Entailment and Proofs
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XI*—ENTAILMENT AND PROOFS

by N. Tennant

1. A common observation in discussions of entailment is that entailment is "the converse of deducibility". The observation goes back at least to Moore, and has been attributed to him ever since. It is remarkable, however, that no writer has taken it seriously enough to make it the basis of an adequate theory of entailment. We frequently encounter, in writings on entailment, a transition or modulation from the deductive to the consequential key. An obscure semantical notion of "genuine validity" of arguments enters the picture, and entailment is then conceived of as that relation which holds between premisses and conclusion of a "genuinely valid" argument. The arguments concerned need not be deductions, understood as carried out according to fixed rules which in every application are obviously, intuitively and "genuinely" valid. Rather, the arguments concerned may consist only of stated premisses, an inference marker, and a conclusion. The problem of entailment, in this consequential key, becomes the problem of explicating what *semantical* conditions must in general be satisfied for any argument of the bare form

$P_1, \ldots, P_n$; therefore Q

to be "genuinely" valid. Some writers have appealed to meaning connections or facts for solution of the problem thus posed. Others have explored the combinatorial possibilities afforded by the method of substitution-cum-truth-tables. Even those who did not begin with the problem in the semantical key, nor yet with the properly posed problem in the deductive key, and who have arrived at calculi of entailment by one route or another, have produced formally adequate algebraic semantics which do not reflect or represent any philosophical insight into the proper nature of entailment. They have tried to capture a non-standard consequence relation by using a standard form of definition relying on a non-standard notion of "truth-in-a-model". Adequacy with

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respect to any independently motivated syntactic characterization is algebraic accident.

The problem of entailment, I shall contend, must be posed in the deductive key. Moreover, for reasons which are not best stated beforehand, but will emerge below, I shall place the problem in its proper perspective by offering two extended observations as supplements to Moore's original one.

(a) "Entailment logic" (whatever that may turn out to be) should not be thought of as a rival alongside the usual rivals such as classical and intuitionistic logic. Rather, a general theory of entailment should produce a philosophically motivated and uniform method for extracting the *entailment fragments* of these. (For the present we may conceive a fragment of a logic to be a proper subrelation of its deducibility relation, with appropriate closure properties.) We need a uniform theoretical account of the "intuitions about entailment" of both the classical and the intuitionistic logician. Adherence to either a classical semantical theory based on truth conditions of sentences, or an intuitionistic one based on assertability conditions, should not matter in the least when one approaches the problem of accounting for how, within their respective systems, the classical and intuitionistic logicians might wish to distinguish the "genuinely" valid arguments (i.e. the entailments) from the "non-genuinely" valid ones. Thus, as a classicist, I might wish to distinguish the "genuinely" valid argument

P or Q, if P then R, if Q then S; therefore R or S

from the "non-genuinely" valid argument

P and Q, if P then R, if Q then not R; therefore S.

Likewise, as an intuitionist I might wish to make the same distinction. I believe it will be a fruitful assumption that in each case my reasons for wishing to make the distinction are the same. That is, entailment considerations are invariant with respect to preferred semantics. A general theory of entailment will, in response, describe what considerations are appropriate and how they are applied, in determining the entailment fragment of one's preferred logic. (Neglect of this consideration renders Anderson and Belnap's treatment of tautological entailments parochial to the classical case. Intuitionistic logic does not sanction all the steps required for the transformation of formulae into normal forms of the desired kind.)

(b) The general theory will, therefore, call for no semantical justification over and above what is shown by a correct analysis of the appropriate considerations just mentioned. In the theory to be put forth below these considerations involve certain structural transformations available for determining the subclass of entailment proofs within the class of proofs of any reasonable system. The non-entailment residue of such a system, consisting as it does of proofs which are valid, but not "genuinely" so, within the system, is not to be regarded as semantically deviant or deficient, an embarrassment to be excised by some future, and better, semantics. Rather, the non-entailment proofs in such a residue will be possessed of whatever objectionable features they might have as a consequence of a theory which, in a manner analogous to that of Grice's theory of implicature, will explain what is "argumentatively" or, as I would like to say, "suasively" inappropriate about such proofs. A conspicuous example, by both classical and intuitionistic lights, is the argument (or proof thereof)

P, not $\sim$ P; therefore Q,

which is not semantically deviant in either system, but is suasively inappropriate in both. More will be said about the notion of suasive inappropriateness below.

2. The reader will already have gathered that I intend, in the sequel, to address myself to the problem in the context of first order logic. I shall be concerned, not with lexical or conceptual entailments such as

This is red; therefore this is coloured
This is coloured; therefore this is extended,

but rather with entailments arising from logical structure alone.

Moreover, Moore's observation points the way, but not the whole way. I shall not, except where it is appropriate, be concerned (as, for example, Smiley was) primarily with global or holistic features of the deducibility relation of any logic, conceived of extensionally. Instead I shall try to locate entailment *by analysis of the proofs in the system* which give rise, in the well-known way, to the deducibility relation. No serious attempt has yet been made to analyse those features of proof which preserve (constitute, respect etc.) entailment or destroy it (impair it, block it, etc.).

What, then, are the main ingredients of the logician's conception of proof? In answering this question I shall regard proofs as pieces of discourse which faithfully represent the structure, or best possible refinement of the structure, of arguments as actually presented or presentable in mathematical and everyday contexts. We may concede that an informal mathematical argument is no more than a "protocol" for finding the corresponding formal proof. There can be no doubt, however, that only certain kinds of formal proof could be the products of the appropriate search, prompted by the protocol, for its "refinement" or "perfected version". Indeed, I would go so far as to claim that only one kind of formal proof can serve this purpose. This is the kind known as "natural deduction". Prawitz has given a lucid account of how, in any piece of discourse purporting to establish some conclusion, we can discern statements immediately supporting the conclusion and then statements immediately supporting those, and so on, until one has uncovered a rough tree-like structure of dependence of the conclusion upon those statements which, apparently having no immediate support in the context of the argument, serve as its premisses. More importantly, the relation of immediate support in each case turns on a single specific aspect of the logical structure of one of the statements immediately involved. Immediate support is conferred, that is, by applications of rules of inference sensitive to this single aspect, namely, the dominant occurrence of a particular logical operator. The introduction rule for a logical operator describes the conditions under which a conclusion with that operator dominant is supported. The elimination rule describes the conditions under which a statement with that operator dominant may support others.

Thus a formal proof is a tree-like array of sentence occurrences, with the conclusion at the bottom, the premisses at the tops of branches, and branchings constrained by rules of inference. Various assumptions made "for the sake of argument" (such as in conditional proof or in *reductio ad absurdum*) might appear also at tops of branches, but it will be possible to say at what step in the proof they are "discharged", being assumptions upon which the conclusion of that step does not depend.

Subsequently I take for granted the primacy of that mode of representation of the course of reasoning which has come to be known as natural deduction. It has a claim to a high level of generality in the representation of argumentative structure, accommodating intuitionistic and classical reasoning alike. Moreover it represents this structure in an agreeably and predictably direct way. The formal proofs provided by systems of natural deduction are prefigured in the informal arguments whose structure they refine and perfect. Henceforth, therefore, by "proof" I shall always mean proof in a system of natural deduction.

There are several formulations of natural deduction, differing not so much in their stocks of basic *versus* derived rules of inference, but in their different ways of representing, via their formal proofs, the same process of reasoning. For example, the most familiar textbook presentation treats proofs as linear arrangements of sentences with marginal annotations to keep track of dependencies. On the other hand, there is a competing treatment in which proofs have the form of tree-like arrays of sentence occurrences.

For compelling reasons of meta-theoretic elegance, simplicity and perspicuity I shall concern myself with proofs in tree-like form in subsequent discussion. Note that in such proofs statements are repeated, or re-derived, every time they are used in immediate support of subsequent statements. It is only in this respect that we encounter a difference between informal arguments and their formal codifications. It is a difference, however, yielding proof-theoretic insights rather than counting against tree proofs as the best codification of informal reasoning. Replication of subtrees in this way provides a toehold for essential "pruning" and "grafting" operations in the course of normalizing proofs.

3. Prawitz's normalization theorem says that every proof can be converted into a unique normal form in which no sentence occurrence stands as the conclusion of an application of an introduction rule and as the major premiss of an application of the corresponding elimination rule. Such sentence occurrences are called maximal, and they represent unnecessary detours in the course of reasoning. They are removed in the course of normalization by repeated application of so-called "reduction procedures" for the logical operators. (For details, see my book *Natural Deduction* (Edinburgh, 1978).) Further reduction procedures are available

that enable one to remove from a proof any sentence occurrence standing as the conclusion of an application of a classical negation rule and as the major premiss of an elimination rule. I shall assume that we use these reduction procedures as well when normalizing a proof. A normalized proof establishes the same conclusion from premisses that are among those of the original, unnormalized proof.

Familiarity with the broad features of Prawitz's treatment will henceforth be assumed. I now resume the discussion of the main ingredients of the logician's conception of proof.

Firstly, every proof must be normalizable. This is a partial explication of the requirement that one should be able to reason straightforwardly from premisses to conclusion without making any of the detours which the reduction procedures are designed to eradicate.

Secondly, deducibility is a transitive relation. Informally, this means that conclusions drawn from premisses which in turn are conclusions drawn from earlier premisses, are deducible from the latter. Note that this is a weak requirement in so far as a proof in virtue of which the final statement of deducibility holds need bear no discernible relation to any proofs in virtue of which the preceding statements of deducibility hold. This circumstance, however, need not count against a deducibility relation's adequacy for mathematics. If I can conclude from my axioms to some lemmata, and then from those lemmata to a theorem, it matters not that I might not be able to put the corresponding proofs together in some sensible way to obtain a proof of the theorem from the axioms. For, provided the deducibility relation *is* transitive, I know that the theorem will be deducible from the axioms by means of some proof or other. A "sensible" way of finding such a proof would be simply to enumerate all proofs. Since *some* proof establishes the theorem from the axioms we shall eventually find one. This, however, does leave something to be desired – something, moreover, that is not missing in actual deductive practice. What is wanted is a stronger form of transitivity:

Deductions are transitive.

That is, conclusions of proof trees can be grafted over assumptions (of the same form) of other proof trees to produce bigger proof trees. The resulting tree is a proof which establishes the bottom-most conclusion from the topmost undischarged assumptions. In this way the earlier proofs are obviously "relevant" in the construction of the final proof. Our stronger transitivity principle may be rephrased as follows:

> Proofhood is preserved under accumulation.

In this form it admits of a stronger and a weaker version:

(i) Proofhood is preserved under accumulation at *any* top formula occurrence, and
(ii) Proofhood is preserved under accumulation at any *undischarged* top formula occurrence.

There is no reason to prefer the strong version to the weak one, since the latter ensures transitivity of deducibility and ensures also that by accumulating proofs of lemmata from axioms over a proof of a theorem from those lemmata, one obtains a proof of the theorem from the axioms.

Normalizability and transitivity of deductions are the two main ingredients of the logician's conception of proof that I wish to highlight. A third is the following:

> Proofhood is preserved upon uniform substitution of more complex expressions for placeholders.

In other words, a proof establishes the validity of an argument at a certain level of logical structure once and for all. No further refinement of logical structure, no further internal articulation, can alter the force of the proof.

The three principles above are fundamental to the notion of proof or deduction. Any new theory of proof resulting from an attempt to solve the problem of entailment that gave up any of them would have to adduce compelling reasons for doing so.

4. Consider now the rules of inference for classical and intuitionistic natural deduction, as given in *Natural Deduction* (*op. cit.*) Each familiar logical connective (and, in the first order case, quantifier) has its own introduction rule and corresponding elimination rule. A proof-theoretic constant Λ is used to record the occurrence of a contradiction. If we add to the introduction and elimination rules the *absurdity rule*

$$\frac{\Lambda}{A}$$ (ex falso quodlibet)

we obtain the full set of rules for intuitionistic logic. If, further, we add any one of the following four *classical rules of negation*, we obtain a full set of rules for classical logic :

$$\frac{\begin{array}{c}\overline{\sim A}\\ \vdots\\ \Lambda\end{array}}{A}$$ (classical reductio)

$$\frac{\begin{array}{cc}\overline{A} & \overline{A}\\ \vdots & \vdots\\ B & B\end{array}}{B}$$ (dilemma)

$$\frac{\sim\sim A}{A}$$ (double negation)

$$\frac{}{A \vee {-}A}$$ (excluded middle)

Each of these classical rules may be derived from any of the others within the system of intuitionistic logic.

5. Now let us look at the operation of discharging assumptions in natural deduction. The rules v–E and ∃–E permit discharge of all assumption occurrences of the indicated forms on which the respective subordinate conclusions depend. But these rules may be applied even where there are no assumptions of the indicated forms to be discharged. For example, we could construct a formal proof with a final step of v–E :

$$\frac{\begin{array}{ccc} & & \overline{B}\\ \vdots & \vdots & \vdots\\ A\vee B & C & C\end{array}}{C}$$

in which the subordinate conclusion C at its left occurrence does not depend on any occurrence of A. In this case the subordinate proof in question already establishes C from undischarged premisses of the whole proof. We readily see that in general "vacuous" proof by cases, like "vacuous" proof by instantiation, does not extend the deducibility relation arising from proofs admitting only "non-vacuous" applications of these two rules. We can therefore demand that applications of v–E and ∃–E actually discharge assumptions as indicated, without loss of deducibility. That is, we require that there be assumptions of the indicated form available for discharge in any application of these rules.

Similarly for negation introduction, for classical reductio (in the presence of the absurdity rule) and for the rule of dilemma.

Only one "discharge rule" remains for scrutiny in the light of these considerations: the rule of conditional proof, or → –I. The supposedly lax maxim of classical and intuitionistic natural deduction according to which discharge of assumptions as indicated is a permissible, not an obligatory operation in the construction of proofs, gives rise in the case of →–I to fallacies according to some entailment theorists. For we can construct the one-step proof

$$\frac{A}{B \to A}$$

without discharging any assumptions of the form B. We may add an innocent further step of → –I, discharging A, to prove the theorem

$$A \to (B \to A),$$

one of the well-known 'paradoxes' of material implication.

Let us consider how a critic may argue against the one-step proof above:

The argument "from" A to (B → A) commits a fallacy of relevance. It purports to establish an implication (B → A) on grounds which supply no indication whatever of the relevance of B to A. It is *suasively inappropriate*. For, if a colleague sincerely asserts (B → A) in a mathematical discussion, say, on the strength of certain other assumptions Δ that surely gives one reason to believe that, using Δ *and B* in a cogent way he can prove A. If he really knows that B is not needed as an assumption in such a proof, it is suasively inappropriate for him to drag it in as a qualifying antecedent to what he can prove outright from Δ alone.

Now in reply to this standard line of argument, I would point out that relevance is not a semantical notion. I see no need yet to incorporate it as an ingredient of a semantical account of either truth conditions or assertability conditions of *conditionals*. I am not convinced that

$$\frac{A}{B \rightarrow A}$$

is suasively inappropriate. For the premiss is relevant to the conclusion. We adopt as a slogan that relevance is an inferential, not a sentential notion. In the interests of preserving relevance of assumptions to conclusions in the "entailment" proofs of both classical and intuitionistic logic we do not have to call for tighter restrictions upon the discharge effected by → –I.

In reply to the critic's main point about mathematical discussion, I would point out that many unobjectionable mathematical demonstrations begin with "over-determining" premisses. It is common to conclude, after reaching the conclusion and inspecting the whole proof "Oh, that's interesting: note that we didn't need/use such-and-such premisses/conditions after all." What turns out retrospectively to have been a diluted statement of deducibility seems quite acceptable.

6. We turn now to the remaining source of fallacies of relevance, the absurdity rule. Note that in a system in which $\sim A$ is defined as $(A \rightarrow \Lambda)$ the step

$$\frac{\Lambda}{\sim A}$$

(*sans* discharge) could count either as an application of the absurdity rule or as a "vacuous" application of → –I. So if we accepted the criticism of vacuous → –I above, two sources of fallacies of relevance would overlap. More centrally, however, the absurdity rule permits the construction of the proof

$$\dfrac{\dfrac{A \quad \sim A}{\Lambda}}{B}$$

of the first of the so-called "Lewis paradoxes". It purports to establish any statement B as a conclusion from the contradictory premisses A, ~A. Lewis argued that his proof was acceptable. He gave the following "independent" proof of B from (A&–A), claiming that each step was valid, and that the sequence they formed established an entailment:

1 A&~A
2 A from (1)
3 ~A from (1)
4 AvB from (2)
5 B from (3), (4)

The step in this proof from ~A, AvB to B is an application of *disjunctive syllogism*, which is not a primitive inference in our system but which may be derived as follows:

$$\dfrac{A \vee B \qquad \dfrac{\dfrac{\overset{(1)}{\overline{A}} \quad \sim A}{\Lambda}}{B} \qquad \overline{B}^{(1)}}{B}(1)$$

This proof involves a conspicuous application of the absurdity rule. If we now use this derivation of disjunctive syllogism to translate the natural deduction enshrined in Lewis' argument (1–5) we obtain the following formal proof:

$$\dfrac{\dfrac{\dfrac{A\&\sim A}{A}}{A \vee B} \qquad \dfrac{\dfrac{\overset{(1)}{\overline{A}} \quad \dfrac{A\&\sim A}{\sim A}}{\Lambda}}{B} \qquad \overline{B}^{(1)}}{B}(1)$$

This proof is not in normal form. Upon normalization we obtain

$$\dfrac{\dfrac{\dfrac{A\&{\sim}A}{A} \quad \dfrac{A\&{\sim}A}{{\sim}A}}{\Lambda}}{B}$$

in which the absurdity rule is still conspicuously applied. This gives good reason to suspect that it is really the absurdity rule which is at fault if we reject the proof as establishing that (A&~A) *entails* B. *Prima facie* rejecting the absurdity rule means rejecting disjunctive syllogism as well. I do not, however, think that disjunctive syllogism is irretrievable in pursuit of a sensible relation of entailment. My reasons for saying so will emerge below.

Consider what is suasively inappropriate about the proof

$$\dfrac{\dfrac{A \quad {\sim}A}{\Lambda}}{B}$$

It is this: In general a proof of Ψ from Δ is suasively appropriate only if a person who believes Δ can reasonably decide, on the basis of the proof, to believe Ψ. But if the proof shows his belief set Δ to be inconsistent "on the way to proving" Ψ from Δ then the reasonable reaction is to *suspend* belief in Δ rather than acquiesce in the doxastic inflation administered by the absurdity rule. Quite obviously our little proof above shows A, ~A to be inconsistent "on the way to proving" B.

Unfortunately this is too trivial an example. The point at issue virtually disappears in deductive degeneracy. For it is highly implausible to think of a man professing to believe A and ~A at the outset and only later "coming to see" by way of the proof

$$\dfrac{A \quad {\sim}A}{\Lambda}$$

that his beliefs are inconsistent!

With more complex examples, however, the point I am making is readily borne out. Indeed, we have only to turn to Bennett's amusing tale of his quaint native reasoner for a case in point. Ironically the native's argument is advanced by Bennett as providing an example of rational and valid use of disjunctive syllogism. I believe, however, that Bennett is hoist with his own petard. His poor native is the best possible evidence for the prelogicality of primitives that I have yet come across in the post-Quinean literature.

Let us set his argument forth.

7. The Oracle says

There will be rain this month or the king will die (RvK)

The native does not know whether the Oracle has spoken truly. But he accepts that

If there is rain this month, the harvest will be ruined (R→H)

and

If the harvest is ruined, the sky gods will be angry (H→S)

So, the native reasons, *if* the Oracle spoke truly then

Either the sky gods will be angry or the King will die (SvK)

The native's reasoning here is formalizable thus:

$$\dfrac{RvK \qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{R} \quad R \rightarrow H}{H} \qquad H \rightarrow S}{S}}{SvK} \qquad \dfrac{\overset{(1)}{K}}{SvK}}{SvK}\,(1)$$

Terrified at the prospect of the sky gods' anger, the native takes precautions. He believes

If I sacrifice a goat, the sky gods won't be angry (G → ~S)

and accordingly sacrifices a goat (G). He concludes with relief

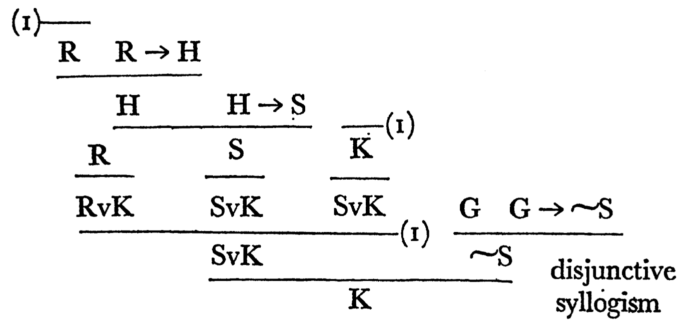
Now the sky gods won't be angry (~S)

Presumably his reasoning here is formalizable thus:

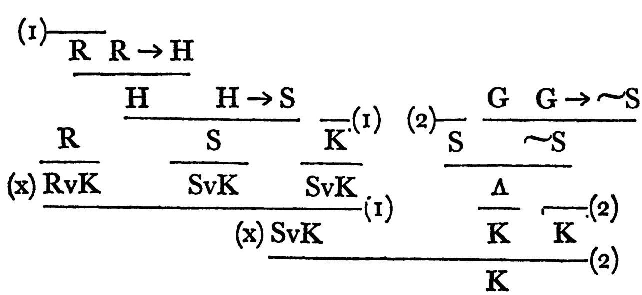
$$\frac{G \qquad G \rightarrow {\sim}S}{{\sim}S}$$

Now for the misfortune: it starts raining (R). Whence (RvK). So the Oracle did speak truly after all! So, by the first proof above, (SvK). Having ~S by the second little proof, Bennett's native now uses disjunctive syllogism to conclude K. "Whereupon he bitterly accuses himself of regicide."

It never rains but it pours. Bennett's native is on a slippery deductive slope. For his reasoning is "valid" *whatever* K stands for. He might equally well accuse himself of *preserving* the king's life, were his political affiliations and the oracular pronouncement's second disjunct appropriately different. Let us put all his reasoning together, to see the doxastic mess accumulation brings:



If we derive the last step in full, we obtain



in which the formula occurrences marked (x) are maximal. If we get rid of them by applying the reduction procedure for v we obtain the following proof in normal form:

$$\dfrac{\dfrac{\dfrac{\dfrac{R \quad R\to H}{H} \qquad H\to S}{S} \qquad \dfrac{G \quad G\to \sim S}{\sim S}}{\Lambda}}{K}$$

Here we see clearly the complete irrelevance of the conclusion K to the inconsistent set of premisses. This is why I said above that the native could accuse himself of anything; all he needs is a variant application of the absurdity rule to end this proof.

On my theory of suasive appropriateness (which is normative for, as well as descriptive of the deductive practices of reasonable believers) Bennett's native should have suspended one of his beliefs (R → H), (H → S) or (G → ~S) as soon as it began raining (R) once he had killed the goat (G). (Or, in Quinean fashion, he might believe the caprine corpse to be a living goat, or believe the drenched earth still to be drought-stricken.) If he had attended to his own reasoning closely enough—that is, if he had normalized it—then the inconsistency of his beliefs would have been clear. It would then be more reasonable to suspend belief than to advance to the irrelevant conclusion K, or any other similarly irrelevant conclusion.

K had been given a spurious relevance in Bennett's story by appearing as an oracular disjunct. Note, however, that when the native finally placed his belief in the Oracle's pronouncement (RvK), it was on the grounds of the *other* disjunct R. With R as his ground for believing what the Oracle said, it would equally have been ground for believing an oracular pronouncement of the form (RvK) *for any K whatever*. Herein lies the genuine irrelevance of K, intuitions concerning which are precisely explicated by the technique of normalization.

Ironically, though, I think Bennett missed a good chance to show that disjunctive syllogism really can be a "valid slice of argumentative life". Let us change his story by imposing a drought. The Oracle says (RvK). The native immediately believes this (which is far more plausible!). But the native, thirsting in these hard times, kills a goat for sustenance. The tribal elders shriek "Fool! Thou hast killed a goat on Oraclesday! Hence the sky gods won't be angry! (via G → ∼S). And have you forgotten the hard-won truth that if it rains the harvest will be ruined? (R → H). And have you forgotten that if the harvest is ruined the sky gods will be angry? Now, dolt, the King will die, and you are to blame!"

Our unfortunate native acquiesces and gouges his eyes out. (They have a stern moral code—no special pleading for actions under different descriptions.) Indeed, the reasoning all round has been impeccable. It may be formalized thus:

$$\dfrac{R\vee K \qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{R}} \quad R\to H}{H} \qquad H\to S}{S} \qquad \dfrac{G \quad G\to\sim S}{\sim S}}{\Lambda} \qquad \overset{(1)}{\overline{K}}}{K}\,(1)$$

where I have taken the liberty of effecting disjunctive syllogism by v–E modified so as to allow the "closing off" of one or both cases by Λ. This way we have disjunctive syllogism without the absurdity rule.

So far, logically so good. But suppose now that the heavens open and the rain pours down. Then the tribal elders will have to revise tribal lore. For they have the subproof:

$$\dfrac{\dfrac{\dfrac{R \quad R\to H}{H} \qquad H\to S}{S} \qquad \dfrac{G \quad G\to\sim S}{\sim S}}{\Lambda}$$

showing that tribal lore and goat-slaughter are inconsistent with rain. Our native, however, acted in good faith when he gouged his eyes out. His logic, at least, was faultless. Too bad for him that faith is not a good basis for action. Such misfortune is the price of believing what an oracle tells you.

8. What emerges from our discussion? Most importantly, a suggestion to modify v–E (proof by cases) to the following:

$$\dfrac{A\vee B \quad \begin{matrix}\overline{A}^{(i)}\\ \vdots \\ \Lambda/C\end{matrix} \quad \begin{matrix}\overline{B}^{(i)}\\ \vdots\\ \Lambda/C\end{matrix}}{C}\,(i)$$

This is understood as follows. If both subordinate conclusions are of the same form, bring it down as the main conclusion in the usual way. (This of course includes the case where both subordinate conclusions are Λ.) If exactly one subordinate conclusion is of the form Λ, bring down the other one as the main conclusion. Thus we may prove disjunctive syllogism as follows:

$$\dfrac{A\vee B \quad \dfrac{\overline{A}^{(1)} \quad {\sim}A}{\Lambda} \quad \overline{B}^{(1)}}{B}\,(1)$$

One drawback, however, is that the result of normalizing away a maximal occurrence of AvB might not produce a proof with the same conclusion as the original one:

$$\dfrac{\dfrac{\begin{matrix}|\\A\end{matrix}}{A\vee B} \quad \begin{matrix}\overline{A}^{(1)}\\\vdots\\\Lambda\end{matrix} \quad \begin{matrix}\overline{B}^{(1)}\\\vdots\\C\end{matrix}}{C}\,(1) \quad\to\quad \begin{matrix}|\\A\\\vdots\\\Lambda\end{matrix}$$

This however, is not at all regrettable. For consider how much better our interests will be served by the explicit derivation of an inconsistency from the very premisses from which we would have been "deriving" C. Transitivity of deduction is lost where it least matters; arguably, just where it ought to be lost;—that is, where the "new" set of premisses is inconsistent. Our desire for transitivity in mathematics is of a piece with our desire for consistency of foundations.

It should now be clear what rules I think should be allowed or rejected for the construction of entailment proofs (or Proofs, as I shall henceforth call them). We reject the absurdity rule. We modify v–E as above. We require all applications of discharge rules except →–I actually to discharge assumptions of the appropriate form. These modifications affect both intuitionistic and classical logic alike, and are therefore within the scope of a general theory of entailment. Note that by allowing vacuous →–I but rejecting the absurdity rule and vacuous ~–I, it is no longer possible to define ~A as (A → Λ).

9. Our brief characterization of Proofs, however, is still incomplete. It can be shown that in order to retain the principle of substitutivity and to be able to normalize any Proof, we must allow Proofs with *irregular* discharge. That is, we must allow an application of a discharge rule to discharge some, but not necessarily all eligible occurrences of assumptions concerned. As a further consequence we must now regard the "discharge annotation" as an integral part of Proofs. One and the same tree-like array of formula occurrences, annotated in different ways, may constitute two distinct Proofs. The annotation really represents a "discharge function" which maps to each step in a Proof just those assumption occurrences discharged at that step.

10. A further problem to be noted is that spuriously relevant premisses can be smuggled into a proof via maximal occurrences of formulae. For example, in the proof

```
|
|   ——(1)
A    B
———————
  A&B
———————
   A
   .
   .
   .
   Λ
 —————(1)
  ~B
```

the premiss B is made spuriously relevant to A (for subsequent discharge by $\sim$–I) via the maximal occurrence of A&B.

Now this problem has been noted before, particularly by Prawitz. Since he prohibits vacuous $\rightarrow$–I in his system of relevance logic, and defines $\sim$A as A $\rightarrow$ Λ, his solution is to require any application of $\rightarrow$–I below one of &–I to discharge only such assumptions as are relevant to both conjuncts, in the sense that they are undischarged in both subordinate proofs. Anderson and Belnap's solution is more drastic. They allow &–I only when both conjuncts depend on the same set of assumptions. Thus not even

$$\frac{A \quad B}{A \& B}$$

is a Proof for them!

Our solution will share the spirit of Prawitz's, but is distinct from it. It is this: before applying $\sim$–I to obtain $\sim$A we must first determine whether A really is relevant to Λ on the basis of the given proof. The way to do this, without requiring the proof to be in normal form already, is as follows:

> See whether A would be among the undischarged assumptions of the proof which would result by normalizing the given proof of Λ.

Thus in the sub-proof above, since normalization produces the proof

$$\begin{array}{c} | \\ | \\ A \\ \vdots \\ \Lambda \end{array}$$

in which B does not occur as an undischarged assumption, we see that B is irrelevant to Λ. Therefore a final step of $\sim$–I to obtain $\sim$ B would be illicit.

Since maximal formulae are major premisses of eliminations, we may regard applications of elimination rules as 'dischargers' of assumptions. Thus

$$\frac{A \quad B}{A\&B}$$

is a proof of A&B from the assumptions A,B each of whose occurrences is undischarged; but if we continue with &–E :

$$\cfrac{\cfrac{A \quad B}{A\&B}}{A}\,(\alpha)$$

we may regard the application (α) of &–E as "discharging" the top occurrence of B, since the latter does not have any undischarged occurrences in the normalized version A of the proof. It would be appropriate to call this type of 'discharge' of assumption occurrences *elimination*, since it occurs at applications of the elimination rules for & and v that produce maximal occurrences of formulae.

Let us call an assumption occurrence which is neither discharged nor eliminated *live*. Our normalization criterion can eliminate assumption occurrences at applications of elimination rules for & and v, and applications of discharge rules can discharge assumption occurrences (irregularly).

At every stage in the construction of a Proof we therefore keep track (via discharge—and elimination-functions) of
(a) which assumption occurrences still live are discharged if the present step is an application of a discharge rule,
(b) which assumption occurrences still live are eliminated if the present step is an application of an elimination rule creating a new maximal formula, and therefore
(c) which occurrences of assumptions are still live (i.e. still available for possible subsequent discharge or elimination).

We may now summarize our findings informally as follows.
(i) In each Proof, considered as a formal object, there will be a discharge function δ and an elimination function ε. The domain of each will consist of assumption occurrences (that is, formulae occurrences at tops of branches in the proof tree). δ will map each assumption occurrence in its domain to that application of a discharge rule at which it was discharged. Any such assumption

occurrence X must be live in the immediate subproof for δ(X) in which it occurs. ε will map each assumption occurrence in its domain to that application of an elimination rule at which it is eliminated (as determined by the normalization criterion—see below). In the case of v–E, which is both a discharge rule and an elimination rule we can countenance simultaneous discharge and elimination of an assumption occurrence.
(ii) In the manipulations and transformations of proof trees, especially in the pruning and grafting involved in applying the reduction procedures for normalization and in accumulating proofs above live assumption occurrences, the various discharge and elimination functions can be amalgamated for the resulting proof (by set-theoretic union etc.) in an obvious way.
(iii) We can ensure that the normalization theorem for Proofs holds by allowing applications of elimination rules that produce maximal formulae only when the resulting proof normalizes to a Proof. Then normalization is a mapping defined for all Proofs, producing a unique result in every case. A Proof in normal form has a null elimination function (there being no maximal formulae) and a discharge function which, in any process of normalization, would be determined from that of the original Proof according to (ii) above.

(iv) The normalization criterion enables us to determine which assumption occurrences in a Proof π whose last step is an elimination α, are eliminated by α. The criterion does this as follows. For an assumption occurrence X to be eliminated by α, it must first of all be live in the immediate sub-Proof of π concerned. Secondly X (by criteria of replica-of-formula-occurrence-after-proof-transformations that are intuitively clear) must not occur undischarged in the normalized version of π, whose discharge function is duly determined as in (ii) above.

11. It might be alleged that our system is defective because of the following: deduction is intransitive in the sense that

> Accumulation of Proofs at live assumption occurrences of other Proofs can fail to produce a Proof.

This can be illustrated even in the entailment fragment of intuitionistic logic. Take the obvious intuitionistic Proofs of

$$\frac{\sim B}{\sim(A\&B)} \qquad \frac{B \quad \sim(A\&B)}{\sim A}$$

Grafting the first above the live occurrence of $\sim$(A&B) in the second does not produce a Proof. For in the tree resulting from this accumulation the penultimate step of $\sim$–E would, by the normalization criterion, eliminate the occurrence of A required for subsequent discharge by the final step of $\sim$–I. In general, accumulation can produce "new" maximal formulae at the point of accumulation, which call for "new" applications of the normalization criterion, which in turn might eliminate certain assumption occurrences required for discharge lower down in the original Proof.

This failure of transitivity in our example is not too drastic because the new combination of premisses is inconsistent, and, moreover, inconsistent in virtue of a Proof directly available from the Proofs involved in the attempted accumulation. I feel that the present system would be adequate as a reconstruction of our pre-systematic intuitions if the following conjecture were true. In the classical case $\rightarrow$ is not primitive and, for definiteness, we take dilemma as our only classical negation rule.

Conjecture. Accumulation of Proofs fails to produce a Proof of the main conclusion only when the new set of premisses is inconsistent. Indeed, accumulation of Proofs fails if and only if this inconsistency would be clear from normalization of the new "Proof". Moreover, if accumulation fails for any Proofs then it fails for *all* Proofs of the entailment statements concerned.

In standard logic the closure of an inconsistent set of mathematical axioms is the whole language. In the logic set forth here, it need not be. We may have formally inconsistent but undecidable theories (an observation I owe to T. J. Smiley). It would therefore be interesting to investigate naive set theory as developed in this logic, to discover whether the paradoxes could be deductively insulated from the main results used in the development of higher mathematics. It would also be interesting to discover how much of first order arithmetic can be developed from Peano's axioms on the basis of the logic here.

Note, finally, that if one insists on representing "entails" by a new connective arrow in the object language in order to pursue considerations of relevance and modality in the manner of Anderson and Belnap, the general methods developed in this paper lend themselves to a systematic survey of the resulting systems. In this sense these methods seem to me to be the most appropriate for a general theory of entailment. In this paper, however, I have concentrated on one system, arguably preferable to others.

NOTES

A. R. Anderson and N. D. Belnap, *Entailment*, Vol. 1 (Princeton, 1975).
T. J. Smiley, 'Entailment and Deducibility', *Proceedings of the Aristotelian Society, n.s., 59*, 1959, pp. 233–254.
D. Prawitz, 'On the Idea of a General Proof Theory', *Synthese, 27*, 1974, pp. 63–77.
J. F. Bennett, 'Entailment', *The Philosophical Review, 78*, 1969, pp. 197–236.

ERRATA

Page 79 line 9: *for* Q itself cannot be true *read* Q itself cannot but be true

Page 171 line 37:
Page 173 line 30: } *for Natural Deduction read Natural Logic*

Page 174 line 8: *the last occurrence of* A *should read* ∼A