Perfect Proofs at First Order

by

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Abstract

In this note we extend a remarkable result of Brauer [2024] concerning propositional Classical Core Logic. We show that it holds also at first order. This affords a soundness and completeness result for Classical Core Logic. The $\mathbb{C}^+$-provable sequents are exactly those that are uniform *substitution instances* of perfectly valid sequents, i.e. sequents that are valid and that need every one of their sentences in order to be so. Brauer [2020] showed that the notion of perfect validity itself is unaxiomatizable. In the Appendix we use his method to show that our notion of relevant validity in Tennant [2024] is likewise unaxiomatizable. It would appear that the taking of substitution instances is an essential ingredient in the construction of a semantical relation of consequence that will be axiomatizable—and indeed, by the rules of proof for Classical Core Logic.

1 Introduction

Core Logic $\mathbb{C}$ is both constructive and relevant. Its classicized extension is Classical Core Logic $\mathbb{C}^+$. Both systems have been—and are—defined as *proof systems* based on rules of inference.[1] In this study we concentrate on $\mathbb{C}^+$.

Unlike Classical Logic **C**, which contains fallacies of relevance, the deducibility relation $\vdash_{\mathbb{C}^+}$ has not heretofore been shown to be coextensive with any known semantical relation of logical consequence relating premise-sets to conclusions. The standard adequacy equivalence

$$\Delta \vdash_{\mathbf{C}} \varphi \;\Leftrightarrow\; \Delta \models \varphi$$

(of soundness and completeness combined) has been achieved by dint of having $\vdash_{\mathbf{C}}$ 'beefed up' by the rule of Ex Falso Quodlibet in the case of natural deduction, and by the so-called structural rules of Thinning and Cut in the case of the sequent calculus. These deductive features ensure the exact match of $\vdash_{\mathbf{C}}$ to the Tarskian relation $\models$ of semantic consequence, which is *explosive*—every Tarskian model making true every sentence in an unsatisfiable set Δ will make *any* conclusion φ true. (This is because the default reading of 'Every F is G' is that such a claim is true when there are no Fs.) The deducibility and consequence relations for **C**, thanks to the aforementioned features, turn out to be exactly coextensive.

Those deductive features are ones that the core systems of proof eschew. They do so in pursuit of *relevance* between premise-sets and conclusions of proofs. Until now, however, it has been difficult to furnish for the deducibility relation $\vdash_{\mathbb{C}^+}$ an exactly coextensive semantical relation that would yield for $\mathbb{C}^+$ a result directly analogous to the above adequacy equivalence for **C**. It is the *soundness* half of that equivalence for $\mathbb{C}^+$ that has proved to be elusive.

One possible candidate semantic notion that one might consider in this connection is that of perfect validity. A perfectly valid sequent is a sequent that is valid in the Tarskian sense and, moreover, needs every one of its sentences for its validity. All perfectly valid sequents can be proved in $\mathbb{C}^+$. So we would have the 'completeness half' of the sought equivalence for $\mathbb{C}^+$. But the converse 'soundness half' is unattainable. This is for the deep reason that the notion of perfect validity (for finite sequents) is unaxiomatizable.[2]

In Tennant [2024] we proposed a 'double-double turnstile' relation $\Vert\!\models$ for the semantical consequence relation that $\vdash_{\mathbb{C}^+}$ might arguably be aiming

[1] For a full account, see Tennant [2017]. The Law of Excluded Middle, note, is a zero-premise rule of inference.

[2] See Brauer [2020].

to capture. But while $\Vdash$ yields the completeness half of the equivalence sought, and can be understood as motivating the deductive changes that $\mathbb{C}^+$ makes to **C**, $\Vdash$ is nevertheless (like the notion of perfect validity) unaxiomatizable;[3] so the soundness half remained unattainable. $\Vdash$ cannot match $\vdash_{\mathbb{C}^+}$ exactly.

The prospect of finding an alternative notion of semantic consequence to match to $\vdash_{\mathbb{C}^+}$ is now realizable, however, owing to a remarkable result of Brauer [2024]. He established it for *propositional* $\mathbb{C}^+$. In this study we extend Brauer's result to *first-order* $\mathbb{C}^+$. The result is that any given $\mathbb{C}^+$-proof is 'perfectible'—it can be transformed into a $\mathbb{C}^+$-proof every one of whose subproofs establishes a *perfectly valid* sequent; and the sequent established by the given $\mathbb{C}^+$-proof is a substitution instance of the sequent established by its perfected version. The perfectibility theorem for first-order $\mathbb{C}^+$-proofs provides the sought notion of semantical consequence to match exactly to the deducibility relation of Classical Core Logic. The new notion of validity for sequents is this: a sequent is valid just in case it is a substitution instance of a perfectly valid sequent.

2 Preliminaries

We work with a first-order language whose primitive extralogical expressions are names, predicates of arbitrary finite adicity, and parameters. The parameters are name-like, and are to feature in the familiar way in quantifier rules.[4] Sentences have no free variables. Proofs consist only of sentences. Δ, Φ will be sets of sentences. ϕ, ψ, θ will be sentences.

Definition 1. Single-conclusion *sequents have the form* $\Delta : \Phi$ *where* Φ *is at most a singleton. That is, either* $\Phi = \emptyset$ *or* $\Phi = \{\phi\}$ *for some sentence* ϕ.

We shall be dealing here only with single-conclusion sequents. Henceforth 'sequent' will mean 'single-conclusion sequent'.

Definition 2. $\Delta_1 : \Phi_1$ *is a* subsequent *of* $\Delta_2 : \Phi_2$ *just in case* $\Delta_1 \subseteq \Delta_2$ *and* $\Phi_1 \subseteq \Phi_2$. *If either of these containments is proper, then* $\Delta_1 : \Phi_1$ *is a* proper *subsequent of* $\Delta_2 : \Phi_2$.

[3] See the Appendix for a proof of the result that $\Vdash$ is unaxiomatizable.

[4] See, for example, Prawitz [1965] and Tennant [1978].

Definition 3. *A sequent is* perfectly *valid just in case it is valid and has no valid proper subsequent.*

Definition 4. *An* atomic *formula is one of the form $P(t_1, \ldots, t_n)$, where P is an n-place primitive predicate, and $t_1, \ldots, t_n$ are names or parameters or variables.*

Definition 5. *A* substitution *is a mapping on formulas induced in the obvious way by the base move of uniformly replacing atomic formulas with formulas, subject to the constraint that variables, names, and parameters be preserved. If a substitution σ deals with the atomic formulas in φ, and ψ is the result of replacing each atomic formula-occurrence of α in φ with an occurrence of $\sigma(\alpha)$, then we write $\sigma(\varphi) = \psi$. We shall also say that ψ is a* coarsening *of φ induced by the substitution σ.*

We shall take proofs to be proofs in the system of natural deduction for Classical Core Logic $\mathbb{C}^+$. It is an important feature of this system that applications of elimination rules have their premises standing proud, with no non-trivial proof-work above them.

Definition 6. *Following Brauer [2024], we identify as* subproofs *of a proof Π those subtrees of the proof-tree Π whose roots are not major premises for eliminations; and we say that a proof is* perfect *just in case it, and every one of its subproofs, is a proof of a perfectly valid sequent.*

Brauer (*loc. cit.*, at p. 9) also introduced the notion of one formula being a *contraction variant* (abbreviated 'c-variant') of another formula. We can explain this notion as follows.

Definition 7. *φ is an* immediate c-variant *of ψ (formally: $\varphi \rhd_c \psi$) just in case ψ can be obtained from φ by replacing in φ an occurrence of a subformula of the form $\theta \vee \theta$ or of the form $\theta \wedge \theta$ with an occurrence of θ.*

Observation 1. *If $\varphi \rhd_c \psi$, then φ is logically equivalent to ψ. Moreover, ψ will be less complex than φ. Thus the relation $\varphi \rhd_c \psi$ is asymmetric.*

Definition 8. *φ is a* c-variant *of ψ (formally: $\varphi \blacktriangleright_c \psi$) just in case there is a finite sequence*

$$\varphi = \chi_1 \rhd_c \ldots \rhd_c \chi_n = \psi \ .$$

Observation 2. *If $\varphi \blacktriangleright_c \psi$, then φ is logically equivalent to ψ. Moreover, ψ will be less complex than φ. Thus the relation '$\varphi \blacktriangleright_c \psi$' is asymmetric.*

3 Perfectibility

Brauer's crucial insight was that a classical core proof could be *coarsened* to yield a perfect proof of the 'same' overall result. The sameness consists in equivalence *modulo* c-variance. His THEOREM 3, p. 10, for *propositional* $\mathbb{C}^+$, reads:

> If Π is a [classical core] proof of $\Delta : \Phi$, then there is a perfect proof Π' of $\Delta' : \Phi'$ and a substitution σ [on atoms] such that $\sigma\Delta'$ consists of c-variants of the members of Δ and $\sigma\Phi'$ is a c-variant of Φ. That is, if $\Phi = \emptyset$, then $\Phi' = \emptyset$ and if $\Phi = \{\phi\}$ then $\Phi' = \{\varphi'\}$ and $\sigma\phi'$ is a c-variant of ϕ.

Brauer's proof of this result was by induction on the height of Π. Throughout the process of perfecting a classical core proof one remains within the space of classical core proofs.

The picture is this: Given a $\mathbb{C}^+$-proof Π of $\Delta : \Phi$, it can be *perfected* so as to yield a *perfect* $\mathbb{C}^+$-proof Π' of $\Delta' : \Phi'$. This process of perfecting involves 'coarsening' assumptions, which is (roughly) the inverse of taking substitution instances.

We stress that Brauer proved his result for *propositional* $\mathbb{C}^+$ because one would wish the result to hold at first order. For then one could make serious application of it in arguing for the adequacy of first-order $\mathbb{C}^+$, *with an accompanying semantics*, for deductive reasoning in mathematics.

We shall show here how to extend Brauer's THEOREM 3 to first-order Classical Core Logic, employing the methods he introduced. The present author views this as an improvement on his own earlier result in Tennant [1984]. That earlier paper concerned a *multiple-conclusion* version of sequent calculus at first order;[5] but, alas, one *without* the conditional '→' primitive. The latter deficiency meant that Intuitionistic Logic **I** could not be identified as a subsystem of Classical Logic **C**; hence also Core Logic $\mathbb{C}$ could not be identified as a subsystem of Classical Core Logic $\mathbb{C}^+$. Brauer [2024] has fixed that deficiency in the propositional case. But since natural deduction is perforce a *single-conclusion* proof system, it is worthwhile verifying that the proof-perfection method in our earlier paper can be prosecuted fully *à la* Brauer in the natural-deduction setting at first-order, with

[5] To be precise: only $\exists$ was dealt with; but that was acceptable at the time because of the expressive completeness of $\neg$, $\wedge$, and $\exists$ in the classical case.

the full set of primitive logical operators for the constructive case, namely $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, and $\exists$.

All we need to supply is the reasoning in the inductive step of Brauer's proof, to deal with the cases involving terminal applications of introduction and elimination rules for the quantifiers $\forall$ and $\exists$. Brauer has already dealt with the full set of connectives.

4 The formal work

We are treating a first-order language that contains only predicates and names as extra-logical primitives. In this setting substitutions will be mappings on the atomic formulas, whose values are formulas of the same adicity (and same alphabetical choice of free variables) and with the same names and parameters occurring in them. It should be clear that when we speak of *terms t* within sentences within a proof (which consists only of *sentences*), we can only be countenancing names and parameters, not variables. Moreover, terms remain invariant and 'in place' under substitutions.

Definition 9. *The left-associated conjunction of sentences* $\phi_1, \ldots, \phi_n$ *is defined inductively as follows:*

$$\bigwedge_{j=1}^{1} \phi_j = \phi_1$$

$$\bigwedge_{j=1}^{k} \phi_j = \bigwedge_{j=1}^{k-1} \phi_j \wedge \phi_k$$

Lemma 1. *For all* $n > 0$ *and for all* Δ *not containing any of* $\phi_1, \ldots, \phi_n$*, from a proof*

$$\begin{array}{c} \underbrace{\Delta, \phi_1, \ldots, \phi_n} \\ \Pi \\ \theta \end{array}$$

one can construct a proof

$$\underbrace{\Delta\ ,\ \bigwedge_{j=1}^{n}\phi_j}_{\vdots}\\ \theta$$

by appending $(n-1)$ *terminal steps of* $\wedge$*E.*

Proof. By induction on n. The basis ($n = 1$) is obvious, since the sought proof is already in hand. For the inductive step (to prove the claim for $n + 1$), the inductive hypothesis takes the form of the sought result. For the inductive step, suppose Δ does not contain any of $\phi_1, \ldots, \phi_n, \phi_{n+1}$. Invoke IH with its universally quantified Delta instantiated to $\Delta \cup \{\phi_{n+1}\}$. This vouchsafes the proof

$$\underbrace{\Delta\ ,\ \phi_{n+1}\ ,\ \bigwedge_{j=1}^{n}\phi_j}_{\Pi^*}\\ \theta$$

which can then be extended with one more application of $\wedge$E to yield the proof sought in the inductive step:

$$\dfrac{\begin{array}{c}\bigwedge_{j=1}^{n+1}\phi_j,\ \text{i.e.,}\\ \bigwedge_{j=1}^{n}\phi_j \wedge \phi_{n+1}\end{array} \qquad \begin{array}{c}\underbrace{\Delta\ ,\ \overline{\phi_{n+1}}^{(i)}\ ,\ \overline{\bigwedge_{j=1}^{n}\phi_j}^{(i)}}_{\Pi^*}\\ \theta\end{array}}{\theta}{\scriptstyle (i)}$$

□

We have made the necessary adjustments to the definitions of atomic formulas and of substitutions to enable us to make the transition from the propositional case to the first-order case. (See Definition 2 and Definition 5.) The basis step in the proof of Brauer's THEOREM 3 for the propositional case is straightforwardly effected for the first-order case. Note also that the re-letterings called for in the inductive cases dealing with steps such as $\wedge$I can be systematically effected by, say, adding numerical superscripts to primitive predicates to ensure vocabulary disjointness of the perfected subproofs.

We turn now to the task of carrying out the four cases in the inductive step (in extending Brauer's theorem to deal with first-order Classical Core Logic) that deal with the quantifier rules. We shall take them in order of increasing 'difficulty': $\forall$I, $\exists$I, $\exists$E. and $\forall$E.

We shall follow the usual convention of writing ψ^x_t as ψt; and we shall use primes, as Brauer did, to indicate c-variant coarsenings. Thus $\psi t'$ is a c-variant coarsening of ψt; and, by Observation 2, $\sigma(\psi t')$ is logically equivalent to ψt. If necessary, to make the scope of the priming clearer, we shall say $\psi'(t)$ is a coarsening of $\psi(t)$. This holds, of course, when there is a uniform non-trivial substitution σ on lexical primitives (here, just: predicates) such that $\sigma[\psi'(t)] = \psi(t)$.

As Brauer has observed[6],

> one difference between the propositional case and the quantified case is that the coarsening procedures will not in general ensure that two parallel subproofs[, after coarsening, will be] disjoint in all their non-logical vocabulary. They *will* ensure that the relation symbols occurring in parallel subproofs will be disjoint, but there might be an individual constant that occurs in both subproofs.

This does not, however, compromise the reasoning in the inductive cases below that deal with the quantifier rules.

The case where Π ends with $\forall$I.

Let the classical core proof Π end with an application of $\forall$I, whose immediate subproof we shall call Π_1:

$$\Pi \quad = \quad \cfrac{\begin{matrix}\Delta_1\\ \Pi_1\\ \psi\end{matrix}}{\forall x \psi^a_x}$$

[6]Personal communication, May 14, 2024.

Suppose IH holds for Π_1:

$$\begin{array}{c}\Delta_1\\ \Pi_1\\ \psi\end{array}$$

So there is a perfect proof that coarsens Π_1, whose premises and conclusion are c-variants of those of Π_1. Let the perfect proof in question be called Π_1'. The conclusion of Π_1' is the coarsened conclusion ψ', which, we must remember, contains the parameter a. The premises of Π_1' are the coarsenings (forming the set Δ_1') of the members of Δ_1. So we have

$$\Pi_1' \quad = \quad \begin{array}{c}\Delta_1'\\ \Pi_1'\\ \psi'\end{array}$$

Since a did not occur in any member of Δ_1, it follows that a does not occur in any member of Δ_1'. Therefore we can apply ∀I to obtain

$$\Pi' \quad = \quad \dfrac{\begin{array}{c}\Delta_1'\\ \Pi_1'\\ \psi'\end{array}}{\forall x[\psi']^a_x}$$

The members of Δ_1' are c-variants of the members of Δ_1; and the new conclusion $\forall x[\psi']^a_x$ is a c-variant of $\forall x\psi^a_x$.

The case where Π ends with ∃I.

Let the classical core proof Π end with an application of $\exists I$, whose immediate subproof we shall call Π_1:

$$\Pi \quad = \quad \dfrac{\begin{array}{c}\Delta_1\\ \Pi_1\\ \psi^x_t\end{array}}{\exists x\psi}$$

Suppose IH holds for Π_1:

$$\begin{array}{c}\Delta_1\\ \Pi_1\\ \psi^x_t\end{array}$$

So there is a perfect proof that coarsens Π_1, whose premises and conclusion are c-variants of those of Π_1. Let the perfect proof in question be called Π_1'. The conclusion of Π_1' is the coarsened conclusion $[\psi^x_t]'$. The premises of Π_1' are the coarsenings (forming the set Δ_1') of the members of Δ_1. So we have

$$\Pi_1' \quad = \quad \begin{array}{c} \Delta_1' \\ \Pi_1' \\ [\psi^x_t]' \end{array}$$

We can now apply ∃I to obtain

$$\Pi' \quad = \quad \dfrac{\begin{array}{c} \Delta_1' \\ \Pi_1' \\ [\psi^x_t]' \end{array}}{\exists x[\psi']}$$

This is because

$$[\psi']^x_t = [\psi^x_t]'.$$

The members of Δ_1' are c-variants of the members of Δ_1; and the new conclusion $\exists x[\psi^x_t]'$ is a c-variant of $\exists x\psi^x_t$.

The case where Π ends with ∃E.

Let the classical core proof Π end with an application of ∃E, whose immediate subproof we shall call Π_1:

$$\Pi \quad = \quad \dfrac{\exists x\psi \qquad \begin{array}{c} \underbrace{\overset{(i)}{\overline{\psi^x_a}} \ , \ \Delta_1} \\ \Pi_1 \\ \theta \end{array}}{\theta}{\scriptstyle(i)}$$

Suppose IH holds for Π_1:

$$\begin{array}{c} \underbrace{\psi^x_a \ , \ \Delta_1} \\ \Pi_1 \\ \theta \end{array}$$

So there is a perfect proof that coarsens Π_1, whose premises and conclusion are c-variants of those of Π_1. Let the perfect proof in question be called Π_1'. The conclusion of Π_1' is the coarsened conclusion θ'. The premises of Π_1' are the coarsenings (forming the set Δ_1') of the members of Δ_1, along with the coarsenings (note the plural) of the parametric assumption (for subsequent discharge) that instantiated $\exists x\psi$. So we have, for some $n \geq 1$,

$$\Pi_1' \quad = \quad \begin{array}{c} \underbrace{[\psi^x_a]_1', \ldots, [\psi^x_a]_n' \,,\ \Delta_1'} \\ \Pi_1' \\ \theta' \end{array}$$

Note that each $[\psi^x_a]_k'$ $(1 \leq k \leq n)$ involves the parameter a; and the parameter a occurs in no member of Δ_1' and does not occur in θ'.

By Lemma 1, we can construct a proof

$$\begin{array}{c} \underbrace{\Delta_1' \,,\ \bigwedge_{j=1}^{n}[\psi^x_a]_j'} \\ \vdots \\ \theta' \end{array}$$

We can now apply $\exists$I to obtain

$$\dfrac{\exists y\left(\bigwedge_{j=1}^{n}[\psi^x_a]_j'\right)^a_y \qquad \begin{array}{c} \underbrace{\Delta_1' \,,\ \overline{\bigwedge_{j=1}^{n}[\psi^x_a]_j'}^{(i)}} \\ \vdots \\ \theta' \end{array}}{\theta'}(i)$$

Note that the major premise of this terminal step of $\exists$E is a c-variant of $\exists x\psi$, and does not contain the parameter a.

The case where Π ends with $\forall$E.

Let the classical core proof Π end with an application of $\forall E$, whose immediate subproof we shall call Π_1:

$$\Pi \quad = \quad \cfrac{\forall x\psi \qquad \begin{array}{c} \underbrace{\overline{\psi^x_{t_1}}^{(i)} \;,\; \ldots \;,\; \overline{\psi^x_{t_n}}^{(i)} \;,\; \Delta_1}_{\Pi_1} \\ \theta \end{array}}{\theta}\,(i)$$

Suppose IH holds for Π_1:

$$\begin{array}{c} \underbrace{\psi^x_{t_1} \;,\; \ldots \;,\; \psi^x_{t_n} \;,\; \Delta_1}_{\Pi_1} \\ \theta \end{array}$$

So there is a perfect proof that coarsens Π_1, whose premises and conclusion are c-variants of those of Π_1. Let the perfect proof in question be called Π'_1. The conclusion of Π'_1 is the coarsened conclusion θ'. The premises of Π'_1 are the coarsenings (forming the set Δ'_1) of the members of Δ_1, along with the coarsenings (to be described in greater detail presently) of the various assumptions (for subsequent discharge) that were instantiations of $\forall x\psi$. For each k ($1 \leq k \leq n$) the assumption $\psi^x_{t_k}$ will have a certain number of coarsenings (call that number m_k) functioning as undischarged assumptions in Π'_1. So the assumption $\psi^x_{t_k}$ in Π_1 gets replaced, upon coarsening of Π_1, by these m_k coarsened assumptions

$$\psi'_1(t_k), \ldots, \psi'_{m_k}(t_k)$$

within Π'_1. We can form the t_k-specific left-associated conjunction of these coarsenings of $\psi^x_{t_k}$:

$$\bigwedge_{i=1}^{m_k} \psi'_i(t_k)$$

Recall $1 \leq k \leq n$. Now we can form the 'big' left-associated conjunction of *all* the coarsened assumptions of Π'_1:

$$\bigwedge_{k=1}^{n} \left(\bigwedge_{i=1}^{m_k} \psi'_i(t_k) \right)$$

We shall use

$$\forall x \bigwedge_{k=1}^{n}\left(\bigwedge_{i=1}^{m_k}\psi'_i(x)\right)$$

as the major premise of the terminal application of ∀E that is needed in order to form the coarsened proof Π'. Note that this last displayed universal is a c-variant of $\forall x\psi$.

The terminal step of ∀E in Π' will be immediately preceded, of course, by a chain of applications of $\wedge E$ appended to the coarsened proof Π'_1 vouchsafed by Inductive Hypothesis. These applications of $\wedge E$ will bring down the (coarsened) conclusion θ' of Π'_1 as their own conclusions; and they will discharge one-by-one all the coarsened assumptions in the foregoing big conjunction—in the way justified by Lemma 1, which secures the existence of the proof Σ embedded in the following display.[7] The fully coarsened proof in this case will therefore be

$$\Pi' \qquad = \qquad \dfrac{\forall x \bigwedge_{k=1}^{n}\left(\bigwedge_{i=1}^{m_k}\psi'_i(x)\right) \qquad \begin{array}{c} \underbrace{\Delta' \;,\; \overline{\bigwedge_{k=1}^{n}\left(\bigwedge_{i=1}^{m_k}\psi'_i(t_k)\right)}^{\,(i)}} \\ \Sigma \\ \theta' \\ \hline \text{multiple } \wedge\text{Es} \\ \hline \theta' \end{array}}{\theta'}\,{\scriptstyle (i)}$$

A Appendix

Brauer's result establishing the unaxiomatizability of the notion of *perfect* validity (which we mentioned in footnote 2) can be extended to the notion ⊫ of *relevant* validity. It is impossible effectively to enumerate all ***and only*** the finite ***relevantly valid*** sequents, i.e. the finite sequents of the following three forms:

[7]The application of Lemma 1 is a matter of some subtlety, given how the conjuncts of the 'big' conjunction have been arranged. One will apply Lemma 1 for each of the innermost left-associated conjunctions, and thereafter apply Lemma 1 again for the outermost left-associated conjunction of the innermost ones. The reader will be spared pictorial details.

1. $\Delta : \bot$
2. $\emptyset : \varphi$ where φ is logically true
3. $\Delta : \varphi$ where Δ is satisfiable and $\Delta \models \varphi$

It is worth noting that this impossibility proof proceeds by considering, as Brauer did, sequents of the form $\varphi \wedge A : A$, where A is an atom not occurring in φ. It would be appropriate to call these *Brauerian* sequents.

Metatheorem 1. *There is no effective enumeration of all and only the finite relevantly valid sequents.*

Proof. Adopt as the main *reductio* assumption that there is an effective enumeration of all and only the finite relevantly valid sequents. Call this effective enumeration π.

Consider the Brauerian sequent-form $\varphi \wedge A : A$, for arbitrary φ and atomic A not occurring in φ. Extract from π an effective sub-enumeration (call it ε) of all and only the relevantly valid sequents of this (effectively decidable) Brauerian form.

A is an atom and is not identical to $\bot$. Since the conclusion A of the relevantly valid sequent $\varphi \wedge A : A$ is not $\bot$, it follows that $\varphi \wedge A$ is satisfiable. That in turn requires that φ be satisfiable. Thus we have

$$\text{if } \varphi \wedge A : A \text{ is relevantly valid, then } \varphi \text{ is satisfiable.} \qquad (1)$$

For the converse of (1), suppose that φ is satisfiable. Let M be an interpretation of the non-logical vocabulary in φ that makes φ true. Since A is an atom not occurring in φ, one can extend M to an interpretation making $\varphi \wedge A$ true simply by assigning the truth-value T to the atom A. Thus $\varphi \wedge A$ is satisfiable. Moreover, $\varphi \wedge A \models A$. It follows by the third definitional clause above for $\Vdash$ that $\varphi \wedge A \Vdash A$.

We have therefore shown that

$$\text{if } \varphi \text{ is satisfiable, then } \varphi \wedge A : A \text{ is relevantly valid.} \qquad (2)$$

It follows, then—combining (1) and (2)—that

$$\varphi \wedge A : A \text{ is relevantly valid if and only if } \varphi \text{ is satisfiable.} \qquad (3)$$

Thus one can extract from our effective enumeration ε of all and only the relevantly valid sequents of the form $\varphi \wedge A : A$, for arbitrary φ and atomic A

not occurring in φ, an effective enumeration of all and only the satisfiable sentences φ.

Result: the satisfiable sentences can be effectively enumerated.

Next we show (still entertaining our main *reductio* assumption) that the *un*satisfiable sentences can be effectively enumerated.

The following is obvious:

$$\text{if } \varphi : \emptyset \text{ is relevantly valid, then } \varphi \text{ is unsatisfiable.} \tag{4}$$

So too is the following:

$$\text{if } \varphi \text{ is unsatisfiable, then } \varphi : \emptyset \text{ is relevantly valid.} \tag{5}$$

It follows, then—combining (4) and (5)—that

$$\varphi : \emptyset \text{ is relevantly valid if and only if } \varphi \text{ is unsatisfiable.} \tag{6}$$

A sequent's being of the form $\varphi : \emptyset$ is an effectively decidable matter. Thus from the effective enumeration π one can extract an effective sub-enumeration of all and only the relevantly valid sequents of the form $\varphi : \emptyset$; and therefore, in effect, of all and only the unsatisfiable sentences φ.

Result: the unsatisfiable sentences can be effectively enumerated.

Summarizing: we have that *the satisfiable sentences can be effectively enumerated*, and *the unsatisfiable sentences can be effectively enumerated*.

Suppose now that one is given an arbitrary sentence θ, and asked whether it is unsatisfiable. One can effectively decide whether this is the case by going down the two enumerations until one discovers θ on one of them. (The sentence θ *will* eventually turn up on one of the enumerations.)

Now suppose that one is given an arbitrary sentence ξ and asked whether it is logically true. One can determine the correct answer by answering the question whether its negation $\neg\xi$ is unsatisfiable. We can answer this question effectively, as just shown. Hence there is an effective method for determining, of an arbitrary given sentence, whether it is logically true.

This contradicts Church's Undecidability Theorem (Church [1936]). Thus there can be no such enumeration as π, which was supposed to be an effective enumeration of all and only the finite relevantly valid sequents.

□

References

Ethan Brauer. Relevance for the classical logician. *Review of Symbolic Logic*, 13:436–457, 2020.

Ethan Brauer. Coarsening Natural Deduction Proofs I: Finding Perfect Proofs. *Journal of Logic and Computation*, 2024. doi: https://doi.org/10.1093/logcom/exad077.

Alonzo Church. A note on the Entscheidungsproblem. *Journal of Symbolic Logic*, 1(1):40–41. *Correction, ibid.*, pp. 101–2, 1936.

Dag Prawitz. *Natural Deduction: A Proof-Theoretical Study*. Almqvist & Wiksell, Stockholm, 1965.

Neil Tennant. *Natural Logic*. Edinburgh University Press, 1978.

Neil Tennant. Perfect Validity, Entailment and Paraconsistency. *Studia Logica*, XLIII:179–198, 1984.

Neil Tennant. *Core Logic*. Oxford University Press, Oxford, 2017. doi: https://doi.org/10.1093/oso/9780198777892.001.0001.

Neil Tennant. The Logic for Mathematics Without Ex Falso Quodlibet. *Philosophia Mathematica*, 2024. doi: https://doi.org/10.1093/philmat/nkae001.