The Philosophical Quarterly Vol. 00, No. 0 *2022*
ISSN 0031-8094 https://doi.org/10.1093/pq/pqac001

ON THE ADEQUACY OF A SUBSTRUCTURAL LOGIC FOR MATHEMATICS AND SCIENCE

By Neil Tennant*

Williamson argues for the contention that substructural logics are 'ill-suited to acting as background logics for science'. That contention, if true, would be very important, but it is refutable, given what is already known about certain substructural logics. Classical Core Logic is a substructural logic, for it eschews the structural rules of Thinning and Cut and has Reflexivity as its only structural rule. Yet it suffices for classical mathematics, and it furnishes all the proofs and disproofs one needs for the hypothetico-deductive method in science. We explain exactly what Classical Core Logic is, why it is a substructural logic par excellence, and what the basic requirements would be for a logic to be 'suited to acting as [a] background logic for science'. We also explain how Classical Core Logic meets all these requirements. We end by examining Williamson's argument in order to expose where its error lies.

Keywords: sequent calculus, substructural logic, adequacy for science and mathematics, cut, thinning, Intuitionistic Logic, Classical Logic, Core Logic, Classical Core Logic.

I. INTRODUCTION

Our aims here are twofold:

(1) We shall show that Williamson's recent critique of substructural logics in general is seriously flawed.
(2) We shall show that the substructural system of Classical Core Logic is adequate unto all the deductive demands of science and mathematics.

As to aim (1): Williamson has recently argued for the contention that substructural logics are 'ill-suited to acting as background logics for science'. That contention is refutable, given what is already known about the substructural systems of Classical Core Logic $\mathbb{C}^+$ and its constructive subsystem Core Logic $\mathbb{C}$. Relevant details about the two Core systems can be found in Tennant (2017).

The reader may rest assured, however, that this study will provide all the expository detail that could possibly be needed for the dialectical purpose of refuting Williamson's contention and locating the fallacy in his argument for it. It is enough, for the purposes of this critique, to speak only about Classical Core Logic. As it happens, its constructive subsystem Core Logic suffices both for constructive (intuitionistic) mathematics and for the hypothetico-deductive method in science. However, limitations of space preclude us from reprising that further case here, which can be found in the book just cited.

As to aim (2): We intend to inform and persuade the 'general philosopher' of the reasons for, and importance of, any claim of the form 'Such-and-such logical system, or kind of logical system, is adequate unto all the deductive demands of science and mathematics'. So we shall need to explain the difference between structural and substructural logics, and show how Classical Core Logic $\mathbb{C}^+$, which (like its constructive subsystem $\mathbb{C}$ of Core Logic) is substructural *par excellence*, is indeed adequate unto all the deductive demands of science and mathematics. The two Core systems are not just very good exemplars of the family of substructural logics, they are *extreme* exemplars. For they are *maximally* substructural, *qua* set-sequent calculi (a description of which will be provided presently).

Since the notion of a substructural logic involves recourse to the so-called structural rules of Gentzen-style sequent-calculus, our discussion will confine itself to that setting. As it happens, there is a deep isomorphism—and for deep reasons—between the sequent calculi for the Core systems, and their respective systems of natural deduction. In the Core systems, natural deductions have directly corresponding sequent proofs of the same results, possessed of the same macro-inferential structure, and conversely.[1] The generality of our conclusions will therefore not be compromised by confining our attention to just the sequent-calculus setting.

We shall focus on *set* sequents, in the setting of *single-conclusion* logic. This means we shall be considering only sequents of the form $\Delta : \Gamma$, where Δ is a finite set of sentences, and Γ is empty or contains exactly one sentence. We stress again that the generality of our conclusions will not thereby be compromised. The Core systems $\mathbb{C}$ and $\mathbb{C}^+$ are single-conclusion, set-sequent calculi.

Single-conclusion calculi are the most natural choice for the regimentation of mathematical and scientific reasoning.[2] That is why we are confining our attention to them.

Moreover, *set-sequent* calculi are by far the simplest ones. They involve at most the three possible structural rules of REFLEXIVITY, THINNING, and CUT (for

[1] See Tennant (2017: 131–2) for further details.

[2] Cf. Steinberger (2011: 334): '... there are no episodes in our ordinary modes of deductive reasoning that can be said to be more faithfully represented in a multiple-conclusion framework than in a single-conclusion system.'

which, see Section II).[3] Anyone who challenges this 'at most' claim will be making it *even harder* to deny that $\mathbb{C}$ and $\mathbb{C}^+$ are substructural logics.

II. GENTZEN SEQUENT CALCULUS FOR $\mathbb{C}$ AND $\mathbb{C}^+$

This section is devoted entirely to setting out formal details about the sequent calculus for Classical Core Logic.

Sequents are complex singular terms (in the metalanguage) of the form

$$\Delta : \Gamma,$$

where, as already mentioned, Δ is a finite set of sentences (the so-called 'antecedent' of the sequent) and Γ (the so-called 'succedent' of the sequent) is either the empty set $\emptyset$ or the singleton $\{\varphi\}$ of a sentence φ. The antecedent Δ can also be the empty set $\emptyset$. The succedent is 'at most' a singleton because the Core systems are single conclusion logics. The colon can be read as 'therefore'.

In place of $\Delta : \{\varphi\}$ we usually write $\Delta : \varphi$. In place of $\Delta \cup \Delta'$, we usually write Δ, Δ'.

An interpretation (or model) of the non-logical vocabulary that makes every sentence in a given set of sentences true is said to *satisfy* that set, and a set of sentences is said to be *satisfiable* just in case some interpretation satisfies it.

The logical or semantic import of a sequent $\Delta : \Gamma$ is the usual one, as follows:

> Any interpretation that satisfies Δ makes at least one sentence in Γ true.

If this holds of every sequent $\Delta : \Gamma$ provable in a system, then the system is said to be *sound*.

Interesting special cases of sequents are the following, or are of the following forms. The reader should bear in mind the semantic import of a sequent when reading the following comments.

Sequent	**Comment**
$\emptyset : \emptyset$	Not provable in a sound system
$\emptyset : \varphi$	If provable in a sound system, then φ is logically true
$\Delta : \emptyset$	If provable in a sound system, then Δ is not satisfiable
$\Delta : \varphi$	If provable in a sound system, then Δ logically implies φ

[3] The main alternatives to using set-sequents are to take antecedents and succedents of sequents to be either multisets or sequences of sentences, and, accordingly, to adopt further structural rules—CONTRACTION for multisets, and, in addition, INTERCHANGE for sequences. (See Gentzen (1934, 1935: 192), where CONTRACTION was stated as ZUSAMMENZIEHUNG, and INTERCHANGE as VERTAUSCHUNG.) These alternatives (i.e., sequent calculi whose sequents are composed of multisets or sequences) can safely be ignored, however, for the dialectical purposes of this study. They also become irrelevant—indeed, inapposite—to the logician who works with a *set-sequent* calculus.

When sentences are separated by commas from sets of sentences in the antecedent of a sequent, the presumption is that those sentences are not members of those sets.

A blank to the right of a colon (for an empty succedent) could be occupied by the symbol $\bot$ (absurdity) or the symbol $\emptyset$ for the empty set. The upshot is the same: if one has a proof of a sequent of the form

$$`\Delta : \quad ' \quad \text{(which can also be written } `\Delta : \emptyset', \text{ or } `\Delta : \bot'),$$

then one has thereby shown that Δ is not satisfiable.

The Core systems, because they are based on *set* sequents, have only one structural rule:

$$\text{REFLEXIVITY} \qquad \overline{\varphi : \varphi}.$$

Proofs, after all, have to get started.

Rule-labels of the forms (@:) and (:@) can be written as (@L) and (@R) respectively—'@ on the left' and '@ on the right' (of the colon). These rules are sometimes called 'operational' or 'logical' rules.

Here now are the logical rules—the Left and Right rules—of Core Logic $\mathbb{C}$ for the logical expressions (the connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$; the quantifiers $\exists$ and $\forall$; and the identity predicate $=$). They ought to strike the reader as very familiar. They are all either primitive or derivable in the system of Intuitionistic Logic. Their occasional differences from the original Left and Right rules of Gentzen are delicate and subtle—but have profound and welcome effects. They afford both *relevance* of premises to conclusions of proofs, and transitivity of deduction *with epistemic gain*, without having to appeal to a structural rule of CUT. Instead, the logical rules below are framed in such a way that one can prove (at the meta-level, of course) that CUT is *admissible*.[4] The proof of this result (Metatheorem 1, Section IV) is constructive. It can be carried out in Core Logic at the meta-level, even for Classical Core Logic as the logic of the object language.[5]

[4] For the distinction between admissible and derivable rules, see Hiż (1959).
[5] Metatheorem 1 was constructively established in Tennant (2015b).

The Official Sequent Rules of Core Logic $\mathbb{C}$

$$(:\neg)\quad \frac{\varphi,\ \Delta : \emptyset}{\Delta : \neg\varphi} \qquad\qquad (\neg :)\quad \frac{\Delta : \varphi}{\neg\varphi,\ \Delta : \emptyset}$$

$$(:\wedge)\quad \frac{\Delta : \varphi \quad \Gamma : \psi}{\Delta, \Gamma : \varphi\wedge\psi} \qquad\qquad (\wedge :)\quad \frac{\Delta : \theta}{\varphi\wedge\psi,\ \Delta\setminus\{\varphi, \psi\} : \theta}$$

where $\Delta \cap \{\varphi, \psi\} \neq \emptyset$

$$(:\vee)\quad \frac{\Delta : \varphi}{\Delta : \varphi\vee\psi} \quad \frac{\Delta : \psi}{\Delta : \varphi\vee\psi} \qquad\qquad (\vee :)\quad \frac{\varphi, \Delta : \Theta \quad \psi, \Gamma : \Xi}{\varphi\vee\psi,\ \Delta, \Gamma : \Theta \cup \Xi}$$

where $\Theta \cup \Xi$ is at most a singleton

$$(:\rightarrow)(a)\quad \frac{\Delta, \varphi : \emptyset}{\Delta : \varphi\rightarrow\psi} \qquad\qquad (\rightarrow :)\quad \frac{\Delta : \varphi \quad \psi, \Gamma : \theta}{\varphi\rightarrow\psi,\ \Delta, \Gamma : \theta}$$

$$(:\rightarrow)(b)\quad \frac{\Delta : \psi}{\Delta\setminus\{\varphi\} : \varphi\rightarrow\psi}$$

Some syntactic definitions for the quantifier rules. ψ^x_t is the result of substituting occurrences of the closed term t for all free occurrences of the variable x in the formula ψ; and ψ^a_x is the result of substituting occurrences of the variable x for all occurrences of the parameter a in the formula ψ, where a is not within the scope of any quantifier in ψ that binds x; and a parameter a is a special case of a closed term. A sequent with the symbol @ next to it contains no occurrences of a.

$$(:\exists)\quad \frac{\Delta : \varphi^x_t}{\Delta : \exists x\varphi} \qquad\qquad (\exists :)\quad \frac{\varphi^x_a,\ \Delta : \psi}{\exists x\varphi,\ \Delta : \psi}\ @$$

$$(:\forall)\quad \frac{\Delta : \varphi}{\Delta : \forall x\varphi^a_x}\ @ \qquad\qquad (\forall :)\quad \frac{\varphi^x_{t_1}\ ,\ \ldots\ ,\ \varphi^x_{t_n},\ \Delta : \theta}{\forall x\varphi,\ \Delta : \theta}$$

$$(:=)\quad \frac{}{\emptyset : t = t} \qquad\qquad (=:)\quad \frac{\Delta : \varphi}{\Delta,\ t = u : \psi}$$

where $\varphi^t_u = \psi^t_u$

Classical Core Logic $\mathbb{C}^+$ is obtained from Core Logic $\mathbb{C}$ by adding the rule of Classical Reductio:

$$(\mathrm{CR})\quad \frac{\neg\varphi,\ \Delta :}{\Delta : \varphi}$$

What sets Core Logic apart from Intuitionistic Logic, and Classical Core Logic apart from Classical Logic, is that the Core systems eschew the structural rules of THINNING and CUT:

THINNING $\dfrac{\Delta : \Gamma}{\varphi, \Delta : \Gamma}$ (on the left); $\dfrac{\Delta :}{\Delta : \varphi}$ (on the right)

CUT $\dfrac{\Delta : \varphi \quad \varphi, \Gamma : \Omega}{\Delta, \Gamma : \Omega}$

The Core systems eschew them because each leads to *irrelevance*. They are the sole culprits, respectively, in the following two sequent proofs of Lewis's First Paradox A, $\neg A$: B, which is anathema to the relevantist:

$$\dfrac{\dfrac{A : A}{A, \neg A : \emptyset}}{A, \neg A : B}\text{THINNING} \qquad \dfrac{\dfrac{A : A}{A : A \vee B} \quad \dfrac{A, \neg A : \emptyset \quad B : B}{A \vee B, \neg A : B}}{A, \neg A : B}\text{CUT}$$

By not incorporating these culprit structural rules, the Core systems $\mathbb{C}$ and $\mathbb{C}^+$ *relevantize* Intuitionistic Logic and Classical Logic, respectively, 'at the level of the turnstile'. For the exact sense in which the premises of proofs are relevant to their conclusions (in both the core systems), see Tennant (2015c).

The Core systems' lack of both THINNING and CUT, however, does not prevent them—as we shall see in due course—from accomplishing all that one could require of a (constructive, respectively, classical) logic for the formalization of deductive reasoning in mathematics and science.

Both $\mathbb{C}$ and $\mathbb{C}^+$ are obviously *substructural* logics. Indeed, one could say: *substructural with a vengeance*. One can advance this claim even in the absence of any detailed explication of what, in general, is to count as a substructural logic. All that one needs to do is re-read Gentzen (1934, 1935), where the contrast between structural and logical rules was first introduced and clarified, to be persuaded that any logical system whose *sole* structural rule is REFLEXIVITY is indeed *sub*structural. This interpretative point will be further confirmed in Section III.1.

Point of terminology for the non-specialist in formal logic:

When a sequent $\Delta : \{\varphi\}$ is the conclusion of a proof in the sequent calculus, one writes the 'single-turnstile' statement '$\Delta \vdash \varphi$', which one can read as 'φ is deducible from Δ'. Likewise, when a sequent $\Delta : \emptyset$ is the conclusion of a proof in the sequent calculus, one writes '$\Delta \vdash \emptyset$' or '$\Delta \vdash$ ' or '$\Delta \vdash \bot$', each of which one can read as 'absurdity is deducible from Δ'. When it is important to mention the logical system whose deducibility relation is in question, one can use a subscript on the single turnstile:

Core Logic $\mathbb{C}$	$\Delta \vdash_{\mathbb{C}} \varphi$	$\Delta \vdash_{\mathbb{C}} \emptyset$	$\Delta \vdash_{\mathbb{C}} \bot$
Classical Core Logic $\mathbb{C}^+$	$\Delta \vdash_{\mathbb{C}^+} \varphi$	$\Delta \vdash_{\mathbb{C}^+} \emptyset$	$\Delta \vdash_{\mathbb{C}^+} \bot$
Classical Logic $\mathbf{C}$	$\Delta \vdash_{\mathbf{C}} \varphi$	$\Delta \vdash_{\mathbf{C}} \emptyset$	$\Delta \vdash_{\mathbf{C}} \bot$

III. THE STRUCTURE OF THE DIALECTIC

Some less formal discussion will now be in order about the general logical issues that we need to be clear about in our critique of Williamson, for these are philosophically important. Two issues need to be clarified: What do logicians and philosophers mean by the term 'substructural logic'? And what is it for a logic (of any kind) to be well-suited for science and mathematics? We shall address these in turn.

III.1 On logics being substructural

The designation 'substructural logic' is due to Došen and Schroeder-Heister (1993). On p. 2 of his historical introduction (Došen 1993) to that jointly edited volume, Došen remarks that

> A very important discovery in Gentzen's thesis [1935] is that in logic there are rules of inference that don't involve any logical constant. Gentzen called such rules *structural*.[6]

Došen went on to write (p. 6)

> Our proposal is to call logics that can be obtained . . . by *restricting* structural rules, *substructural logics*. (First emphasis added.)

This understanding of the designation 'substructural logic' is endorsed also by Restall (2000: 1), where he writes

> Substructural logics focus on the behaviour and presence—or more suggestively, the *absence*—of *structural rules*.' (Both emphases in the original.)[7]

It is this understanding of the designation 'substructural logic' on which we are proceeding here,[8] in addressing Williamson's contention that such logics are ill-suited for science and mathematics.

Without the structural rule of REFLEXIVITY, no sequent proof in any logical system could even get started. The two main systems of first-order extensional logic that are of interest to the philosopher, however—Classical Logic and Intuitionistic Logic—are, in the Gentzenian tradition of sequent calculus, '*fully* structural'. By this we mean that their sequent systems contain, in addition to REFLEXIVITY, the structural rules of THINNING *and* of CUT.

One can profitably investigate, however—and it was this line of investigation that led to the Core systems—whether the two traditional structural rules

[6] The discovery of structural rules was actually due to Hertz (1928); see Gentzen (1932) and Tennant (2015a).

[7] Restall's attribution of the designation to Schroeder-Heister and Došen, in fn. 1 across pp. 1–2 is of course correct, but slightly misleading in making the reader think that the introduction to their jointly edited volume was jointly written. When Došen wrote of 'Our proposal' on p. 6, he was speaking for both editors; but in his own, singly authored, historical introduction.

[8] Unfortunately, Došen (1993: 1) was simply in error in saying that '[t]he most important substructural logic is intuitionistic logic'. Intuitionistic Logic, formulated with sequents whose antecedents are sets of sentences and whose conclusions are at most singletons, has all three structural rules that such a sequent system might employ: REFLEXIVITY, THINNING, and CUT.

of THINNING and CUT are really necessary in order for a logical system, in its sequent-calculus formulation, to furnish what is required for adequate regimentation of our deductive reasoning within mathematical and scientific theorizing.

III.2 On logics being well-suited or ill-suited for science

An explication of the just-mentioned requirement is crucial for any assessment of a particular logical system, or logical systems of some general kind, as either 'well-suited' or 'ill-suited' for such theorizing.

A claim of such ill-suitedness is what a logician would call an informal claim, not susceptible of formal proof. This is because neither well-suitedness nor its contradictory, ill-suitedness (for science), has been precisely explicated.

Think of the analogous negative claim that there is no *effective method* (the pre-formal notion in need of explication) for determining, of a given sentence, whether it is logically true. To establish this kind of negative claim in a rigorous fashion, one requires, first, an explication of the notion of 'effective method'. One requires, second, a formal, rigorous proof that there is no method of the *explicated* kind that will enable one to determine, of any given sentence, whether it is logically true. This is exactly what was required for Alonzo Church's famous theorem on the undecidability of first-order logic. Church proved the formal result that there is no *recursive function* that yields the value 1 on (the code number for) any sentence that is logically true and yields the value 0 on (the code number for) any sentence that is not logically true. This serves to establish the non-existence of any effective method for determining whether a given sentence is logically true—but only by courtesy of 'Church's Thesis' in the background, to the effect that every effective method can be expressed as a recursive function defined on the natural numbers. Kurt Gödel's famous First Incompleteness Theorem for arithmetic, on our contemporary understanding of its most general form, relies on Church's Thesis in the same way. The Gödel result is nowadays stated in this general form: there is no *recursive* enumeration of (the code numbers for) all and only the true sentences of arithmetic. By Church's Thesis, it follows that one cannot *effectively* enumerate all and only the true sentences of arithmetic.[9]

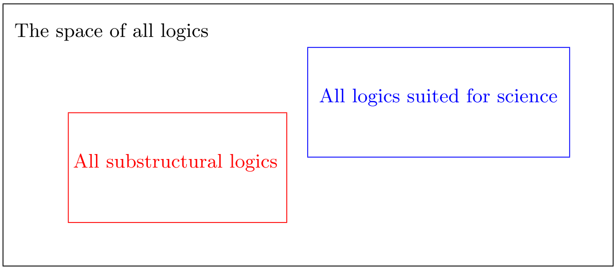
It therefore behooves Williamson, when he says that (all) substructural logics are 'ill-suited to acting as background logics for science', to be prepared to furnish an explication of the notion of suitedness or ill-suitedness that is at issue. The burden of explication of this notion is on *him*, for he is the one making the negative claim. Here, however, we shall relieve him of this burden by providing such explication in Section VI.

[9] Gödel's original theorem was that a particular axiomatic system for Peano Arithmetic, if it is consistent, neither proves nor refutes a certain sentence that he constructed. See Tarski et al. (1953) for the more general and powerful form of the First Incompleteness Theorem.

Moreover, there was a second burden of explication on Williamson, which went undischarged. He needed also, for the same reason, to explicate the very notion of a *substructural logic*, in order to make his negative claim stick. We have already explicated this notion for him in Section III.1.

III.3 How explications can clarify what is at issue

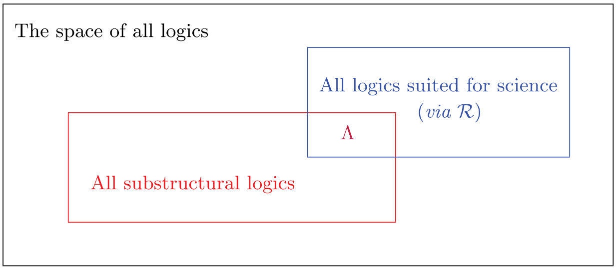
Only when both notions have been explicated—*substructural logic*, and *logic suited for science*—and thereby sharply delineated for technical investigation, can Williamson offer any dispositive proof that their extensions do not overlap. He asks his reader to be persuaded that the picture is like this:



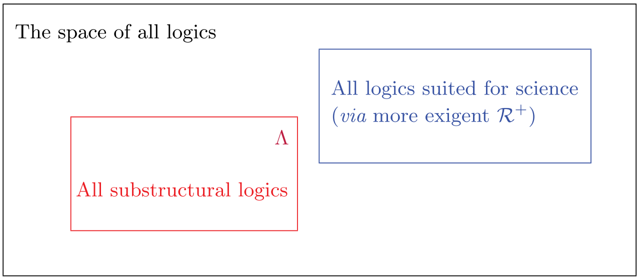
In the absence of the required explications (which is the situation at present), Williamson remains vulnerable to the following dialectical move from his critic:

> *Here* is a logic (call it Λ) that, in advance of any precise explication of the notion of a substructural logic, is *intuitively and obviously* substructural. (Indeed, any proffered explication of 'substructural logic' would have to be rejected out of hand if this particular logic Λ were not thereby counted as substructural.)
> And *here* is a set (call it $\mathcal{R}$—see Section VI) of requirements on the deducibility relation of any logic that is to be regarded as 'suited for science'. These requirements are individually necessary and jointly sufficient for a chosen logic to be suited for science. They have been formulated with careful attention to what mathematicians and scientists actually require of a formal logic that is to serve the deductive needs of their discipline.
> Now note this: the logic Λ satisfies the requirements $\mathcal{R}$—indeed, *provably* so. See the rigorously established metatheorems in Section V.

Williamson is therefore confronted with what we maintain is the *actual* situation, which looks more like this:



Moreover, the only way out that Williamson might have, to get the depicted situation to look like *this*:



is to propose some additional requirement that ought to be, but is not at present, in $\mathcal{R}$, and demonstrate that the logic Λ does not satisfy that additional requirement.

The critic would therefore be seeking from Williamson an account of how he (Williamson) would propose to strengthen our explication $\mathcal{R}$ (of suitedness for science—see Section VI) to such a would-be $\mathcal{R}^+$.

IV. ADMISSIBILITY OF CUT

The following metatheorem for $\mathbb{C}$ and $\mathbb{C}^+$ shows that these systems afford transitivity of deduction in an *epistemically gainful* fashion, despite not containing the rule of CUT. This metatheorem establishes the *admissibility* of CUT in a very strong form, even though CUT is not a rule *of* the systems.[10]

Metatheorem 1 (CUT FOR EPISTEMIC GAIN).

There is an effective method (a binary operation) [,] *on proofs such that for any proof* Π *of the sequent* $\Delta : \varphi$ *and any proof* Σ *of the sequent* $\varphi, \Gamma : \psi$*, the object* $[\Pi, \Sigma]$ *is a proof of some subsequent of the sequent* $\Delta, \Gamma : \psi$.

The metalogical proof of Metatheorem 1, even for the classical system $\mathbb{C}^+$, was carried out using only Core Logic $\mathbb{C}$ in the metalanguage.[11] Note that

[10] A weaker form of admissibility would be the metalogical inference

$$\frac{\Delta \vdash \varphi \qquad \varphi, \Gamma \vdash \psi}{\Delta, \Gamma \vdash \psi}.$$

(Note that this is *not* a rule of CUT in the system, because it has single turnstiles in place of the colons.) But the displayed metalogical inference, unlike Metatheorem 1, does not guarantee that a proof-witness for the deducibility statement that is the conclusion of this inference can be determined from proof-witnesses for the deducibility statements that are the premises of the same. This effective determination, however, is exactly what Metatheorem 1 guarantees.

[11] See Tennant (2012) for the proof of Metatheorem 1 for Core Logic, and Tennant (2015b) for its extension to Classical Core Logic.

in general $\Theta : \bot$ (i.e., $\Theta : \emptyset$) is a subsequent of $\Theta : \theta$. So if the effectively determined proof $[\Pi, \Sigma]$ does not establish ψ from Δ, Γ, it establishes the *inconsistency* of Δ, Γ. That is why this form of transitivity of deduction can yield epistemic gain.

Metatheorem 1 holds for Core Logic, and for Classical Core Logic. Likewise, the following obvious corollary holds for each of those systems. One may read the turnstile as core deducibility, or as classical core deducibility, respectively.

Corollary 1 (CUT FOR ABSURDITY).
If $\Delta \vdash \varphi$ *and* $\Gamma, \varphi \vdash \bot$, *then* $\Delta, \Gamma \vdash \bot$.

V. SOME IMPORTANT METATHEOREMS

We now turn to a consideration of some standard semantical fare: models, logical consequence, and (importantly exploiting the semantical notion of logical consequence) soundness and completeness theorems. This is to prepare the ground for our explication in Section V of what it would be for a logic to be adequate unto the deductive demands of mathematics and science.

In the following metatheorems, the notion of 'model' employed is the standard one due to Kemeny (1948), building on the seminal work of Tarski (1956; first published in 1936), subsequently deployed in standard texts such as Bell and Slomson (1969) and Chang and Keisler (1977), and now widely used in the formal semantics for extensional first-order languages. The central semantical notion of logical consequence, represented by the familiar 'double turnstile' $\models$, is defined by generalizing over models in the familiar way:

Definition 1. $\Delta \models \varphi \Leftrightarrow_{df}$ *for every model M, if every sentence in* Δ *is true-in-M, then* φ *is true-in-M also.*

Metatheorem 2. $\Delta \nvdash_{\mathbb{C}^+} \bot \quad \Rightarrow \quad \Delta$ *has a countable model.*

Proof. By easy application of Henkin's method. It turns out that the only structural feature of the deducibility relation that is needed for this result is CUT FOR ABSURDITY (Corollary 1), which holds for Classical Core Logic $\mathbb{C}^+$. □

Metatheorem 3. Δ *has no model* $\quad \Rightarrow \quad \Delta \vdash_{\mathbb{C}^+} \bot$.

Proof. Suppose for conditional proof that Δ has no model. Suppose for classical *reductio ad absurdum* that $\Delta \nvdash_{\mathbb{C}^+} \bot$. By Metatheorem 2, Δ has a (countable) model. This contradicts our supposition for conditional proof. Hence by classical *reductio ad absurdum* we may conclude $\Delta \vdash_{\mathbb{C}^+} \bot$. □

Metatheorem 3 tells us that Classical Core Logic $\mathbb{C}^+$ deductively reveals all incoherencies (non-satisfiabilities).

Metatheorem 4. (Soundness of $\mathbb{C}^+$)

$$\Delta \vdash_{\mathbb{C}^+} \varphi \Rightarrow \; \Delta \models \varphi \,.$$

Metatheorem 4 is no surprise, since $\mathbb{C}^+$ is a subsystem of Classical Logic. Classical Core Logic $\mathbb{C}^+$ also enjoys the following form of completeness.

Metatheorem 5. (Completeness of $\mathbb{C}^+$)

$$(\Delta \textit{ has a model and } \Delta \models \varphi) \;\Rightarrow\; \Delta \vdash_{\mathbb{C}^+} \varphi \,.$$

VI. EXPLICATION OF A LOGIC'S BEING SUITED FOR SCIENCE

We shall now assemble $\mathcal{R}$, our explication of the properties required of a logic for it to be 'suited for science'. Soundness of proof is an obvious *sine qua non*:

$$\Delta \vdash_{\mathbb{C}^+} \varphi \Rightarrow \; \Delta \models \varphi.$$

Recall Metatheorem 3:

$$\Delta \text{ has no model} \;\Rightarrow\; \Delta \vdash_{\mathbb{C}^+} \bot \,.$$

This marks one end of a spectrum of 'logical needs'. Our logical system must enable us to detect all incoherencies. Classical Core Logic $\mathbb{C}^+$ does this.

At the other end of the spectrum, we have the following special case of Metatheorem 5, by setting $\Delta = \emptyset$:

$$\varphi \text{ is true in every model} \;\Rightarrow\; \vdash_{\mathbb{C}^+} \varphi \,.$$

Our logical system must enable us to prove all logical truths. Classical Core Logic $\mathbb{C}^+$ does this.

As another special case of Metatheorem 5 we have, in the middle range of the spectrum, so to speak, for non-empty Δ:

$$\left.\begin{array}{c} \Delta \text{ has a model} \\ \varphi \text{ is falsifiable} \\ \Delta \models \varphi \end{array}\right\} \;\Rightarrow\; \Delta \vdash_{\mathbb{C}^+} \varphi \,.$$

Our logical system must enable us to derive, from any satisfiable set Δ of axioms—as is typically the case in *mathematics*—every sentence φ that is not logically true but follows logically from Δ. (Any proof of such a sentence φ from Δ must accordingly have as a premise at least one of the axioms in Δ.) Classical Core Logic $\mathbb{C}^+$ does this.

Likewise, in the *empirical testing of scientific theories* (assuming these are expressed at first order—Quine's 'Grade A idiom'), we need to be able to derive all possible predictions φ on the assumption that certain boundary and initial conditions will hold. The latter must at least be logically consistent with the theories being tested. The set Δ involved would therefore contain both the scientific hypotheses being tested, and the statements of the relevant boundary and initial conditions for the experiment to be carried out. Moreover, any such prediction φ must itself be falsifi*able*—even if it turns out to be true. We have seen that both the mathematical case and the empirical scientific case are instances of the special case of Metatheorem 5 in the 'middle range' of the logical spectrum we have described.

It now follows that Classical Core Logic $\mathbb{C}^+$ enables one to carry out all the derivations that might be called for, both in mathematics and in empirical science.

There is nothing more (that this author can see) that one could demand of a 'background logic for science' than its soundness and these last three spectrum-straddling completeness results. Collectively they satisfy all our epistemic needs in the deductive reasoning involved in mathematics and science. Clearly, the system $\mathbb{C}^+$ caters for all those needs.

We offer the foregoing properties of $\mathbb{C}^+$ as the explicating set $\mathcal{R}$ of requirements on a logic for it to be suited for science. The present author invites rigorous statements of yet further requirements from those who believe they are entitled to require yet more of a logic for it to be 'suited for science'. We also offer the system $\mathbb{C}^+$ as a perfect instance of the counterexample Λ as it figures in the second diagram in Section III.

Note that any defender of Williamson's thesis, if they were to produce some further requirement ρ to augment $\mathcal{R}$, would still face the task of establishing that $\mathbb{C}^+$ does *not* satisfy ρ. Given the track record thus far, however, of all the exigent demands that $\mathbb{C}^+$ *does* satisfy, there is reason to be sanguine that such a defender would not, in the end, prevail. We also note that it would be fruitless for any defender of Williamson's thesis to advance, as a further requirement ρ, that the logic be *relevant*—that is, that it avoid the infamous first and second Lewis Paradoxes, and ensure that the premises of any of its proofs be suitably *relevantly* connected with its conclusion (in a way that would itself require explication). For, by the explicative work and main result of Tennant (2015c), the Core systems already have that covered. Moreover—and ironically—if such a 'relevance requirement' ρ *were* to be proferred, it would be to the great embarrassment of the other side (that is, the anti-substructuralists). Classical Logic fails to satisfy it. So too does Intuitionistic Logic. And this is because these two standard systems are *fully structural*!

VII. WILLIAMSON'S WORRY ABOUT SUBSTRUCTURAL LOGICS

Let us now examine more closely the argument against substructural logics that Williamson (2018) advances. On the strength of this argument, Williamson concludes that substructural logics are 'ill-suited to acting as background logics for science'. Since the Core system $\mathbb{C}^+$ is a substructural logic as explained above, this conclusion would imply that that system too is 'ill-suited to acting as [a] background [logic] for science'. This latter contention, however, collides with the foregoing metatheorems concerning $\mathbb{C}^+$. The question therefore arises: where is the mistake in Williamson's argument?

Let us focus on the relevant part of Williamson's reasoning. At p. 413 one reads the following:

> Closure operations have some standard structural features:[fn]
>
> i. If $\Gamma \subseteq \Delta$ then $\mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Delta)$
> ii. $\Gamma \subseteq \mathrm{Cn}(\Gamma)$
> iii. $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) \subseteq \mathrm{Cn}(\Gamma)$
>
> Arguably, all three features are assumed in the ordinary testing of scientific theories. There is no limit to the length of permitted chains of reasoning, so consequences of consequences of Γ are consequences of Γ, as (iii) says. Even a zero-length chain of reasoning counts, so members of Γ are consequences of Γ, as (ii) says. Finally, if $\Gamma \subseteq \Delta$, then we can trivially reason from Δ to each member of Γ; since we can reason from Γ to each member of $\mathrm{Cn}(\Gamma)$, $\mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Delta)$, as (i) says.[fn]
> Together, (i)–(iii) imply all the standard structural rules for a consequence relation, including Cut, which in this notation says: if $A \in \mathrm{Cn}(\Gamma)$ and $B \in \mathrm{Cn}(\Delta \cup \{A\})$ then $B \in \mathrm{Cn}(\Gamma \cup \Delta)$[fn]
> . . .
> In general, substructural logics are ill-suited to acting as background logics for science.

The defect in this argument is where Williamson writes

> There is no limit to the length of permitted chains of reasoning, *so consequences of consequences of* Γ *are consequences of* Γ, as (iii) says. [Emphasis added.]

We note with interest that the way the operation Cn is being deployed here by Williamson makes it clear that it is to be understood as being generated by the *deducibility relation* of the logic in question, presumably generated by the rules of inference that the system affords. With the important phrase 'permitted chains of reasoning', Williamson is clearly talking of inference and proof within a deductive system. Our ensuing critique of Williamson's argument can therefore, in all fairness, be carried out with reference only to the (syntactic) deducibility relation, rather than with reference to any accompanying *semantic* relation of logical consequence.[12]

Let us now examine how what we have set out above about the Core systems applies to what Williamson has argued. For Williamson's Γ in the quote above, the reader is invited to take $\{P, \neg P\}$. One $\mathbb{C}$-consequence of this Γ is $P \vee Q$.

[12] Should the defender of Williamson wish to resort to the semantic construal of 'chains of reasoning', however, we shall content ourselves here with entering the comment that *that* move on Williamson's behalf can be blocked as well.

Another $\mathbb{C}$-consequence is $\neg P$ (trivially). A $\mathbb{C}$-consequence of these two $\mathbb{C}$-consequences of this Γ is Q. By Williamson's principle (iii), Q should be a $\mathbb{C}$-consequence of this Γ, i.e., of $\{P, \neg P\}$. But it is not. Indeed, Q is not even a $\mathbb{C}^+$-consequence of $\{P, \neg P\}$. The whole point of the Core systems is that they avoid Lewis's notorious First Paradox (at the level of the turnstile).

The reader might say: 'Did not Williamson give a completely general argument for his principle (iii)?' The answer is affirmative, and it remains to enter a judgement as to the validity of that argument.

Bear in mind that the argument was intended to render the conclusion that (unrestricted) CUT should hold. The very least that one requires of an argument of this kind is that it not be question-begging. It is Williamson's claim that *consequences of consequences of* Γ *are consequences of* Γ that is in error. The sole reason he offered for this claim was that *there is no limit to the length of permitted chains of reasoning*. *So*, he said, *consequences of consequences of* Γ *are consequences of* Γ. What this amounts to is the following non-sequitur, in the context where sequent calculi are chosen as the formal systems for regimenting the reasoning involved:

> Sequent proofs can be as long as one likes; *so*, one can apply (unrestricted) CUT in forming them.

Against this, we simply point out: sequent proofs in Classical Core Logic can be as long as one likes, but the system contains no rule of CUT.

What the observation about unbounded length of sequent proofs really entails is only that one might wish to 'string proofs together' (not necessarily always linearly), by superimposing the conclusion of one onto a premise of another. According to Metatheorem 1, however, the epistemological point of doing so is more than adequately accommodated in our use of the Core systems. Proofs Π and Σ that are strung together (the conclusion φ of the sequent proved by Π being a premise of the sequent proved by Σ),

$$\begin{array}{cc} \Pi & \Sigma \\ \Delta : \varphi & \varphi, \Gamma : \psi, \end{array}$$

effectively deliver a proof $[\Pi, \Sigma]$ of (a potentially even stronger) *subsequent* of the target sequent $\Delta, \Gamma : \{\psi\}$ that one is concerned to prove.

Note that we are not saddling Williamson here with an overly 'linear' conception of the structure of deductive reasoning, a structure that in general involves *partial* orderings of sentences in tree-like fashion. The reader should be alerted to the fact that the use of multiple interpolated lemmas (in mathematical reasoning, for example) is wholly accommodated by the Core systems. Suppose one uses sets $\Delta_1, \ldots, \Delta_n$ of one's available set Δ of mathematical axioms to furnish Core proofs $\Pi_1, \ldots, \Pi_n$ of lemmas $\lambda_1, \ldots, \lambda_n$, respectively, and then one uses a further set Δ_0 of axioms, along with the lemmas, to furnish a Core proof Σ of the conclusion θ. On the usual conception of 'stringing proofs together', one would be able to form a tree-like proof (a natural deduction) like this:

$$\underbrace{\begin{array}{ccc} & \Delta_1 & & \Delta_n \\ & \Pi_1 & \cdots & \Pi_n \\ \Delta_0, & (\lambda_1), & & ,(\lambda_n) \end{array}}_{\begin{array}{c}\Sigma \\ \theta\end{array}} .$$

This would be a proof of the conclusion θ from the combined set of axioms $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$. In the standard Gentzen–Prawitz system of natural deduction, one can form the proof by accumulation exactly as depicted. And if one were working instead in the standard *sequent* calculus of Gentzen, which contains the rule of CUT, one could apply the latter rule n times to the sequent proofs involved, to the same overall effect:

$$\dfrac{\begin{array}{c}\Pi_n\\ \Delta_n : \lambda_n\end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\Pi_2\\ \Delta_2 : \lambda_2\end{array} \qquad \dfrac{\begin{array}{c}\Pi_1\\ \Delta_1 : \lambda_1\end{array} \qquad \begin{array}{c}\Sigma\\ \Delta_0, \lambda_1, \ldots, \lambda_n : \theta\end{array}}{\Delta_1, \Delta_0, \lambda_2, \ldots, \lambda_n : \theta}\,\text{CUT}}{\Delta_2, \Delta_1, \Delta_0, \lambda_3, \ldots, \lambda_n : \theta}\,\text{CUT} \\ \vdots \\ \Delta_{n-1}, \ldots, \Delta_1, \Delta_0, \lambda_n : \theta \end{array}}{\Delta_n, \Delta_{n-1}, \ldots, \Delta_1, \Delta_0 : \theta}\,\text{CUT}$$

Now, the natural-deduction and sequent-calculus formulations of the Core systems, as already remarked, are deeply isomorphic.[13] And they deliver the same cumulative result (indeed, very often, something *even better*—see the italicized phrases in the sentence that follows) in the following sense. As Metatheorem 1 tells us, there is an effective binary operation $[\Pi, \Sigma]$ on Classical Core proofs that ensures that

$$[\Pi_n, [\Pi_{n-1}, \ldots, [\Pi_2, [\Pi_1, \Sigma]] \ldots]]$$

is a Classical Core proof either of θ or of $\perp$ from *some (possibly proper) subset of* $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$.

The admissibility of CUT therefore guarantees deductive progress via 'transitivity' of deduction in a way that is, if anything, *even stronger* than what is afforded the Classical logician by having the rule of CUT *in* their logical system. Whenever the 'exact' target sequent that is to be 'had' by unrestricted applications of CUT *within* the Classical Logician's system happens to elude the Classical Core logician, the sequent that the latter comes up with, as a result of applying the operation $[\,,]$ in the manner illustrated above, will be

[13] See footnote 1.

a possibly *logically stronger*, and certainly no weaker, subsequent of the 'exact' target sequent.[14]

Summary: The foregoing considerations justify the following methodological and metalogical conclusions. *Contra* Williamson,

(i) the substructural logic $\mathbb{C}^+$ is adequate for the regimentation of the deductive reasoning involved in the hypothetico-deductive method in the natural sciences (as a matter of fact, so too is $\mathbb{C}$);

(ii) the substructural logic $\mathbb{C}^+$ is adequate for the regimentation of the deductive reasoning involved in classical mathematics; and

(iii) the substructural logic $\mathbb{C}$ is adequate for the regimentation of the deductive reasoning involved in constructive and intuitionistic mathematics.

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The Ohio State University, USA

* *The paper has benefited from constructive comments by referees.*

[14] I owe to Ethan Brauer the suggestion that this important point be driven home.