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Contracting Intuitionistic Theories*

Abstract. I reformulate the *AGM*-account of contraction (which would yield an account also of revision). The reformulation involves using introduction and elimination rules for relational notions. Then I investigate the extent to which the two main methods of partial meet contraction and safe contraction can be employed for theories closed under *intuitionistic* consequence.

Keywords: contraction; revision; classical logic; intuitionistic logic; partial meet contraction; safe contraction

1. Changes of mind

From the classical point of view, there are three ways one can change one's mind about a contingent proposition φ:

Switching: believe $\varphi \rightsquigarrow$ believe $\neg\varphi$

Surrendering: believe $\varphi \rightsquigarrow$ neither believe φ nor believe $\neg\varphi$

Adopting: neither believe φ nor believe $\neg\varphi \rightsquigarrow$ believe φ.

When switching a belief, one *revises* one's system of belief. When surrendering a belief, one *contracts* one's system of belief. And when adopting a belief, one *expands* one's system of belief.

According to the well-known Levi Identity (see below), revision involves contraction followed by expansion. The agent who wishes to switch from A to $\neg A$ should first surrender A and then adopt $\neg A$. The act of surrender can be thought of as 'opening one's mind' to the possibility that $\neg A$ is the case, rather than A. Immediately after such surrender, the agent will be committed neither to A nor to $\neg A$. She will then be free consistently to adopt $\neg A$, thereby completing the switch from A to $\neg A$.

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2. Motivation: in search of an *AGM*-style treatment of the intuitionistic case

By and large, *AGM*-theory has not paid sufficient attention to the intuitionistic case—that is, the case where the belief-sets in question are closed only under *intuitionistic*, rather than classical, consequence. Different *AGM*-theorists have also been curiously at odds over the exact status of intuitionistic logic within their theory of theory-contraction and -revision. Here, for example, is a recent quote from Makinson [11]:

> *There is nothing wrong with classical logic*
>
> It is important to bear in mind that when we devise logics for belief change or nonmonotonic reasoning, we are not objecting to any classical principles. In this, the enterprise is quite different from that of the relevantists, or the intuitionists. We do not see ourselves as fabricating non-classical logics, but rather as offering a more imaginative use of classical logic ... In the AGM presentation, the background monotonic consequence operation may be taken to be classical consequence itself, but not only so. It can be any supraclassical consequence operation satisfying all the Tarski conditions (inclusion, monotony, idempotence) plus disjunction in the premises plus compactness. Equivalently: any operation formed from classical consequence by adding a fixed set of extra premises.

Makinson, therefore, seems to be on the defensive against the intuitionist, and not anxious to acknowledge that it might count as a *virtue* of *AGM*-theory if it *could* somehow accommodate the intuitionist.

How does the intuitionist fare according to other *AGM*-theorists? Fuhrmann states ([4], p. 20) that the standard *AGM*-postulates 'provide a *stable* characterisation of contractions and revisions.' The word 'stable' here is a term of approbation: ' ... there are a number [of] independently motivated approaches ... which turn out to agree in their pronouncing a mapping from theories *cum* formulae into theories a contraction, respectively a revision operation.' Earlier, in [5], Fuhrmann had claimed, in connection with his account of 'theory-contraction through base-contraction'

> ... theories are not assumed to be closed under *classical* consequence. Thus, we are free to apply the results obtained here to theories that are closed under a consequence relation induced by a non-classical logic.

What Fuhrmann calls the 'stable postulates' of AGM-theory, however, allow the following dismaying situation for the intuitionistic theory-reviser. Suppose one holds the theory $[A]$ that consists of the logical consequences of the single (atomic) sentence A. Intuitively, in order to revise $[A]$ with respect to $\neg A$ one would need to *contract* $[A]$ with respect to A, then *expand* the result by adding $\neg A$ (and finally close under intuitionistic consequence). But (as the reader can easily check) one 'contraction' of $[A]$ with respect to A that is *permitted by the AGM-postulates for contraction*, is $[\neg\neg A]$. And the expansion of *this* theory by $\neg A$ would be *inconsistent*.

Why is $[\neg\neg A]$ permitted as a contraction of $[A]$ with respect to A? Answer: Because the AGM-postulate of *success* requires only that contracting with respect to A produce a theory that does not logically imply A. But, as the intuitionist will assure you, the theory $[\neg\neg A]$ is very serviceable in this regard! This is because $\neg\neg A$ does not imply A in intuitionistic logic.

It would appear, then, that we need to re-think matters in order to accommodate the intuitionistic belief-reviser. This undertaking differs from two other recent forays within the AGM-tradition in non-classical directions. [7] treats contraction and revision when the underlying logic of the object language is that of tautological entailment, a fragment of relevance logic. But the logic of tautological entailment has classical features, such as double-negation elimination, that are not acceptable to the intuitionist. [3] treat of disbeliefs on a par with beliefs, in such a way that disbelieving a proposition is not believing its negation. This too would be unacceptable to the intuitionist, for whom disbelieving φ remains a matter of believing $\neg\varphi$, and for whom the important question is how best to analyze the logical behavior of negation.

The deducibility relation $\vdash$ used for closure of a rational agent's beliefs need not be that of classical logic. My own view is that a logical saint should be using only *intuitionistic relevant* logic! Be that as it may, it is indisputable that considerable interest attaches to the question whether there is a theory of contraction (and revision) that is general enough to accommodate the logical saint who *does* use a more restricted logic than full classical logic. Surely the precepts and principles involved in rational changes of mind should be invariant across choice of *object* logic? And is not intuitionistic logic the best-known and most widely advocated sub-classical logic? Why, then, have AGM-theorists not sought to apply their account of rational theory-change to the intuitionistic case?

Suppose—as I shall do here—one uses classical logic in the metatheory. Surely that metatheory should nevertheless speak to the concerns of the *intuitionistic* reasoner who wishes to be told how best to change her mind?

So-called AGM-theory is a *classical* theory of saintly changes of mind on the part of saints who use *classical* logic. My present project is that of carefully re-working the central results of AGM-theory, keeping an eye out for logical moves *imputed to the saint* that are strictly classical, hence unacceptable to the (saintly) intuitionist. I readily own that I shall not be too concerned to keep the metatheory itself intuitionistic. Rather, I just want to make it apply to the intuitionistic saint. So it is quite a modest project. This paper can regarded as a friendly amendment to AGM-theory, showing that at least one of its main methods is of wider application than one of its founders was inclined to think (see §10 below).

In an alternative approach to contraction, not to be essayed upon here, I would seek to eliminate this source of methodological embarrassment, and provide a *constructive* theory of contraction that would be invariant across different choices of logic for the object-language. Indeed, one goal would be to provide an account of contraction that would enable one to consider revising the very logic used for closure of the theories being contracted. I venture the philosophical conjecture that there will prove to be an 'equilibrium logic'—that is, a logic none of whose basic rules could be surrendered without thereby rendering its user unable to provide a metatheory of contraction and revision. And I venture also the conjecture that intuitionistic relevant logic is an equilibrium logic in this sense. Perhaps also it is the *only* equilibrium logic.

3. Taking the intuitionistic case seriously

In the intuitionistic case, A logically implies $\neg\neg A$; *but the converse does not hold.* Hence the various kinds of change of mind could be more nuanced. One could switch from $\neg A$ to A; but one could *also* switch from $\neg A$ to $\neg\neg A$, which is weaker than A. For the classicist, these switches would be indistinguishable. Also, the intuitionist could surrender A while holding on to $\neg\neg A$—something the classicist cannot do. And the intuitionist could adopt $\neg\neg A$ without incurring commitment to A—again, something the classicist cannot do.

One has to be careful when formulating the conditions for SUCCESS in the project of contracting a system K of beliefs with respect to any of its consequences A. Any contraction with respect to A must not only fail to imply A ; it must also be consistent with $\neg A$. Against the background of classical logic, these two requirements are equivalent; against the background of intuitionistic logic, however, they come apart. In the intuitionistic case the way to express the condition for SUCCESS is to insist that any contraction be consistent with $\neg A$. This will guarantee that any contraction fails to imply A. Intuitionistically, merely insisting that a contraction fail to imply A is not enough to guarantee that it will also be consistent with $\neg A$.

The task of contracting a belief-system K with respect to one of its consequences A is this:

1. make the result consistent with $\neg A$; and
2. make the result as extensive a sub-system of K as you reasonably can.

The first requirement is that of SUCCESS; the second requirement is that of MINIMUM MUTILATION.[1]

4. Some basic metatheory

4.1. Theories

K is a *theory* in the language L just in case K is a logically closed set of sentences of L. The following two rules exactly capture this condition:

$$\frac{K \vdash p \quad p \in L}{p \in K} \qquad \frac{p \in K}{p \in L}$$

Consistent theories may be thought of as corresponding to 'logically saintly' sets of beliefs.

If X is a set of sentences in a language L, then $[X]_L$ is its *logical L-closure*, that is, $\{\varphi \in L \mid X \vdash \varphi\}$. For ease of expression, I take reference to L to be implicit in what follows.

LEMMA 4.1. *Let $\mathcal{F}$ be a non-empty set of theories (in L). Then the following are equivalent:*

(i) *for every $K \in \mathcal{F}$, $K \vdash \varphi$;*

(ii) $\cap\mathcal{F} \vdash \varphi$;

(iii) $\varphi \in \cap\mathcal{F}$.

[1] The modifier 'reasonably' is doing a lot of work in the statement of MINIMAL MUTILATION. This informal statement involves the unexplicated notions of 'system' and 'sub-system'. A system, however, need not be a logically closed theory. Also, sub-systems of K that are 'as extensive as they reasonably can be' should not necessarily be taken to be maximal in the set-theoretic sense. For an explication of this last quoted phrase in the context of finitary belief-systems, see [12]. For a different explication in the AGM-framework of infinitary, logically closed theories, see [13].

PROOF. (i) ⇒(iii):

$$\dfrac{\mathcal{F} \neq \emptyset \qquad \dfrac{\dfrac{\dfrac{}{J \in \mathcal{F}}{\scriptstyle (1)} \qquad \forall K \in \mathcal{F}\ K \vdash \varphi}{J \vdash \varphi}}{\varphi \in J}{\scriptstyle (1)}}{\varphi \in \cap\mathcal{F}}$$

(ii)⇒(i):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{J \in \mathcal{F}}{\scriptstyle (1)}}{\cap\mathcal{F} \subseteq J} \qquad \cap\mathcal{F} \vdash \varphi}{J \vdash \varphi}}{\forall K \in \mathcal{F}\ K \vdash \varphi}{\scriptstyle (1)}$$

(iii)⇒(ii):

$$\dfrac{\varphi \in \cap\mathcal{F}}{\cap\mathcal{F} \vdash \varphi}$$

■

4.2. A relational approach

I shall bring out the essential features of AGM-theory *without* talking of contraction *functions*. Instead, I take as central the *relational* notion

$$J \text{ is a contraction of } K \text{ with respect to } A,$$

where A is a sentence (indeed, a theorem of K).[2] This is our *explicandum*, which we need to characterize by means of a rigorous theory. I shall abbreviate it to

$$\downarrow(J, K, A).$$

(Think of this notation as indicating a 'thumbs down' for the proposition A.) By adopting the relational notation $\downarrow(J, K, A)$, I make it clear that for fixed K and A, there can in general be more than one satisfier J. Note that AGM-theorists prefer to write

$$J = K - A \text{ (or } J = K_A^-),$$

[2]Lindström and Rabinowicz ([8], [9]) have given a similar treatment of *revision*, but with slightly different aims and methods. I focus here on the proper relational treatment of *contraction*. (A similar treatment will extend to revision, but that is material for another paper.) Lindström and Rabinowicz, in [9], touched on a relational treatment of contraction in connection with Grove spheres, but did not provide the kind of detailed study undertaken here. Moreover, I shall be analyzing the relation of contraction in such a way that EXTENSIONALITY will be *derivable*, not explicitly postulated.

and are thereby forced to acknowledge the existence of more than one *contraction function* in order to accommodate the possible multiplicity of contractions. I prefer to speak instead of admissible *contractions*, in the plural, and avoid any untoward hint of assumed uniqueness that might mistakenly be imputed from use of the functional locution '*the* contraction of K with respect to A'.

5. Inferential postulates for theory-contraction

In keeping with the basic AGM-approach, K will be a *theory* in some fixed language L. Likewise, any contraction J will be a subtheory of the theory K being contracted. I shall be assuming throughout that the initial theory K is consistent. Moreover, the logic that is used for closure of the changed theory J is the *same* as the logic that is used for the closure of K. This is an important condition whose significance is often not remarked. It means that one is debarred from contracting on the *logic* when trying to avoid commitment to any K-theorem A; hence that the turnstiles in the following rules (as indeed with all our rules) represent the same deducibility relation, regardless whether it is K or J that appears to the left of the turnstile:[3]

$$\frac{\downarrow(J,K,A) \quad J \vdash p}{p \in J} \quad \text{CLOSURE}$$

$$\frac{\downarrow(J,K,A) \quad J \vdash p}{K \vdash p} \quad \text{INCLUSION}$$

AGM-theory relies tacitly on the understanding that the logic used for the closure of J is the same as that used for the closure of K. This is the only way that AGM-theory can secure their result (also known as the 'failure postulate') that if $\vdash A$ then $(K-A) = K$.[4]

In the non-trivial case—which is the only case I shall countenance—the sentence A with respect to which the contraction takes place is *not* a theorem of logic, and *is* a theorem of the theory K being contracted. Any contraction J of K with respect to A will be a *proper* sub-theory of K, for it will fail

[3] Despite its focus on such contractions and revisions as leave logic itself intact, AGM-theory is inadequate in various ways. A better theory of contraction and revision should, however, also be able to say something interesting about the revisability (or otherwise) of logic itself. But that is beyond the scope of this paper.

[4] Note that AGM-theory allows for the contraction of K with respect to any logical truth A. The result, of course, must be K itself. In my re-working of AGM-theory here, I do not allow contractions of this trivial kind. One simply does not need them.

to prove A. If we are dealing with closure under a logic (such as classical or intuitionistic logic) in which absurdity implies everything, then of course the theory J must be consistent. But if we are dealing with closure under a *paraconsistent* logic, this might not be the case. In the paraconsistent case, a proper subtheory J of an inconsistent theory K could still itself be inconsistent.

The condition of SUCCESS discussed above can be formulated as the following rule:

$$\frac{\downarrow(J,K,A) \qquad J,\neg A \vdash \bot}{\bot} \quad \text{SUCCESS}$$

Note now the following simple derivation:

$$\frac{\downarrow(J,K,A) \qquad \dfrac{\dfrac{\vdash A}{J \vdash A}}{J,\neg A \vdash \bot}}{\bot}$$

of the rule

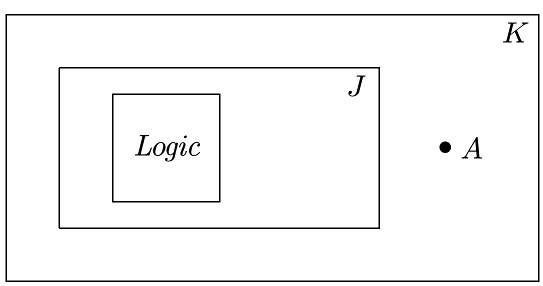
$$\frac{\downarrow(J,K,A) \qquad \vdash A}{\bot}$$

This rule says that *nothing* can count as a contraction of a theory K with respect to a *logical theorem* A. So the stronger rule that expresses the condition of SUCCESS of contractions also ensures that those contractions are non-trivial—that is, that one cannot contract with respect to logical theorems.

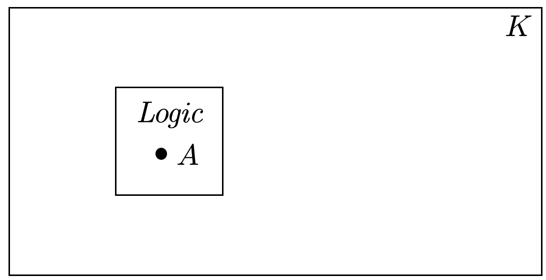
Just as one cannot contract with respect to logical theorems, so too one cannot contract with respect to sentences that are not theorems of K :

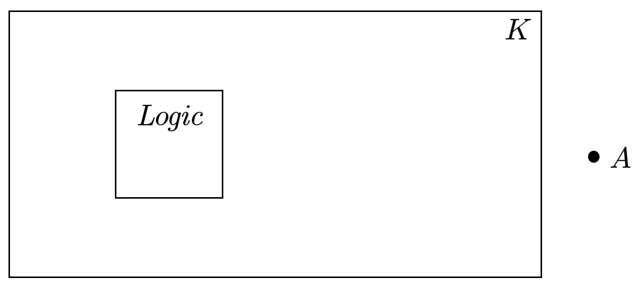
$$\frac{\downarrow(J,K,A)}{K \vdash A} \quad \text{VACUITY}$$

In summary, then, contractions are possible only in situations like this:



Contractions are impossible (i.e., not defined) in situations like these:





The next principle is both powerful and controversial. It is called the Postulate of Recovery. It is an attempt to express the idea that a contraction is a 'minimal mutilation' of the theory being contracted.[5] Controversy has arisen over whether Recovery indeed captures this idea. In its attempt to do so, Recovery says that any theorem of K is implied by A plus any contraction of K with respect to A:

$$\frac{\downarrow(J, K, A) \quad K \vdash p}{J, A \vdash p} \qquad \text{RECOVERY}$$

6. Summary of postulates

Since I am adopting a relational approach to contraction, as explained above, it will be useful to summarize the postulates that hold for that approach.

[5]In [6], at p. 67, Gärdenfors wrote

> The criterion of informational economy requires that [the contraction of a theory] be a 'large' subset of [the theory]. Formally this can be expressed as [the axiom of recovery].

They are very similar to the standard postulates in presentations of AGM-theory, as in Gärdenfors [6]. Where my rules differ, they do so only because of my conventional decision to say that if A is not a theorem of K, or is a theorem of logic, then there are no contractions of K with respect to A. Note also that I do not state any rule of EXTENSIONALITY (a.k.a. PRESERVATION). We shall see in a moment that EXTENSIONALITY will be derivable from other rules we shall have stated.

$$\frac{\downarrow(J,K,A) \quad J\vdash p}{p\in J} \qquad \text{CLOSURE}$$

$$\frac{\downarrow(J,K,A) \quad J\vdash p}{K\vdash p} \qquad \text{INCLUSION}$$

$$\frac{\downarrow(J,K,A) \quad J,\neg A\vdash \bot}{\bot} \qquad \text{SUCCESS}$$

$$\frac{\downarrow(J,K,A)}{K\vdash A} \qquad \text{VACUITY}$$

$$\frac{\downarrow(J,K,A) \quad K\vdash p}{J,A\vdash p} \qquad \text{RECOVERY}$$

Thus stated, these five postulates for contraction express the *commitments* that one makes by claiming that $\downarrow(J,K,A)$. Thus they are, in effect, *elimination rules* for the notion. One needs to state a corresponding *introduction rule* that exactly balances this set of elimination rules.[6] In principle, this is straightforward, even if the result is made prolix by the fact that there are, in all, five conditions that need to be verified before one may infer the conclusion $\downarrow(J,K,A)$.

In what follows, it will be important to limit the sideways spread of notation. I shall therefore abbreviate '$K,\neg A\vdash\bot$' to '$K\gg A$'. Accordingly, '$K,\neg A\nvdash\bot$' becomes '$K\not\gg A$'. Note that $\gg$ is compact because $\vdash$ is.

The sought introduction rule would be[7]

[6]By requiring that the introduction rule exactly balance this set of elimination rules, I am in effect saying that the elimination rules exhaust the content of the notion. But of course certain supplementary postulates have been proposed in the literature on AGM-theory, which would need to be framed as yet further elimination rules (and as corresponding premises or subordinate proofs in the matching introduction rule). By confining myself to the five elimination rules above, I accept that I am considering contraction in the AGM-sense which AGM-theory itself explicates as 'partial meet contraction'.

[7]Note that I have used p as a *placeholder for sentences* in the elimination rules stated above. Here, the Greek lowercase φ is to be understood as a *sentence-parameter* ensuring the required degree of generality within the subproofs of the introduction rule.

$$\dfrac{\begin{matrix}\overline{J \vdash \varphi} \\ \vdots \\ \varphi \in J\end{matrix} \quad \begin{matrix}\overline{J \vdash \varphi} \\ \vdots \\ K \vdash \varphi\end{matrix} \quad \begin{matrix}\overline{J \gg A} \\ \vdots \\ \bot\end{matrix} \quad K \vdash A \quad \begin{matrix}\overline{K \vdash \varphi} \\ \vdots \\ J, A \vdash \varphi\end{matrix}}{\downarrow(J, K, A)}$$

Here is a trivial proof using the foregoing introduction rule:

$$\dfrac{\dfrac{\overline{X \vdash \varphi}^{(1)}}{\varphi \in [X]} \quad \dfrac{\overline{X \vdash \varphi}^{(1)}}{X, A \vdash \varphi} \quad \dfrac{X \not\gg A \quad \overline{X \gg A}^{(1)}}{\bot} \quad \overline{[X, A] \vdash A} \quad \dfrac{\overline{[X, A] \vdash \varphi}^{(1)}}{X, A \vdash \varphi}}{\downarrow([X], [X, A], A)}{}^{(1)}$$

Note that the proof goes through even if $X, A \vdash \bot$. It is a commonsense result: any theory surely counts as *a* contraction, with respect to any non-theorem A, of its expansion by A, whether or not this expansion is consistent. (This does not, of course, entail that this is the *only* such contraction.)

Given the foregoing introduction and elimination rules for the ternary predicate $\downarrow$, the condition of extensionality:

$$\dfrac{\downarrow(J, K, A) \quad A \vdash B \quad B \vdash A}{\downarrow(J, K, B)} \qquad \text{EXTENSIONALITY}$$

will obtain, given only that the deducibility relation satisfies CUT. In the following proof, all the assumption-discharges are effected by the final step of $\downarrow$-introduction:

$$\dfrac{\dfrac{\downarrow(J, K, A) \quad \overline{J \vdash \varphi}}{\varphi \in J} \quad \dfrac{\downarrow(J, K, A) \quad \overline{J \vdash \varphi}}{K \vdash \varphi} \quad \dfrac{\downarrow(J, K, A) \quad \dfrac{\overline{J \gg B} \quad B \vdash A}{J \gg A}}{\bot} \quad \text{CUT}\,\dfrac{\dfrac{\downarrow(J, K, A)}{K \vdash A} \quad A \vdash B}{K \vdash B} \quad \text{CUT}\,\dfrac{\dfrac{\downarrow(J, K, A) \quad \overline{K \vdash \varphi}}{J, A \vdash \varphi} \quad B \vdash A}{J, B \vdash \varphi}}{\downarrow(J, K, B)}$$

7. The Levi 'Identity'

Contraction is important because of how it facilitates *revision* of a theory. Suppose that the theory K (representing our system of beliefs) implies the contingent belief A (i.e., A is not a *logical* theorem). Suppose further that one now wishes to adopt the belief $\neg A$. The resulting *revision* of K with respect to $\neg A$ is called $K * \neg A$ (or $K^*_{\neg A}$) by those in the AGM-tradition, with their emphasis on contraction- and revision-*functions*. By contrast, on the relational approach that I prefer, we would use a notation such as $\uparrow(J, K, B)$ to abbreviate 'J is a revision of K with respect to B'. (One can think of this as indicating a 'thumbs up' for the proposition B.)

There is a fundamental and widely shared intuition as to how best to revise K with respect to $\neg A$ (or, at the very least, how one *might* so revise).

The intuition was stated in functional manner by Isaac Levi:

$$K * \neg A = [(K - A) \cup \{\neg A\}]$$

Note that for fixed φ the operation

$$[X \cup \{\varphi\}]$$

is single-valued. Hence the multiplicity of revisions of K with respect to $\neg A$ is wholly attributable to the multiplicity of contractions of K with respect to A.

I shall take the Levi identity as unproblematic, and not to be questioned. But this is only on the strict understanding that contraction with respect to A is an operation that opens one's mind to the possibility that $\neg A$. So the SUCCESS condition for contraction must be expressed as '$J, \neg A \nvdash \bot$', rather than the weaker '$J \nvdash A$'.

8. Partial meet contraction for the intuitionist

8.1. Maximal non-($\neg A$-contradicting) subsets of K

This heading is a bit of a mouthful; it takes over from the classicist's notion of 'maximal non-(A-implying) subsets of K'. This change is occasioned by our need to accommodate the saintly intuitionist.

DEFINITION 8.1. J is a maximal subset of K with property Φ if and only if: J is a subset of K with property Φ, but for any K-theorem B not in J, $(J \cup \{B\})$ lacks property Φ.

I shall abbreviate 'J is a maximal subset of K that fails to contradict the negation of the K-theorem A' as $J \lhd_A K$). Introduction and elimination rules for $J \lhd_A K$ are as follows:

$$\dfrac{K \vdash A \qquad \begin{matrix}\overline{J \vdash \varphi} \\ \vdots \\ K \vdash \varphi\end{matrix} \qquad \begin{matrix}\overline{J \gg A} \\ \vdots \\ \bot\end{matrix} \qquad \begin{matrix}\underbrace{\overline{K \vdash \psi}\ ,\ \overline{\psi \notin J}} \\ \vdots \\ J, \psi \gg A\end{matrix}}{J \lhd_A K}$$

$$\dfrac{J \lhd_A K}{K \vdash A} \qquad \dfrac{J \lhd_A K \quad J \vdash \varphi}{K \vdash \varphi} \qquad \dfrac{J \lhd_A K \quad J \gg A}{\bot} \qquad \dfrac{J \lhd_A K \quad K \vdash \psi \quad \psi \notin J}{J, \psi \gg A}$$

DEFINITION 8.2. $K \perp A \ =_{\mathrm{df}} \ \{J \mid J \lhd_A K\}$.

LEMMA 8.1. $\dfrac{J \lhd_A K \quad J \vdash \varphi}{\varphi \in J}$

PROOF.

$$\dfrac{\dfrac{J \lhd_A K \qquad \dfrac{J \vdash \varphi \qquad \dfrac{J \lhd_A K \qquad \dfrac{J \lhd_A K \quad J \vdash \varphi}{K \vdash \varphi} \qquad \dfrac{}{\varphi \notin J}{\scriptstyle (1)}}{J, \varphi \gg A}}{J \gg A}}{\perp}}{\varphi \in J}{\scriptstyle (1)}$$

■

The final step of the preceding proof is classical reductio; but it is a step in the metalogic, not the object logic. Remember that I am concerned here only to avoid strictly classical inferences in the *object* logic.

I note the following proof-schema in intuitionistic (relevant) logic:

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{}{A}{\scriptstyle (1)} \qquad \neg A}{\perp}}{\underbrace{J \ , \ A \to B \ , \ \neg A}_{}}{\scriptstyle (1)} \\ \vdots \\ \perp \end{array}$$

which establishes the rule $\dfrac{J, A \to B \gg A}{J \gg A}$, to be used in the proof of the next Lemma.

LEMMA 8.2. *If $J \lhd_A K$ and $K \vdash B$, then $J \vdash A \to B$.*

PROOF.

$$\dfrac{\dfrac{J \lhd_A K \qquad \dfrac{\dfrac{J \lhd_A K \qquad \dfrac{K \vdash B}{K \vdash A \to B} \qquad \dfrac{}{J \nvdash A \to B}{\scriptstyle (1)}}{J, A \to B \gg A}}{J \gg A}}{\perp}}{J \vdash A \to B}{\scriptstyle (1)}$$

■

COROLLARY 8.3. $\dfrac{J \lhd_A K \quad K \vdash B}{J, A \vdash B}$

PROOF. Immediate from Lemma 8.2. ■

LEMMA 8.3. *Let* $\emptyset \neq \mathcal{J} \subseteq (K \perp A)$. *Then* $\cap\mathcal{J}$ *satisfies* RECOVERY.

PROOF.

$$\dfrac{\dfrac{\emptyset \neq \mathcal{J} \qquad \dfrac{\dfrac{\dfrac{\overline{J \in \mathcal{J}}^{(1)} \qquad \mathcal{J} \subseteq K \perp A}{J \lhd_A K} \qquad K \vdash \varphi}{J \vdash A \to \varphi}\,\text{Lemma 8.2}}{\forall J \in \mathcal{J}\; J \vdash A \to \varphi}\,(1)}{\cap\mathcal{J} \vdash A \to \varphi}\,\text{Lemma iii, Lemma 8.1}}{\cap\mathcal{J}, A \vdash \varphi}$$

■

THEOREM 8.4.

$$\exists\mathcal{J}(\emptyset \neq \mathcal{J} \subseteq (K \perp A) \;\wedge\; J = \cap\mathcal{J}) \;\;\Rightarrow\; \downarrow(J, K, A)$$

PROOF. Suppose

$$\exists\mathcal{J}(\emptyset \neq \mathcal{J} \subseteq (K \perp A) \;\wedge\; J = \cap\mathcal{J}).$$

Call such a set $\mathcal{F}$. So $\emptyset \neq \mathcal{F} \subseteq (K \perp A)$ and $J = \cap\mathcal{F}$. We now verify the five conditions required by the introduction rule for $\downarrow(\cap\mathcal{F}, K, A)$—whence, by the last identity, it will follow that $\downarrow(J, K, A)$. By Lemma iii,

CLOSURE: $\cap\mathcal{F}$ is logically closed.

Since $\cap\mathcal{F}$ is the intersection of a family of subsets of K,

INCLUSION: Any sentence implied by $\cap\mathcal{F}$ is implied by K.

Since $\cap\mathcal{F}$ is the intersection of a family of non-($\neg A$-contradicting) sets, $\cap\mathcal{F}$ does not contradict $\neg A$. Hence we have

SUCCESS: $\cap\mathcal{F}, \neg A \nvdash \perp$.

By Lemma 8.3,

RECOVERY is satisfied by $\cap\mathcal{F}$.

Since $\mathcal{F}$ is non-empty, there is some J such that $J \lhd_A K$; whence, by the first elimination rule for $J \lhd_A K$, we have

VACUITY: $K \vdash A$.

So we have established the implication. ■

LEMMA 8.4. *Suppose* $J \subseteq K$, $K \vdash A$ *and* $J \not\gg A$. *Then for some* M, $M \lhd_A K$ *and* $J \subseteq M$.

PROOF. Let $\kappa_0, \kappa_1, \ldots$ be an enumeration of all the members of K. Define

$$\begin{array}{ll} J_0 =_{df} J; & \\ J_{n+1} =_{df} J_n \cup \{\kappa_n\} & \text{if } J \cup \{\kappa_n\} \not\gg A; \\ J_{n+1} =_{df} J_n & \text{if } J \cup \{\kappa_n\} \gg A. \end{array}$$

Clearly by construction $J \subseteq \bigcup_i J_i$. By induction, $\forall i\ J_i \not\gg A$. Hence by compactness of $\gg$, $\bigcup_i J_i \not\gg A$. By construction, $\bigcup_i J_i \subseteq K$. Finally, suppose $\varphi \notin \bigcup_i J_i$, and that $\varphi = \kappa_m$. Then $\varphi \notin J_{m+1}$. By construction it follows that $J_m \cup \{\varphi\} \gg A$; whence $\bigcup_i J_i, \varphi \gg A$. Thus we have shown $\bigcup_i J_i \lhd_A K$. ■

I note the following proof-schema in intuitionistic (relevant) logic:

$$\begin{array}{c} \underbrace{J\ ,\ \overline{A}^{(1)}} \\ \vdots \\ \dfrac{B \quad \neg B}{\bot} \\ \underbrace{J\ ,\ \psi\ ,\ \overline{\neg A}^{(1)}} \\ \vdots \\ \bot \end{array}$$

which establishes the rule $\dfrac{J, \psi \gg A \quad J, A \vdash B}{J, \psi \gg B}$, to be used in the proof of the next Lemma.

LEMMA 8.5. $\dfrac{J \lhd_A K \quad J \not\gg B \quad K \vdash B}{J \lhd_B K}$

PROOF.

$$\dfrac{K \vdash B \quad \dfrac{J \lhd_A K \quad \overline{J \vdash \varphi}^{(1)}}{K \vdash \varphi} \quad \dfrac{J \not\gg B \quad \overline{J \gg B}^{(1)}}{\bot} \quad \dfrac{\dfrac{J \lhd_A K \quad \overline{K \vdash \psi}^{(1)} \quad \overline{\psi \notin J}^{(1)}}{J, \psi \gg A} \quad \dfrac{J \lhd_A K \quad K \vdash B}{J, A \vdash B}\text{Cor. 8.3}}{J, \psi \gg B}}{J \lhd_B K}(1)$$

■

I note the following proof-schema in intuitionistic (relevant) logic:

$$\begin{array}{c}\underbrace{N\ ,\ \dfrac{\dfrac{\dfrac{\dfrac{}{A}{\scriptstyle(1)}}{A\vee B}\quad \neg(A\vee B)}{\bot}}{\neg A}{\scriptstyle(1)}}\\ \vdots \\ \bot\end{array}$$

which establishes the rule $\dfrac{N\gg A}{N\gg A\vee B}$, to be used in the proof of the next Lemma.

LEMMA 8.6. $$\dfrac{\downarrow(J,K,A)\quad B\in\cap\{M\mid J\subseteq M\ \wedge\ M\lhd_A K\}}{J\gg A\vee B}$$

PROOF.

$$\dfrac{\dfrac{\dfrac{\dfrac{\downarrow(J,K,A)}{J\subseteq K}\quad \dfrac{}{J\not\gg A\vee B}{\scriptstyle(4)}\quad \dfrac{\dfrac{\downarrow(J,K,A)}{K\vdash A}}{K\vdash A\vee B}}{\exists M(J\subseteq M\wedge M\lhd_{A\vee B}K)}\quad \dfrac{{\scriptstyle(3)}\dfrac{}{J\subseteq N\wedge N\lhd_{A\vee B}K}\quad \dfrac{{\scriptstyle(2)}\dfrac{}{N\lhd_{A\vee B}K}\quad \dfrac{\dfrac{{\scriptstyle(2)}\dfrac{}{J\subseteq N}\quad \dfrac{{\scriptstyle(2)}\dfrac{}{N\lhd_{A\vee B}K}\quad \dfrac{\dfrac{{\scriptstyle(2)}\dfrac{}{N\lhd_{A\vee B}K}\quad \dfrac{\dfrac{}{N\gg A}{\scriptstyle(1)}}{N\gg A\vee B}}{\bot}}{N\not\gg A}{\scriptstyle(1)}\quad \dfrac{\downarrow(J,K,A)}{K\vdash A}}{N\lhd_A K}\quad B\in\cap\{M\mid J\subseteq M\wedge M\lhd_A K\}}{B\in N}}{N\vdash A\vee B}}{\bot}}{\bot}{\scriptstyle(2)}}{\bot}{\scriptstyle(3)}}{J\gg A\vee B}{\scriptstyle(4)}$$

■

LEMMA 8.7. $$\dfrac{J\gg A\vee B\quad \downarrow(J,K,A)\quad B\in\cap\{M\mid J\subseteq M\ \wedge\ M\lhd_A K\}}{J\gg B}$$

PROOF.

$$\dfrac{J\gg A\vee B\qquad \dfrac{\dfrac{\text{RECOVERY}\ \downarrow(J,K,A)\quad \dfrac{B\in\cap\{M\mid J\subseteq M\wedge M\lhd_A K\}}{K\vdash B}}{J,A\vdash B}\qquad B\vdash B}{J,A\vee B\vdash B}}{J\gg B}$$

■

LEMMA 8.8. $$\dfrac{\downarrow(J,K,A)\quad B\in\cap\{M\mid J\subseteq M\ \wedge\ M\lhd_A K\}}{J\gg B}$$

PROOF. Immediate from Lemma 8.6 and Lemma 8.7, by CUT on $J\gg A\vee B$.

■

8.2. The central representation theorem

When theories are closed under *classical* consequence, one has the classical AGM representation theorem on partial meet contractions:

$$\downarrow(J, K, A) \;\;\Leftrightarrow\;\; \exists \mathcal{J}(\emptyset \neq \mathcal{J} \subseteq (K \perp A) \;\wedge\; J = \cap \mathcal{J}).$$

Theorem 8.4 has established the implication from right to left in the case of theories closed only under *intuitionistic* consequence. We shall now prove Theorem 8.6, which, with Theorem 8.4, will approximate the classical representation theorem for theories closed under intuitionistic consequence. First we exploit the classical deductibility relation $\vdash_C$ in the following definition.

DEFINITION 8.5. $[J]_C \;=_{\text{df}}\; \{\varphi \mid J \vdash_C \varphi\}$. We call $[J]_C$ the *classical closure* of J.

THEOREM 8.6.

$$\downarrow(J, K, A) \;\;\Rightarrow\;\; \exists \mathcal{J}(\emptyset \neq \mathcal{J} \subseteq (K \perp A) \;\wedge\; J \subseteq \cap \mathcal{J} \subseteq [J]_C)$$

PROOF. Suppose that $\downarrow(J, K, A)$. It follows that $J \subseteq K$, $K \vdash A$, and $J \not\gg A$. Let $\mathcal{F}$ be

$$\{M \mid J \subseteq M \;\wedge\; M \lhd_A K\}.$$

By Lemma 8.4, $\mathcal{F} \neq \emptyset$. Obviously also

$$\{M \mid J \subseteq M \;\wedge\; M \lhd_A K\} \subseteq \{M \mid M \lhd_A K\} = K \perp A.$$

It remains to establish the two inclusions

$$J \;\subseteq\; \cap\{M \mid J \subseteq M \;\wedge\; M \lhd_A K\} \;\subseteq\; [J]_C.$$

Since by Lemma 8.4 there is some M such that $J \subseteq M$ and $M \lhd_A K$, it is trivial that

$$J \;\subseteq\; \cap\{M \mid J \subseteq M \;\wedge\; M \lhd_A K\}.$$

So the first inclusion is established. Recall that we are assuming $\downarrow(J, K, A)$. So by Lemma 8.8 we have the implication

$$B \in \cap\{M \mid J \subseteq M \;\wedge\; M \lhd_A K\} \;\;\Rightarrow\;\; J \gg B.$$

But we also have the implication

$$J \gg B \;\;\Rightarrow\;\; J \vdash_C B.$$

Thus the second inclusion is established. ■

Note that the Postulate of Recovery for contractions was used to prove Lemma 8.7; which was used to prove Lemma 8.8; which was used to prove the Representation Theorem. Without Recovery, one can at best claim that every contraction of K with respect to A is *included in* (but does not necessarily include in its classical closure) the intersection of some non-empty family of maximal non-($\neg A$-contradicting) subsets of K. This, however, would be a rather trivial result, since it is so easily witnessed—one need only take the singleton of a maximal non-($\neg A$-contradicting) subset of K. The second inclusion claimed in Theorem 8.6 eliminates this trivialization. Note that in the classical case we have $J = [J]_C$, so the two-link chain of inclusions

$$J \ \subseteq\ \cap\{M \mid J \subseteq M \ \wedge\ M \lhd_A K\} \ \subseteq\ [J]_C$$

compresses to yield

$$J \ =\ \cap\{M \mid J \subseteq M \ \wedge\ M \lhd_A K\}.$$

We thereby obtain the classical AGM representation theorem for partial meet contractions of classical theories.

It would appear, then, that one can prove an interesting approximation of the main representation theorem of AGM-theory ([1]) even when the belief systems concerned are closed under only intuitionistic logic.

9. Safe contraction for the intuitionist

9.1. Minimal $\neg A$-contradicting sets

Definition 9.1. Δ is a minimal set with property Φ if and only if Δ has property Φ, and no proper subset of Δ has property Φ.

The second conjunct of the right-hand side of this definition is equivalent to: for every ψ in Δ, $\Delta\backslash\{\psi\}$ lacks property Φ.

I shall abbreviate 'Δ is a minimal $\neg A$-contradicting set' as $\Delta \blacktriangleleft A$. Introduction and elimination rules for $\Delta \blacktriangleleft A$ are as follows:

$$\frac{\Delta\gg A \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\varphi\in\Delta}\ ,\ \overline{\Delta\backslash\{\varphi\}\gg A}^{(i)}}\\ \vdots \\ \bot\end{array}}{\Delta \blacktriangleleft A}\,(i) \qquad \frac{\Delta \blacktriangleleft A}{\Delta\gg A} \qquad \frac{\Delta \blacktriangleleft A \quad \varphi\in\Delta \quad \Delta\backslash\{\varphi\}\gg A}{\bot}$$

LEMMA 9.1. *Suppose* $\Delta \gg A$ *and* $\nvdash A$. *Then there is a non-empty finite subset* Γ *of* Δ *such that* $\Gamma \blacktriangleleft A$. *Stated as a rule:*

$$\frac{\Delta \gg A \qquad\qquad \nvdash A}{\exists\Gamma(\Gamma \neq \emptyset \wedge \Gamma \textit{ finite} \wedge \Gamma \subseteq \Delta \wedge \Gamma \blacktriangleleft A)}$$

PROOF. By compactness, there is a finite subset Γ_0, say, of Δ such that $\Gamma_0 \gg A$. Consider the following iterable pattern of reasoning, for $k \geq 0$:

> Since $\nvdash A$, Γ_k is non-empty. If no proper subset of Γ_k contradicts $\neg A$, take Γ_k for Γ. Otherwise, choose some non-empty proper subset Γ_{k+1}, say, of Γ_k such that $\Gamma_{k+1} \gg A$.

By iterating this pattern of reasoning, one will eventually reach some non-empty subset Γ_n of Δ ($n \leq \overline{\overline{\Delta}}$) that contradicts $\neg A$, but none of whose proper subsets does; and one will be able to take it for Γ. ■

9.2. An ordering relation

The method of so-called safe contraction is due to Alchourrón and Makinson [2]. It exploits an extra primitive relation $<$ among the theorems of K. The informal interpretation of '$\varphi < \psi$' is 'φ is more vulnerable than ψ'.

DEFINITION 9.2. Suppose $\varphi \in \Delta$. Then φ is $<$-least in Δ if and only if for no ψ in Δ do we have $\psi < \varphi$.

I shall abbreviate 'φ is $<$-least in Δ' as $\varphi < \Delta$. Note that in this context $<$ cannot be confused with the relation 'is more vulnerable than', since Δ is a *set* of sentences, not a sentence. Introduction and elimination rules for $\varphi < \Delta$ are as follows:

$$\dfrac{\varphi \in \Delta \qquad \begin{array}{c}\underbrace{{}^{(i)}\overline{\psi \in \Delta}\,,\ \overline{\psi < \varphi}^{(i)}}\\ \vdots \quad \psi \text{ parametric}\\ \bot\end{array}}{\varphi < \Delta}\,(i) \qquad\qquad \frac{\varphi < \Delta}{\varphi \in \Delta} \qquad\qquad \frac{\varphi < \Delta \quad \psi \in \Delta \quad \psi < \varphi}{\bot}$$

The only structural condition imposed on $<$ is

[C] Every non-empty finite subset of K has at least one $<$-least member.

[C] can be stated as the following rule of inference:

$$\frac{\Delta \neq \emptyset \quad \Delta \text{ finite} \quad \Delta \subseteq K}{\exists \varphi \in \Delta \;\; \varphi < \Delta}$$

Given the intended meaning of $<$, it must be an irreflexive relation. That much having been said, there is considerable slack in how best to understand it. There are two kinds of extreme for the relation $<$. One kind of extreme is the null relation. The other kind of extreme is a linear ordering of K. With the second kind of extreme, there is further variation possible, in the order-type of the ordering. In between the two extremes are cases where the relation $<$ is a partial ordering; and cases where it is not connected; and even cases where it is not transitive. When $<$ is transitive, however, we do at least know that in order for [C] to be satisfied, there cannot be any finite $<$-loops of the form $k_1 < k_2 < \ldots < k_n < k_1$.

When $<$ is the null relation, *every* member of every finite subset Δ of K is $<$-least in Δ. When $<$ is a linear ordering, every finite subset Δ of K has *exactly one* $<$-least member. With the other kinds of ordering just mentioned, finite subsets of K can have different multiplicities of $<$-least members within them. Such variations should be borne in mind when considering the following definition.

DEFINITION 9.3. Let A be a theorem of K that is not a logical theorem. A theorem φ of K is $<$-*unsafe* with respect to A if and only if φ is $<$-least in some minimal $\neg A$-contradicting subset of K.

I shall abbreviate 'the K-theorem φ is $<$-unsafe with respect to the non-logical K-theorem A' as $U_<(\varphi, A, K)$. Introduction and elimination rules for $U_<(\varphi, A, K)$ are as follows:

$$\frac{\varphi \in \Delta \quad \Delta \subseteq K \quad \Delta \blacktriangleleft A \quad \varphi < \Delta}{U_<(\varphi, A, K)}$$

$$\frac{U_<(\varphi, A, K) \qquad \begin{array}{c} \underbrace{\overline{\varphi \in \Delta}^{(i)}\ ,\ \overline{\Delta \subseteq K}^{(i)}\ ,\ \overline{\Delta \blacktriangleleft A}^{(i)}\ ,\ \overline{\varphi < \Delta}^{(i)}} \\ \vdots \quad \varphi \text{ parametric} \\ G \end{array}}{G}\,(i)$$

Obviously any non-logical K-theorem A is itself $<$-unsafe with respect to A, since $\{A\}$ is a minimal $\neg A$-contradicting subset of K, and A is $<$-least in $\{A\}$.

Definition 9.4. Let A be a theorem of K that is not a logical theorem. A theorem φ of K is $<$-*safe* with respect to A if and only if φ is not $<$-unsafe with respect to A, i.e., φ is not $<$-least in any minimal $\neg A$-contradicting subset of K.

Note that $\vdash$ is compact. Thus every minimal $\neg A$-contradicting subset of K is finite, hence has at least one $<$-least member. Another consequence of Definitions 9.3 and 9.4 is that every logical theorem in the language of K is $<$-safe with respect to A—for the simple reason that no logical theorem can be a member of any minimal $\neg A$-contradicting subset of K.

Definition 9.5. $K^{<}A \ =_{\mathrm{df}}\ \{\varphi \mid K \vdash \varphi \wedge \neg U_{<}(\varphi, A, K)\}$.

Note that Alchourrón and Makinson [2] use the notation K/A instead of $K^{<}A$. I prefer the latter notation because it makes explicit the fact that 'the' safe contraction depends on the relation $<$. Change any details of the relation $<$, and one is likely to change the identity of the corresponding safe contraction. Hence one should speak of 'the $<$-safe contraction of K with respect to A', rather than 'the safe contraction of K with respect to A'.

Definition 9.6. Let A be a non-logical theorem of K. The $<$-*safe* contraction of K with respect to A $\ =_{\mathrm{df}}\ [K^{<}A]$.

The idea behind $<$-safe contraction is simply this: in order to contract K with respect to A, take the logical closure of $K^{<}A$, i.e. the logical closure of the set of all and only those K-theorems that are $<$-safe with respect to A. Note that this logical closure $[K^{<}A]$ can, in general, contain K-theorems that are $<$-*unsafe* with respect to A. Nevertheless, every minimal $\neg A$-contradicting set Δ of K-theorems will have an $<$-unsafe member *not* in $[K^{<}A]$.

Lemma 9.2. *$K^{<}A$ satisfies* success*:* $\dfrac{K^{<}A \gg A \qquad \nvdash A}{\bot}$

Proof.

$$\dfrac{\dfrac{K^{<}A \gg A \qquad \nvdash A}{\exists\Gamma(\Gamma \neq \emptyset \wedge \Gamma \text{ finite} \wedge \Gamma \subseteq K^{<}A \wedge \Gamma \blacktriangleleft A)} \qquad \dfrac{[\mathrm{C}]\dfrac{\overline{\Delta \neq \emptyset}^{(2)} \quad \overline{\Delta \text{ finite}}^{(2)} \quad \dfrac{\overline{\Delta \subseteq K^{<}A}^{(2)}}{\Delta \subseteq K}}{\exists\varphi \in \Delta\ \varphi < \Delta} \qquad \dfrac{\dfrac{\overline{\psi \in \Delta}^{(1)} \quad \dfrac{\overline{\Delta \subseteq K^{<}A}^{(2)}}{\Delta \subseteq K} \quad \overline{\Delta \blacktriangleleft A}^{(2)} \quad \overline{\psi < \Delta}^{(1)}}{U_{<}(\psi, A, K)} \qquad \dfrac{\dfrac{\overline{\psi \in \Delta}^{(1)} \quad \overline{\Delta \subseteq K^{<}A}^{(2)}}{\psi \in K^{<}A}}{\neg U_{<}(\psi, A, K)}}{\bot}}{\bot}{\scriptstyle(1)}}{\bot}{\scriptstyle(2)}$$

■

Lemma 9.3. $\dfrac{K \vdash \varphi \qquad (\neg A) \vdash \varphi}{K^{<}A \vdash \varphi}$

PROOF.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{U_<(\varphi,A,K)}(2) \qquad \dfrac{\dfrac{}{\Delta \blacktriangleleft A}(1) \quad \dfrac{}{\varphi \in \Delta}(1) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \blacktriangleleft A}(1)}{\Delta \gg A}}{\Delta\setminus\{\varphi\},\varphi \gg A} \quad \neg A \vdash \varphi}{\Delta \setminus \{\varphi\} \gg A}\text{CUT}}{}}{\bot}}{\bot}(1)}{\neg U_<(\varphi,A,K)}(2) \qquad K \vdash \varphi}{\dfrac{\varphi \in K^<A}{K^<A \vdash \varphi}}
$$

Note that step (1) discharges no assumptions of the form $\varphi < \Delta$.

■

LEMMA 9.4. *$K^<A$ satisfies* RECOVERY*:* $\dfrac{K \vdash \varphi}{K^<A, A \vdash \varphi}$

PROOF. $$\dfrac{\dfrac{\dfrac{K \vdash \varphi}{K \vdash A \rightarrow \varphi} \quad \dfrac{}{\neg A \vdash A \rightarrow \varphi}}{K^<A \vdash A \rightarrow \varphi}\text{Lemma 9.3}}{K^<A, A \vdash \varphi}$$

■

The contraction-postulates of CLOSURE, INCLUSION and VACUITY are trivially satisfied by $K^<A$. Lemma 9.2 shows that SUCCESS is satisfied, and Lemma 9.4 shows that RECOVERY is satisfied. Hence we have shown

THEOREM 9.7. *For all $<$ satisfying [C], and for all non-logical K-theorems A, $\downarrow(K^<A, K, A)$*

10. Conclusion, and a note of caution

We have now shown that the method of safe contraction, unlike the method of partial meet contraction, transfers exactly to the intuitionistic case.

Nevertheless, AGM-theory, even though it can now accommodate the intuitionistic case *by its own lights*, still suffers from the defect of not properly illuminating the requirement of minimal mutilation. The objections that have been raised against RECOVERY as a theoretical explication of the requirement of minimal mutilation are as problematic for the intuitionistic as they are for the classical AGM-er.

There is a certain irony that emerges from our comparison above of the methods of partial meet contraction and of safe contraction, in regard to how well they generalize to the intuitionistic case. In [10], Makinson wrote

> [Safe contraction's] lack of public success compared to AGM['s partial meet contraction] may be due in part to its place of publication (the Polish journal *Studia Logica*, undeservedly little read in the USA, as contrasted with *The Journal of Symbolic Logic*); compared to contraction via epistemic entrenchment it may be due to the kind of readership (mainly grantless logician philosophers in the case of *Studia Logica* as contrasted with well-funded computer scientists in the case of *Proceedings of the Second Conference on Theoretical Aspects of Reasoning about Knowledge*). Or, perhaps, our favorite baby is just less attractive.

The foregoing investigation has been of the extent to which the two main methods of contraction can be made to work for *intuitionistic* theories. The results of that investigation might make Makinson's (and Alchourrón's) favorite baby of safe contraction appear more attractive—at least, compared to its nursery playmates, the partial meet contractions—than its proud parents seem to have realized. As a grantless logician philosopher I am happy to be able to point this out in the pages of *Studia Logica*.

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