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The Transmission of Truth and the Transitivity of Deduction*

Neil Tennant

Frege assigned to logic the task of 'discovering the laws of truth'[1] Logical proofs preserve truth from their premisses to their conclusions. Logic is also said to be a matter of form, not content. I intend in this essay t o stay with considerations of just truth and logical form, in order to add ress the question of the transitivity of deduction.

I shall investigate in very general terms the principles that we might be concerned to lay down for a system of logic whose deducibility (and consequence) relation is not unrestrictedly transitive. The choice of principles we eventually make might not only rule some systems out and some systems in; it might also serve to constrain exactly how a system of rules of inference is to be set up in order to generate the field of its deducibility relation.

We shall draw throughout only on notions of great generality: that of *being true* (according to whatever semantics is chosen); that of *uniform substitution*, an effective operation by which syntactic ob jects of a logical kind (formulae, proofs) are produced from other syntactic o bjects of the same kind; and other notions that can be defined in terms of the se, such as *perfect validity, soundness* and *perfect proof.* These notions have application across all logical systems.

A word of warning in advance: I shall be defining a very special sense of soundness, whose importance will emerge in the course of my discussion. Standardly, one says an argument is sound just in case it is truth-preserving (under all interpretations of non-logical expressions) *and* its premisses (hence also its conclusion) are true (under interpretation in the 'ac-

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[1]'Thoughts', at p. 1 in Gottlob Frege, *Logical Investigations*, edited by P. T. Geach, Blackwell, Oxford, 1977.

tual world'). This is not, however, the sense in which I shall be using the term 'sound'. Details will emerge below.

Another word of warning in advance: when speaking of proofs I shall be confining myself to systems of natural deduction, even when the notions in play might find application in other systems of proof as well.

Before defining the notion of a sound argument, let us speak informally of *good* arguments.

A good argument transmits truth from its premisses to its conclusion. But its premisses may be false, while yet the argument still be good. So a good argument is one that *would* transmit truth from its premisses to its conclusion, whenever the premisses turned out to be true. But one cannot transmit what one could not acquire. So one is tempted to say of a good argument: it must be possible for its premisses to turn out true.

Prima facie, then, we have a problem if we regard as good an argument whose premisses cannot all be true.

Unless, of course, the argument establishes just that: namely, that the premisses form an inconsistent set. Such an argument is called a *reductio ad absurdum* (its conclusion is absurdity). It has every right to be regarded as good (or, anticipating: as sound). So we must be more cautious in stating our preceding conclusion: *prima facie* we have a problem if we regard as good an argument *for a conclusion other than absurdity* whose premisses cannot all be true. The principle of proof by *reductio ad absurdum* must remain intact.

A problem for the logical analyst is emerging here. Two desiderata sit uneasily together:

1. you cannot transmit what you do not have
2. sometimes you can show you cannot have it

Satisfiability of premisses and validity of *reductio* pull in opposite directions— or so it would seem.

Logic, as noted above, is a matter of form, not content. Form can be gross or fine. Grossly, one is obviously justified in holding that $A\&B$ logically implies B. This justification survives upon substitution for finer discriminations: $A\&\sim A$ logically implies $\sim A$ even though $A\&\sim A$ cannot be true. So *some* arguments with conclusions other than absurdity proceed soundly from inconsistent premisses. But one may still be able to ensure that the way they do so can be represented at a grosser level in such a way that the inconsistency of the premisses is masked and the reasoning can be seen to make no illicit use of it.

It would be very different, however, with the unsound argument

$A, \sim A$ *therefore* B.

With this argument, there is no way of ascending to a g rosser level to mask the inconsistency of the premisses while still maintaining soundness of the resulting grosser argument.

Dually, one would not wish to obtain logical truths from falsifiable premisses. This would rule out the argument A *therefore* $A \vee \sim A$; until one observed that this argument is a substitution instance of A *therefore* $A \vee B$, whose conclusion is *not* logically true. At the grosser level the logical truth of the conclusion might be masked, and the reasoning be seen to make no illicit use of it. It would be very different, however, with the unsound argument B *therefore* $A \vee \sim A$. With this argument, there is no way of ascending to a grosser level to mask the logical truth of the conclusion while still maintaining soundness of the resulting grosser argument.

Perfectly sound reasoning would make no use of unnecessary assumptions. Consider the argument $(A \vee B) \supset C, A$ *therefore* $B \supset C$. It has the following two proofs:

$$\dfrac{\dfrac{\dfrac{A}{A \vee B} \qquad (A \vee B) \supset C}{C}}{B \supset C} \qquad\qquad \dfrac{\dfrac{\dfrac{\overline{\;B\;}^{\,(1)}}{A \vee B} \qquad (A \vee B) \supset C}{C}}{B \supset C}\,(1)$$

The first proof may be regarded as somehow defective in that it uses both premisses when (as one can see from the second proof) the second premiss will do. But this defect is only apparent. For the first proof is a substitution instance of the proof

$$\dfrac{\dfrac{\dfrac{A}{A \vee B} \qquad (A \vee B) \supset C}{C}}{D \supset C}$$

This proof establishes the argument $(A \vee B) \supset C, A$ *therefore* $D \supset C$; and this argument *does* need both its premisses. For both the proper sub-argument $(A \vee B) \supset C$ *therefore* $D \supset C$ and the proper sub-argument A *therefore* $D \supset C$ admit of counterexamples (that is, interpretations making their premisses true and their conclusions false). Such a proof I shall call a *perfect* proof.

Let a *non-trivial* substitution on a proof be one that is not induce d by a one-one mapping of atomic sentences to atomic sentences. A *perfect proof* in a system S is then defined as one that is not a non-trivial substitution instance of any proof in system S. The idea is that a perfect proof captures the 'bare bones' of a piece of reasoning, ignoring (that is, not representing) any occurrence of a logical operator that is not actually exploited in the reasoning involved. Imagine doing the following with a ny proof. Highlight the dominant occurrence of the logical operator on which the formal correctness of the application of any rule of inference turns. Let that highlighting be inherited at the appropriate occurrences of that formula as a subformula either above or below; ... and so on. The idea is that in a perfect proof every operator occurrence in every formula occurrence would end up being highlighted. So in a perfect proof there are no otiose operator occurrences. A perfect proof works, moreover, with the *shallowest possible* level of logical analysis of the premisses and conclusion that will enable one to make the moves that make up the proof. In particular, it minimizes repetitions of occurrences of propositional variables.

The notion of perfect proof, therefore, is a very general one, and can be applied across widely different systems (systems of natural deduction for, say, minimal, intuitionistic or classical logic). It is therefore apt for the formulation of independently motivated constraints on one's choice of such a logical system, or of principled re-formulation of it. I have in mind here constraints that might incline one, say, to the choice of some 'relevantized' version of one of these systems.

It was emerging that soundness is more than standard validity. I want to explicate the notion of soundness adumbrated here; and consider what might happen to deducibility when we require deductions to be sound, that is, to establish only sound sequents (or arguments).

One subsidiary notion that is in play already is that of an argument *needing all its premisses* in order to secure its conclusion. Another subsidiary notion is that of an argument (whose conclusion is not absurdity) *needing its premisses to form a satisfiable set.* These notions apply most naturally to what one might call 'stand-alone' arguments. There is no requirement that they should apply to any of the *sub-arguments* of a stand-alone argument. That, indeed, is the implicit problematic of this chapter: to fashion a notion of soundness that is compositional across argument structure.

Think for a moment in terms of classical sequents $X : Y$, where X and Y are sets of sentences. Dropping one or more premisses in X yields a *proper subsequent* of $X : Y$. To say that the argument $X : Y$ *needs all its premisses* is to say that any proper subsequent of this kind admits of counterexample. Likewise, dropping any members of Y yields a proper subsequent of $X : Y$. In the special case where Y is a singleton $\{A\}$, dropping its sole member A produces the proper subsequent $X : \emptyset$, which states that X is unsatisfiable. To say that the argument $X : A$ *needs its conclusion* is to say that the proper subsequent $X : \emptyset$ admits of counterexample. We now define the main notions involved:

$X : Y$ *admits of counterexample* iff some interpretation make s all of X true and makes all of Y false[2]

$X : Y$ is *valid* iff $X : Y$ admits of no counterexample

$X : Y$ is *perfectly valid* iff $X : Y$ is valid and every *proper* subsequent of $X : Y$ is *invalid*, that is, admits of counterexample

$X : Y$ is *sound* iff $X : Y$ is a substitution instance of a *perfectly* valid sequent

These definitions capture the spirit of what has come to be known as the Geach–Smiley–von Wright condition, but which Smiley himself attributes to W. E. Johnson.[3]

Thus $A\&{\sim}A : A$ and $A\&{\sim}A :{\sim}A$ are sound; while $A\&{\sim}A : B$ is not. This looks promising.

Also, $A : A \vee B$ is sound, and $A \vee B, {\sim}A : B$ is sound; while $A, {\sim}A : B$ is not. While each of these results in itself is promising, their combination may appear unsettling. For what it tells us is that sound arguments (or their proofs) *do not always accumulate.* Another way of putting this is to say that soundness (and deducibility) *is not unrestrictedly transitive.* Or, more succinctly: *Cut fails. That is, the structural rule of Cut is not admissible for sound arguments*:

$$\frac{X : A \quad Y, A : B}{X, Y : B}\,\text{Cut}$$

One aim of this essay is to justify the reaction 'So what?!' Such failure of Cut is to be *welcomed.* When Cut fails, it *ought* to fail. Failure of Cut is *logically enlightening.* It represents *epistemic gain.* I share Smiley's diagnosis[4] of the paradox of irrelevance as resulting

> from ascribing to an intuitive concept of entailment (which requires a co nnexion of meanings between premisses and conclusion that is quite lacking in the 'paradoxical' [$A\& \sim A$ entails B]) the property of unrestricted transitivity ...

but demur from his further claim that this property is

> ... admittedly essential to any entailment-relation that is to be put to sustained use as a logical instrument.

[2] Or, in case there are alternatives to truth besides falsity: '... makes all of Y not true '.

[3] T. J. Smiley, 'Entailment and deducibility', *Proceedings of the Aristotelian Society 1959*, pp. 233–253; at p. 239.

[4] T. J. Smiley, *loc. cit.*, p. 233.

Smiley writes[5] of

> (t)he need for an unrestrictedly transitive entailment-relation for serious logical work

and claims that

> the need itself is undeniable: the whole point of logic as an instrument, and the way in which it brings us new knowledge, lies in the contrast between the transitivity of 'entails' and the non-transitivity of 'obviously entails', *and all this is lost if transitivity cannot be relied on.* Of course if there is an effective way of predicting when transitivity will hold then most of the objection vanishes; there is such a way where the ... system ... is decidable (so that inferences can be checked directly); but I do not see how the thing might be done in, say, predicate logic ... [my emphasis].

Smiley's worry emphasized in this last quotation is, I submit, mistaken. What is needed of logic as an instrument of knowledge is only that chains of obvious entailments should yield entailments on *satisfiable sets of premisses*; and that all unsatisfiable sets should provably be so (that is, should be *inconsistent*). This will suffice for mathematics (on the assumption that it is not inconsistent) and for the hypothetico-deductive method in science. And *this* kind of guarantee can be forthcoming regardless of the decidability or otherwise of the logic in question. We simply do not need Cut in order for our logic to be a useful (indeed, completely adequate) instrument of knowledge.

By Cut I mean the principle above claiming the *unrestricted* transit ivity of sound argumentation. With soundness defined above as a semantic notion, it is apparent that we would need (pending a metatheorem to the effect that the sound arguments were exactly those that admitted of proof) to consider the correlative principle of Cut for deducibility. That principle would read

$$\frac{\text{There is a proof of } X : A \quad \text{There is a proof of } Y, A : B}{\text{There is a proof of } X, Y : B}$$

This principle of Cut for deducibility would follow from, but not in general imply, the following principle of Cut for proofs, which provides for *accumulation of proofs* in a system of proof:

[5] *loc. cit.*, p. 242.

Cut for Proofs by Proof Accumulation

Given any proof Π of A from X, and any proof Σ o f B from Y, A the result of grafting (copies of) Π onto appropriate assumption occurrences of A in Σ is itself a proof of B from X, Y:

$$\begin{array}{c} \quad X \\ \quad \Pi \\ Y,\ (A) \\ \Sigma \\ B \end{array}$$

This indeed is more or less the way in which Gentzen incorporated Cut as a structural rule of his sequent systems. The point of his *Hauptsatz* was then to show that applications of Cut could be eliminated from proofs, and their results still be secured by means of the remaining rules of the system.

It may turn out for some systems that the straightforward operation of ac cumulation may not be enough to secure the sought proof of $X, Y : B$. That is, one would not in general find a proof of the overall result $X, Y : B$ by chaining the proofs Π and Σ together in any straightforward sense. This would happen if, for example, proofs were defined so as to be in normal form. The result Φ of straightforward accumulation might not itself be in normal form, hence not, on this definition, a proof. But one might be able to *normalize* such an object Φ so as to obtain a proof Θ, in normal form, that establishes the overall result $X, Y : B$. So what we are envisaging in general here is the possibility that there may be some more complicated (but still effective) operation fA6 that can be performed on the given proofs Π and Σ so as to furnish a proof Θ of that overall result. Θ need not in any way be a straightforward composite (*qua* labelled tree) of the two proofs Π and Σ, or even of multiple copies thereof. Yet Θ would be obtained effectively from Π and Σ. We may summarize this line of thought as follows.

Principle of Cut for Proofs by Proof Conversion

There is an effective operation f that will turn any proof Pi of A from X and any proof Σ of B from Y, A into a proof Θ of $X, Y : B$:

$$\begin{array}{ccc} X & & \\ \Pi & & \\ A & f & X \cup Y \\ & \rightarrow & \Theta \\ Y, A & & B \\ \Sigma & & \\ B & & \end{array}$$

Note that the crucial thing about both Principles of Cut for Proofs (that by Accumulation and that by Conversion) is that they provide for proofs *of the overall result* $X, Y : B$. This is precisely what unrestricted transitivity of deduction consists in. If a system's deducibility relation does not obey dilution (thinning) it is possible for that deducibility relation not to be *unrestrictedly* transitive. Such, indeed, is the case with the systems of truth table logic, intuitionistic relevant logic and classical relevant logic.[6]

The principle that I shall recommend we should choose instead of Cut (for proofs) will be called Paste. Paste will be formulated below, in an *ordinary* form and in a *perfect* form. The way that Paste differs from Cut is that Paste provides in general only for a proof of *some subsequent* of the erstwhile overall result $X, Y : B$. That is, the proof Θ obtained by Paste may have for its set of undischarged assumptions some *proper subset* of X, Y; and it may have absurdity ($\perp$) for its conclusion .

Cut, indeed, should never have been laid down as a feature of the deducib ility relation of any decent logical system. It would be an interesting topic, but not one for this chapter, to enquire into the reasons why Cut ever became a respectable part of orthodox logic. There may have been an interesting historical interplay between closure operations in algebra and nascent notions of logical consequence in the early twentieth century. Topologically, the closure of the closure of a set is just its closure: $[[X]] = [X]$. To say, by way of analogy, that the deductive closure of the deductive closure of a set of sentences is just its closure is just to say that deducibility is unrestrictedly transitive, that is, that Cut holds.

Another notion that we shall have occasion to bring into play is that of a (uniform) substitution instance. The substitution instance itself is the finer object; that of which it is a substitution instance is the grosser object. Generally I shall write $\Phi \Leftarrow \mathbf{\Phi}$ to indicate that the finer object Φ is a substitution instance of the grosser object $\mathbf{\Phi}$. These objects may be sentences or sequents or proofs.

Let us return to the point that Cut fails for a more stringent notion of soundness than is involved in the usual semantical treatments of logical consequence, be they intuitionistic or classical. I shall work towards a careful statement of exactly how and where Cut might fail (both for soundness and for the associated notion(s) of deducibility) and exactly how and where it might hold. I shall argue that the new Paste principle of *epistemically gainful but restricted transitivity* should be adopted for deducibility. This principle, which I shall state for single-conclusion arguments, is as follows:

[6] For sources, see footnotes that follow.

Paste for Proofs by Conversion

If $X : A$ is sound and $Y, A : B$ is sound then *some subsequent* of $X, Y : B$ is sound. Indeed, there is an effective operation f that will turn any proof Π of A from X and any proof Σ of B from Y, A into a proof Θ of some subsequent of $X, Y : B$:

$$\begin{array}{cccccc}
X & & & & & \\
\Pi & & & & & \\
A & f & Z & & Z & \\
 & \rightarrow & \Theta & \text{or} & \Theta & \text{where } Z \subseteq (X \cup Y) \\
Y, A & & \perp & & B & \\
\Sigma & & & & & \\
B & & & & &
\end{array}$$

Here A is called the *paste formula.* We require only that the operation f be effective, not feasible. Thus when I speak below of epistemic gain, I should, strictly, speak of *potential* epistemic gain. Note that the principle states that one can (in principle, but not necessarily feasibly) paste proofs together and thereby obtain proof of a result *at least as good as, and often better than*, the result that the old rule of Cut would have furnished. This we have seen with the attempted proof-by-accumulation of the Lewis paradox. Taking Π as the proof

$$\frac{A}{A \vee B}$$

and taking Σ as the proof

$$\dfrac{A \vee B \qquad \dfrac{\sim A \quad \dfrac{}{A}\,(1)}{\perp} \qquad \dfrac{}{B}\,(1)}{\underline{B}}\,(1)$$

all we get by pasting them in the system of intuitionistic relevant logic, and then converting,[7] is the proof

$$\frac{\sim A \quad A}{\perp}$$

[7]which process would involve *normalization* and *extraction.* Extraction is the process of getting rid of applications of the absurdity rule. An extraction theorem is in effect a dilution-elimination theorem.

We do *not* get any proof of B from $\sim A, A$.

So far so good with blocking the Lewis paradox itself. But what assurance do we have that the transforms that we *do* obtain via f applied to proofs Π and Σ never *themselves* involve anything rum from the relevantist's point of view? How can we be sure that $f(\Pi, \Sigma)$ does not itself trade illicitly on some hidden inconsistency among its undischarged assumptions in order to establish its conclusion, or trade illicitly on the logical truth of its conclusion while yet making it look as though it follows from those assumptions?

The guarantee that there is nothing rum on the part of the transform would have to take the form of this strengthened Paste principle:

Principle of Paste for Proofs by Conversion and with Perfection

There is an effective operation f that will turn any proof Pi of A from X and any proof Σ of B from Y, A into a proof Θ of some subsequent of $X, Y : B$; and Θ is, moreover, a substitution instance of a proof $\boldsymbol{\Theta}$ that establishes a perfectly valid sequent:

$$\begin{array}{cccccc} X & & & & & \\ \Pi & & & & & \\ A & f & Z & & Z & \\ & \rightarrow & \Theta & \text{or} & \Theta & \text{where } Z \subseteq (X \cup Y) \text{ and } \Theta \Leftarrow \boldsymbol{\Theta} \\ Y, A & & \bot & & B & \\ \Sigma & & & & & \\ B & & & & & \end{array}$$

The new second part in this statement of the principle goes exactly as far as we need. It assures us that whatever stronger result $Z : B$ we may thus obtain (where B is not $\bot$), we may rest assured that it in turn *will not have been obtained by any sort of irrelevance within the proof* Θ *that establishes it.* Its proof Θ will exploit only such logical structure as will fail to reveal the inconsistency of its set Z of premisses, or the logical truth of its conclusion B; and will, moreover, need each of its premisses to obtain that conclusion. This is guaranteed by the existence of the perfect proof $\boldsymbol{\Theta}$ of which Θ is a substitution instance. The inferential moves made in Θ are, so to speak, acceptable via their schematization as the corresponding steps in $\boldsymbol{\Theta}$; so since $\boldsymbol{\Theta}$ by definition contains nothing rum (how could it?—it establishes a perfectly valid sequent), nor, then, will Θ. So the result of pasting and transforming by means of the mapping f will give a *bona fide* sound result—even if it does happen to be a proper subsequent of $X, Y : B$. And if it is a proper subsequent of $X, Y : B$ then we can at least claim *epistemic gain.*

Logic will have been an adequate instrument of knowledge—indeed, *more* than adequate in the usual way. For the conversion of Π and Σ will have produced a stronger logical result than $X, Y : B$. We might learn that not all of X,Y is needed in order to obtain the conclusion B—perhaps even that B is itself a logical truth; or we might learn that (some subset of) X, Y is inconsistent.

The motivation for the Principle of Paste for Proofs by Conversion (PPPC) comes, somewhat surprisingly, from a very familiar and apparently orthodox sourc e: the truth tables. The system of *truth table logic*[8] provides a natural setting for our main idea. It is very easy to establish that PPPC holds for this system. Part of the reason why this is so easy is that accumulation and conversion of proofs in truth table logic can only ever take place when the bottom conclusion (that is, the conclusion of Σ) is absurdity ($\perp$). This is because the paste formula A has to stand as the major premiss for an application of an elimination rule within the proof Σ; and all elimination rules in truth table logic have absurdity as their overall conclusion.

It would be useful, however, to generalize our means of reasoning so that applications of elimination rules can lead to conclusions other than absurdity. By modifying the rules of inference of the system of truth table logic in a principled and controlled way we can achieve this end. We obtain thereby the system of *intuitionistic relevant logic*.[9] I have proved elsewhere[10] that PPPC holds for intuitionistic relevant logic. But the newly modified rules considerably complicate attempts to prove analogously that the Principle of Paste for Proofs by Conversion **and with Perfection** also holds for this system. I simply conjecture this at present.

PPPC holds also for *classical relevant logic*.[11] And if the Principle of Paste for Proofs by Conversion **and with Perfection** holds for intuitionistic relevant logic, then it holds also for classical relevant logic.

[8] N. Tennant, 'Truth table logic , with a survey of embeddability results', *Notre Dame Journal of Formal Logic*, **30**, 1989, p 459–484.

[9] N. Tennant, *Autologic*, Edinburgh University Press, 1992, pp. 6–11, p. 39. See also N. Tennant, 'Delicate proof theory', in J. Copeland (ed.), *Logic and Reality: Essays in Pure and Applied Logic, In Memory of Arthur Prior* , Oxford University Press, forthcoming.

[10] N. Tennant, 'Intuitionistic Mathematics Does Not Need *Ex Falso Quodlibet*', forthcoming in *Topoi*.

[11] N. Tennant, 'Perfect validity, entailment and paraconsistency', *Studia Logica*, XLIII, 1984, pp 179–198.

Suppose we have materials for a paste:

$$\begin{array}{c} X \\ \Pi \\ A \end{array}$$

$$\begin{array}{c} Y,\ (A) \\ \Sigma \\ B \end{array}$$

and suppose moreover that there *is* a proof of B from (some subset Z of) $X \cup Y$. It is possible, however, that the mapping f in our statement of PPPC (with or without Perfection) might go to work on the proofs Π and Σ in such a way as to miss this fact. $f(\Pi, \Sigma)$ might turn out to be a proof of $\bot$ from some other subset W of $X \cup Y$. Thus as a special case, there might be as supposed a proof of B from $X \cup Y$ itself (that is, $Z = X \cup Y$) while yet $f(\Pi, \Sigma)$ turn out to be a proof of $\bot$ from some subset W of $X \cup Y$; perhaps even from $X \cup Y$ itself. Or, Z could be $X \cup Y$, and $f(\Pi, \Sigma)$ turn out to be a proof of B from some *proper* subset W of $X \cup Y$. That B follows from Z would here go unremarked by the operation of f.

The worry, then, is that there might be some kind of *transitive lacuna* even within an epistemically gainful system. We could have $X : A$ sound, $Y, A : B$ sound and $X, Y : B$ sound. We could even have the assurance that every proof in general is a substitution instance of a perfect proof, and that perfect proofs establish only perfectly valid sequents. Thus in having a proof Π of $X : A$, we learn that $X : A$ is sound; in having a proof Σ of $Y, A : B$ we learn that $Y, A : B$ is sound; but we might be una ble to operate on Π and Σ with f so as to obtain a proof that would enable us to learn that $X, Y : B$ is sound. We might miss some gross fact about deducibility when the materials (the proof Π and Σ) constituting some of the patchwork facts are in hand. Our Paste principle has the overall form

$$\exists f\, \forall \begin{array}{c} X \\ \Pi \\ A \end{array} \quad \forall \begin{array}{c} Y, A \\ \Sigma \\ B \end{array} \quad \exists Z \subseteq (X \cup Y) \quad f(\Pi, \Sigma) \text{ is a proof of } B \text{ or of } \bot \text{ from } Z$$

$$\text{and}$$

$$\exists \mathbf{\Theta}(\mathbf{\Theta} \text{ is perfect and } f(\Pi, \Sigma) \Leftarrow \mathbf{\Theta})$$

and the worry is now that there *could well be* some proof Ξ of B from $X \cup Y$ itself but that $f(\Pi, \Sigma)$ might not establish the same sequent (namely, $X, Y : B$) as Ξ!

This is a worry that one may simply have to live with. The point still s tands that $f(\Pi, \Sigma)$, if it establishes a result other than $X, Y : B$, represents epistemic gain. For that result will have the form $Z : B$, for some *proper* subset Z of $X \cup Y$, or the form $Z : \bot$ (that is, $Z : \emptyset$), for some subset Z of $X \cup Y$.

One can see how the transitive *lacuna* arises.[12] We supposed above that we had a result to the effect that every proof is a substitution instance of a perfect proof. Thus the proof Π of $X : A$ could be a substitution instance of some perfect proof $\mathbf{\Pi}$ of $\mathbf{X} : \mathbf{A}_1$, and the proof Σ of $Y, A : B$ could be a substitution instance of some perfect proof $\mathbf{\Sigma}$ of $\mathbf{Y}, \mathbf{A}_2 : \mathbf{B}$; but $\mathbf{A}_1$ and $\mathbf{A}_2$ could be so different that a straightforward cut or paste would not be possible. $\mathbf{A}_1$ and $\mathbf{A}_2$ could be distinct formulae. One could try to find a unifier A^* (that is, a common substitution instance of $\mathbf{A}_1$ and $\mathbf{A}_2$) for these two formulae, which would induce a substitution instance Π_1 of Π and a substitution instance Σ_2 of Σ, and *then* use A^* as the paste formula in order to operate with f (on Π_1 and Σ_2, hence, in some sense, on Π and Σ via $\mathbf{A}_1$ and $\mathbf{A}_2$); but would there be any guarantee that we would th ereby avoid unwanted epistemic gain, as it were? Would there be any guarantee that we could obtain an f-transform $\Theta(=_{df}\ (\Pi_1, \Sigma_2))$ whose perfected version $\mathbf{\Theta}$ established a (perfectly valid) sequent of which $X, Y : B$ was a substitution instance?

Note that epistemic gain can occur even when both Π and Σ are perfect. Thus

$$\frac{A}{A \vee B}$$

is perfect, as is the IR proof

$$\dfrac{A \vee B \qquad \dfrac{\sim A \quad \overline{A}^{\,(1)}}{\bot} \qquad \overline{B}^{\,(1)}}{\underline{B}}\;(1)$$

but there is no perfect IR proof establishing a sequent of which $A, \sim A : B$ would be a substitution instance. The result of pasting is converted instead to the simple proof

$$\frac{\sim A \quad A}{\bot}$$

But suppose that Π is a perfect proof of A from X and that Σ is a perfect proof of B from Y, A *and*, moreover, that $X, Y : B$ is *sound*. Should we then require of our mapping f that it produce from Π and Σ some proof Θ of the result $X, Y : B$? *Can* we so require? If we can, should we also require that Θ itself be perfect? This latter requirement *could* be imposed

[12] cf. Smiley, *loc. cit.*, p. 242.

(so there would be no loss in requiring that it *should*) if we had a general method for turning pr oofs Θ into the perfect proofs $\boldsymbol{\Theta}$ of which they were substitution instances. That general method could then be tacked on as the last part of the mapping f itself, in our statement of the Paste principle.

The ideas that are emerging here call at this point for summary. I shall state some candidate principles that it would be good to have. But first, three preliminaries :

1. two formulae A_1, A_2 are said to *unify* via the substitution σ if and only if $\sigma A_1 = \sigma A_2$
2. $\mathbf{X} : \mathbf{A}$ is a *suprasequent* of $X : A$ if and only if $X : A$ is a substitution instance of $\mathbf{X} : \mathbf{A}$
3. $\boldsymbol{\Pi}$ is a *supraproof* of Π if and only if Π is a substitution instance of $\boldsymbol{\Pi}$

Perfectibility
There is an operation π such that for every proof $\Pi, \pi(\Pi)$ is a perfect supraproof of Π

Perfect soundness[13]
Every perfect proof establishes a perfectly valid sequent.

Soundness
Every proof establishes a sound sequent.

Completeness
Every sound sequent has a proof.

Perfect Pasting of Perfect Proofs
There is an effective operation f such that:
for every perfect proof Π of $X : A_1$ and for every perfect proof Σ of $Y, A_2 : B$ if A_1 and A_2 unify via σ
and there is a proof of $\sigma X, \sigma Y : \sigma B$
then $f(\Pi, \Sigma)$ is a perfect proof of a suprasequent of $\sigma X, \sigma Y : \sigma B$.

Note that there is no perfect completeness theorem to the effect that every perfectly valid sequent has a perfect proof. As a counterexample, consider the perfectly valid sequent $A, A \supset (B\&C) : B\&C$.

Perfectibility with the weaker Principle of Paste for Proofs by Conversion implies the stronger Principle of Paste for Proofs by Conversion and with Perfection. The weaker principle has already been established for the

[13] Note that this soundness requirement embodies a requirement of relevance. It rules out standard classical and intutionistic logic, since they contain the perfect proof of Lewis's first paradox that is constructed by one step of negation elimination followed by one step of *ex falso quodlibet*.

full system of first-order intuitionistic relevant logic.[14] Thus Perfectibility is an important goal as a metatheorem.

Even if we attain that goal, however, we will not necessarily thereby have Perfect Pasting of Perfect Proofs. With Perfectibility, however, it will be enough, for Perfect Pasting of Perfect Proofs, to prove the slightly weaker result.

Perfectible Pasting of Perfect Proofs

There is an effective operation f such that:
for every perfect proof Π of $X : A_1$ and for every perfect proof Σ of $Y, A_2 : B$
if A_1 and A_2 unify via σ
and there is a proof of $\sigma X, \sigma Y : \sigma B$
then $f(\Pi, \Sigma)$ is a(n ordinary) proof of a suprasequent of $\sigma X, \sigma Y : \sigma B$.

Both Principles of Pasting of Perfect Proofs involve the hypothesis that $\sigma X, \sigma Y : \sigma B$ is *sound*. They might therefore be called principles of Pasting of Perfect Proofs for a Sound Result. Of course, we already have the

Principle of Pasting of Perfect Proofs for a Perfect Result

There is an effective operation f such that:
for every perfect proof Π of $X : A_1$ and for every perfect proof Σ of $Y, A_2 : B$
if A_1 and A_2 unify via σ
and there is a perfect proof of $\sigma X, \sigma Y : \sigma B$
then $f(\Pi, \Sigma)$ is a perfect proof of a suprasequent of $\sigma X, \sigma Y : \sigma B$.

The stronger hypothesis here to the effect that there is a perfect proof of $\sigma X, \sigma Y : \sigma B$ rules out the prospect of epistemic gain by proof of a proper subsequent. For, since perfect proofs establish only perfectly sequents, $\sigma X, \sigma Y : \sigma B$ will be perfectly valid. Thus one will be forced, via the operation f of PPPC, to find a proof of $\sigma X, \sigma Y : \sigma B$ itself. By perfectibility, we can then turn this proof into a perfect proof of a suprasequent of $\sigma X, \sigma Y : \sigma B$.

But why should we go so far as to require that we have Perfect Pasting for Sound Results, if these results are not themselves perfect? Why should the way of getting *to* A embodied in the proof Π and the way of getting *from* A embodied in the proof Σ always guarantee a way to the overall (sound but not perfect) result $X, Y : B$? Choice of lemmata A after all serve only a strategic purpose. They can reduce dramatically the computational effort involved in finding the proofs Π and Σ as opposed to finding an outright and

[14]'Intuitionistic Mathematics Does Not Need *Ex Falso Quodlibet', op. cit.*

direct proof Θ of the overall result. But the logical *quid pro quo* exacted for this saving of investigative effort is that when we come to paste the proofs Π and Σ and convert the result, we have to be prepared to make do with whatever (epistemically gainful) result the process delivers.

In serious scientific work, when logic is being used as an instrument to discover the *truth*, this is all that matters. Better a subsequent of $X, Y : B$ than $X, Y : B$ it self! Since X and Y will both be subsets of our set of axioms, or of our set of scientific hypotheses and observational statements, we will always be interested in (i) their mutual consistency, and (ii) how little of their combined forces are needed, should they be consistent, to deliver the con sequence B.

It is only a very special and narrow concern (even in the presence of perfectibility) that is catered for by the demand that we should design our logical system in such a way as to guarantee the truth of the Principle of Pasting of Perfect Proofs for a Sound Result. The sound result will still be *discoverable* by *some* means or other anyway, since it admits of (direct) proof. Why insist that someone who has chosen a judicious lemma A and has found a proof Π of $X : A$ and a proof Σ of $Y, A : B$, should, simply by virtue of the soundness (but not perfect validity) of $X, Y : B$ itself, be able to bring out *that* fact of soundness by putting together those particular proofs Π and Σ and operating on them in some effective way?

Lemmata are but points of logical access; each furnishes a pair of perspectives on the overall problem. Through a lemma, one sees, Janus-like, two aspects of truth-transmission, looking up and looking down. Looking up at the premisses X from the lemma A, one sees A as having been derived by virtue of a particular delineation of logical structure. Looking down to the conclusion B from the lemma A and other premisses Y, one sees B as following from Y, A by virtue of a possibly different delineation of logical structure. The lemma A conforms to both delineations by being the unification of its counterparts in the upper proof and the lower proof. If the overall argument $X, Y : B$ is indeed sound, and one wants to see *this*, and *not* rest content with any epistemically gainful proof of some proper subsequent of $X, Y : B$, then one must work to do so directly, from X, Y to B without breaking the job down into a passage *to* some lemma A and a passage *from* it.

The job of logic is to transmit truth as *economically* as possible. There is economy in the 'proper subsetting of sequents' that yields epistemic gain. There is economy in proof-search when premisses have to be relevant to their conclusions.[15] And there is computational economy in seeking proofs

[15] Or *should* be. This is why I object to the Anderson–Belnap system R of propositional logic—it is undecidable. By contrast, my method of relevantising a logical system does not increase the complexity of its decision problem. See my book *Autologic*, Edinburgh University Press, 1992, for a full discussion of all aspects of this consideration.

of and from judiciously chosen lemmata. If that makes deduction fail to be unrestrictedly transitive, then so be it.