\Diamond \diamond K\varphi}{\diamond K\varphi}$$

By way of counterexample, consider the Cartesian proposition 'Grass is purple' as an instance of φ. It is *metaphysically* possible that grass be purple; but it happens, in our world, not to be. In any other possible world in which grass *were* purple, however, it would be possible (in that world) to know that it was purple. Hence such possible knowledge would also be a metaphysical possibility. That is, the premise $\Diamond \diamond K\varphi$ of the last displayed rule of inference, for this choice of φ, is true. Its conclusion $\diamond K\varphi$, however, is false—for grass's being purple is not anything we could come to know in *this* world.

Since the rule of inference in question is invalid, we can prohibit any application of it in proofs. The purported 'proof' Ξ is not a proof.

3.1.5 Summary of discussion of how to restrict Factiveness

We see, then, that there are still two ways of restricting the Rule of Factiveness, so as to avert the Fitch paradox in the presence of the KK-rule. The first restriction is easy to apply: simply ban applications of the Rule of Factiveness that involve conclusions containing K. The second restriction, however, seems at this stage to be just as effective. But we need to bear in mind that it stops the rot only when we have converted the proof to be appraised into normal form. In this regard, the second restriction is like the global restriction proposed earlier for applications of the Knowability Principle. These applications, too, can be assessed for correctness only when the proof in question has been transformed into normal form.

This is not the first philosophical problem for which the technique of converting proofs into normal form has afforded useful insights. Dag Prawitz used normalization techniques to frame a fertile conception of intuitionistic consequence. (See [7] and [8].) Reduction procedures also form the centerpiece of Michael Dummett's inferential theory of meaning, and his arguments in favor of intuitionistic logic as the right logic. (See [3], the two famous essays 'The Philosophical Basis of Intuitionistic Logic' and 'The Justification of Deduction' in [4], and [5].) Normalization lies at the heart also of the present author's characterization of *relevance* in deduction. (See [12] for an exposition and for relevant sources.)

4 Conclusion

We have undertaken here only the most preliminary explorations of proof-theoretic measures designed to stave off certain threats of paradox in an anti-realist epistemic logic. We are still a long way, of course, from having a fully adequate proof-theory governing the interaction among $\Diamond$, $\diamond$ and K (let alone a formal semantics, with respect to which one might be able to establish the soundness and completeness of whatever proof system is devised). The aim here has been to clear the way for an eventual proof (if such can be found) of a metatheorem to the effect that a suitable system of proof (in epistemic modal logic) embodying the globally restricted Knowability Principle is *Fitch-free*: that is, it affords no proof of φ from $K\varphi$. Establishing such a result, however, is beyond the scope of the present paper.

This discussion has suggested a proof-theoretic path for the anti-realist to follow, without being Fitched. The way forward is to formulate the Cartesian restriction on the Knowability Principle by reference to the *ultimate grounds* that *one could know* for the truth of a proposition. As our discussion revealed, the technique of normalizing proofs is crucial for detecting incorrect applications of the Knowability Principle. And the use of a natural deduction format affords us structural insights into proofs, and the resources by means of which one can frame some of the delicate but philosophically motivated restrictions that might be called for on applications of certain rules of inference. With an eye to such resources, other logical or epistemic principles, besides the Knowability Principle, can be adopted in more liberal or more exigent formulations, by way of cautious development of an anti-realist, epistemic logic. The guiding requirement will be that every truth be knowable, without implying that it need ever be known.

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