# Pythagoras meets Peano, courtesy of Core Logic

by

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**Abstract**

We present a *completely formalized* proof, down to the last primitive number-axiomatic and logical-inferential details, *in Core Logic*, of the statement that no square of a natural number is twice any such square.

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## 1 The Challenge

In late August and early September 2015 there was a lively discussion, on the moderated email list fom@cs.nyu.edu, of Core Logic and the question whether the rule *Ex Falso Quodlibet*—conspicuously eschewed by Core Logic—is indispensable for formalizing mathematical proofs.[1] Harvey Friedman issued as a challenge to the core logician the formalization of the 'usual proofs' of two well-known results in number theory.[2] He asked, in particular,

> What would a detailed analysis of Tennantism look like for, say, the usual proofs of
>
> $n^2 = 2m^2$ has no solution in nonzero integers ...

It seemed clear, reading between the lines, that Friedman was of the opinion that this could not be done. Here we address this particular challenge problem—showing rigorously, in Core Logic, and from the *Peano Axioms* for arithmetic, that no square of a *natural number* is twice any such square (hence that the square root of 2 is irrational).

This, according to legend, is the discovery, made by some student or associate of Pythagoras, that shook the Pythagorean dogma that the world is made up out of whole numbers. One story has it that the proof led to its discover's expulsion from the cult; another, that it led to his execution by the same. I trust, then, that merely *formalizing* the proof will not be considered any essential advance; for the metaphysical cat is already out of the mathematical bag.

The extension of this result to the nonzero integers is of course straightforward, once one makes the move to the integers. The result is made all the more difficult to obtain, however, by initially restricting oneself to the Peano Axioms, and not helping oneself axiomatically to the usual algebraic laws (commutativity and associativity of addition and multiplication, for example, as well as distributivity of multiplication over addition) that are usually laid down as axioms for the ring (or integral domain) of the integers. In Peano Arithmetic, such properties of addition and multiplication have to be derived as theorems. This study ventures to present an *absolutely formal, fully detailed* proof, using only Core Logic, of the statement

$$\forall x \neg \exists y (y \neq 0 \wedge x.x = 2.(y.y))$$

[1]Everything that the reader might need to know about Core Logic can be found in the three publications Tennant [2012], Tennant [2015a] and Tennant [2015b].

[2]See http://www.cs.nyu.edu/pipermail/fom/2015-September/019105.html.

from Peano's axioms for the natural numbers. Or, rather: it presents a replete set of chunks of core proof, that collectively make up a single formal proof of the target result. (See Theorem 1.) This is simply because I am working within the confines of the A4 page. So I have had to break the deductive reasoning down into manageable chunks for the reader. In order to pull this off, there has been the occasional lapse into landscape mode.

The resounding *theoretical* answer to the aforementioned challenge problem of Friedman is that *Ex Falso Quodlibet* is *not* needed for formalizing mathematical proofs. That much is established by metatheorems. All that this study contributes is a single, sustained and important example of how this can be so. I do not usually take single-case inductions to be dispositive in foundational matters. But, in light of the metatheorems in the cited publications on Core Logic, a core-logical formalization of the proof that $\sqrt{2}$ is irrational struck me as invitingly apt, illustrative, timely and worthwhile.

It may be (for all I know) that this is the first time in the history of humankind that such a proof has ever been presented. For, in any mathematics textbook that proves this result 'rigorously' (yet, strictly speaking, *informally*) the proof takes a scant half-page or so. Ironically, one is more morally certain of the truth of the result on the basis of the *informal* proof than one can be (unaided by any automated proof-checker) on the basis of the fully formalized proof. This is because the fully formalized proof is *very* long, and it is psychically draining to check it for correctness. But such epistemic ironies are beside the point here. It is enough to appreciate that fulfilling the hand-waving promise by the formal logician that mathematical proofs can be fully formalized is no easy task. What follows should go some way to convince the reader that this is so (both that it is possible and that it is, nevertheless, no easy task). I am naturally relying on the orthodox logician—especially Friedman—who is keen to fault the core logician, to check the core proof offered here for formal correctness down to the very last detail. This is no exercise in *falsche Spitzfindigkeit*. For it is undertaken to meet Friedman's challenge head-on, to show him (and anyone else who may be interested) that Core Logic has what it takes to formalize *in*formal expert mathematical reasoning directly, naturally, and homologously.

## 2 Peano's axioms for the natural numbers

The theory of natural numbers is expressed in the first-order language with identity based on the name 0 (zero), the one-place function sign $s$ (successor), and the two-place function signs + (plus) and . (times). For definiteness, we take the theory to be axiomatized by the now famous axioms

$$\begin{array}{l} \forall x \neg\, 0 = sx \\ \forall x \forall y (sx = sy \rightarrow x = y) \\ \forall x\; x + 0 = x \\ \forall x \forall y\; x + sy = s(x + y) \\ \forall x\; x.0 = 0 \\ \forall x \forall y\; x.sy = (x.y) + x \end{array}$$

plus all (countably) infinitely many instances of the following axiom schema of *Mathematical Induction*:

$$(P0 \wedge \forall x (Px \rightarrow Psx)) \rightarrow \forall y Py$$

Whatever formula $Px$ is used in order to obtain a substitution instance of this axiom schema is called the *induced predicate* for the instance in question.

Note that this choice of axioms means that certain number-theoretic statements that the average mathematician would take as so obvious as not to stand in need of proof will actually have to be proved—indeed, in some cases, at quite considerable length. But that is just part of the bracing challenge to be faced anyway.

## 3 Definitions of non-primitive notions

**Definition 1.** $1 =_{df} s0$

**Definition 2.** $2 =_{df} ss0$

**Definition 3.** *$m$ is less than $n$ (in symbols: $m < n$)*
$\equiv_{df} \exists k\, m + sk = n$

**Definition 4.** *$m$ is less than or equal to $n$ (in symbols: $m \leq n$)*
$\equiv_{df} m < n \vee m = n$

**Definition 5.**
*$k$ divides $n$ with remainder $r$ (in symbols: $k|n;r$)*
$\equiv_{df} r < k \;\wedge\; \exists m\, n = (k.m + r)$

**Definition 6.**
*$k$ divides $n$ with no remainder (in symbols: $k|n;0$, abbreviated further to $k|n$)*
$\equiv_{df} \exists m\, n = km$

**Definition 7.**
*n is even (in symbols: En)* $\equiv_{df}$ $2|n$. *Equivalently,* $ss0|n$. *Equivalently,* $\exists m\, n = ss0.m$

**Definition 8.**
*n is odd (in symbols: On)* $\equiv_{df}$ $2|n;1$ *Equivalently,* $ss0|n;s0$. *Equivalently, in light of Lemma 1:* $\exists m\, n = ss0.m + s0$

In presenting our formal proofs below, we shall frequently resort to the serial forms of certain elimination rules. We do so in order to prevent sideways spread; and also because the serial forms are likely to be more familiar to the reader than the parallelized forms. The occasional exception, when parallelized forms are used, will be included in order to familiarize the reader with how these forms of the rules are applied. We shall be at pains, however, to ensure that all the formal proofs we provide are *in normal form*. Also, they *do not use Ex Falso Quodlibet*. And *sometimes* they use the 'liberalized' rules of $\rightarrow$I and $\vee$E of Core Logic. What these investigations reveal is just how naturally the resources of Core Logic *directly* formalize the expert informal reasoning employed in the proof that $\sqrt{2}$ is not a ratio of whole numbers.

## 4 On Formalizing Uses of the Principle of Mathematical Induction

We coin the description 'incremental induction' for the kind of Mathematical Induction whose axiom schema was stated above. Its being incremental is a matter of showing that the property in question is *transmitted under the successor operation*. In proofs by induction, this corresponds to the familiar *inductive step* that appeals to the *inductive hypothesis* $Pa$ to derive the conclusion $Psa$ (for a suitably chosen individual parameter $a$).

Suppose that, when proceeding informally, one proves a lemma $\forall y Py$ by using an instance of (incremental) Mathematical Induction. That is, one proves the 'basis step' $P0$; then one effects the 'inductive step' from the inductive hypothesis $Pa$ to the conclusion $Psa$; and finally one invokes the instance

$$(P0 \wedge \forall x(Px \rightarrow Psx)) \rightarrow \forall y Py$$

of Mathematical Induction to conclude

$$\forall y Py$$

The formalization of *this* stretch of reasoning would have the following overall form:

$$
\dfrac{(P0 \wedge \forall x(Px \rightarrow Psx)) \rightarrow \forall y Py \qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{(1)}}\\ \Xi\\ Psa\end{array}}{Pa \rightarrow Psa}{\scriptstyle (1)}}{\forall x(Px \rightarrow Psx)}}{P0 \wedge \forall x(Px \rightarrow Psx)} \qquad \overline{\forall y Py}^{(2)}}{\forall y Py}{\scriptstyle (2)}
$$

with the lemma $\forall y Py$ as the overall conclusion. Here $\Pi$ is the proof of the 'basis step' $P0$ for the proof by induction; and $\Xi$ is the proof of the 'inductive step', using the inductive hypothesis $Pa$ to deduce the conclusion $Psa$.

Two clarifying remarks are in order here.

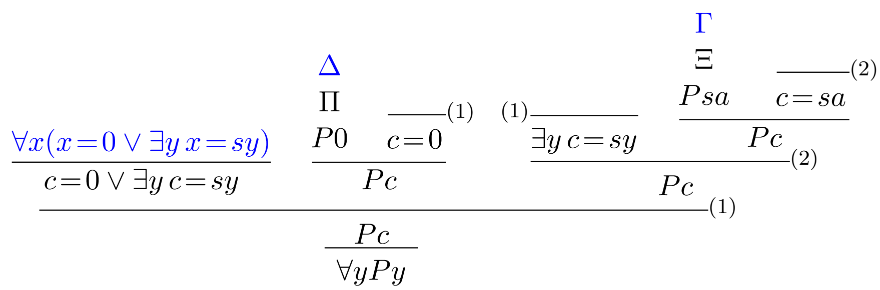
**Remark 1:** One does not *have* to make use of the inductive hypothesis $Pa$; such use is *permissible*, not obligatory. The application of the rule of $\rightarrow$I at the step marked (1) ensures that the assumption $Pa$, if used, is discharged. But, to stress once again: it may turn out that there is no assumption of the form $Pa$ to *be* discharged! The step of $\rightarrow$I would still be in good order; the overall proof by induction would simply look like this:

$$
\dfrac{(P0 \wedge \forall x(Px \rightarrow Psx)) \rightarrow \forall y Py \qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Gamma\\ \Xi\\ Psa\end{array}}{Pa \rightarrow Psa}}{\forall x(Px \rightarrow Psx)}}{P0 \wedge \forall x(Px \rightarrow Psx)} \qquad \overline{\forall y Py}^{(2)}}{\forall y Py}{\scriptstyle (2)}
$$

Note, however, that if one is equipped with the subproofs

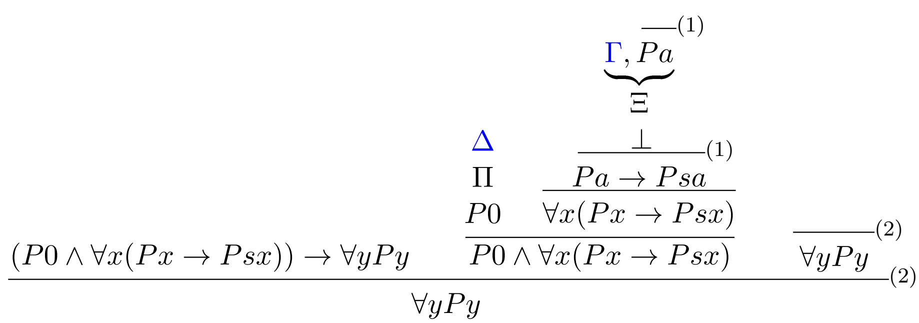
$$
\begin{array}{c}\Delta\\ \Pi\\ P0\end{array} \qquad \text{and} \qquad \begin{array}{c}\Gamma\\ \Xi\\ Psa\end{array}
$$

as indicated, then the conclusion $\forall y Py$ could be obtained as follows, using the premise $\forall x(x=0 \vee \exists y\, x=sy)$:



This extra premise is Lemma 8 below. It is the special axiom that, in Robinson's finitely axiomatized theory of arithmetic, replaces the Axiom Schema of Mathematical Induction. As we shall presently see, there is a proof of $\forall x(x=0 \vee \exists y\, x=sy)$ in Peano Arithmetic, using an instance of the Axiom Schema of Mathematical Induction, that eschews any use of the inductive hypothesis.

**Remark 2**, complementary to Remark 1: One does not *have* to produce $Psa$ as the conclusion of the inductive step! It would suffice to simply reduce the inductive hypothesis $Pa$ to absurdity. The step of $\rightarrow$I would still be in good order; the overall proof by induction would then look like this:



Note that the final step, marked (2), is an application of the *parallelized* rule of $\rightarrow$E, with a *degenerate* major subproof (which proves $\forall yPy$ from $\forall yPy$). The major premise of that application of $\rightarrow$E is the chosen instance of the axiom schema of Mathematical Induction. The minor subproof for the step of $\rightarrow$E in question is the subproof 'in the middle', of $P0 \wedge \forall x(Px \rightarrow Psx)$. The two subproofs $\Pi$, $\Xi$ can of course use, in addition, any of the Peano axioms, along with other suppositions. These respectively form the two sets $\Delta$, $\Gamma$ indicated in blue. When $\Delta$, $\Gamma$ contain *only* axioms, then (what the mathematicians call the lemma) $\forall yPy$ is (what logicians would call) a *theorem* of Peano arithmetic. Otherwise, $\forall yPy$ is a result following, within the theory of arithmetic, conditionally upon the extra suppositions in $\Delta$, $\Gamma$ that are not axioms.

Suppose now that one subsequently appeals to the mathematicians' lemma $\forall yPy$ as a *premise* in some further proof (call it $\Sigma$) of a conclusion $\theta$ on which the mathematicians are willing to bestow the honorific label 'theorem' (of Peano arithmetic). Then in the formalization of *this* overall stretch of reasoning *there is no call for a so-called 'cut'* with the lemma in question ($\forall yPy$) as the cut sentence. This is because the *overall* formal proof of the mathematical theorem $\theta$ in such circumstances will be able to take the following shape:

$$\dfrac{(P0\wedge\forall x(Px\to Psx))\to\forall yPy \qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma,\overline{Pa}^{(1)}}\\ \Xi\\ Psa\end{array}}{Pa\to Psa}{\scriptstyle(1)}}{\forall x(Px\to Psx)}}{P0\wedge\forall x(Px\to Psx)} \qquad \begin{array}{c}\underbrace{\Omega,\overline{\forall yPy}^{(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$$

Here $\Pi$ and $\Xi$ are as before; but now the major subproof for the final step of $\to$E is one's proof $\Sigma$ of $\theta$, which uses the lemma $\forall yPy$ as a premise. So the final step is still an application of the parallelized rule of $\to$E, but now with a *non*-degenerate major subproof, namely $\Sigma$. The major premise of the final step is still the chosen instance of the axiom schema of Mathematical Induction. The three subproofs $\Pi$, $\Xi$ and $\Sigma$ can of course use, in addition, any of the Peano axioms, along with other suppositions. These respectively form the three sets $\Delta$, $\Gamma$ and $\Omega$ indicated in blue. When $\Delta$, $\Gamma$ and $\Omega$ contain *only* axioms, then $\theta$ is a theorem of Peano arithmetic. Otherwise, $\theta$ is a result following, within the theory of arithmetic, conditionally upon the extra suppositions in $\Delta$, $\Gamma$ and $\Omega$ that are not axioms.

In an effort to prevent sideways spread it would be quite in order to suppress the major premise for →E on the left:

$$\dfrac{\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{\,(1)}}\\ \Xi\\ Psa\end{array}}{Pa\rightarrow Psa}{\scriptstyle(1)}}{\forall x(Px\rightarrow Psx)}}{P0\wedge\forall x(Px\rightarrow Psx)}\qquad \begin{array}{c}\underbrace{\Omega, \overline{\forall yPy}^{\,(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$$

For it can be efectively determined what the induced predicate is, for such an application of induction.

The Principle of Mathematical Induction can be parallelized even further, as follows:

$$\dfrac{(P0\wedge\forall x(Px\rightarrow Psx))\rightarrow\forall yPy\qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\qquad \dfrac{\dfrac{\begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{\,(1)}}\\ \Xi\\ Psa\end{array}}{Pa\rightarrow Psa}{\scriptstyle(1)}}{\forall x(Px\rightarrow Psx)}}{P0\wedge\forall x(Px\rightarrow Psx)}\qquad \begin{array}{c}\underbrace{\Omega\ ,\ \overline{Pt_1}^{\,(i)}\ ,\ldots,\ \overline{Pt_n}^{\,(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$$

since there will only over be finitely many appeals to the lemma $\forall yPy$ that has been established by induction. These appeals will involve singular terms $t_1,\ldots,t_n$ (which may be, or contain, parameters). Indeed, the parallelized rule just stated can be 'inferentialized' even further, *and its major premise suppressed*, so as to become the *Rule* of Mathematical Induction

$$\textsc{rmi}\qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\qquad \begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{\,(i)}}\\ \Xi\\ \bot/Psa\end{array}\qquad \begin{array}{c}\underbrace{\Omega\ ,\ \overline{Pt_1}^{\,(i)}\ ,\ldots,\ \overline{Pt_n}^{\,(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$$

Note how we have designated the conclusion of the proof $\Xi$ of the inductive step as '$\bot/Psa$'. This is pursuant to Remarks 1 and 2 above. In the foregoing statement of the rule RMI, it is to be understood that the proof $\Xi$ of the inductive step satisfies exactly one of the following conditions:

1. $\Xi$ has $Pa$ as an undischarged assumption, and has $\perp$ as its conclusion;
2. $\Xi$ has $Pa$ as an undischarged assumption, and has $Psa$ as its conclusion;
3. $\Xi$ does not have $Pa$ as an undischarged assumption, and has $Psa$ as its conclusion.

In each of the first two cases, the application of RMI discharges all assumption-occurrences of $Pa$ in $\Xi$. In the third case, such discharge is not called for, since $Pa$ is not used as an assumption.

Note that with applications of RMI each of $\Pi$, $\Xi$ and $\Sigma$ is a *proof*. This should go without saying, since rules of inference enable one to form proofs, but only *from* (simpler) proofs. There is a special need here, however, to stress that the major subproof $\Sigma$ has to be well formed. In particular, if any of the terms $t_1, \ldots, t_n$ is (or contains) a *parameter* $a$, then $a$ cannot occur in such a way as to violate any of the parametric restrictions on applications, within $\Sigma$, of the two rules $\exists$E and $\forall$I, applications of which might well *have* to involve $a$ as a parameter. This places a limitation on the extent to which one might be able to defer applications of RMI to points 'lower down' within a proof. They may instead have to be applied 'higher up', so as to discharge those assumptions $Pt_i$ that contain parameters that would otherwise, if allowed to occur in those same assumptions *un*discharged, render illegitimate an application, within $\Sigma$, of either $\exists$E or $\forall$I.

As a special case (for $n=1$) we have

$$\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\qquad \begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{(i)}}\\ \Xi\\ \perp/Psa\end{array}\qquad \begin{array}{c}\\ \\ \overline{Pt}^{(i)}\end{array}}{Pt}\,(i)$$

And as a further special case of *that* we have, with parameter $b$ as one's choice for the term $t$, the proof-schema

$$\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\qquad \begin{array}{c}\underbrace{\Gamma, \overline{Pa}^{(i)}}\\ \Xi\\ \perp/Psa\end{array}\qquad \begin{array}{c}\\ \\ \overline{Pb}^{(i)}\end{array}}{Pb}\,(i)$$

With $b$ chosen so as to meet the requirements for $\forall$I, we can then obtain the usual conclusion $\forall y Py$ of the proof by mathematical induction:

$$\dfrac{\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\quad\begin{array}{c}\underbrace{\Gamma,\overline{Pa}^{(i)}}\\ \Xi\\ \bot/Psa\end{array}\quad\overline{Pb}^{(i)}}{Pb}\,{\scriptstyle (i)}}{\forall y P y}$$

If in fact one did this, and subsequently appealed to $\forall y Py$ as a major premise for $\forall$E in a proof of $\theta$:

$$\dfrac{\forall y P y\qquad\begin{array}{c}\underbrace{\Omega\;,\overline{Pt_1}^{(1)},\ldots,\overline{Pt_n}^{(1)}}\\ \Sigma\\ \theta\end{array}}{\theta}\,{\scriptstyle (1)}$$

one would have a *prime-facie* violation of the requirement of normality for one's overall proof of $\theta$ from $\Delta, \Gamma, \Omega$. But such an appearance of abnormality is just that: a mere appearance. For one can always take for the genuinely underlying proof the *reduct*

$$\left[\begin{array}{ccc}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\quad\begin{array}{c}\underbrace{\Gamma,\overline{Pa}^{(i)}}\\ \Xi\\ \bot/Psa\end{array}\quad\overline{Pb}^{(i)}}{Pb}\,{\scriptstyle (i)}}{\forall y P y} & , & \dfrac{\forall y P y\qquad\begin{array}{c}\underbrace{\Omega\;,\overline{Pt_1}^{(1)},\ldots,\overline{Pt_n}^{(1)}}\\ \Sigma\\ \theta\end{array}}{\theta}\,{\scriptstyle (1)}\end{array}\right]$$

which is simply the form RMI (Rule of Mathematical Induction) stated above. As a convenient reminder:

$$\text{RMI}\qquad\dfrac{\begin{array}{c}\Delta\\ \Pi\\ P0\end{array}\quad\begin{array}{c}\underbrace{\Gamma,\overline{Pa}^{(i)}}\\ \Xi\\ \bot/Psa\end{array}\quad\begin{array}{c}\underbrace{\Omega\;,\overline{Pt_1}^{(i)},\ldots,\overline{Pt_n}^{(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}\,{\scriptstyle (i)}$$

*There is no blowup in length of proof, when taking the reduct in place of the two proofs between the square brackets.* That much is absolutely obvious by inspection.

## 5 Results proved without using Mathematical Induction

Because we are restricting our primitive means of mathematical expression to the name 0 (zero), the one-place function sign $s$ (successor), and the two-place function signs + (plus) and . (times), we have had to *define* certain other expressions that mathematicians conveniently take as expressively primitive. We saw this in §3.

**Lemma 1.** $1 < 2$*, i.e.,* $s0 < ss0$*.*

*Proof.*[3]

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, s0+sy=s(s0+y)}}{s0+s0=s(s0+0)} \qquad \dfrac{\forall x\, x+0=x}{s0+0=s0}}{\dfrac{s0+s0=ss0}{\exists k\, s0+sk=ss0}}$$

$$\text{i.e., } s0 < ss0$$

□

**Pause for a moment's reflection** ... We have just taken *five* primitive steps of inference to establish the trivial truth that $0 < 1$. The alarmed reaction might be 'To what dreadful lengths will we have to go in order to show that no square of a natural is twice any such square'? The answer, reassuringly, is that the proof of the latter can be broken down into manageable chunks, all of them *formal proofs in Core Logic*, using only the Peano axioms. The rest of this study shows how.

**Lemma 2.** *0 is not a successor; in symbols, expressed inferentially:*

$$\dfrac{0=st}{\bot}$$

[3] This proof is due to Ben Cleary.

*Proof.*

$$\dfrac{\dfrac{\forall x\,\neg 0=sx}{\neg 0=st}\qquad 0=st}{\bot}$$

Trivially, also, we have

$$\dfrac{st=0}{\bot}$$

□

**Lemma 3.** $\dfrac{st=su}{t=u}$

*Proof.*

$$\dfrac{\dfrac{\dfrac{\forall x\forall y(sx=sy\to x=y)}{\forall y(st=sy\to t=y)}}{st=su\to t=u}\qquad st=su\qquad \dfrac{}{t=u}{}^{(1)}}{t=u}{}^{(1)}$$

Note that the last step of this proof is an application of the parallelized rule $\to$I, with a degenerate major subproof. We shall frequently use the rule of inference stated in this lemma as a primitive rule, since it saves a great deal of sideways spread. Likewise with any other inferential rules that we have established formally, such as those of Lemma 2. □

**Lemma 4.** $\dfrac{\lambda.\lambda\neq 0}{\lambda\neq 0}$

*Proof.*

$$\dfrac{\lambda.\lambda\neq 0\qquad \dfrac{\dfrac{\dfrac{}{\lambda.\lambda=\lambda.\lambda}\qquad \dfrac{}{\lambda=0}{}^{(1)}}{\lambda.\lambda=\lambda.0}\qquad \dfrac{\forall x\,x.0=0}{\lambda.0=0}}{\lambda.\lambda=0}}{\dfrac{\bot}{\lambda\neq 0}{}^{(1)}}$$

□

**Lemma 5.** $\dfrac{\lambda=n.\rho\qquad \lambda\neq 0}{\rho\neq 0}$

*Proof.*

$$\dfrac{\dfrac{\lambda\neq 0 \qquad \dfrac{\lambda=n.\rho \qquad \dfrac{\dfrac{\dfrac{}{n.\rho=n.\rho}\quad \dfrac{}{\rho=0}{\scriptstyle(1)}}{n.\rho=n.0} \qquad \dfrac{\forall x\, x.0=0}{n.0=0}}{n.\rho=0}}{\lambda=0}}{\bot}}{\rho\neq 0}{\scriptstyle(1)}$$

□

**Lemma 6.** *From the assumption that $a$ is even it follows that $sa$ is odd*

*Proof.*

$$\dfrac{\exists m\, a=ss0.m \qquad \dfrac{\dfrac{\begin{matrix}\text{Lemma 1:}\\ s0<ss0\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{sa=sa}\quad \dfrac{}{a=ss0.b}{\scriptstyle(1)}}{sa=s(ss0.b)} \qquad \dfrac{\forall x\, x=x+0}{ss0.b=ss0.b+0}}{sa=s(ss0.b+0)} \qquad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, ss0.b+sy=s(ss0.b+y)}}{ss0.b+s0=s(ss0.b+0)}}{sa=ss0.b+s0}}{s0<ss0 \;\wedge\; sa=ss0.b+s0}}{ss0|sa;s0}}{ss0|sa;s0}{\scriptstyle(1)}$$

□

Note that Lemma 1 is not a cut sentence here of the kind that would, upon accumulation of proofs, produce an *abnormal* proof. Rather, the earlier proof of Lemma 1 could be inserted above its 'premise occurrence' in the last proof just given, and the resulting proof would still be a proof in Core Logic. We have broken the reasoning down into these last two chunks (proof of Lemma 1 followed by proof of Lemma 6) solely in order to avoid unmanageable sideways spread on an A4 page. This is a theme that will be reprised quite frequently below, and we shall not take the trouble to remark on it any further.

**Lemma 7.** $ss0 = ss0.s0$

*Proof.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\ s(x+y)=x+sy}{\forall y\ s(0+y)=0+sy}}{s(0+s0)=0+ss0} \qquad \dfrac{\dfrac{\dfrac{\forall x \forall y\ s(x+y)=x+sy}{\forall y\ s(0+y)=0+sy}}{s(0+0)=0+s0} \quad \dfrac{\forall x\ x+0=x}{0+0=0}}{s0=0+s0}}{ss0=0+ss0} \qquad \dfrac{\dfrac{\dfrac{\forall x \forall y\ x.sy=x.y+x}{\forall y\ ss0.sy=ss0.y+ss0}}{ss0.s0=ss0.0+ss0} \quad \dfrac{\forall x\ x.0=x}{ss0.0=0}}{ss0.s0=0+ss0}}{ss0=ss0.s0}
$$

□

## 6 Results proved using Mathematical Induction

For the formal proofs to follow, if we were to cite the actual instances to be used of the axiom schema of Mathematical Induction, it would be prohibitively difficult to accommodate sideways spread on the page. We shall therefore offer proofs in which Mathematical Induction takes the last-stated form of a rule of inference, namely RMI.

**Lemma 8.** *Every number is either 0 or a successor; in symbols:*

$$\forall y(y=0 \vee \exists x\, y=sx)$$

*Proof.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{0=0}}{0=0 \vee \exists x\, 0=sx} \qquad \dfrac{\dfrac{\dfrac{}{a=0 \vee \exists x\, a=sx}{}^{(3)} \quad \dfrac{\dfrac{\dfrac{}{a=0}{}^{(2)}}{sa=s0}}{\exists x\, sa=sx} \quad \dfrac{\dfrac{}{\exists x\, a=sx}{}^{(2)} \quad \dfrac{\dfrac{\dfrac{}{a=sb}{}^{(1)}}{sa=ssb}}{\exists x\, sa=sx}}{\exists x\, sa=sx}{}^{(1)}}{\exists x\, sa=sx}{}^{(2)}}{sa=0 \vee \exists x\, sa=sx} \qquad \dfrac{}{c=0 \vee \exists x\, c=sx}{}^{(3)}}{c=0 \vee \exists x\, c=sx}{}^{(3)}}{\forall y(y=0 \vee \exists x\, y=sx)}
$$

The final step (of RMI) in this proof appears to involve an inductive step that actually uses the inductive hypothesis

$$a=0 \vee \exists x\, a=sx$$

to derive

$$sa = 0 \vee \exists x\, sa = sx.$$

There is, though, an even shorter proof by induction which does not use the inductive hypothesis at all:

$$\dfrac{\dfrac{\dfrac{0=0}{0=0 \vee \exists x\, 0=sx} \qquad \dfrac{\dfrac{sa=sa}{\exists x\, sa=sx}}{sa=0 \vee \exists x\, sa=sx} \qquad \dfrac{}{c=0 \vee \exists x\, c=sx}{}^{(3)}}{c=0 \vee \exists x\, c=sx}{}^{(3)}}{\forall y(y=0 \vee \exists x\, y=sx)}$$

□

Having established the '$Q$-axiom' (Lemma 8), we can re-state it as an atomicized rule of inference, which we shall label $QR$ (for '$Q$-*Rule*'):

$$QR \qquad \dfrac{\begin{matrix} \square\overline{\phantom{t=0}}^{(i)} \\ t=0 \\ \vdots \\ \psi/\bot \end{matrix} \qquad \begin{matrix} \square\overline{\phantom{t=sa}}^{(i)} \\ t=sa \\ \vdots \\ \psi/\bot \end{matrix}}{\psi/\bot}{}^{(i)} \qquad \text{where } a \text{ is parametric}$$

We shall now use the rule $QR$ to prove the 'zero-cancellation' law.

**Lemma 9.** $\dfrac{su.t=0}{t=0}$

*Proof.*

$$\dfrac{\overline{t=0}^{(1)} \qquad \dfrac{\dfrac{\dfrac{su.t=0 \qquad \overline{t=sa}^{(1)}}{su.sa=0} \qquad \dfrac{\dfrac{\forall x \forall y\, x.sy=x.y+x}{\forall y\, su.sy=su.y+su}}{su.sa=su.a+su}}{su.a+su=0} \qquad \dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, su.a+sy=s(su.a+y)}}{su.a+su=s(su.a+u)}}{\dfrac{s(su.a+u)=0}{\bot}}}{t=0}{}^{(1)\ QR}$$

□

**Lemma 10.** $\dfrac{m<sa}{m\le a}$

*Proof.* Unpacking the definitions of $<$ and $\leq$, the rule to be derived amounts to

$$\frac{\exists y\, m+sy=sa}{\exists z\, m+sz=a \lor m=a}$$

and its derivation, using *QR*, is as follows:

$$\dfrac{\exists y\, m+sy=sa \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{m+sb=sa}{}^{(1)} \quad \dfrac{}{b=0}{}^{(2)}}{m+s0=sa} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, m+sy=s(m+y)}}{m+s0=s(m+0)}}{s(m+0)=sa} \quad \dfrac{\forall x\, x+0=x}{m+0=m}}{sm=sa}\,\mathbf{L3}}{\dfrac{m=a}{\exists z\, m+sz=a \lor m=a}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{m+sb=sa}{}^{(1)} \quad \dfrac{}{b=sc}{}^{(2)}}{m+ssc=sa} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, m+sy=s(m+y)}}{m+ssc=s(m+sc)}}{s(m+sc)=sa}\,\mathbf{L3}}{\dfrac{\dfrac{m+sc=a}{\exists z\, m+sz=a}}{\exists z\, m+sz=a \lor m=a}}}{\exists z\, m+sz=a \lor m=a}{}^{(2)\ (QR)}}{\exists z\, m+sz=a \lor m=a}{}^{(1)}$$

□

**Corollary 1.** $\dfrac{m \leq sa}{m \leq a \lor m=sa}$

*Proof.*

$$\dfrac{\begin{array}{c} m\leq sa \\ \text{i.e., } m<sa \lor m=sa \end{array} \qquad \dfrac{\dfrac{\dfrac{}{m<sa}{}^{(1)}}{m\leq a}\,\mathbf{L10}}{m\leq a \lor m=sa} \qquad \dfrac{\dfrac{}{m=sa}{}^{(1)}}{m\leq a \lor m=sa}}{m\leq a \lor m=sa}{}^{(1)}$$

□

**Lemma 11.** $(b+b)+s0 = b+(b+s0)$

*Proof.* The induced predicate is $(x+x)+s0 = x+(x+s0)$. For the basis we need to prove

$$(0+0)+s0 = 0+(0+s0)$$

The following formal proof does the job:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(0+0)+s0=(0+0)+s0} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, (0+0)+sy=s((0+0)+y)}}{(0+0)+s0=s((0+0)+0)}}{(0+0)+s0=s((0+0)+0)} \quad \dfrac{\forall x\, x+0=x}{(0+0)+0=0+0}}{(0+0)+s0=s(0+0)} \quad \dfrac{\forall x\, x+0=x}{0+0=0}}{(0+0)+s0=s(0+(0+0))} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, 0+sy=s(0+y)}}{0+s(0+0)=s(0+(0+0))}}{(0+0)+s0=0+s(0+0)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, 0+sy=s(0+y)}}{0+s0=s(0+0)}}{(0+0)+s0 = 0+(0+s0)}$$

For the inductive step we do not need to use the inductive hypothesis

$$(a+a)+s0 = a+(a+s0).$$

Instead, we prove

$$(sa+sa)+s0 = sa+(sa+s0)$$

directly as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, s(sa+a)+sy=s(s(sa+a)+y)}}{s(sa+a)+s0=s(s(sa+a)+0)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sa+sy=s(sa+y)}}{sa+sa=s(sa+a)}}{(sa+sa)+s0=s(s(sa+a)+0)} \quad \dfrac{\forall x\, x+0=x}{s(sa+a)+0=s(sa+a)}}{(sa+sa)+s0=ss(sa+a)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sa+sy=s(sa+y)}}{sa+sa=s(sa+a)}}{(sa+sa)+s0=s(sa+sa)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sa+sy=s(sa+y)}}{sa+ssa=s(sa+sa)}}{(sa+sa)+s0=sa+ssa} \quad \dfrac{\forall x\, x+0=x}{sa+0=sa}}{(sa+sa)+s0=sa+s(sa+0)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sa+sy=s(sa+y)}}{sa+s0=s(sa+0)}}{(sa+sa)+s0 = sa+(sa+s0)}$$

□

**Lemma 12.** *$b+s0=s0+b$.*

*Proof.* The induced predicate is $x+s0=s0+x$. For the basis we need to prove

$$0+s0=s0+0$$

The following formal proof does the job:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, 0+sy=s(0+y)}}{0+s0=s(0+0)} \quad \dfrac{\forall x\, x+0=x}{0+0=0}}{0+s0=s0} \quad \dfrac{\forall x\, x+0=x}{s0+0=s0}}{0+s0=s0+0}$$

For the inductive step we use the inductive hypothesis

$$a+s0=s0+a.$$

From it we prove

$$sa+s0=s0+sa$$

as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\; x+sy=s(x+y)}{\forall y\; sa+sy=s(sa+y)}}{sa+s0=s(sa+0)} \quad \dfrac{\forall x\; x+0=x}{sa+0=sa}}{sa+s0=ssa} \quad \dfrac{\forall x\; x+0=x}{a+0=a}}{sa+s0=ss(a+0)} \quad \dfrac{\dfrac{\forall x \forall y\; x+sy=s(x+y)}{\forall y\; a+sy=s(a+y)}}{a+s0=s(a+0)}}{sa+s0=s(a+s0)} \quad \text{IH: } a+s0=s0+a}{sa+s0=s(s0+a)} \quad \dfrac{\dfrac{\forall x \forall y\; x+sy=s(x+y)}{\forall y\; s0+sy=s(s0+y)}}{s0+sa=s(s0+a)}}{sa+s0=s0+sa}$$

□

**Lemma 13.** $\forall x\; 0+x=x$

*Proof.*

$$\dfrac{\dfrac{\dfrac{\forall x\; x+0=x}{0+0=0} \quad \dfrac{\dfrac{\dfrac{\forall x \forall y\; x+sy=s(x+y)}{\forall y\; 0+sy=s(0+y)}}{0+sa=s(0+a)} \quad \dfrac{}{0+a=a}{\scriptstyle(1)}}{0+sa=sa} \quad \dfrac{}{0+b=b}{\scriptstyle(1)}}{0+b=b}{\scriptstyle(1)}}{\forall x\; 0+x=x}$$

□

**Lemma 14.** $\forall x\; 0+x=x+0$

*Proof.*

$$\dfrac{\dfrac{0+0=0+0 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\; x+sy=s(x+y)}{\forall y\; 0+sy=s(0+y)}}{0+sa=s(0+a)} \quad \dfrac{}{0+a=a+0}{\scriptstyle(1)}}{0+sa=s(a+0)} \quad \dfrac{\forall x\; x+0=x}{a+0=a}}{0+sa=sa} \quad \dfrac{\forall x\; x+0=x}{sa+0=sa}}{0+sa=sa+0} \quad \dfrac{}{0+b=b+0}{\scriptstyle(1)}}{0+b=b+0}{\scriptstyle(1)}}{\forall x\; 0+x=x+0}$$

□

**Lemma 15.** $t.s0 = t$

*Proof.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\, x.sy = x.y + x}{\forall y\, t.sy = t.y + t}}{t.s0 = t.0 + t} \qquad \dfrac{\forall x\, x.0 = 0}{t.0 = 0}}{t.s0 = 0 + t} \qquad \begin{array}{c}\text{Lemma 13:}\\ 0 + t = t\end{array}}{t.s0 = t}$$

□

**Lemma 16.** $0.t = 0$

*Proof.*

$$\dfrac{\dfrac{\forall x\, x.0 = 0}{0.0 = 0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\, x.sy = x.y + x}{\forall y\, 0.sy = 0.y + 0}}{0.sa = 0.a + 0} \qquad \dfrac{\forall x\, x + 0 = x}{0.a + 0 = 0.a}}{0.sa = 0.a} \qquad 0.a = 0}{0.sa = 0} \qquad \dfrac{}{0.t = 0}{\scriptstyle (1)}}{0.t = 0}{\scriptstyle (1)}$$

□

**Lemma 17.** $s0.t = t$

*Proof.*

$$\dfrac{\dfrac{\forall x\, x.0 = 0}{s0.0 = 0} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x \forall y\, x.sy = x.y + x}{\forall y\, s0.sy = s0.y + s0}}{s0.sa = s0.a + s0} \qquad \dfrac{}{s0.a = a}{\scriptstyle (1)}}{s0.sa = a + s0} \qquad \dfrac{\dfrac{\forall x \forall y\, x + sy = s(x + y)}{\forall y\, a + sy = s(a + y)}}{a + s0 = s(a + 0)}}{s0.sa = sa} \qquad \dfrac{\forall x\, x + 0 = x}{a + 0 = a}}{s0.sa = sa} \qquad \dfrac{}{s0.t = t}{\scriptstyle (1)}}{s0.t = t}{\scriptstyle (1)}$$

□

The following result is called the 'additive cancellation' law. We shall innovate by casting not only its statement, but also its inductive proof, in 'rule-inferential' form.

**Lemma 18.**

$$\frac{t + k = u + k}{t = u}$$

*Proof.* By induction, with the induced rule

$$\frac{t{+}n{=}u{+}n}{t{=}u}$$

Because we are doing this inductive proof inferentially, the task for the basis is that of deriving the 'basis *rule*'

$$\frac{t{+}0{=}u{+}0}{t{=}u}$$

Moreover, the task for the inductive step is that of using the 'inductive hypothesis' *rule*

$$\frac{t{+}a{=}u{+}a}{t{=}u}$$

to derive the *rule*

$$\frac{t{+}sa{=}u{+}sa}{t{=}u}$$

With that much by way of preparation we proceed to the 'rule-inductive' proof itself.

The basis proof is as follows:

$$\cfrac{\cfrac{t{+}0{=}u{+}0 \qquad \cfrac{\forall x\, x{+}0{=}x}{t{+}0{=}t}}{t{=}u{+}0} \qquad \cfrac{\forall x\, x{+}0{=}x}{u{+}0{=}u}}{t{=}u}$$

The proof of the inductive step (using the *rule version* of the Inductive Hypothesis) is

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\forall x \forall y\, x{+}sy{=}s(x{+}y)}{\forall y\, t{+}sy{=}s(t{+}y)}}{t{+}sa{=}s(t{+}a)} \qquad t{+}sa{=}u{+}sa}{s(t{+}a){=}u{+}sa} \qquad \cfrac{\cfrac{\forall x \forall y\, x{+}sy{=}s(x{+}y)}{\forall y\, u{+}sy{=}s(u{+}y)}}{u{+}sa{=}s(u{+}a)}}{s(t{+}a){=}s(u{+}a)}\,\textbf{L3}}{t{+}a{=}u{+}a}\,\text{IH}}{t{=}u}$$

□

**Lemma 19.** $\forall x \forall y\, x+sy=sx+y$

*Proof.*

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, a+sy=s(a+y)}}{a+s0=s(a+0)}
\quad
\dfrac{\forall x\, x+0=x}{a+0=a}
}{a+s0=sa}
\quad
\dfrac{\forall x\, x+0=x}{sa+0=sa}
}{a+s\color{red}{0}\color{black}{}=sa+\color{red}{0}}
\quad
\dfrac{
\dfrac{
\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, a+sy=s(a+y)}}{a+ssb=s(a+sb)}
\quad
\dfrac{}{a+s\color{red}{b}\color{black}{}=sa+\color{red}{b}}(1)
}{a+ssb=s(sa+b)}
\quad
\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sa+sy=s(sa+y)}}{sa+sb=s(sa+b)}
}{a+s\color{red}{sb}\color{black}{}=sa+\color{red}{sb}}
\quad
\dfrac{}{a+sc=sa+c}(1)
}{
\dfrac{\dfrac{a+sc=sa+c}{\forall y\, a+sy=sa+y}}{\forall x\forall y\, x+sy=sx+y}
}(1)
$$

□

**Lemma 20.** $\forall x \forall y\, x+y=y+x$

*Proof.*

$$
\dfrac{
\begin{matrix}\text{Lemma 14:}\\ a+\color{red}{0}\color{black}{}=\color{red}{0}\color{black}{}+a\end{matrix}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, a+sy=s(a+y)}}{a+sb=s(a+b)}
\quad
\dfrac{}{a+\color{red}{b}\color{black}{}=\color{red}{b}\color{black}{}+a}(1)
}{a+sb=s(b+a)}
\quad
\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, b+sy=s(b+y)}}{b+sa=s(b+a)}
}{a+sb=b+sa}
\quad
\begin{matrix}\text{Lemma 19:}\\ b+sa=sb+a\end{matrix}
}{a+\color{red}{sb}\color{black}{}=\color{red}{sb}\color{black}{}+a}
\quad
\dfrac{}{a+\color{red}{c}\color{black}{}=\color{red}{c}\color{black}{}+a}(1)
}{
\dfrac{\dfrac{a+\color{red}{c}\color{black}{}=\color{red}{c}\color{black}{}+a}{\forall y\, a+y=y+a}}{\forall x\forall y\, x+y=y+x}
}(1)
$$

□

**Corollary 2.** $t+(u+t) = (t+u)+t$

*Proof.*

$$
\dfrac{
\begin{matrix}\text{Lemma 20:}\\ t+(u+t)=(u+t)+t\end{matrix}
\quad
\begin{matrix}\text{Lemma 20:}\\ t+u=u+t\end{matrix}
}{t+(u+t)=(t+u)+t}
$$

□

**Lemma 21.** $t+(u+v) = (t+u)+v$

*Proof.*

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{}{a+b=a+b} \quad \dfrac{\forall x\, x+0=x}{(a+b)+0=a+b}}{(a+b)+0=a+b}
\quad
\dfrac{\forall x\, x+0=x}{b=b+0}
}{(a+b)+0=a+(b+0)}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, (a+b)+sy=s((a+b)+y)}}{(a+b)+sc=s((a+b)+c)}
\quad
\dfrac{}{(a+b)+c=a+(b+c)}{\scriptstyle (1)}
}{a+(b+sc)=s(a+(b+c))}
\quad
\dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, a+sy=s(a+y)}}{a+s(b+c)=s(a+(b+c))}
}{(a+b)+sc=a+s(b+c)}
\quad
\dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, b+sy=s(b+y)}}{b+sc=s(b+c)}
}{(a+b)+sc=a+(b+sc)}
}{t+(u+v)=(t+u)+v}{\scriptstyle (1)}
$$

□

**Lemma 22.** $st.u=(t.u)+u$

*Proof.* By induction, with the induced predicate $sa.x=a.x+x$. The basis proof is as follows:

$$\dfrac{\dfrac{\forall x\, x.0=0}{sa.0=0} \qquad \dfrac{\dfrac{\forall x\, x+0=x}{0+0=0} \quad \dfrac{\forall x\, x.0=0}{a.0=0}}{a.0+0=0}}{sa.0=a.0+0}$$

The proof of the inductive step is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, sa.sy=sa.y+sa)}}{sa.sc=sa.c+sa} \quad \text{IH: } sa.c=a.c+c}{sa.sc=(a.c+c)+sa} \quad \text{Lemma 21: } (a.c+c)+sa=a.c+(c+sa)}{sa.sc=a.c+(c+sa)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, c+sy=s(c+y)}}{c+sa=s(c+a)}}{sa.sc=a.c+s(c+a)} \quad \text{Lemma 20: } a+c=c+a}{sa.sc=a.c+s(a+c)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, a.c+sy=s(a.c+y)}}{a.c+s(a+c)=s(a.c+(a+c))}}{sa.sc=s(a.c+(a+c))} \quad \text{Lemma 21: } a.c+(a+c)=(a.c+a)+c}{sa.sc=s((a.c+a)+c)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, (a.c+a)+sy=s((a.c+a)+y)}}{(a.c+a)+sc=s((a.c+a)+c)}}{sa.sc=(a.c+a)+sc} \quad \dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, a.sy=a.y+a}}{a.sc=a.c+a}}{sa.sc=a.sc+sc}$$

□

**Lemma 23.** $t.u = u.t$

*Proof.* By induction, with the induced predicate $a.x == x.a$. The basis proof is as follows:

$$\dfrac{\dfrac{\forall x\, x.0=0}{a.0=0} \qquad \begin{array}{c}\text{Lemma 16}:\\ 0.a=0\end{array}}{a.0 == 0.a}$$

The proof of the inductive step is

$$\dfrac{\dfrac{\dfrac{\forall x \forall y\, x.sy=x.y+x}{\forall y\, a.sy=a.y+a}}{a.sb=a.b+a} \qquad \dfrac{\begin{array}{c}\text{Lemma 22}:\\ sb.a=b.a+a\end{array} \qquad \begin{array}{c}\text{IH}:\\ a.b=b.a\end{array}}{sb.a=a.b+a}}{a.sb == sb.a}$$

$\square$

**Lemma 24.** $t.(u+v)=(t.u)+(t.v)$

*Proof.* By induction, with the induced predicate $a.(b+x)=a.b+a.x$. The basis proof is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a.b=a.b} \qquad \dfrac{\forall x\, x+0=x}{b+0=b}}{a.(b+0)=a.b} \qquad \dfrac{\forall x\, x+0=x}{a.b+0=a.b}}{a.(b+0)=a.b+0} \qquad \dfrac{\forall x\, x.0=0}{a.0=0}}{a.(b+0)=a.b+a.0}$$

The proof of the inductive step is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a.(b+sc)=a.(b+sc)} \qquad \dfrac{\dfrac{\forall x \forall y\, x+sy=s(x+y)}{\forall y\, b+sy=s(b+y)}}{b+sc=s(b+c)}}{a.(b+sc)=a.s(b+c)} \qquad \dfrac{\dfrac{\forall x \forall y\, x.sy=x.y+x}{\forall y\, a.sy=a.y+a}}{a.s(b+c)=a.(b+c)+a}}{a.(b+sc)=a.(b+c)+a} \qquad \begin{array}{c}\text{IH:}\\ a.(b+c)=a.b+a.c\end{array}}{a.(b+sc)=(a.b+a.c)+a} \qquad \begin{array}{c}\text{Lemma 21:}\\ (a.b+a.c)+a=a.b+(a.c+a)\end{array}}{a.(b+sc)=a.b+(a.c+a)} \qquad \dfrac{\dfrac{\forall x \forall y\, x.sy=x.y+x}{\forall y\, a.sy=a.y+a}}{a.sc=a.c+a}}{a.(b+sc)=a.b+a.sc}$$

$\square$

**Lemma 25.** $t.(u.v) = (t.u).v$

*Proof.* By induction, with the induced predicate $(a.b).x = a.(b.x)$. The basis proof is as follows:

$$\dfrac{\dfrac{\forall x\, x.0=0}{(a.b).0=0} \qquad \dfrac{\dfrac{\forall x\, x.0=0}{a.0=0} \quad \dfrac{\forall x\, x.0=0}{b.0=0}}{a.(b.0)=0}}{(a.b).0=a.(b.0)}$$

The proof of the inductive step is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\,(a.b).sy=(a.b).y+(a.b)}}{(a.b).sc=(a.b).c+(a.b)} \quad \overset{\text{IH:}}{(a.b).0=a.(b.0)}}{(a.b).sc=a.(b.c)+(a.b)} \quad \overset{\text{Lemma 24:}}{a.(b.c+b)=a.(b.c)+(a.b)}}{(a.b).sc=a.(b.c+b)} \qquad \dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, b.sy=b.y+b}}{b.sc=b.c+b}}{(a.b).sc=a.(b.sc)}$$

□

**Lemma 26.** $ss0.u = u+u$

*Proof.* The induced predicate is $ss0.x = x+x$. For the basis we need to prove

$$ss0.0 = 0+0$$

The following formal proof does the job:

$$\dfrac{\dfrac{\forall x\, x.0=0}{ss0.0=0} \qquad \dfrac{\forall x\, x+0=x}{0+0=0}}{ss0.0 = 0+0}$$

For the inductive step we use the inductive hypothesis

$$ss0.a = a+a.$$

From it we prove

$$ss0.sa = sa+sa$$

as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, ss0.sy=ss0.y+ss0}}{ss0.sa=ss0.a+ss0} \quad \overset{\text{IH:}}{ss0.a=a+a}}{ss0.sa=(a+a)+ss0} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\,(a+a)+sy=s((a+a)+y)}}{(a+a)+ss0=s((a+a)+s0)}}{ss0.sa=s((a+a)+s0)} \quad \overset{\text{Lemma 11:}}{(a+a)+s0=a+(a+s0)}}{ss0.sa=s(a+(a+s0))} \quad \overset{\text{Lemma 12:}}{a+s0=s0+a}}{ss0.sa=s(a+(s0+a))} \quad \overset{\text{Corollary 2:}}{a+(s0+a)=(a+s0)+a}}{ss0.sa=s((a+s0)+a)} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\,(a+s0)+sy=s((a+s0)+y)}}{(a+s0)+sa=s((a+s0)+a)}}{ss0.sa=(a+s0)+sa} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, a+sy=s(a+y)}}{a+s0=s(a+0)}}{ss0.sa=s(a+0)+sa} \quad \dfrac{\forall x\, x+0=x}{a+0=a}}{ss0.sa=sa+sa}$$

$\square$

Another result we can prove using $QR$ is that only zero has a zero square.

**Lemma 27.**

$$\frac{t.t=0}{t=0}$$

*Proof.*

$$\dfrac{\dfrac{}{t=0}{\scriptstyle (1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{t=sa}{\scriptstyle (1)} \quad t.t=0}{sa.sa=0}}{sa=0}\,\mathbf{L9}}{\bot}\,\mathbf{L2}}{}}{t=0}\,{\scriptstyle (1)}$$

$\square$

We can now prove the law of *multiplicative cancellation*.

**Lemma 28.**

$$\frac{sk.v=sk.w}{v=w}$$

*Proof.* The proof of this lemma will be given in the rule-inferential fashion that was exhibited with Lemma 18. In the case at hand the induced rule is

$$\frac{sk.n=sk.w}{n=w}$$

Because we are doing this inductive proof inferentially, the task for the basis is that of deriving the 'basis *rule*'

$$\frac{sk.0=sk.w}{0=w}$$

Moreover, the task for the inductive step is that of using the 'inductive hypothesis' *rule*

$$\frac{sk.a=sk.w}{a=w}$$

to derive the *rule*

$$\frac{sk.sa=sk.w}{sa=w}$$

The basis proof is

$$\dfrac{\dfrac{sk.0=sk.w \qquad \dfrac{\forall x\, x.0=0}{sk.0=0}}{0=sk.w}\ \mathbf{L9}}{0=w}$$

The proof of the inductive step has the overall form

$$\dfrac{\begin{matrix}\underbrace{sk.sa=sk.w\ ,\ \overline{w=0}^{\,(1)}} \\ \Pi_1 \\ \bot\end{matrix} \qquad \begin{matrix}\underbrace{sk.sa=sk.w\ ,\ \overline{w=sb}^{\,(1)}} \\ \Pi_2 \\ sa=w\end{matrix}}{sa=w}\ (1)\ (QR)$$

The embedded subproof $\Pi_1$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, sk.sy=sk.y+sk}}{sk.sa=sk.a+sk} \qquad sk.sa=sk.w}{sk.w=sk.a+sk} \qquad w=0}{sk.0=sk.a+sk} \qquad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sk.a+sy=s(sk.a+y)}}{sk.a+sk=s(sk.a+k)}}{sk.0=s(sk.a+k)} \qquad \dfrac{\forall x\, x.0=0}{sk.0=0}}{s(sk.a+k)=0}\ \mathbf{L2}}{\bot}$$

and the embedded subroof $\Pi_2$ is

$$\dfrac{\dfrac{\overline{sa=sa} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{sk.sa=sk.w \quad w=sb}{sk.sa=sk.sb} \quad \dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, sk.sy=sk.y+sk}}{sk.sa=sk.a+sk}}{sk.a+sk=sk.sb} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sk.a+sy=s(sk.a+y)}}{sk.a+sk=s(sk.a+k)}}{s(sk.a+k)=sk.sb} \quad \dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, sk.sy=sk.y+sk}}{sk.sb=sk.b+sk} \quad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sk.b+sy=s(sk.b+y)}}{sk.b+sk=s(sk.b+k)}}{sk.sb=s(sk.b+k)}}{s(sk.a+k)=s(sk.b+k)}\ \mathbf{L3}}{sk.a+k=sk.b+k}\ \mathbf{L18}}{sk.a=sk.b}\ \text{IH}}{a=b}}{sa=sb} \qquad w=sb}{sa=w}$$

□

**Lemma 29.** $\dfrac{(2.t).(2.t) = 2.(u.u)}{u.u = 2.(t.t)}$

*Proof.* We allow ourselves the luxury of doing without multiplicative dots, since multiplication is the only operation in play.

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\text{Lemma 25:}\\ (2t)(2t)=2(t(2t))\end{array} \quad \begin{array}{c}\text{Lemma 23:}\\ t(2t)=(2t)t\end{array}}{(2t)(2t)=2((2t)t)} \quad \begin{array}{c}\text{Lemma 25:}\\ (2t)t=2(tt)\end{array}}{(2t)(2t)=2(2(tt))} \qquad (2t)(2t)=2(uu)}{\cfrac{2(uu)=2(2(tt))}{uu=2(tt)}\ \mathbf{L28}}$$

□

A particularly useful consequence of Lemmas 9 and 27 is that twice the square of a nonzero number is nonzero. We state this as Lemma 30, whose special form will prove useful in due course.

**Lemma 30.** $\dfrac{\mu\neq 0 \wedge \lambda.\lambda=2.(\mu.\mu)}{\lambda.\lambda\neq 0}$

*Proof.*

$$\cfrac{\cfrac{\mu\neq 0 \wedge \lambda.\lambda=2.(\mu.\mu) \qquad \cfrac{\overline{\mu\neq 0}^{\,(2)} \qquad \cfrac{\cfrac{\cfrac{\overline{\lambda.\lambda=2.(\mu.\mu)}^{\,(2)} \quad \overline{\lambda.\lambda=0}^{\,(1)}}{2.(\mu.\mu)=0}}{\mu.\mu=0}\ \mathbf{L9}}{\mu=0}\ \mathbf{L27}}{\bot}}{\bot}\ (2)}{\lambda.\lambda\neq 0}\ (1)$$

□

**Lemma 31.** $\dfrac{t=ss0.u \quad \neg u=0}{u<t}$

*Proof.*

$$
\dfrac{\begin{array}{c}\text{Lemma 8 :}\\ u=0\lor\exists x\,u=sx\end{array}\qquad \dfrac{\neg u=0\qquad \dfrac{\phantom{x}}{u=0}{\scriptstyle(1)}}{\bot}\qquad \dfrac{\exists x\,u=sx\qquad \dfrac{\dfrac{\dfrac{t=ss0.u\qquad \begin{array}{c}\text{Lemma 26:}\\ ss0.u=u+u\end{array}}{t=u+u}\qquad \dfrac{\phantom{x}}{u=sa}{\scriptstyle(2)}}{t=u+sa}}{\exists y\,t=u+sy}}{\exists y\,t=u+sy}{\scriptstyle(1)}}{\begin{array}{c}\exists y\,t=u+sy\\ \text{i.e., } t<u\end{array}}{\scriptstyle(2)}
$$

∎

**Lemma 32.** $\forall n\ s(ss0.n+s0)=ss0.(n+s0)$

*Proof.* By induction on $n$. For the Basis Step $n=0$, we reason as follows, using Lemma 7 as a premise:

$$
\dfrac{\dfrac{ss0=ss0\qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\ s(x+y)=x+sy}{\forall y\ s(0+y)=0+sy}}{s(0+0)=0+s0}\qquad \dfrac{\forall x\ x+0=x}{0+0=0}}{s0=0+s0}\qquad \dfrac{\forall x\ x.0=0}{ss0.0=0}}{s0=ss0.0+s0}}{ss0=s(ss0.0+s0)}\qquad \dfrac{ss0=ss0.s0\qquad \dfrac{\dfrac{\dfrac{\forall x\forall y\ s(x+y)=x+sy}{\forall y\ s(0+y)=0+sy}}{s(0+0)=0+s0}\qquad \dfrac{\forall x\ x+0=x}{0+0=0}}{s0=0+s0}}{ss0=ss0.(0+s0)}}{s(ss0.0+s0)=ss0.(0+s0)}
$$

For the Inductive Step we assume the Inductive Hypothesis (IH):

$$s(ss0.k+s0)=ss0.(k+s0)$$

and proceed to derive the conclusion

$$s(ss0.sk+s0)=ss0.(sk+s0)$$

We do so by means of the following two chunks of proof, intended to be joined (so as to make a core proof) at the green sentence-occurrences. This division into two chunks is solely in order to avoid sideways spread.

$$
\dfrac{\dfrac{\dfrac{\forall x\forall y\,x+sy=s(x+y)}{\forall y\,ss0.(k+s0)+sy=s(ss0.(k+s0)+y)}}{ss0.(k+s0)+ss0=s(ss0.(k+s0)+s0)}\qquad \dfrac{\begin{array}{c}\text{IH:}\\ s(ss0.k+s0)=ss0.(k+s0)\end{array}\qquad \dfrac{\dfrac{\dfrac{\phantom{x}}{s(ss0.sk+s0)=s(ss0.sk+s0)}\qquad \dfrac{\dfrac{\forall x\forall y\,x.sy=x.y+x}{\forall y\,ss0.sy=ss0.y+ss0}}{ss0.sk=ss0.k+ss0}}{s(ss0.sk+s0)=s((ss0.k+ss0)+s0)}\qquad \dfrac{\dfrac{\forall x\forall y\,x+sy=s(x+y)}{\forall y\,ss0.k+sy=s(ss0.k+y)}}{ss0.k+ss0=s(ss0.k+s0}}{s(ss0.sk+s0)=s(s(ss0.k+s0)+s0)}}{s(ss0.sk+s0)=s(ss0.(k+s0)+s0)}}{s(ss0.sk+s0)=ss0.(k+s0)+ss0}
$$

$$\dfrac{\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, sk+sy=s(sk+y)}}{sk+s0=s(sk+0)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, ss0+sy=s(ss0+y)}}{ss0.ssk=ss0.ssk+ss0} \quad \dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, k+sy=s(k+y)}}{k+s0=s(k+0)} \quad s(ss0.sk+s0)=ss0.(k+s0)+ss0}{s(ss0.sk+s0)=ss0.s(k+0)+ss0} \quad \dfrac{\forall x\, x+0=x}{k+0=k}}{s(ss0.sk+s0)=ss0.sk+ss0}}{s(ss0.sk+s0)=ss0.ssk} \quad \dfrac{\forall x\, x+0=x}{sk+0=sk}}{s(ss0.sk+s0)=ss0.s(sk+0)}}{s(ss0.sk+s0)=ss0.(sk+s0)}$$

$\square$

**Lemma 33.** *From the assumption that a is odd it follows that sa is even*

*Proof.*

$$\dfrac{\exists m\; a = ss0.m+s0 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{s=ss0.b+s0}{}^{(1)}}{sa=s(ss0.b+s0)} \qquad \dfrac{\text{Lemma 32}: \; \forall n\; s(ss0.n+s0)=ss0.(n+s0)}{s(ss0.b+s0)=ss0.(b+s0)}}{sa=ss0.(n+s0)}}{\exists k\; sa = ss0.k}}{\exists k\; sa = ss0.k}\,{}^{(1)}$$

$\square$

**Lemma 34.** *Every number is either even or odd—i.e.,* $\forall n(ss0|n \vee ss0|n; s0)$

*Proof.* By induction. For the basis step we provide the following proof:

$$\dfrac{\dfrac{\forall x\, x.0 = 0}{0=ss0.0}}{\dfrac{\exists m\, 0=ss0.m}{\dfrac{\text{i.e., } ss0|0}{ss0|0 \vee ss0|0; s0}}}$$

Inductive Hypothesis (IH): $ss0|a \vee ss0|a; s0$
Inductive Step:

$$\dfrac{\text{IH: } ss0|a \vee ss0|a; s0 \qquad \dfrac{\dfrac{\dfrac{}{ss0|a}{}^{(1)}}{ss0|sa; s0}\,\mathbf{L6}}{ss0|sa \vee ss0|sa; s0} \qquad \dfrac{\dfrac{\dfrac{}{ss0|a; s0}{}^{(1)}}{ss0|sa}\,\mathbf{L33}}{ss0|sa \vee ss0|sa; s0}}{ss0|sa \vee ss0|sa; s0}\,{}^{(1)}$$

$\square$

**Lemma 35.** *Given any two numbers, neither one when doubled is double the other plus 1. In symbols:*

$$\forall m \forall n(\neg(2.m = 2.n + 1) \wedge \neg(2.m + 1 = 2.n))$$

*Proof.* By induction on $m$. For the basis we need to prove

$$\forall n(\neg(2.0 = 2.n + 1) \wedge \neg(2.0 + 1 = 2.n))$$

We invoke the abbreviations

$$\Psi(b, k) \; : \; \neg(ss0.k + s0 = ss0.b)$$

$$\Phi(b, k) \; : \; \neg(ss0.k = ss0.b + s0)$$

So the basis has the form

$$\forall n(\Phi(n, 0) \wedge \Psi(n, 0))$$

The inductive step in the proof will have the overall form

$$\dfrac{\dfrac{\dfrac{\forall n(\Phi(n,k) \wedge \Psi(n,k)) \qquad \dfrac{{}^{(3)}\overline{\Phi(b,k) \wedge \Psi(b,k)} \qquad \begin{matrix}\overline{\Phi(b,k)}^{(1)} \\ \Sigma \\ \Psi(b,sk)\end{matrix}}{\Psi(b,sk)}{\scriptstyle(1)}}{\Psi(b,sk)}{\scriptstyle(3)} \qquad \dfrac{\forall n(\Phi(n,k) \wedge \Psi(n,k)) \qquad \dfrac{{}^{(4)}\overline{\Phi(b,k) \wedge \Psi(b,k)} \qquad \begin{matrix}\overline{\Psi(b,k)}^{(2)} \\ \Pi \\ \Phi(b,sk)\end{matrix}}{\Psi(b,sk)}{\scriptstyle(2)}}{\Phi(b,sk)}{\scriptstyle(4)}}{\Phi(b,sk) \wedge \Psi(b,sk)}}{\forall n(\Phi(n,sk) \wedge \Psi(n,sk))}$$

The overall result, proved by induction, will therefore be

$$\forall m \forall n(\Phi(n, m) \wedge \Psi(n, m)),$$

that is,

$$\forall m \forall n(\neg(ss0.m = ss0.n + s0) \wedge \neg(ss0.m + s0 = ss0.n)),$$

or, writing 1 for $s0$ and 2 for $ss0$,

$$\forall m \forall n(\neg(2.m = 2.n + 1) \wedge \neg(2.m + 1 = 2.n)).$$

In words, as stated in the Lemma:

*Given any two numbers, neither one when doubled is double the other plus 1.*

We now have the task of providing the two embedded subproofs

$$\begin{array}{c}\underbrace{\Delta, \neg(ss0.k = ss0.b + s0)}_{\Sigma} \\ \neg(ss0.sk + s0 = ss0.b)\end{array} \qquad \begin{array}{c}\underbrace{\Gamma, \neg(ss0.k + s0 = ss0.b)}_{\Pi} \\ \neg(ss0.sk = ss0.b + s0)\end{array}$$

for appropriate selections $\Delta$ and $\Gamma$ of axioms for arithmetic. As intimated earlier, the axioms we use will be highlighted in blue. We present first the embedded subproof $\Pi$:

$$\dfrac{\neg(ss0.k+s0=ss0.b) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, ss0.sy=ss0.y+ss0}}{ss0.sk=ss0.k+ss0} \qquad \dfrac{}{ss0.sk=ss0.b+s0}{}^{(1)}}{ss0.k+ss0=ss0.b+s0}{}^{(1)} \qquad \dfrac{\dfrac{\forall x\forall y\, s(x+y)=x+sy}{\forall y\, s(ss0.k+y)=ss0.k+sy}}{s(ss0.k+s0)=ss0.k+ss0}}{s(ss0.k+s0)=ss0.b+s0} \qquad \dfrac{\dfrac{\forall x\forall y\, s(x+y)=x+sy}{\forall y\, s(ss0.b+y)=ss0.b+sy}}{s(ss0.b+0)=ss0.b+s0}}{s(ss0.k+s0)=s(ss0.b+0)}}{ss0.k+s0=ss0.b+0} \qquad \dfrac{\forall x\, x+0=x}{ss0.b+0=ss0.b}}{ss0.k+s0=ss0.b}}{\dfrac{\bot}{\neg(ss0.sk=ss0.b+s0)}{}^{(1)}}$$

Finally we present the embedded subproof $\Sigma$:

$$\dfrac{\neg(ss0.k=ss0.b+s0) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall x\forall y\, x.sy=x.y+x}{\forall y\, ss0.sy=ss0.y+ss0}}{ss0.sb=ss0.b+ss0} \qquad \dfrac{}{ss0.k+s0=ss0.sb}{}^{(1)}}{ss0.b+ss0=ss0.k+s0}{}^{(1)} \qquad \dfrac{\dfrac{\forall x\forall y\, s(x+y)=x+sy}{\forall y\, s(ss0.b+y)=ss0.b+sy}}{s(ss0.b+s0)=ss0.b+ss0}}{s(ss0.b+s0)=ss0.k+s0} \qquad \dfrac{\dfrac{\forall x\forall y\, s(x+y)=x+sy}{\forall y\, s(ss0.k+y)=ss0.k+sy}}{s(ss0.k+0)=ss0.k+s0}}{s(ss0.b+s0)=s(ss0.k+0)}}{ss0.b+s0=ss0.k+0} \qquad \dfrac{\forall x\, x+0=x}{ss0.k+0=ss0.k}}{ss0.k=ss0.b+s0}}{\dfrac{\bot}{\neg(ss0.k+s0=ss0.sb)}{}^{(1)}}$$

□

**Lemma 36.** *Every number is not both even and odd. In symbols:*

$$\forall x \neg (Ex \wedge Ox)$$

*i.e.,*

$$\forall x \neg (\exists y\, x{=}2y \wedge \exists z\, x{=}2z{+}1)$$

*Proof.* We use Lemma 35 as a premise in the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{(1)}\overline{\exists y\, x{=}2y \wedge \exists z\, x{=}2z{+}1}}{\exists y\, x{=}2y} \qquad \dfrac{\dfrac{{}^{(1)}\overline{\exists y\, x{=}2y \wedge \exists z\, x{=}2z{+}1}}{\exists z\, x{=}2z{+}1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall m \forall n(\neg(2m=2n{+}1) \wedge \neg(2m{+}1=2n))}{\forall n(\neg(2a=2n{+}1) \wedge \neg(2a{+}1=2n))}}{\neg(2a=2b{+}1) \wedge \neg(2a{+}1=2b)}}{\neg(2a=2b{+}1)} \qquad \dfrac{\overline{c=2a}^{\,(2)} \qquad \overline{c=2b{+}1}^{\,(3)}}{2a=2b{+}1}}{\bot}}{\bot}{\scriptstyle(1)}}{\bot}{\scriptstyle(2)}}{\neg(\exists y\, c{=}2y \wedge \exists z\, c{=}2z{+}1)}{\scriptstyle(1)}}{\forall x \neg(\exists y\, x{=}2y \wedge \exists z\, x{=}2z{+}1)}$$

□

That use of Lemma 35 (which was proved by induction) does *not* make Lemma 35 into a cut sentence, for the reasons explained in §4.

**Lemma 37.** *The square of an odd is odd. In symbols:*

$$\frac{O(t)}{O(t.t)}$$

In proving this result, we shall make use of Associativity of Addition (Lemma 21), Commutativity of Multiplication (Lemma 23), Associativity of Multiplication (Lemma 25), Distributivity (Lemma 24) and the fact that $t.1{=}1$ (Lemma 15). We shall avail ourselves of the usual abbreviatory conventions in algebra, whereby, for example, the term

$$((ss0.m) + s0).((ss0.m) + s0)$$

is rendered more readably as

$$(2m + 1)(2m + 1).$$

That is, we often suppress multiplication signs and simply juxtapose the two multiplicanda. Explicit dots (multiplication signs), however, have greater scope than implicit ones. Thus '$t.2m$', for example, is to be read as '$t.(2.m)$'. We also take successor to bind more tightly than multiplication, which in turn binds more tightly than addition. This enables us to use parentheses less frequently. Order of arguments in operations matters, however, as does 'order of bracketing'.

*Proof.* The overall form of the formal proof is

$$\dfrac{\begin{array}{c}O(t),\\ \text{i.e., } \exists x\, t=2x+1\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\Pi\\ (2m+1)(2m+1)=2(m[(2m+1)+1])+1\end{array} \qquad \dfrac{}{t=2m+1}{}^{(1)}}{t.t=2(m[(2m+1)+1])+1}}{\exists x\, t.t=2x+1}}{\begin{array}{c}\exists x\, t.t=2x+1\\ \text{i.e., } O(t.t)\end{array}}{}^{(1)}$$

and the embedded subproof $\Pi$, exploiting the abbreviation $\mu =_{df} (2m+1)(2m+1)$, is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\text{Lemma 24:}\\ (2m+1)(2m+1)=(2m+1)2m+(2m+1)1\end{array} \quad \begin{array}{c}\text{Lemma 15:}\\ (2m+1)1=2m+1\end{array}}{\mu=(2m+1)2m+(2m+1)} \quad \begin{array}{c}\text{Lemma 21:}\\ (2m+1)2m+(2m+1)=((2m+1)2m+2m)+1\end{array}}{\mu=((2m+1)2m+2m)+1} \quad \begin{array}{c}\text{Lemma 23:}\\ (2m+1)2m=2m(2m+1)\end{array}}{\mu=(2m(2m+1)+2m)+1} \quad \begin{array}{c}\text{Lemma 15:}\\ 2m.1=2m\end{array}}{\mu=(2m(2m+1)+2m.1)+1} \quad \begin{array}{c}\text{Lemma 24:}\\ 2m(2m+1)+2m.1=2m((2m+1)+1)\end{array}}{\mu=2m((2m+1)+1)+1} \quad \begin{array}{c}\text{Lemma 25:}\\ 2m((2m+1)+1)=2(m[(2m+1)+1])\end{array}}{\mu=2(m[(2m+1)+1])+1}$$

□

**Lemma 38.** *The following is a valid argument-form:*

$$\begin{array}{l} Fht \\ \forall x(Fx \vee Gx) \\ \forall x\neg(Fx \wedge Gx) \\ \underline{\forall x(Gx \rightarrow Ghx)} \\ Ft \end{array}$$

*Proof.*

$$\dfrac{\dfrac{\forall x\neg(Fx\wedge Gx) \qquad \dfrac{\dfrac{}{\neg(Fht\wedge Ght)}{\scriptstyle(2)} \qquad \dfrac{Fht \qquad \dfrac{\forall x(Gx\rightarrow Ghx) \qquad \dfrac{\dfrac{}{Gt\rightarrow Ght}{\scriptstyle(3)} \qquad \dfrac{\forall x(Fx\vee Gx) \qquad \dfrac{\dfrac{}{Ft\vee Gt}{\scriptstyle(5)} \qquad \dfrac{\dfrac{}{\neg Ft}{\scriptstyle(1)} \quad \dfrac{}{Ft}{\scriptstyle(5)}}{\bot} \qquad \dfrac{}{Gt}{\scriptstyle(6)}}{Gt}{\scriptstyle(6)}}{Gt}{\scriptstyle(5)} \qquad \dfrac{}{Ght}{\scriptstyle(4)}}{Ght}{\scriptstyle(4)}}{Ght}{\scriptstyle(3)}}{Fht\wedge Ght}}{\bot}}{\bot}{\scriptstyle(2)}}{Ft}{\scriptstyle(1)}$$

□

**Lemma 39.** *Only evens have even squares. In symbols:*

$$\frac{E(t.t)}{E(t)}$$

*Proof.* The proof is a substitution instance of the foregoing proof of Lemma 38. We have

$$\begin{array}{rl} & Et.t \\ \text{Lemma 34:} & \forall x(Ex \vee Ox) \\ \text{Lemma 36:} & \forall x\neg(Ex \wedge Ox) \\ \text{Lemma 37:} & \underline{\forall x(Ox \rightarrow Ohx)} \\ & Et \end{array}$$

□

### 6.1 Complete Induction

In §4 we discussed what we called *incremental* Mathematical Induction. (It is sometimes also called *simple* or *weak* induction.) There is a closely related principle to which we now turn, called *complete* or *strong* Mathematical Induction. The axiom schema in question is

$$\forall x(\forall y(y<x \rightarrow Py) \rightarrow Px) \rightarrow \forall z Pz$$

### 6.1.1 Deriving Complete Induction

**Lemma 40.** *Any proof* $\Pi$ *using Complete Induction can be turned into a proof* $\Pi^\dagger$ *using only incremental Mathematical Induction.*

*Proof.* We proceed by induction in the metalanguage, on the complexity of proofs $\Pi$. (So *this* induction, in the structural theory of proofs, has nothing to do with the two kinds of induction in formal arithmetic—complete and incremental—involved in the statement being proved.)

For the basis step: clearly if one is given a proof $\Pi$ involving no applications of Complete Induction, then it can be turned (by doing nothing to it) into a proof using only incremental Mathematical Induction. That is, for $\Pi^\dagger$ take $\Pi$.

Inductive hypothesis: Suppose the result holds for all proofs simpler than the proof $\Pi$ under consideration.

Inductive Step: Show by cases that the result holds for $\Pi$. If the terminal step of $\Pi$ is an application of any rule other than $\to$E with an instance of Complete Induction as major premise, then the result obviously holds for $\Pi$. (For $\Pi^\dagger$ take $\Pi$.) The only real work that needs to be done is when $\Pi$ *does* end with an application of $\to$E that has as its major premise an instance of Complete Induction. In such a case, $\Pi$ takes the form

$$\dfrac{\forall x(\forall y(y<x\to Py)\to Px)\to\forall zPz \qquad \begin{array}{c}\Sigma\\ \forall x(\forall y(y<x\to Py)\to Px)\end{array}}{\forall zPz}{\scriptstyle (1)}$$

Here we can take the minor subproof $\Sigma$ and embed it as follows, to produce a proof $\Theta$ of $P0$.

$$\Theta:\quad \dfrac{\dfrac{\begin{array}{c}\Sigma\\ \forall x(\forall y(y<x\to Py)\to Px)\end{array}}{\forall y(y<0\to Py)\to P0} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}{\scriptstyle (1)}\dfrac{}{b<0}\\ \text{i.e., }\exists x\,0=b+sx\end{array} \qquad \dfrac{\dfrac{{\scriptstyle (2)}\dfrac{}{0=b+sa} \qquad \dfrac{\dfrac{\forall x\forall y\, x+sy=s(x+y)}{\forall y\, b+sy=s(b+y)}}{b+sa=s(b+a)}}{0=s(b+a)}}{\bot}}{\bot}{\scriptstyle (2)}}{b<0\to Pb}{\scriptstyle (1)}}{\forall y(y<0\to Py)}}{P0}$$

The conclusion of $\Sigma$ might stand as the conclusion of an application of $\forall$I—in which case $\Theta$ would not be in normal form. But as we know, there is an effective method of transforming $\Theta$ into a core proof, which is in normal form. So, if necessary, we apply that method, in order to ensure that the proof $\Pi_0$ below, of the basis step for our proposed use of incremental induction, is a core proof.

Using $\Theta$, we can construct the proof $\Pi_0$ for the basis step for incremental induction, using the induced predicate $\forall y(y \leq x \rightarrow Py)$:

$$\begin{array}{c}\Pi_0\\ \forall y(y \leq 0 \rightarrow Py)\end{array} \quad : \quad \dfrac{\dfrac{\dfrac{\overset{(2)}{\overline{b \leq 0}} \qquad \dfrac{\begin{array}{c}\Theta\\ P0\end{array} \quad \overset{(1)}{\overline{b=0}}}{Pb} \qquad \dfrac{\overset{(1)}{\overline{b<0}}}{\bot}}{Pb}\,(1)}{b \leq 0 \rightarrow Pb}\,(2)}{\forall y(y \leq 0 \rightarrow Py)}$$

Here is the proof $\Pi_1$ of the inductive step, from the inductive hypothesis $\forall y(y \leq a \rightarrow Py)$ to the incremental conclusion $\forall y(y \leq sa \rightarrow Py)$.

$$\begin{array}{c}\forall y(y \leq a \rightarrow Py)\\ \Pi_1\\ \forall y(y \leq sa \rightarrow Py)\end{array} \quad :$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{(2)}{\overline{m \leq sa}}}{m \leq a \vee m = sa}\,(\text{C1}) \qquad \dfrac{\dfrac{\forall y(y \leq a \rightarrow Py)}{m \leq a \rightarrow Pm} \quad \overset{(1)}{\overline{m \leq a}}}{Pm} \qquad \dfrac{\overset{(1)}{\overline{m = sa}} \quad \dfrac{\dfrac{\forall x(\forall y(y<x \rightarrow Py) \rightarrow Px)}{\forall y(y<sa \rightarrow Py) \rightarrow Psa} \atop \text{i.e., } \forall y(y \leq a \rightarrow Py) \rightarrow Psa \quad \forall y(y \leq a \rightarrow Py)}{Psa}}{Pm}}{Pm}\,(1)}{m \leq sa \rightarrow Pm}\,(2)}{\forall y(y \leq sa \rightarrow Py)}$$

Equipped with the proofs $\Pi_0$ and $\Pi_1$, we can now complete the sought proof $\Pi^{\dagger}$, by incremental Mathematical Induction, of the conclusion $\forall z Pz$. Its penultimate step is an application of the rule RMI. Note that the parameter $c$ in the major subproof for that application is playing the role of $t_1$ in the statement of the rule.

$$\Pi^\dagger : \quad \dfrac{\dfrac{\begin{matrix}\Pi_0\\ \forall y(y\le 0\to Py)\end{matrix} \qquad \begin{matrix}\overline{\forall y(y\le a\to Py)}^{\,(1)}\\ \Pi_1\\ \forall y(y\le sa\to Py)\end{matrix} \qquad \dfrac{\dfrac{\overline{\forall y(y\le c\to Py)}^{\,(1)}}{c\le c\to Pc} \quad \dfrac{\overline{c=c}}{c\le c}}{Pc}}{Pc}\,{\scriptstyle(1)}}{\forall z Pz}$$

□

So we have seen that any instance of the Axiom schema of Complete Induction can be derived within Peano arithmetic based on the axiom schema of incremental Mathematical Induction. Note that we are *not* claiming that applications in the derivation of the latter schema will involve the same induced predicate P as does the instance of Complete Induction to be derived. For, as we have seen, incremental Mathematical Induction yields Complete Induction on the induced predicate $Px$ courtesy of the related, but still distinct, induced predicate $\forall y(y\le x\to Py)$.

### 6.1.2 The Least Number Principle

Closely related (indeed: classically equivalent) to complete Mathematical Induction is the following *Least Number Principle*:

$$\forall x(\neg Px \to \exists y(\neg Py \wedge \forall z(z<y\to Pz)))$$

This tells us that if the universal claim $\forall xPx$ *has* a counterexample at all, then there is a *least* number that serves as such a counterexample. Note that the uniqueness of such a number is not explicitly claimed (even though it would be unique, should it exist).

**Lemma 41.** *Complete Induction classically implies the Least Number Principle; in symbols:*

$$\frac{\forall x(\forall y(y<x\to Py)\to Px) \to \forall zPz}{\forall x(\neg Px \to \exists y(\neg Py \wedge \forall z(z<y\to Pz)))}$$

*Proof.* Steps of Classical Reductio are marked (1) and (3):

$$
\dfrac{\dfrac{\dfrac{\dfrac{{}^{(4)}\dfrac{}{\neg Pa} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{(3)}\dfrac{}{\neg\exists y(\neg Py\wedge Qy)} \quad \dfrac{\dfrac{{}^{(1)}\dfrac{}{\neg Pa} \quad \dfrac{}{Qa}{}^{(2)}}{\neg Pa\wedge Qa}}{\exists y(\neg Py\wedge Qy)}}{\bot}}{Pa}{}^{(1)}}{Qa\to Pa}{}^{(2)}}{\forall x(Qx\to Px)} \qquad \forall x(Qx\to Px)\to\forall zPz}{\forall zPz}}{Pa}}{\bot}}{\exists y(\neg Py\wedge Qy)}{}^{(3)}}{\neg Pa\to\exists y(\neg Py\wedge Qy)}}{\forall z(\neg Pz\to\exists y(\neg Py\wedge Qy))}
$$

Now for $Qx$ take $\forall y(y<x\to Py)$, in order to obtain the desired proof.

If the predicate $Px$ is effectively decidable, then the step marked (1) is constructively acceptable. But the step marked (3) would then be an application of Markov's Rule. For, given any natural number $x$, there are only *that many* (*finitely* many) numbers $y$ less than $x$ that need to be checked for $P$-hood, in order to decide whether the complex predicate applies to $x$. So, if $P(x)$ is effectively decidable, then so too is the slightly more complex predicate $Qx$, i.e. $\forall y(y<x\to Py)$; whence also $\neg Px\wedge\forall y(y<x\to Py)$. That would make the existence of some $x$ such that $\neg Px\wedge\forall y(y<x\to Py)$ a $\Sigma^0_1$ matter. □

**Lemma 42.** *Provided that $Px$ is effectively decidable, the Least Number Principle constructively implies Complete Induction; in symbols:*

$$
\dfrac{\forall x(\neg Px\to\exists y(\neg Py\wedge\forall z(z<y\to Pz)))}{\forall x(\forall y(y<x\to Py)\to Px)\to\forall zPz}
$$

*Proof.* Once again let the formula $\forall y(y<x\to Py)$ be abbreviated to $Qx$. Then the problem becomes that of proving the argument

$$
\dfrac{\forall x(\neg Px\to\exists y(\neg Py\wedge Qy))}{\forall x(Qx\to Px)\to\forall zPz}
$$

The following proof uses Classical Reductio just once, at the step marked (2). It is constructively acceptable if $Px$ is effectively decidable.

$$
\dfrac{
  \dfrac{\dfrac{}{\neg Pa}\,(2) \quad \dfrac{\forall x(\neg Px \to \exists y(\neg Py \wedge Qy))}{\neg Pa \to \exists y(\neg Py \wedge Qy)}}{\exists y(\neg Py \wedge Qy)}
  \qquad
  \dfrac{
    \dfrac{\dfrac{\dfrac{}{\neg Pb \wedge Qb}\,(1)}{Qb} \quad \dfrac{\dfrac{}{\forall x(Qx \to Px)}\,(3)}{Qb \to Pb}}{Pb}
    \quad
    \dfrac{\dfrac{}{\neg Pb \wedge Qb}\,(1)}{\neg Pb}
  }{\bot}
}{
  \dfrac{\dfrac{\bot}{Pa}\,(2)}{\dfrac{\forall z Pz}{\forall x(Qx \to Px) \to \forall z Pz}\,(3)}
}\,(1)
$$

□

# 7 The main argument, given informally

Suppose $\frac{p}{q} = \sqrt{2}$ — equivalently, $p^2 = 2.q^2$ for some non-zero $q$. We shall derive a contradiction. Consider the property $Px$ that $p$ is being supposed to enjoy:

$$\exists y(y \neq 0 \wedge x.x = 2.(y.y))$$

We shall show that the assumption $Pa$, for arbitrary $a$, leads to a contradiction.

So suppose $Pa$. By the Least Number Principle there is a least number $y$ with property $P$:

$$\exists y(Py \wedge \forall z(z < y \to \neg Pz))$$

Let $\lambda$ be such a number; that is, suppose we have

$$P\lambda \wedge \forall z(z < \lambda \to \neg Pz)$$

Assuming nothing else about $\lambda$, we shall derive a contradiction.

$P\lambda$ means that

$$\exists y(y \neq 0 \wedge \lambda.\lambda = 2.(y.y))$$

Let $\mu$ be such a number:

$$\mu \neq 0 \wedge \lambda.\lambda = 2.(\mu.\mu)$$

Since $\mu$ is non-zero, so too is $2.(\mu.\mu)$; whence also $\lambda.\lambda$ is non-zero. It follows that $\lambda$ itself is non-zero:

$$\lambda \neq 0$$

Also, $\lambda.\lambda$ is even. By Lemma 39 it follows that $\lambda$ itself is even:

$$\exists y\, \lambda = 2.y$$

Suppose that $\rho$ is such a number:

$$\lambda = 2.\rho$$

Since $\lambda$ is non-zero, so too is $\rho$:

$$\rho \neq 0$$

By Lemma 31,

$$\rho < \lambda$$

Recall that we have

$$\lambda.\lambda = 2.(\mu.\mu)$$

Substituting $2.\rho$ for $\lambda$, we obtain

$$(2.\rho).(2.\rho) = 2.(\mu.\mu)$$

By Lemma 29 we have

$$\mu.\mu = 2.(\rho.\rho),$$

whence, by Lemma 39, $\mu$ itself is even:

$$\exists y\, \mu = 2.y$$

Let $\sigma$ be such a number:

$$\mu = 2.\sigma$$

So we have

$$(2.\sigma).(2.\sigma) = 2.(\rho.\rho),$$

whence by Lemma 29 again we have

$$\rho.\rho = 2.(\sigma.\sigma)$$

Since $\mu$ is non-zero, so too is $\sigma$:

$$\sigma \neq 0$$

So we have

$$\sigma \neq 0 \;\wedge\; \rho.\rho = 2.(\sigma.\sigma),$$

whence

$$\exists y(y \neq 0 \land \rho.\rho = 2.(y.y))$$

Recall that we are supposing

$$P\lambda \land \forall z(z < \lambda \to \neg Pz)$$

So we have

$$\forall z(z < \lambda \to \neg Pz)$$

Instantiating with respect to $\rho$, we have

$$\rho < \lambda \to \neg P\rho$$

It follows that

$$\neg P\rho,$$

i.e.

$$\neg\exists y(y \neq 0 \land \rho.\rho = 2.(y.y)).$$

Contradiction.

## 8 The main argument, given formally

**Theorem 1.** $\forall x \neg\exists y(y \neq 0 \land x.x = 2.(y.y))$

*Proof.* We seek to show

$$\forall x \neg Px$$

where

$$Px \;\equiv_{df}\; \exists y(y \neq 0 \land x.x = 2.(y.y))$$

Overall, the formal proof will be of the following form:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\text{Least Number Principle :}\\ \forall x(Px \to \exists y(Py \land \forall z(z<y \to \neg Pz)))\end{matrix} \quad \dfrac{{}^{(2)}\overline{Pa \to \exists y(Py \land \forall z(z<y \to \neg Pz))} \quad \overline{Pa}^{(1)} \quad \dfrac{{}^{(3)}\overline{\exists y(Py \land \forall z(z<y \to \neg Pz))} \quad \begin{matrix}\overline{P\lambda \land \forall z(z<\lambda \to \neg Pz)}^{(4)}\\ \Sigma\\ \bot\end{matrix}}{\bot}(1)}{\bot}(3)}{\bot}(2)}{\neg Pa}(1)}{\forall x \neg Px}$$

Clearly for the construction of the outstanding embedded proof $\Sigma$ it will suffice to find a proof of the form

$$\begin{array}{c}\underbrace{P\lambda\ ,\ \forall z(z<\lambda\to\neg Pz)}\\ \Pi\\ \bot\end{array}$$

The following can serve as $\Pi$, provided we can supply its embedded subproof $\Omega$. Note that within this display of $\Pi$ we can see only the parameters $\lambda$, $\mu$ and $\rho$ invoked. The role of the parameter $\sigma$ is confined to the embedded subproof $\Omega$.

$$\dfrac{\begin{array}{c}P\lambda\\ \text{i.e., }\exists y(y\neq0\wedge\lambda.\lambda=2.(y.y))\end{array}\quad \dfrac{\dfrac{\dfrac{\overline{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}^{(1)}}{\lambda.\lambda=2.(\mu.\mu)}}{\exists y\,\lambda=2.y}\,\text{L39}\quad \dfrac{\forall z(z<\lambda\to\neg Pz)\quad \dfrac{\overline{\rho<\lambda\to\neg P\rho}^{(3)}\quad \dfrac{\overline{\lambda=2.\rho}^{(2)}\quad \dfrac{\overline{\lambda=2.\rho}^{(2)}\quad \dfrac{\dfrac{\overline{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}^{(1)}}{\lambda.\lambda\neq0}\,\text{L30}}{\lambda\neq0}\,\text{L4}}{\rho\neq0}\,\text{L5}}{\rho<\lambda}\,\text{L31}\quad \dfrac{\overline{\neg P\rho}^{(4)}\quad \begin{array}{c}\underbrace{\overline{\lambda=2.\rho}^{(2)}\ ,\ \overline{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}^{(1)}}\\ \Omega\\ P\rho:\ \exists y(y\neq0\wedge\rho.\rho=2.(y.y))\end{array}}{\bot}}{\bot}\,(4)}{\bot}\,(3)}{\bot}\,(2)}{\bot}\,(1)$$

Finally we supply the embedded subproof $\Omega$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}{\lambda.\lambda=2.(\mu.\mu)}\quad \lambda=2.\rho}{(2.\rho).(2.\rho)=2.(\mu.\mu)}}{\mu.\mu=2.(\rho.\rho)}\,\text{L29}}{\exists y\,\mu=2.y}\,\text{L39}\quad \dfrac{\dfrac{\dfrac{\dfrac{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}{\mu\neq0}\quad \overline{\mu=2.\sigma}^{(1)}}{\sigma\neq0}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mu\neq0\wedge\lambda.\lambda=2.(\mu.\mu)}{\lambda.\lambda=2.(\mu.\mu)}\quad \lambda=2.\rho}{(2.\rho).(2.\rho)=2.(\mu.\mu)}}{\mu.\mu=2.(\rho.\rho)}\,\text{L29}\quad \overline{\mu=2.\sigma}^{(1)}}{(2.\sigma).(2.\sigma)=2.(\rho.\rho)}}{\rho.\rho=2.(\sigma.\sigma)}\,\text{L29}}{\sigma\neq0\wedge\rho.\rho=2.(\sigma.\sigma)}}{\exists y(y\neq0\wedge\rho.\rho=2.(y.y))}}{\exists y(y\neq0\wedge\rho.\rho=2.(y.y))}\,(1)$$

□

## References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