GP's LP

by

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Abstract

This study takes a careful inferentialist look at Graham Priest's Logic of Paradox (LP). I conclude that it is sorely in need of a proof-system that could furnish formal proofs that would regiment faithfully the 'naïve logical' reasoning that could be undertaken by a rational thinker *within* LP (if indeed such reasoning could ever take place).

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1 Introduction

Graham Priest first put forward his Logic of Paradox, now known by the acronym LP, in Priest [1979]. Subsequently LP was one of many paraconsistent logics that he considered in his survey article Priest [2002]. (Unsourced page references will be to the latter.) By definition, a paraconsistent logic is one that does not admit (hence certainly does not allow one to derive, let alone have as a primitive rule) the inference known as *Explosion*:

$$A, \neg A : B.$$

Priest's LP was a creature of formal semantics. At its inception it had no proof system. All it had was a logical consequence relation $\models_{\text{LP}}$ defined in the usual way—the usual way, that is, for a *many*-valued logic. In such logics, there are more than two truth-values, and one is concerned with preserving 'designatedness' of truth value. Many-valued logics therefore evince a vestige of classical, two-valued thinking in the way they partition their many values into two classes—the designated and the undesignated. LP employs certain 3-valued truth tables (or perhaps one should say: *truth-value* tables) for the connectives $\neg$, $\wedge$, $\vee$ and $\rightarrow$.

At p. 224 of Priest [1979], we read that 'the logic of naïve proofs [which is what Priest is after—NT] is not classical'. Nevertheless it turns out that LP is 'classical enough' for negation and conjunction to suffice for the definition of disjunction and the conditional. We read at p. 227 *loc. cit.* that Priest is content to define $A \vee B$ de Morgan-wise as $\neg(\neg A \wedge \neg B)$; and thereafter to define $A \rightarrow B$ as the usual disjunction $\neg A \vee B$.

The truth-values that Priest [1979] proposes for LP are *True*, *False*, and *both-True-and-False*. *True*, i.e. *True only*, is abbreviated as t ; *False*, i.e. *False only*, is abbreviated as f ; and *both-True-and-False* is equated with *Paradoxical*, and abbreviated accordingly as p. The row-by-row truth-value tables for the connectives $\neg$ and $\wedge$, as well as the (primitive or defined) $\vee$ and $\rightarrow$, are as follows. We state them so that the rows can be read 'from left to right', i.e. from values of immediate constituents to values of the respective compounds.

A	$\neg A$
t	f
p	p
f	t

A	B	$A \wedge B$	$A \vee B$	$A \to B$
t	t	t	t	t
t	p	p	t	p
t	f	f	t	f
p	t	p	t	t
p	p	p	p	p
p	f	f	p	p
f	t	f	t	t
f	p	f	p	t
f	f	f	f	t

In this form, significant patterns are difficult to discern. This can be remedied by resorting to stating a 3-by-3 matrix for each binary connective and using some font changes to highlight the patterns.

$\wedge$	**t**	*p*	f
t	**t**	*p*	f
p	*p*	*p*	f
f	f	f	f

$\vee$	**t**	*p*	f
t	**t**	**t**	**t**
p	**t**	*p*	*p*
f	**t**	*p*	f

$\to$	**t**	*p*	f
t	**t**	*p*	f
p	**t**	*p*	*p*
f	**t**	**t**	**t**

As Priest notes, these are the truth-value tables of Kleene's (strong) 3-valued logic in Kleene [1952], pp. 332 *ff.*[1] The main difference is that in LP the designated values are t and p, whereas in Kleene's logic only t is designated.

Observation 1. *Priest [1979], at p. 227, describes the truth-value table for* $\to$ *as resulting from* defining $A \to B$ *as* $\neg A \vee B$*; but there should be no objection in principle to anyone who opts to treat* $\to$ *as a* primitive *connective in LP, furnished with that truth-value table. It would then be very natural to pose the question: What rules of inference govern '*$\to$*'?*

By the end of this study it will emerge, perhaps surprisingly, that there appears to be no straightforward answer to this question.

[1] Kleene had u as his third value, standing for 'undefined' (or, at p. 335, 'unknown'). He was concerned with partial recursiveness of compounds, not at all with paradoxes or 'truth-value gluts'.

2 The formal semantics of LP

Definition 1. *An* assignment τ *is a function assigning values in* $\{t, p, f\}$ *to certain atoms. (Which atoms these are will depend on the context.) The value* $\tau(\varphi)$ *is defined in the familiar way, by appeal to the truth-value tables of LP. When* $\tau(\varphi)$ *is a designated value, we shall say that* τ *designates* φ*, and abbreviate this as* $\tau \Vdash \varphi$*. When* τ *designates every member of* Δ*, we shall say that* τ *designates* Δ*, and write* $\tau \Vdash \Delta$*.*

Logical implication (the semantic relation) in LP can now be defined as follows.

Definition 2.

$$\Delta \models_{LP} \psi \quad \textit{if and only if} \quad \forall\tau(\tau \Vdash \Delta \Rightarrow \tau \Vdash \psi),$$

i.e., for every truth-value assignment τ *to the atoms involved, if* τ *designates every member of* Δ*, then* τ *designates* ψ*.*

We shall read $\Delta \models_{\mathrm{LP}} \psi$ as 'the inference $\Delta : \varphi$ is LP-valid'; or 'Δ LP-implies φ'; or 'φ is an LP-consequence of Δ'.

Remember that truth-value assignments are single-valued.

Definition 3. φ *and* ψ *are (semantically)* equivalent *just in case they have the same truth(-value) table.*

For example, $\neg A \vee B$ is equivalent to $A \rightarrow B$:

A	B	$\neg A$	$\neg A \vee B$	$A \rightarrow B$
t	t	f	$\boldsymbol{t}$	$\boldsymbol{t}$
t	p	f	$\boldsymbol{p}$	$\boldsymbol{p}$
t	f	f	$\boldsymbol{f}$	$\boldsymbol{f}$
p	t	p	$\boldsymbol{t}$	$\boldsymbol{t}$
p	p	p	$\boldsymbol{p}$	$\boldsymbol{p}$
p	f	p	$\boldsymbol{p}$	$\boldsymbol{p}$
f	t	t	$\boldsymbol{t}$	$\boldsymbol{t}$
f	p	t	$\boldsymbol{t}$	$\boldsymbol{t}$
f	f	t	$\boldsymbol{t}$	$\boldsymbol{t}$

Observation 2. *That* $\varphi \models_{LP} \psi$ *and* $\psi \models_{LP} \varphi$ *is not in general sufficient to ensure that* φ *and* ψ *are semantically equivalent. This is because mutual LP-implication is compatible with one of the two sentences being* t*, and the other* p*, in some same row.*

Definition 4. *φ is a* daudology *just in case φ takes a designated value on every LP-assignment; i.e., just in case $\emptyset \models_{LP} \varphi$.*

N.B. This is a neologism, not a typo. 'Daudology' takes its initial letter from 'designated'. It is the LP-analog of a tautology in the two-valued case.

2.1 Some low-level explorations of LP-consequence

All the results of this subsection are obtained by using only Core Logic in the metalanguage.

Observation 3. *A good example of a daudology is $A \vee \neg A$:*

A	$\neg A$	$A \vee \neg A$
t	f	$\boldsymbol{t}$
p	p	$\boldsymbol{p}$
f	t	$\boldsymbol{t}$

Note that we do not need a solid column of t-entries; all we need is the absence of any f-entries. This is because both t and p are designated.

For $\Delta \cup \{\varphi\}$ we shall write Δ, φ and typically understand the latter to mean also that $\varphi \notin \Delta$.

Lemma 1. *If $\tau \nVdash \neg\varphi$ then $\tau \Vdash \varphi$.*

Proof. Suppose $\tau \nVdash \neg\varphi$. Then by definition of $\Vdash$ we have $\tau(\neg\varphi) = f$. Hence by the truth-value table for $\neg$ we have $\tau(\varphi) = t$. So by definition of $\Vdash$ again we have $\tau \Vdash \varphi$. □

Note that the proof of Lemma 1 implicitly relies on the following assumption, which could be called the Law of Excluded Fourth: *an assignment τ assigns, to every sentence whose atoms it deals with, a determinate one of the three values t, p or f.* The Law of Excluded Fourth underlies Priest's whole three-valued semantical approach to LP-consequence. The core meta-logician is entitled, therefore, to appeal to that law when investigating the properties of LP that are revealed in this subsection.

Lemma 2. [LP-consequence satisfies Dilution]
If $\Delta \models_{LP} \psi$, then $\Delta, \varphi \models_{LP} \psi$.

Proof. Suppose $\Delta \models_{\mathrm{LP}} \psi$. Suppose μ is an arbitrary assignment dealing with the atoms involved in Δ, φ, ψ. Suppose $\mu \Vdash \Delta, \varphi$. Then $\mu \Vdash \Delta$. Hence by main supposition we have $\mu \Vdash \psi$. But μ was arbitrary. Thus $\Delta, \varphi \models_{\mathrm{LP}} \psi$. □

When one is working with sequents of the form $\Delta : \Gamma$, where Δ and Γ are finite sets of sentences, the natural reading of '$\Delta \models \Gamma$' (for any many-valued logic) is that no assignment designates every sentence Δ but none in Γ. We are adopting that reading here, but treating only of sequents whose succedents are at most a singleton. (We write '$\Delta \vdash \varphi$', however, rather than '$\Delta \vdash \{\varphi\}$'.) We shall follow the convention of rendering '$\Delta \models \emptyset$' as '$\Delta \models \bot$'. This means, on the reading adopted, that no assignment designates every member of Δ.

Lemma 3. [Classical Reductio is LP-valid]
If $\Delta, \neg\varphi \models_{LP} \bot$, then $\Delta \models_{LP} \varphi$.

Proof. Suppose $\Delta, \neg\varphi \models_{\text{LP}} \bot$. It follows that for no τ do we have both $\tau \Vdash \Delta$ and $\tau \Vdash \neg\varphi$. Suppose μ is an arbitrary assignment dealing with the atoms involved in Δ, φ. Suppose $\mu \Vdash \Delta$. Then $\mu \nVdash \neg\varphi$. Hence by Lemma 1 $\mu \Vdash \varphi$. But μ was arbitrary. Thus $\Delta \models_{\text{LP}} \varphi$. □

Observation 4. *We have used constructive reasoning in the metalanguage to show that* Classical *Reductio is LP-valid. This should strike the unsuspecting reader as surprising. It provokes the question 'How does LP pull this off?' (i.e., reveal itself, 'constructively' at the meta-level, to validate a strictly classical, non-constructive, rule of reasoning). The answer must be that an awful lot is packed into the Law of Excluded Fourth. That principle is so powerful that merely constructive reasoning can draw out from it the LP-validity of Classical Reductio.*

Lemma 4. [Double-Negation Elimination is LP-valid]
$\neg\neg\varphi \models_{LP} \varphi$.

Proof. Suppose μ is an arbitrary assignment dealing with the atoms in φ. Suppose $\mu \Vdash \neg\neg\varphi$. Then either $\mu(\neg\neg\varphi) = t$ or $\mu(\neg\neg\varphi) = p$.

Suppose first that $\mu(\neg\neg\varphi) = t$. Then $\mu(\neg\varphi) = f$; whence $\mu(\varphi) = t$.

Suppose secondly that $\mu(\neg\neg\varphi) = p$. Then $\mu(\neg\varphi) = p$; whence $\mu(\varphi) = p$. Either way, φ is assigned a designated value under μ. That is, $\mu \Vdash \varphi$.

But μ was arbitrary. Hence $\neg\neg\varphi \models_{\text{LP}} \varphi$. □

Lemma 5. [Dilemma is LP-valid]
If $\Delta, \varphi \models_{LP} \psi$ and $\Gamma, \neg\varphi \models_{LP} \psi$, then $\Delta, \Gamma \models_{LP} \psi$.

Proof. Suppose that

$$\Delta, \varphi \models_{\text{LP}} \psi \quad \text{and} \quad \Gamma, \neg\varphi \models_{\text{LP}} \psi.$$

We shall show that

$$\Delta, \Gamma \models_{\mathrm{LP}} \psi.$$

Let μ be an arbitrary assignment dealing with the atoms involved in $\Delta, \Gamma, \varphi, \psi$. Suppose that

$$\mu \Vdash \Delta, \Gamma.$$

It follows immediately that

$$\mu \Vdash \Delta \quad \text{and} \quad \mu \Vdash \Gamma.$$

We shall now show that $\mu \Vdash \psi$.

We reason by using proof by cases (Disjunction Elimination) in the metalanguage. We know that μ assigns one of the three values t, p or f to any sentence. So, in particular,

$$(\mu(\varphi) = t \quad \text{or} \quad \mu(\varphi) = p) \quad \text{or} \quad \mu(\varphi) = f.$$

Suppose on the one hand that either $\mu(\varphi) = t$ or $\mu(\varphi) = p$. Then $\mu \Vdash \varphi$. It follows that

$$\mu \Vdash \Delta, \varphi.$$

By main supposition we have

$$\Delta, \varphi \models_{\mathrm{LP}} \psi.$$

So

$$\mu \Vdash \psi.$$

Suppose on the other hand that $\mu(\varphi) = f$. Thus $\mu(\neg\varphi) = t$, whence

$$\mu \Vdash \neg\varphi.$$

It follows that

$$\mu \Vdash \Gamma, \neg\varphi.$$

By main supposition we have

$$\Gamma, \neg\varphi \models_{\mathrm{LP}} \psi.$$

So

$$\mu \Vdash \psi.$$

Either way, we have $\mu \Vdash \psi$.

But μ was an arbitrary assignment such that $\mu \Vdash \Delta, \Gamma$. It follows that

$$\Delta, \Gamma \models_{\mathrm{LP}} \psi.$$

□

Lemma 6. *Suppose* $\Delta, \varphi \models_{LP} \perp$. *Then* $\Delta \models_{LP} \varphi \to \psi$.

Proof. The main supposition amounts to the following:

$$\forall\tau\ \neg(\tau \Vdash \Delta \wedge \tau \Vdash \varphi).$$

Let μ be an arbitrary assignment to the atoms involved in Δ, φ, ψ. Suppose that $\mu \Vdash \Delta$. We shall show that $\mu \Vdash \varphi \to \psi$; and this will establish the lemma. This we do by using Disjunctive Syllogism in the metalanguage. We know that μ assigns one of the three values t, p or f to any sentence. So, in particular,

$$(\mu(\varphi \to \psi) = t \quad \text{or} \quad \mu(\varphi \to \psi) = p) \quad \text{or} \quad \mu(\varphi \to \psi) = f.$$

We are seeking to show that μ designates $\varphi \to \psi$, i.e.

$$\mu(\varphi \to \psi) = t \quad \text{or} \quad \mu(\varphi \to \psi) = p.$$

So all we have to do is rule out the possibility that $\mu(\varphi \to \psi) = f$. Assume for (constructive) *reductio*, then, that $\mu(\varphi \to \psi) = f$. By the 3-valued table for $\to$, we have

$$\mu(\varphi) = t \quad \text{and} \quad \mu(\psi) = f.$$

Thus $\mu \Vdash \Delta \wedge \mu \Vdash \varphi$, contradicting the main supposition. □

Lemma 7. *Suppose* $\Delta, \varphi \models_{LP} \psi$. *Then* $\Delta \models_{LP} \varphi \to \psi$.

Proof. The main supposition amounts to the following:

$$\forall\tau(\tau \Vdash \Delta, \varphi \Rightarrow \tau \Vdash \psi).$$

Let μ be an arbitrary assignment to the atoms involved in Δ, φ, ψ. Suppose that $\mu \Vdash \Delta$. We shall show that $\mu \Vdash \varphi \to \psi$; and this will establish the lemma. This we do by using Disjunctive Syllogism in the metalanguage. We know that μ assigns one of the three values t, p or f to any sentence. So, in particular,

$$(\mu(\varphi \to \psi) = t \quad \text{or} \quad \mu(\varphi \to \psi) = p) \quad \text{or} \quad \mu(\varphi \to \psi) = f.$$

We are seeking to show that μ designates $\varphi \to \psi$, i.e.

$$\mu(\varphi \to \psi) = t \quad \text{or} \quad \mu(\varphi \to \psi) = p.$$

So all we have to do is rule out the possibility that $\mu(\varphi \to \psi) = f$. Assume for (constructive) *reductio*, then, that $\mu(\varphi \to \psi) = f$. By the 3-valued table for $\to$, we have

$$\mu(\varphi) = t \quad \text{and} \quad \mu(\psi) = f.$$

Thus $\mu \Vdash \Delta, \varphi$. Hence by main supposition we have

$$\mu \Vdash \psi \ ;$$

but this contradicts

$$\mu(\psi) = f.$$

□

We are using Modus Ponens here in the metalanguage, even though it is not LP-valid.[2] But the discrepancy, once registered, is innocuous. This is because it is our prerogative as investigators of LP at the meta-level to be able to reason in normal mathematical fashion about the system, since it is mathematically well defined. Both Modus Ponens and Disjunctive Syllogism are part of (even the constructive) mathematician's unrelinquishable tool-kit. Otherwise, how could any useful consequences be drawn out from the various definitions that the LP-theorist provides of such notions as $\models_{\mathrm{LP}}$? The 'naïve logician' would not proceed any differently. For we are talking here about the *inferential* part of the 'logic of naïve *proof*', which *obviously* includes all the usual logical inferences employed by mathematicians. This prescinds from the *axiomatic* basis of the logic of naïve proof in, say, arithmetic. Priest's comments about naïve reasoning in arithmetic, in §II.2 and §II.6 of Priest [1979], in no way commit the formalizer to *not* incorporate inferential rules such as Modus Ponens and Disjunctive Syllogism in whatever formal system of proof will result from one's investigations. See also Observation 6.

Corollary 1. *Suppose* $\Delta \models_{LP} \psi$*. Then* $\Delta \models_{LP} \varphi \to \psi$*.*

Proof. Suppose $\Delta \models_{\mathrm{LP}} \psi$. Then by Lemma 2 we have $\Delta, \varphi \models_{\mathrm{LP}} \psi$. By Lemma 7 it follows that $\Delta \models_{\mathrm{LP}} \varphi \to \psi$. □

Corollary 1 also has this direct proof: Suppose $\Delta \models_{\mathrm{LP}} \psi$. Let μ be an arbitrary assignment dealing with Δ, φ, ψ. Suppose $\mu \Vdash \Delta$. Then $\mu \Vdash \psi$. Hence $\mu \Vdash \varphi \to \psi$.

[2]This point is owed to Matthew Souba.

2.2 Considerations of natural deduction

In general, a natural deduction is a proof of some conclusion θ from some (finite) set Δ of undischarged assumptions (premises). For any system $\mathcal{S}$ of formal proof, we write

$$\mathcal{P}_{\mathcal{S}}(\Pi, \theta, \Delta)$$

for 'Π is an $\mathcal{S}$-proof of the conclusion θ from the set Δ of undischarged assumptions'. Note that 'from the set' means, here, 'using exactly the set'.

Suppose there *is* a natural deduction system to be had for LP, in which all LP-proofs are LP-sound. Expressed as a single metalinguistic sentence, we have:

$$\forall\Pi\,\forall\Delta\,\forall\theta(\mathcal{P}_{\text{LP}}(\Pi, \theta, \Delta) \Rightarrow \Delta \models_{\text{LP}} \theta)$$

Equivalently, we have the metalinguistic rule of inference

$$\frac{\mathcal{P}_{\text{LP}}(\Pi, \theta, \Delta)}{\Delta \models_{\text{LP}} \theta}$$

Lemma 6 forces the following reflection. This rule is admissible in LP:

$$\frac{\begin{array}{c}\underbrace{\Delta\ ,\ \overline{\varphi}^{(i)}}\\ \vdots\\ \bot\end{array}}{\varphi \rightarrow \psi}\,(i)$$

Lemma 7 and Corollary 1 together force a similar reflection. The conventional Rule of Conditional Proof (a.k.a. $\rightarrow$-Introduction), which permits vacuous discharge, is also admissible in LP:

$$\frac{\begin{array}{c}\underbrace{\Delta\ ,\ \overline{\varphi}^{(i)}}\\ \vdots\\ \psi\end{array}}{\varphi \rightarrow \psi}\,(i)$$

Note that these last two rules are the two parts of the rule of $\rightarrow$-Introduction in Core Logic. (See, for example, Tennant [2015b]; and Appendix I below.)

Lemma 8. *Explosion ($A, \neg A : B$) is not LP-valid. That is,*

$$A, \neg A \not\models_{LP} B.$$

Proof. It is clear how to invalidate Explosion. Let A be assigned p. Then $\neg A$ too takes the value p. Now assign B the value f. This yields a counterexample to Explosion: each of the premises A and $\neg A$ enjoys a designated value, whereas the conclusion B does not. □

The form of definition of $\models_{\text{LP}}$ guarantees that it is unrestrictedly transitive:

Lemma 9 (Unrestricted Cut).
If $\Delta \models_{LP} \varphi$ and $\Gamma, \varphi \models_{LP} \psi$, then $\Delta, \Gamma \models_{LP} \psi$.

Proof. Suppose that

$$\Delta \models_{\text{LP}} \varphi \qquad (1)$$

and

$$\Gamma, \varphi \models_{\text{LP}} \psi \qquad (2)$$

Suppose μ is an arbitrary assignment dealing with the atoms involved in $\Delta, \Gamma, \varphi, \psi$. Suppose that

$$\mu \Vdash (\Delta \cup \Gamma) \qquad (3)$$

We shall show that $\mu \Vdash \psi$. From (3) it follows that

$$\mu \Vdash \Delta \qquad (4)$$

and

$$\mu \Vdash \Gamma \qquad (5)$$

From (1) and (4) we have

$$\mu \Vdash \varphi \qquad (6)$$

From (2), (5) and (6), we have $\mu \Vdash \psi$. □

Corollary 2. *As a special case of Lemma 9 we have the following:*

If $\models_{LP} \varphi$ and $\Gamma, \varphi \models_{LP} \psi$, then $\Gamma \models_{LP} \psi$.

Proof. Set $\Delta = \emptyset$ in Lemma 9. □

Observation 5. *Corollary 2 says one can suppress daudologies as premises.*

Now it is well known, from Lewis's famous argument, that Explosion follows from the combination of

1. having $A \vee B$ implied by A (and implied by B);
2. having Disjunctive Syllogism: $\neg A, A \vee B : B$; and
3. having unrestricted transitivity.

This stares one in the face upon arranging the following little bits of proof in an inviting pattern:

$$\frac{A}{A \vee B}$$

$$\frac{A \vee B \quad \neg A}{B}$$

Since LP validates $\vee$I (hence may have it as a rule) and enjoys unrestricted transitivity, but invalidates Explosion, it follows that LP must invalidate Disjunctive Syllogism. And so it does.

Lemma 10. *Disjunctive Syllogism ($\neg A, A \vee B : B$) is not LP-valid. That is,*

$$\neg A, A \vee B \not\models_{LP} B.$$

Proof. It is clear how to invalidate Disjunctive Syllogism. We adduce once again the assignment

$$\tau(A) = p \qquad \tau(B) = f$$

By LP's truth-value tables, we have

$$\tau(\neg A) = p\ , \quad \tau(A \vee B) = p\ ;\ \text{but}\ \tau(B) = f.$$

The value p is designated but the value f is not. So Disjunctive Syllogism is not LP-valid. □

Observation 6. *The proofs of Lemma 6 and of Lemma 7 use Disjunctive Syllogism in the metalogic. But Lemma 10 says Disjunctive Syllogism is not LP-valid. So the advocate of LP cannot undertake the foregoing metalogical reasoning that yields the insights of Lemma 6 and of Lemma 7. This reveals a reflexive instability in the position of the LP-advocate—unless some* alternative *passage of metalogical reasoning can be furnished, which is formalizable in a proof-system for LP.*

I cannot, however, find any ready alternative. I would be happy to be instructed in this regard by an LP-er.

Disjunctive Syllogism as stated above is an inference from the premises $A \vee B$, $\neg A$ to the conclusion B. Metalinguistically, one would state

$$A \vee B, \neg A \models_{\mathcal{S}} B$$

for any logical system $\mathcal{S}$ validating Disjunctive Syllogism. For LP, however, we have, on the one hand,

$$A \vee B, \neg A \not\models_{\text{LP}} B,$$

whence also (by $\wedge$I and unrestricted transitivity)

$$(A \vee B) \wedge \neg A \not\models_{\text{LP}} B.$$

On the other hand, the THEOREM in §III.8 on p. 228 of Priest [1979] tells us that the classical tautologies are exactly the daudologies. Among the latter we have

$$\models_{\text{LP}} ((A \vee B) \wedge \neg A) \rightarrow B$$

Note that this is a *formal semantic* claim, using the double turnstile $\models_{\text{LP}}$.

2.3 The trouble with Modus Ponens

If Priest is correct in thinking that there is a sound and complete system of natural deduction for LP—giving rise to a *single* turnstile $\vdash_{\text{LP}}$ coextensive with $\models_{\text{LP}}$—then we know that to match the last semantic claim (the second of the two results stated at the end of §2.2) there must be some proof in this system of the form

$$\begin{array}{c} \emptyset \\ \Pi \\ ((A \vee B) \wedge \neg A) \rightarrow B \end{array}$$

The first of the two results stated at the end of §2.2—

$$(A \vee B) \wedge \neg A \not\models_{\text{LP}} B$$

—shows that the usual rule of detachment (i.e., Modus Ponens):

$$\dfrac{\begin{array}{c} \Delta \\ \vdots \\ \varphi \end{array} \qquad \begin{array}{c} \Gamma \\ \vdots \\ \varphi \rightarrow \psi \end{array}}{\psi}$$

fails rather conspicuously in LP. Indeed, it can fail even in a context where the newly accumulated premises $\Delta \cup \Gamma$ are jointly *consistent*:

$$\dfrac{(A \vee B) \wedge \neg A \qquad \begin{array}{c}\emptyset \\ \Pi \\ ((A \vee B) \wedge \neg A) \rightarrow B\end{array}}{B}$$

So one *cannot* have, in LP, the following rule, either primitive or derived:[3]

$$\dfrac{\varphi \qquad \varphi \rightarrow \psi}{\psi}$$

There is of course a deep motivating reason for the LP-er to invalidate Modus Ponens. If he *were* entitled to the instance

$$\dfrac{(A \vee B) \wedge \neg A \qquad ((A \vee B) \wedge \neg A) \rightarrow B}{B}$$

then he would be able to construct a proof of Explosion as follows:

$$\dfrac{\dfrac{\dfrac{A}{A \vee B} \qquad \neg A}{(A \vee B) \wedge \neg A} \qquad \begin{array}{c}\emptyset \\ \Pi \\ ((A \vee B) \wedge \neg A) \rightarrow B\end{array}}{B}$$

And this, by Lemma 8, would mean that the would-be natural-deduction system for LP is unsound with respect to $\models_{\mathrm{LP}}$.

[3]Priest, of course, is aware of this; see Priest [1979], p. 228 *infra*, where he points out that $A, A \rightarrow B \not\models B$. (He does not use the subscript LP with $\models$. His $\models$ is our $\models_{\mathrm{LP}}$.)
Note also that the generalized (or parallelized) elimination rule

$$\dfrac{\varphi \rightarrow \psi \qquad \begin{array}{c}\vdots \\ \varphi\end{array} \qquad \begin{array}{c}\overline{\psi}^{(i)} \\ \vdots \\ \theta\end{array}}{\theta}{\scriptstyle(i)}$$

fails *a fortiori*, since it has Modus Ponens as a degenerate instance:

$$\dfrac{\varphi \rightarrow \psi \qquad \varphi \qquad \overline{\psi}^{(i)}}{\psi}{\scriptstyle(i)}$$

Note that this problem for Modus Ponens arises even without inquiring into the nature of any LP-proof Π that would have to be vouchsafed by any LP-proof system establishing all daudologies as theorems. We are entitled to assume that LP-proofs, however they might be constructed, are LP-sound. That granted, we know that the final step of the 'proof' just given must be LP-invalid by the following considerations:

1. the left immediate subproof establishes the following true statement of LP-consequence:

$$A, \neg A \models_{\mathrm{LP}} (A \vee B) \wedge \neg A\,;$$

2. the right immediate subproof Π establishes the following true statement of daudologousness:

$$\models_{\mathrm{LP}} ((A \vee B) \wedge \neg A) \rightarrow B\,;$$

3. $\models_{\mathrm{LP}}$ is transitive;

4. therefore, if the final step (of what looks like Modus Ponens) were LP-valid, it would follow that

$$A, \neg A \models_{\mathrm{LP}} B\,.$$

   But this we already know is impossible.

This trouble with Modus Ponens is hardly surprising. Priest does, after all, define $A \rightarrow B$ as $\neg A \vee B$. Thus the primitive-looking would-be rule of Modus Ponens:

$$\frac{A \quad A \rightarrow B}{B}$$

is really only a form of Disjunctive Syllogism:

$$\frac{A \quad \neg A \vee B}{B}\,,$$

which we have already seen to be LP-invalid.

On further reflection, the puzzle with $\rightarrow$ deepens. Suppose one were seeking to have it as a *primitive* connective in LP. Consider how (in a classical system, at least) the two sentences $A \rightarrow B$ and $\neg A \vee B$ are interdeducible.

Here, for example, are the two proofs establishing their interdeducibility in Classical Core Logic. In each case the LP-invalid step is highlighted.

$$\dfrac{\dfrac{\dfrac{\overline{A}^{(1)} \quad A\to B}{B}}{\neg A\vee B} \qquad \dfrac{\overline{\neg A}^{(1)}}{\neg A\vee B}}{\neg A\vee B}{\scriptstyle(1)} \qquad\qquad \dfrac{\dfrac{\neg A\vee B \qquad \dfrac{\overline{\neg A}^{(1)} \quad \overline{A}^{(2)}}{\bot} \qquad \overline{B}^{(1)}}{B}{\scriptstyle(1)}}{A\to B}{\scriptstyle(2)}$$

Under our supposition, the LP-advocate would not be able to avail himself of either of these proofs. For the proof on the left uses Modus Ponens; while the proof on the right contains a subproof that establishes Disjunctive Syllogism (in the form $\neg A\vee B, A : B$). The latter emerges therein as derived, courtesy of Core Logic's liberalized rule of $\vee$-Elimination. This rule allows one to bring down as the main conclusion of a proof by cases the conclusion of either of the two case proofs should the other case-proof's conclusion be absurdity ($\bot$). If one were (like the intuitionistic logician) to eschew Core Logic's liberalized rule of $\vee$-Elimination, then one would obtain equiform conclusions for the two case proofs by insinuating into the first case proof an application of the rule Ex Falso Quodlibet:

$$\dfrac{\dfrac{\neg A\vee B \qquad \dfrac{\dfrac{\overline{\neg A}^{(1)} \quad \overline{A}^{(2)}}{\bot}}{B} \qquad \overline{B}^{(1)}}{B}{\scriptstyle(1)}}{A\to B}{\scriptstyle(2)}$$

The offending appeal to, in effect, Disjunctive Syllogism, remains highlighted within this intuitionistic version of the right-hand core proof given above. But the advocate of LP would now cast a jaundiced glare at the invocation of EFQ, which is not allowed by his lights. It would be a mistake, however to lay the 'blame' for Disjunctive Syllogism at the door of EFQ. EFQ is reprehensible all on its own; while, in the view of the core logician (who *rejects* EFQ), *Disjunctive Syllogism is perfectly acceptable*. Indeed, Disjunctive Syllogism is *perfectly* valid!—it is valid, and has no valid proper subsequent.

We have seen, then, that even though $A\to B$ and $\neg A\vee B$ *have the same truth-value table* (as we saw above), nevertheless *neither* of them can be deduced from the other in the usual way in any 'standard' system of natural deduction for LP that might seek to subject $\to$ to anything like its 'own' rules as a primitive. (A logical operator \$ has rules of 'its own' just in case the rules in question are stated in such a way that \$ is the only operator explicitly occurring in them.) It is highly unusual for (semantically) equivalent sentences to fail thus to be interdeducible. More familiar are certain systems in which interdeducibility fails to secure synonymy—see Smiley [1962]. Smiley proposed, as necessary and sufficient for the synonymy of any two sentences within a logical calculus, that they be interreplaceable, *salva veritate*, in all statements of deducibility. Smiley's exigent definition of synonymy makes it trivial, therefore, that synonymous sentences are interdeducible. We have seen that what we defined above as LP-equivalence—possession of the same three-valued LP-truth-value table—cannot *even* secure LP-interdeducibility in any natural sense; and therefore certainly cannot guarantee synonymy in Smiley's yet more demanding sense. Despite Observation 1, Priest appears to be denied the option of adopting $A \to B$ as a *primitive* conditional, and making it behave (in a system of natural deduction) exactly the way it ought to behave if it were understood as being captured also as 'merely' an abbreviation for $\neg A \vee B$. With the latter, 'abbreviatory' option, of course, the question of *interdeducibility* would not arise; or, at least, it would be settled trivially in the affirmative: from $\neg A \vee B$ one can deduce $\neg A \vee B$, just by writing it down. And one can construe this either way—that one has deduced the (defined) $A \to B$ from $\neg A \vee B$, or *vice versa*.

Modus Ponens, for the would-be LP-proof-theorist, is *verboten*. But *no* proof-theorist for any system $\mathcal{S}$ can refuse to accept a rule of inference (as a means of constructing $\mathcal{S}$-proofs) that happens to be $\mathcal{S}$-valid. So the would-be LP-proof-theorist cannot refuse 'to be given' any of the four classical rules of negation respectively known as the Law of Excluded Middle, Dilemma, Double-Negation Elimination and Classical Reductio ad Absurdum. By the same token, in light of Lemma 7 and Corollary 1, the would-be LP-proof-theorist also cannot refuse 'to be given' the Rule of Conditional Proof.

This makes the situation with LP rather puzzling for the inferentialist. For Lemma 7 and Corollary 1 together showed that the conventional Rule of Conditional Proof—which is part of the rule of $\to$-Introduction in Core Logic—is LP-admissible (i.e., LP-valid). The remaining part of the rule of $\to$-Introduction that is found in Core Logic is that which allows one to infer the conditional upon refuting its antecedent. And *this* part is *also* LP-admissible, as Lemma 6 shows.

So the conventional Rule of Conditional Proof—indeed, even the full rule of $\to$-Introduction of Core Logic—might as well be adopted as the rule of $\to$-Introduction for LP. In unrestrictedly transitive systems, the usual 'harmoniously balancing' companion rule, the rule of $\to$-Elimination, is taken to be Modus Ponens. But the latter *is not allowed* as a rule of inference in LP. Conclusion: LP cannot regiment the logical behavior of $\to$ by means of introduction and elimination rules that are in harmony with one another.[4]

So: what is the inferentialist to do, when confronted with the strange relation $\models_{\mathrm{LP}}$? What might be left in the inferentialist's toolbox to make the conditional arrow intelligible to one who wishes to learn *how to reason deductively* within, or by means of, LP? Or, if not in LP itself, then at least in some other system with just as good a claim as LP to capturing the 'logic of naïve proofs'?

3 A different route for the inferentialist

Tennant [1979] put forward a different paraconsistent logic, namely Core Logic.[5] (For its natural deduction rules, see Appendix I below; for the corresponding sequent rules, see Appendix II.) One main aim of Core Logic is to track the ways in which the premises of a proof are relevant to its conclusion. This motivating concern was not necessarily to handle the paradoxes, let alone to lay any logical foundations for such an arresting philosophical claim as Priest's, that some *contradictions* are *true*. The interest was only in how logicians might avail themselves of techniques of proof-normalization in order to establish conclusions only from genuinely relevant premises.[6] Not long after our papers appeared, Priest made the humorous remark to me that whereas I was countenancing truth-value *gaps*, he was 'merely' countenancing truth-value *gluts*. But the contrast of course, between our respective systems runs far deeper than that. The contrast is in both output—the field of the consequence or deducibility relation—and the *methodology*—formal-semantical (in Priest's case) and proof-theoretical (in mine). This study dwells on this contrast, and seeks to draw some lessons from it.

It turned out (from the perspective of this Core logician) that by getting the logical rules right—in a perfectly tweaked form, differing slightly but crucially from their form in Gentzen [1934, 1935] and Prawitz [1965]—it was possible to hold that the resulting logical system forged *analytical* logical

[4] I am grateful to a referee for referring me to Priest [2008], at p. 125, where Priest points out that changing just one entry in LP's 3-valued truth table for the conditional turns it into the conditional of RM_3, for which Modus Ponens holds. But, as Priest then points out, this change turns the whole *system* into RM_3. And this would take us right off our titular topic.

[5] The system was originally called Intuitionistic Relevant Logic.

[6] The project was accomplished for the propositional part of Core Logic in Tennant [1992], and reached its full fruition for first-order logic in Tennant [2015c].

connections among sentences in two important and complementary regards. Such connections obtained *both* because of the meanings conferred on the logical operators by the logical rules governing them; *and also* because those rules were so formulated that when their applications were arranged so as to form proofs, relevant connections were preserved among the *extra*-logical expressions occurring in the premises and the conclusion of any core proof. This phenomenon of relevant connection is preserved also in Classical Core Logic, whose strictly classical rules for negation are carefully tweaked in similar fashion.[7] So, to the extent that one is after a 'logic of naïve proof', Core Logic and Classical Core Logic are perfect vehicles for the rigorous formalization of reasoning in constructive and classical mathematics respectively.[8]

At its inception *this* particular logical system (Core Logic) was a system of *proof*. The mode of investigation was thoroughly *inferentialist*. It was only later that it was discovered that the proof-theoretic methods had fruitful application in the diagnosis and defusing of the logico-semantic paradoxes.[9]

The inferentialist logician who inquires after how a particular logical system is characterized wants to be given a *system of proof*—ideally, a set of

[7] For details, see Tennant [2015c].

[8] That the contrast between Gentzen's system of natural deduction and the artificial Frege–Hilbert systems reflects favorably on the former appears to be tacitly conceded at p. 25 in Hilbert and Ackermann [1938]—the second, 'improved' (*verbesserte*) 1938 edition of *Grundzüge der theoretischen Logik*, whose first edition appeared in 1928. There Hilbert and Ackermann write

> Wir erwähnen endlich noch als eine Sonderstellung einnehmend den von G. Gentzen aufgestellten „Kalkül des natürlichen Schließens"[fn], der aus dem Bestreben hervorgegangen ist, das formale Ableiten von Formeln mehr als bisher dem inhaltlichen Beweisverfahren, wie es z. B. in der Mathematik üblich ist, anzugleichen. Der Kalkül enthält keine logischen Axiome, sondern nur Schlußfiguren, die angeben, welche Folgerungen aus gegebenen Annahmen gezogen werden können, sowie solche, die Formeln liefern, bei denen die Abhängigkeit von den Annahmen beseitigt ist.

I would translate this as follows (rather than using the translation provided in Hilbert and Ackermann [1950] at p. 30):

> We mention finally one more system, one occupying a special position, the 'calculus of natural deduction' set up by G. Gentzen, which emerged from the endeavor to make the formal derivation of formulae resemble more closely than it has until now the contentful procedure of proof that is customary, for example, in mathematics. The calculus contains no logical axioms, but only rules of inference, which specify which consequences can be drawn from given assumptions, as well as rules that deliver formulae while rendering them independent of [certain] assumptions.

[9] See Tennant [1982], Tennant [1995] and Tennant [2015a].

primitive rules of inference by means of which proofs can be constructed in the now familiar way. Moreover, it is a special mark in favor of a particular kind of proof system if the proofs it furnishes are *homologous to*, or *smoothly regiment*, the more informal proofs that they formalize and that carry conviction for expert reasoners in areas like mathematics. The preferred format for such description is that of natural deduction and/or the sequent calculus, familiar from the *loci classici* Gentzen [1934, 1935] and Prawitz [1965]. Logical rules in systems of natural deduction deal with single dominant occurrences of logical operators—in conclusions of introduction rules, and in major premises of elimination rules. The introduction and elimination rules in natural deduction are mirrored by the Right and Left rules, respectively, in the sequent calculus.

If thinkers *could* reason in an ordinary, but appropriately adjusted, sort of way within a framework like Priest's once they had liberated themselves from the dogma of consistency, then one would expect their patterns of reasoning—and the primitive steps that are sanctioned within them—to be formalizable 'naturally'. This is probably why the assurance was given, in Priest [1979] at p. 241, that 'It is not difficult to give an axiom or rule system for LP . . . '.

It is difficult, however, to find in the literature either a natural-deduction presentation, or a sequent-calculus presentation, of LP.

In an email exchange in April and May 2016, I received Priest's helpful answer to my question '[W]here, in your estimation, is the best or fullest treatment of a proof system (natural deduction or sequent calculus) for your Logic of Paradox?' He directed me to the rules given in §4.6 of Priest [2002]. These are the rules of the system of First Degree Entailment, plus Excluded Middle:

$$\frac{}{\alpha \vee \neg\alpha}$$

The quantifier rules are in §6.4, and are the usual ones. (LP confines all its significant differences from Classical Logic to the propositional level. The same is true of Core Logic.)

LP of course cannot contain Explosion:

$$\frac{\alpha \quad \neg\alpha}{\beta}$$

because LP is paraconsistent. FDE plus Explosion is Kleene's system K3. Another 'rule' that needs to be mentioned at this stage is Implosion, the dual of Explosion (and which a relevantist would regard as *similarly irrelevant*!):

$$\frac{\beta}{\alpha \vee \neg\alpha}$$

Priest commented

> I think you need LEM, not Implosion. Normally these would be equivalent, but without either you have FDE, which has no logical truths, and so Implosion does not deliver LEM. [FDE plus LEM] is sound and complete [for LP].

So the list of officially approved rules for propositional LP is the one given (in §4 below), these rules being taken from pp. 302, 303 and 309 in Priest [2002].

There emerged from the aforementioned email exchange the following further important points about the rules of inference for LP.

1. There are no rules just for negation, in isolation. The de Morgan-like rules (see §4 below) plus LEM are 'the rules' for negation.

2. Likewise, there are no rules just for the conditional, in isolation.

3. There is no straightforward way to identify the intuitionistic fragment of LP (if there is such a thing) by simply dropping certain primitive rules of (classical) LP.

4. Implosion—despite its inherent irrelevance—is semantically valid in LP.

5. There is no 'normal form' proof of Implosion by means of the rules for LP. (This will be shown at the end of §4.)

Another comment is in order. Among the rules of FDE is the classical rule of Double Negation Elimination. But as Priest observed, FDE has no logical truths; hence by means of its rules one can prove no theorems. Hence despite containing Double Negation Elimination, FDE does not prove the Law of Excluded Middle. This is rather peculiar. For, against the background of Intuitionistic Logic, the four strictly classical rules of Double Negation Elimination, Classical Reductio, Dilemma and the Law of Excluded Middle are interderivable. Moreover, in order to derive DNE using LEM and the standard rule of Proof by Cases, one has to use EFQ. In the absence of EFQ, then (as in any paraconsistent logic) LEM is *prima facie* weaker than DNE. Yet in LP it seems to be the other way round. It appears that DNE is weaker than LEM; for the system containing DNE needs to have LEM *added* in order to yield the full classical system of LP. This terrain, for the logician sensitive to the usual marks of non-constructivity, appears to be full of potholes.

4 Rules for propositional LP

The following set of rules for propositional LP were gleaned, then, from Priest [2002], with helpful guidance from its author.[10]

From p. 302:

$\wedge$I $\qquad \dfrac{\alpha \quad \beta}{\alpha \wedge \beta}$

$\wedge$E $\qquad \dfrac{\alpha \wedge \beta}{\alpha} \qquad \dfrac{\alpha \wedge \beta}{\beta}$

$\vee$I $\qquad \dfrac{\alpha}{\alpha \vee \beta} \qquad \dfrac{\beta}{\alpha \vee \beta}$

From p. 303:

$\vee$E

$$\dfrac{\alpha \vee \beta \quad \begin{matrix} \overline{\alpha} \\ \vdots \\ \gamma \end{matrix} \quad \begin{matrix} \overline{\beta} \\ \vdots \\ \gamma \end{matrix}}{\gamma}$$

So far, so good ... for this is straight out of Gentzen [1934, 1935] and Prawitz [1965]. These are the familiar Introduction and Elimination rules for $\wedge$ and $\vee$. But *where are the rules for* $\neg$ *?* ... All we learn is that the behavior of negation is constrained only *in relation to* conjunction, disjunction and itself, as follows. Such multi-operator constraining of how any single operator behaves is what makes it so difficult to identify an *intuitionistic* subsystem of LP.

[10] The rule $\wedge$I was given the mistaken label $\vee$I on p. 302.

From p. 309:

$$\frac{\neg(\alpha \wedge \beta)}{\neg\alpha \vee \neg\beta} \qquad \frac{\neg\alpha \vee \neg\beta}{\neg(\alpha \wedge \beta)}$$

$$\frac{\neg(\alpha \vee \beta)}{\neg\alpha \wedge \neg\beta} \qquad \frac{\neg\alpha \wedge \neg\beta}{\neg(\alpha \vee \beta)}$$

$$\frac{\alpha}{\neg\neg\alpha} \qquad \frac{\neg\neg\alpha}{\alpha}$$

LEM $\quad \dfrac{}{\alpha \vee \neg\alpha}$

Conspicuously missing here, for the Gentzen–Prawitz proof-theorist, is any account of negation *on its own*, or *in its own right*. LP appears not to provide so much as a rule of constructive *reductio ad absurdum* (i.e., negation introduction), let alone a rule of negation elimination.

Bear in mind that Priest is happy with the following *abnormal* proof of Implosion from LEM, using the time-honored trick of applying $\wedge$E immediately after $\wedge$I:

$$\dfrac{\dfrac{\dfrac{}{A \vee \neg A} \qquad B}{(A \vee \neg A) \wedge B}}{A \vee \neg A}$$

This makes B *spuriously* relevant to the ultimate conclusion $A \vee \neg A$. A simple $\wedge$-Reduction transforms the foregoing 'proof' into a single invocation of LEM, which is all that is 'really going on':

$$\dfrac{\dfrac{\dfrac{}{A \vee \neg A} \qquad B}{(A \vee \neg A) \wedge B}}{A \vee \neg A} \quad \overset{\wedge\text{-Redn.}}{\rightsquigarrow} \quad \dfrac{}{A \vee \neg A}$$

The conclusion $A \vee \neg A$ does not *really* follow *from* an arbitrary, thematically unconnected 'premise' B after all. Implosion implodes.

But not for LP! According to Priest, the would-be proof of Implosion exploiting the abnormality trick really does count as a proof in his 'natural deduction' system for LP. So that system will be devoid of any meaningful normalization theorem concerning its proofs, which is one of the main motivations for using natural deduction as one's format for fully regimented proofs.

5 Upshot; and a proposed agenda for the LP-er

The upshot of all this, for the inferentialist logician, is a sense of both bewilderment and bafflement as to *how one is permitted to reason 'within'* LP. There is no easily recognizable 'body of deductive reasoning' produced by LP-experts that lends itself to faithful and homologous formalization by means of such LP-proofs as are available in the rather sketchy system that this inquiry has uncovered. My own experience is that logicians who think that they understand 'what it would be to reason like an LP-er' are really only reasoning *about* LP (in particular: about its semantic consequence relation $\models_{\mathrm{LP}}$). Their reasoning does not, on any particular occasion when it is directed towards some conclusion, lend itself to formalization as a proof *in* a proof-system for LP, a system that could generate a deducibility relation $\vdash_{\mathrm{LP}}$ that might be shown (at the metalevel) to coincide with $\models_{\mathrm{LP}}$. Rather, they 'reason LP-ishly' at arm's remove, 'one level up', as metalogicians using ordinary mathematical reasoning (as I have above!) to work out whether particular sequents lie in the extension of $\models_{\mathrm{LP}}$. They don't work 'within LP itself', in accordance with *primitive deductive rules* of LP, which could then in turn be studied by a proof-theorist. As we have pointed out above, there is a reflexive instability in making moves at the metalevel, when reasoning about $\models_{\mathrm{LP}}$, that are not themselves catered for within $\models_{\mathrm{LP}}$ itself.

If anything positive is to emerge from these modest investigations, one hopes it will be this: LP-ers ought to devote some of their ingenious energies to the devising of a natural-deduction system, or sequent calculus, for LP that could be offered as 'the' organon of inference in 'naïve logic'. It's just not good enough to stick with $\models_{\mathrm{LP}}$ 'down there', and reason about it like any classical or constructive logician 'up here'. Rather, one must get down there with the requisite logical goggles on, in order to appreciate or better understand the murky currents of thought that the LP-er, by his own lights, can allow himself to be guided by when setting out from certain premises and seeking to arrive at some conclusion LP-implied by them. One should also be mindful of the fact that some naïve reasoners might wish their reasoning to be strictly *constructive*. That means we ought to be prepared to identify a strict subset of the primitive rules of LP that would enable and allow all the inferential moves that the naïve *constructive* reasoner wishes to make.

If such an investigation were to be undertaken by LP-devotees intent on furnishing us at long last with a proof-system for LP worthy of the title '(regimentation of) naïve logic', then the next item on the collective agenda would be to find points of contrast or similarity with the proof-system of Core Logic. For the latter system also lays claim to the title of 'naïve logic', on at least three scores: (i) it avoids altogether EFQ and its ilk (so it is a non-explosive, or paraconsistent, logic); but (ii) it respects every ordinary logical inference (such as Disjunctive Syllogism) which naïve beginners find (correctly!) to be intuitively correct; while (iii) it can be deployed to regiment the reasoning that is actually involved in generating the various logico-semantical paradoxes, and at the same time allows the theorist to point out what it is about (the formalizations of) those passages of reasoning that reveal that one is dealing thereby with the paradoxical, rather than with the straightforwardly inconsistent.[11]

The sort of thing I am asking the LP-er to provide is: some set of logical rules, each, ideally, focusing on just one logical operator at a time, plus—*perhaps*—some structural rules (as in some sequent calculi) that do not mention any operators; something, indeed, like the systems $\mathbb{C}$ of Core Logic, and $\mathbb{C}^+$ of Classical Core Logic. These are laid out in the Appendix, and are furnished here as a guide. Note, however, that the core systems have no structural rules other than the Rule of Initial Sequents (REFLEXIVITY). Both CUT and THINNING are *admissible* only, not derivable (and *a fortiori* not primitive) rules of the systems.

Both $\mathbb{C}$ and its classicized extension $\mathbb{C}^+$ are paraconsistent, and are *transitive* in the following important sense:

> Any pair of proofs of the sequents $\Delta : \varphi$ and $\Gamma, \varphi : \psi$ respectively can be effectively transformed into a proof of a sequent either of the form $\Delta', \Gamma' : \varphi$ or of the form $\Delta', \Gamma' : \bot$, for some $\Delta' \subseteq \Delta$ and $\Gamma' \subseteq \Gamma$.

That is, CUT is admissible.[12]

Appendix I: Natural Deduction Rules for Core Logic $\mathbb{C}$

[11] In support of these closing claims, see Tennant [2015a], Tennant [2016] and Tennant [2017].

[12] We mean here that the following metalinguistic inference holds:

$$\frac{\Delta \vdash \{\varphi\} \qquad \Gamma, \varphi \vdash \{\psi\}}{\Delta, \Gamma : \{\psi\}}$$

where $\Theta \vdash \Xi$ means that there is a proof of some subsequent of $\Theta : \Xi$, i.e. a proof of some sequent of the form $\Theta' : \Xi'$, where $\Theta' \subseteq \Theta$ and $\Xi' \subseteq \Xi$.

A box annotating a discharge stroke indicates that we must have used at least one of the indicated undischarged assumptions in the subordinate proof in question. This is called *non-vacuous* discharge of assumptions. *Vacuous* discharge, by contrast, is indicated by a diamond. Note that the absence of any vertically descending dots above a major premise for elimination indicates that it must 'stand proud', with no proof-work above it.

$$(\neg\mathrm{I})\qquad \dfrac{\begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\overline{\varphi}}\ ,\ \Delta}\\ \Pi\\ \bot\end{array}}{\neg\varphi}\,(i) \qquad\qquad (\neg\mathrm{E})\qquad \dfrac{\neg\varphi \qquad \begin{array}{c}\Delta\\ \Pi\\ \varphi\end{array}}{\bot}$$

$$(\wedge\mathrm{I})\qquad \dfrac{\begin{array}{c}\Delta_1\\ \Pi_1\\ \varphi_1\end{array}\qquad \begin{array}{c}\Delta_2\\ \Pi_2\\ \varphi_2\end{array}}{\varphi_1\wedge\varphi_2} \qquad\qquad (\wedge\mathrm{E})\qquad \dfrac{\varphi_1\wedge\varphi_2 \qquad \begin{array}{c}\underbrace{\overset{(i)-\square-(i)}{\overline{\varphi_1\ ,\ \varphi_2}}\ ,\ \Delta}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

$$(\vee\mathrm{I})\qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ \varphi_1\end{array}}{\varphi_1\vee\varphi_2}\qquad \dfrac{\begin{array}{c}\Delta\\ \Pi\\ \varphi_2\end{array}}{\varphi_1\vee\varphi_2}$$

$$(\vee\mathrm{E}_{\theta\bot})\qquad \dfrac{\varphi_1\vee\varphi_2 \qquad \begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\overline{\varphi_1}}\ ,\ \Delta_1}\\ \Pi_1\\ \theta\end{array}\qquad \begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\overline{\varphi_2}}\ ,\ \Delta_2}\\ \Pi_2\\ \bot\end{array}}{\theta}\,(i)$$

$$(\vee\mathrm{E}_{\bot\theta})\qquad \dfrac{\varphi_1\vee\varphi_2 \qquad \begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\overline{\varphi_1}}\ ,\ \Delta_1}\\ \Pi_1\\ \bot\end{array}\qquad \begin{array}{c}\underbrace{\overset{\square\!-\!(i)}{\overline{\varphi_2}}\ ,\ \Delta_2}\\ \Pi_2\\ \theta\end{array}}{\theta}\,(i)$$

($\vee$E$_{\theta\theta}$)

$$\dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_1}\;,\;\Delta_1} \\ \Pi_1 \\ \theta \end{array} \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_2}\;,\;\Delta_2} \\ \Pi_2 \\ \theta \end{array}}{\theta}\,(i)$$

($\vee$E$_{\perp\perp}$)

$$\dfrac{\varphi_1 \vee \varphi_2 \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_1}\;,\;\Delta_1} \\ \Pi_1 \\ \perp \end{array} \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_2}\;,\;\Delta_2} \\ \Pi_2 \\ \perp \end{array}}{\perp}\,(i)$$

($\rightarrow$I)(a)

$$\dfrac{\begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_1}\;,\;\Delta} \\ \Pi \\ \perp \end{array}}{\varphi_1 \rightarrow \varphi_2}\,(i)$$

($\rightarrow$I)(b)

$$\dfrac{\begin{array}{c} \underbrace{\overset{\Diamond\,\overline{}\,(i)}{\varphi_1}\;,\;\Delta} \\ \Pi \\ \varphi_2 \end{array}}{\varphi_1 \rightarrow \varphi_2}\,(i)$$

($\rightarrow$E)

$$\dfrac{\varphi_1 \rightarrow \varphi_2 \qquad \begin{array}{c} \Delta_1 \\ \Pi_1 \\ \varphi_1 \end{array} \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\varphi_2}\;,\;\Delta_2} \\ \Pi_2 \\ \theta \end{array}}{\theta}\,(i)$$

($\exists$I)

$$\dfrac{\begin{array}{c} \Delta \\ \Pi \\ \psi^x_t \end{array}}{\exists x \psi}$$

($\exists$E)

$$\dfrac{\exists x \psi(x) \qquad \begin{array}{c} \underbrace{\overset{\Box\,\overline{}\,(i)}{\psi(a)}\;,\;\Delta} \\ \Pi \\ \theta \end{array}}{\theta}\,(i)$$

where a does not occur in any sentence in Δ or in $\exists x \psi(x)$ or in θ

($\forall$I)

$$\dfrac{\begin{array}{c} \Delta \\ \Pi \\ \psi(a) \end{array}}{\forall x \psi(x)}$$

where a does not occur in any sentence in Δ or in $\forall x \psi(x)$

(∀E)

$$\dfrac{\forall x\psi \qquad \begin{array}{c} \underbrace{\overset{(i)}{\overline{\psi^x_{t_1}}},\ \ \ldots\ \ \Box\ \ \ldots\ \ ,\overset{(i)}{\overline{\psi^x_{t_n}}}\ ,\ \ \Delta} \\ \Pi \\ \theta \end{array}}{\theta}\,(i)$$

The classicized extension $\mathbb{C}^+$ of Core Logic may be obtained by adding one or other of the following two rules (Classical Reductio or Dilemma).

(CR)

$$\dfrac{\begin{array}{c} \Box\overline{}^{(i)} \\ \neg\varphi \\ \vdots \\ \bot \end{array}}{\varphi}\,(i)$$

(Dil)

$$\dfrac{\begin{array}{c} \Box\overline{}^{(i)} \\ \varphi \\ \vdots \\ \psi \end{array} \qquad \begin{array}{c} \Box\overline{}^{(i)} \\ \neg\varphi \\ \vdots \\ \psi \end{array}}{\psi}\,(i) \qquad \dfrac{\begin{array}{c} \Box\overline{}^{(i)} \\ \varphi \\ \vdots \\ \psi \end{array} \qquad \begin{array}{c} \Box\overline{}^{(i)} \\ \neg\varphi \\ \vdots \\ \bot \end{array}}{\psi}\,(i)$$

Appendix II: Sequent Calculus Rules for Core Logic $\mathbb{C}$

The only structural rule is REFLEXIVITY, or the Rule of Initial Sequents:

$$\dfrac{}{\varphi : \varphi}$$

The logical rules are as follows.

(: ¬) $\dfrac{\Delta, \varphi :}{\Delta : \neg\varphi}$

(¬ :) $\dfrac{\Delta : \varphi}{\Delta, \neg\varphi :}$

$$(:\wedge)\qquad \frac{\Delta_1 : \varphi_1 \qquad \Delta_2 : \varphi_2}{\Delta_1, \Delta_2 : \varphi_1 \wedge \varphi_2}$$

$$(\wedge :)\qquad \frac{\Delta, \varphi_i : \Gamma}{\Delta, \varphi_1 \wedge \varphi_2 : \Gamma} \quad \text{for i=1,2;}$$

$$\left(\text{which affords also the more economical} \quad \frac{\Delta, \varphi, \psi : \Gamma}{\Delta, \varphi \wedge \psi : \Gamma} \right)$$

$$(:\vee)\qquad \frac{\Delta : \varphi_i}{\Delta : \varphi_1 \vee \varphi_2} \quad \text{for i=1,2}$$

$$(\vee :)\qquad \frac{\Delta_1, \varphi_1 : \Gamma_1 \qquad \Delta_2, \varphi_2 : \Gamma_2}{\Delta_1, \Delta_2, \varphi_1 \vee \varphi_2 : \Gamma_1 \cup \Gamma_2} \quad \text{where } \Gamma_1 \cup \Gamma_2 \text{ is at most a singleton}$$

$$(:\rightarrow)\qquad \frac{\Delta, \varphi_1 :}{\Delta : \varphi_1 \rightarrow \varphi_2} \qquad \frac{\Delta : \varphi_2}{\Delta \setminus \{\varphi_1\} : \varphi_1 \rightarrow \varphi_2}$$

$$(\rightarrow :)\qquad \frac{\Delta_1 : \varphi_1 \qquad \Delta_2, \varphi_2 : \Gamma}{\Delta_1, \Delta_2, \varphi_1 \rightarrow \varphi_2 : \Gamma}$$

$$(:\exists)\qquad \frac{\Delta : \psi^x_t}{\Delta : \exists x \psi}$$

$$(\exists :)\qquad \frac{\Delta, \psi^x_a : \Gamma}{\Delta, \exists x \psi : \Gamma} \quad \text{where the conclusion sequent has no occurrences of } a$$

$$(:\forall)\qquad \frac{\Delta : \psi}{\Delta : \forall x \psi^a_x} \quad \text{where } a \text{ occurs in } \psi \text{ but in no member of } \Delta$$

$$(\forall :)\qquad \frac{\Delta, \psi^x_{t_1}, \ldots, \psi^x_{t_n} : \Gamma}{\Delta, \forall x \psi : \Gamma}$$

$$(\mathrm{CR})\qquad \frac{\neg\varphi, \Delta : \emptyset}{\Delta : \varphi}$$

$$\text{(Dil)} \qquad \frac{\varphi, \Delta : \psi \quad \neg\varphi, \Gamma : \psi}{\Delta, \Gamma : \psi} \qquad \frac{\varphi, \Delta : \psi \quad \neg\varphi, \Gamma : \emptyset}{\Delta, \Gamma : \psi}$$

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