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Which 'Intensional Paradoxes' are Paradoxes?

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Abstract

We begin with a brief explanation of our proof-theoretic criterion of paradoxicality—its motivation, its methods, and its results so far. It is a proof-theoretic account of paradoxicality that can be given in addition to, or alongside, the more familiar semantic account of Kripke. It is a question for further research whether the two accounts agree in general on what is to count as a paradox. It is also a question for further research whether and, if so, how the so-called Ekman problem bears on the investigations here of the intensional paradoxes. Possible exceptions to the proof-theoretic criterion are Prior's Theorem and Russell's Paradox of Propositions—the two best-known 'intensional' paradoxes. We have not yet addressed them. We do so here. The results are encouraging. §1 studies Prior's Theorem. In the literature on the paradoxes of intensionality, it does not enjoy rigorous formal proof of a *Gentzenian* kind—the kind that lends itself to proof-theoretic analysis of recondite features that might escape the attention of logicians using non-Gentzenian systems of logic. We make good that lack, both to render the criterion applicable to the formal proof, and to see whether the criterion gets it right. Prior's Theorem is a theorem in an *unfree*, classical, quantified propositional logic. But if one were to insist that the logic employed be *free*, then Prior's Theorem would not be a *theorem* at all. Its proof would have an *undischarged assumption*—the 'existential presupposition' that the proposition $\forall p(Qp \rightarrow \neg p)$ exists. Call this proposition ϑ. §2 focuses on ϑ. We analyse a Priorean *reductio* of ϑ along with the possibilitate $\Diamond\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q))$. The attempted *reductio* of this premise-pair, which is constructive, cannot be brought into normal form. The criterion says we have not straightforward inconsistency, but rather genuine paradoxicality. §3 turns to problems engendered by the proposition $\exists p(Qp \wedge \neg p)$ (call it η) for the similar possibilitate $\Diamond\forall q(Qq \leftrightarrow (\eta \leftrightarrow q))$. The attempted disproof of this premise-pair—again, a constructive one—cannot succeed. It cannot be brought into normal form. The criterion says the premise-pair is a genuine paradox. In §4 we show how Russell's Paradox of Propositions, like the Priorean intensional paradoxes, is to be classified as a genuine paradox by the proof-theoretic criterion of paradoxicality.

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1 Introduction

1.1 Some History by Way of Background

We begin with some broad brush strokes of historical commentary by way of explanation of both the purpose of this study and its choice of methods.

The usual way that paradoxes are set out for thinkers first becoming acquainted with them is to use ordinary language as far as possible, and, at a push, to elevate matters to the level of what is called 'informal rigor'. If this latter route is taken, the informally rigorous deductive reasoning can be taken as raw material for fuller formalization or 'regimentation', using some favored proof system to that end. It is fair to say that the vast majority of such formalizations involve systems of formal proof in which the reasoning is set out in the Lemmon–Mates–Fitch style that is so popular with teachers of Introductory Logic courses for Philosophy students.

Serious work in proof theory, however, is much better carried out in Gentzenian systems of natural deduction or sequent calculus. Their main distinguishing feature (compared to proof-formating in Lemmon–Mates–Fitch style) is the *clarity of exposure* (to the naked eye, as it were) of the essentially *tree-like structure* of dependencies of conclusions on assumptions. Moreover, by focusing on proofs that really *are* trees—both in an abstract mathematical sense and *manifestly* so, on the printed page—proof theorists are able to deal more expertly with concepts and methods such as reduction procedures, normalization of proofs, subformulae properties, and normal forms. The deeper insights of post-Gentzenian proof theory are difficult to acquire if the student's grounding is in the Lemmon–Mates–Fitch tradition.

In the hands of writers such as Dummett, Prawitz, Martin-Löf, and Schroeder-Heister, modern proof theory has been a technical gift that keeps on giving *philosophically*. Its full contemporary flowering into the field of *proof-theoretic semantics* bears testimony to this.[1] The resources of proof theory have grown in stature and scope to the point where they afford the real prospect of being deployable in 'equal partnering', as it were, with model-theoretic semantics. And, just as Kripke 'pushed the envelope' to provide a *semantic* modeling of truth and paradox in the *semantically closed* languages whose treatment Tarski—the founder of model-theoretic semantics and the pioneering *language-stratifier*—refused to undertake, so too one could expect exponents of proof theory in due course to venture beyond their previous confines of formalizing mathematical reasoning and their focus on (relative-)consistency problems in the foundations of mathematics. They could broaden the scope of application of their methods. They could address, say, the problem of paradox in semantically closed languages with their own distinctively proof-theoretic ideas.

[1] See, for example, [5], [14] and [15].

1.2 The Proof-theoretic Criterion of Paradoxicality

It was against this brief historical background that in [18] we proposed what we called a *proof-theoretic criterion* for paradoxicality. At the time of that writing, the paradigm cases of paradox were of course the Liar Paradox and Russell's Paradox in set theory. The seminal tract [8] had indeed sought to get to grips with Russell's Paradox, and offered the observation that its 'proof' (using rules for *naive* set theory that Prawitz formulated) resisted normalization. Our *Dialectica* paper took the germ of that idea and applied it to all the usual logico-semantic paradoxes arising from semantic closure: the Liar; the Wedge Liar (a.k.a. the postcard paradox); Grelling's paradox of heterologicality; the Curry paradox; and Tarski's quotational paradox.

The proof-theoretic criterion that we proposed focused on the disproof (i.e., proof of absurdity) in these cases (or the proof of an arbitrary conclusion in the case of the Curry). The proofs or disproofs were most conveniently taken as Gentzen–Prawitz natural deductions. The test was to see whether the (dis)proof in question could be brought into normal form, by means of allowable reduction procedures. *If it could not*, then one would be dealing with a genuine paradox. This resistance to normalization was the quintessential feature of paradox-because-of-semantic-closure.

Upon illustrating the idea in the case of the Liar, we wrote (*loc. cit.*, p. 270)

> The normalisation sequence … enters a loop, never terminating with a proof in normal form.

That is, all normalization sequences fail to terminate. So we do not have even weak normalization.[2] We then went on to say (p. 271)

> … not every paradox need display this feature so clearly. It is therefore of considerable interest to enquire after techniques for discerning, in less direct fashion, whether something at root similar to this circularity of inference is at work in all paradoxes. I wish to maintain that it is indeed their distinguishing feature. I propose precisely the test of non-terminating reduction sequences.

The clearest cases of resistance to normalization were ones in which the reduction sequence (as in all the aforementioned cases) would enter a loop. But that is merely one special way in which a reduction sequence could fail to terminate.

The study [37] prompted the characterization in [19] of another way in which a reduction sequence could be non-terminating: it could (metaphorically) spiral *ad infinitum*, ratcheting up a numeral with each 'turn'. It would not exactly loop, but it would exhibit a discernible pattern that would prompt a warranted reaction of 'Here we go again—*ad infinitum*!'.

The (then) 'state of the art' of the proof-theoretic approach to paradox was described for a wider philosophical audience in [23]. We have subsequently argued that the proof-theoretic criterion deals satisfactorily with one more challenging agendum: the now well-known Revenge Paradox.[3]

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2 Thanks are owed to Peter Schroeder-Heister for suggesting that this point be stressed.

3 [27]. PDF of presentation available on request.

The criterion that we proposed was also designed to accommodate the kinds of paradox (still arising from semantic closure) that Kripke brought so forcefully to philosophers' attention. What Kripke showed is that paradoxicality can be a property of a set of sentences, not just of a single sentence on its own. Kripke also (and famously) gave examples where the paradoxicality of certain sentences involving semantic vocabulary depended crucially on the contribution of *empirical* contingencies, such as exactly what it was that Jones had said. (See [6] at p. 691.) Let us represent those empirical contingencies as a model M. [18] (at p. 283) ventured the following completeness conjecture (concerning the co-extensiveness of paradoxicality of sets of sentences according to the proof-theoretic criterion with paradoxicality on Kripke's semantic account):

> When … I speak of a proof of a conclusion from Θ(M), I have in mind a proof from assumptions that are truths in (every member of) M, and by means of rules for the logical operators and the truth predicate, as well as the *id est* rules of inference—really, one could say, rules of reference—that are legitimated by (every member of) M. The completeness conjecture is then that
>
> A set of sentences is paradoxical [in Kripke's sense] relative to M
> iff
> there is some proof of Λ [i.e., ⊥] from Θ(M), involving those sentences in *id est* inferences, that has a looping reduction sequence.
>
> Compactness of paradoxicality would follow as a corollary. So would the view that paradox is really only theory-sensitive and not model-sensitive. Clearly some result such as this is worth having, given the ideal and infinitistic nature of Kripke's semantical definition of paradox.

The occurrence of the word 'looping' in this quote should, especially in light of Yablo's subsequent paradox, be an occurrence of 'non-terminating'.

1.3 The Technical Challenge for the Conjecture

The technical challenge of establishing this conjecture (if possible at all) is formidable.

In light of the complexity results in [3] for Kripkean fixed points for the language $\mathcal{L}$ that supplements a language of arithmetic[4] with the truth predicate T, it would be necessary at least to determine the complexity of the criterial predicate about *non-terminating* reduction sequences that is employed in the (re-)formulation of the conjecture.

As Burgess's title says, the truth is never simple. That pithy comment is based on two main results about Kripkean fixed points generated using the 'strong' Kleene three-valued method of evaluation that produces the jumps from stage to stage for the extension A^+ and anti-extension A^- of the truth predicate T in any fixed point. (A^+

[4] The language in question has the usual primitive expressions for zero, successor, addition, and multiplication, plus distinct function symbols for all the other primitive recursive functions. We are talking here of complexity in the sense of the arithmetical hierarchy, not in the sense of computational complexity in the sub-recursive hierarchy that occupies those who work in algorithmic complexity theory.

and A^- are always disjoint. We shall use (A^+, A^-) as a complex sortal variable over fixed points.) The two results in question (*loc. cit.*, p. 668) concern the complexities of these 'plus' and 'minus' sets in the *minimum fixed point* (called $(0^+, 0^-)$) and in the *maximum intrinsic fixed point* (called (I^+, I^-)) respectively.

We concentrate here on the minimum fixed point, whose fuller designation would be $(0^+_K, 0^-_K)$, where the subscript K indicates use of the strong Kleene three-valued method of evaluation. $(0^+_K, 0^-_K)$ is generated by beginning the infinite sequence of (re-) evaluations with both empty extension and empty anti-extension (at stage zero) for the truth predicate T. The eventual extension 0^+_K for T in this fixed point is complete Π^1_1 (Burgess's THEOREM 6.1 at p. 668).

The set of sentences in the language $\mathcal{L}$ on which the paradoxicality theorist would wish to focus, however, is the set of Kripkean *paradoxes*, which Kripke defined as those sentences that receive no truth value in *any* fixed point. Pathological but non-paradoxical sentences like the truth-teller are true in some fixed points, and false in others. So they need somehow to be 'winnowed out' from that part of the language that lies 'in between' 0^+_K and 0^-_K, so as to leave behind only paradoxes. $\mathcal{L} \setminus (0^+_K \cup 0^-_K)$ is the part of the language that is in question here. It will contain, however, both the truth-teller and its negation. And these are not paradoxes.

The elusive set $\mathcal{P}$ (of just the paradoxes) that we are trying to define would be

$$[\mathcal{L} \setminus (0^+_K \cup 0^-_K)] \setminus \Theta$$

where Θ is the set of all sentences θ such that θ is true in some fixed point or θ is false in some fixed point. Thus

$$\mathcal{P} = [\mathcal{L} \setminus (0^+_K \cup 0^-_K)] \setminus \{\theta \in \mathcal{L} | \exists (A^+, A^-)\, \theta \in A^+ \vee \exists (A^+, A^-)\, \theta \in A^-\}.$$

Using the 'V' notation that Burgess introduces at p. 667 (for *codes* of sentences) this becomes

$$\mathcal{P} = [\mathcal{L} \setminus (0^+_K \cup 0^-_K)] \setminus (V^+_K \cup V^-_K).$$

But since $0^+_K \subseteq V^+_K$ and $0^-_K \subseteq V^-_K$, this boils down to

$$\mathcal{P} = \mathcal{L} \setminus (V^+_K \cup V^-_K).$$

Burgess's result 6.2(c) at p. 668 is that

$$\omega \setminus (V^+_K \cup V^-_K) \text{ is complete } \Pi^1_1.$$

We have a decision method for sieving ω for just the codes of L-sentences. Thus $\mathcal{P}$ is complete Π^1_1.[5]

With $\mathcal{P}$ being this complex, can the proof-theoretic criterion of paradoxicality 'get it right' (at least, for this pure language of arithmetic plus the predicate T, with no empirical hostages to fortune)? In support of the foregoing conjecture, the proof-theorist can only rejoin that this level of complexity for Kripkean paradoxes might

[5] Correspondence with John Burgess has been most helpful here.

well be matched by that of the myriad ways in which reduction sequences might fail to terminate.

1.4 A New Grouping of the So-called Paradoxes

We have written further on the proof-theoretic criterion of paradoxicality, most notably to propose that Russell's Paradox in set theory is *not*, after all, a genuine paradox.[6] Instead, it turns out to enjoy the form of straightforward normal-form disproof of the claim that there is such a thing as the set of all sets that are not members of themselves. Russell's Paradox is actually a negative-existential theorem of set theory. The set theory in question must, however, be equipped with 'single-barreled' rules of introduction and elimination for the set-abstraction operator $\{x | \ldots x \ldots\}$, which forms *singular terms* from formulae. And the underlying logic must be a *free* logic, not committed to the Fregean dogma that every singular term must denote.[7]

The proof-theoretic picture that is emerging is that a new distinction ought perhaps to be drawn among the so-called 'paradoxes', one that differs from Ramsey's famous Group A-Group B distinction. There are the genuine paradoxes, whose paradoxicality is revealed by the proof-theoretic criterion; and there are the so-called 'paradoxes', like that of Russell in set theory, that are better understood as straightforward negative existential theorems. This shows how a formal explication of a previously informal idea can furnish reforming theoretical insights. The explication can persuade one to carve things up and classify them rather differently.

2 On $\forall p(Qp \to \neg p)$ [$= \vartheta$]

At this juncture the proof-theoretic criterialist needs to address the less frequently discussed 'intensional paradoxes'. These were not considered in Ramsey's seminal article in 1925.[8] Of course, the intensional paradoxes due to Prior were formulated well after Ramsey's article. Along with Russell's paradox of propositions, Prior's intensional paradoxes deserve serious consideration.[9] They are called intensional because they involve an attitudinal operator Q on propositions. In the context of this study, however, Q's being an *attitudinal* operator is not essential. Q can be taken to be an arbitrary propositional operator; for our proof-theoretic analysis will still go through.[10]

6 See [26], §11.4, and [29].

7 This motivates also further examination (and possible re-classification) of other 'paradoxes' in set theory, such as the Burali–Forti paradox.

8 Russell's paradox of propositions appeared as an appendix to Russell's *Principles of Mathematics*, in 1903 (see §5)). Thanks are owed to an anonymous referee for noting that [33] argued that Ramsey's 'divide and conquer' strategy depended on a reconceptualization of logic that hides the intensional paradoxes from view.

9 A promissory note: deserving also of serious further consideration is the Berry paradox. In unpublished work, we argue that it (perhaps surprisingly) keeps company with the Russell paradox in set theory, in being a straightforwardly provable negative existential. A lot depends on how carefully one formulates it.

10 Thanks to Peter Schroeder-Heister for suggesting that this point be made.

The question to be addressed here is whether these so-called intensional paradoxes are genuine paradoxes according to the proof-theoretic criterion. This study argues that indeed they are. The term ‘intensional paradoxes’ seems to be an informal one, for which hardly any writers seem concerned to have a precise explication. The results in question seem only to need to be intuitively surprising, ‘hard to get one’s head around’—very much in keeping with the etymology of the word ‘paradox’ being the Greek for ‘beyond belief’.

In this section we shall examine one particular ‘intensional paradox’, attributed to Prior. That it enjoys the kind of status just described is evident from [2], at p. 497. In §2 of that study, with the phrase ‘Priorean paradoxes’ appearing in the heading, one reads the following:

> Let’s begin by discussing a puzzling result of Prior (1961). … Armed with a pair of assumptions, namely Ul for the quantifiers into sentence position, and classical propositional logic, one can prove …
>
> PRIOR’S THEOREM: $Q\forall p(Qp \to \neg p) \to (\exists p(Qp \wedge p) \wedge \exists p(Qp \wedge \neg p))$.[11]
>
> The authors go on to say that
>
> this theorem is … puzzling … [I]t is extremely surprising to be informed that this fact has the status of a logical truth. This doesn’t seem like the sort of thing that could be figured out from logic alone.

The word ‘seem’ here is bearing a lot of weight. Many a logical truth does not *seem* to be logically true—or, equivalently, can be *difficult to see as* logically true. Our goal in this section is to show exactly how Prior’s Theorem might be able to be seen to be logically true.

Let us abbreviate the proposition $\forall p(Qp \to \neg p)$ as ϑ. Then Prior’s Theorem takes the form

$$Q\vartheta \to (\exists p(Qp \wedge p) \wedge \exists p(Qp \wedge \neg p)).$$

Consider now the following formal proofs, in free quantified propositional logic, represented with abbreviations on the left of the colon, and fully formal renderings on the right. Note the existential presuppositions $\exists!\vartheta$ called for in applications of $\exists$-I and of $\forall$-E.

First we have the proof Ω.

$$\begin{array}{c} \underbrace{Q\vartheta,\ \vartheta,\ \exists!\vartheta}_{\Omega} \\ \exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p) \end{array} \quad : \quad \dfrac{\dfrac{\dfrac{Q\vartheta \quad \vartheta}{Q\vartheta\wedge\vartheta} \quad \exists!\vartheta}{\exists p(Qp\wedge p)} \qquad \dfrac{\dfrac{Q\vartheta \quad \dfrac{Q\vartheta \quad \dfrac{\begin{array}{c}\vartheta\text{, i.e.}\\ \forall p(Qp\to\neg p)\end{array} \quad \exists!\vartheta}{Q\vartheta\to\neg\vartheta}}{\neg\vartheta}}{Q\vartheta\wedge\neg\vartheta} \quad \exists!\vartheta}{\exists p(Qp\wedge\neg p)}}{\exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p)}$$

[11] We have supplied the parentheses around the consequent of this conditional.

We now use π as a parameter for quantificational reasoning about propositions. Next we have the proof Ξ, employing π in this way. Note the application of Classical Reductio at the final step of Ξ, labeled (3).

$$\begin{array}{c}\neg\vartheta\\ \Xi\\ \exists p(Qp\wedge p)\end{array} \quad : \quad \dfrac{\dfrac{\neg\vartheta\text{, i.e. } \neg\forall p(Qp\to\neg p) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{(3)}{\overline{\neg\exists p(Qp\wedge p)}} \qquad \dfrac{\dfrac{\overset{(2)}{\overline{Q\pi}} \quad \overset{(1)}{\overline{\pi}}}{Q\pi\wedge\pi} \quad \overset{(4)}{\overline{\exists!\pi}}}{\exists p(Qp\wedge p)}}{\bot}\,(1)}{\neg\pi}\,(2)}{Q\pi\to\neg\pi}\,(4)}{\forall p(Qp\to\neg p)}}{\bot}\,\text{(CR)}\,(3)}{\exists p(Qp\wedge p)}$$

Finally we embed our proofs Ω and Ξ to form the following proof Π of Prior's Theorem. Note that Π involves yet another application of a strictly classical rule of inference, namely DILEMMA, at its penultimate step, labeled (6).

$$\begin{array}{c}\exists!\vartheta\\ \Pi\\ Q\vartheta\to(\exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p))\end{array} \quad :$$

$$\dfrac{\dfrac{\begin{array}{c}\underbrace{\overset{(7)}{\overline{Q\vartheta}}\ ,\ \overset{(6)}{\overline{\vartheta}}\ ,\ \exists!\vartheta}\\ \Omega\\ \exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p)\end{array} \qquad \dfrac{\begin{array}{c}\overset{(6)}{\overline{\neg\vartheta}}\\ \Xi\\ \exists p(Qp\wedge p)\end{array} \qquad \dfrac{\dfrac{\overset{(7)}{\overline{Q\vartheta}} \quad \overset{(6)}{\overline{\neg\vartheta}}}{Q\vartheta\wedge\neg\vartheta} \quad \exists!\vartheta}{\exists p(Qp\wedge\neg p)}}{\exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p)}}{\exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p)}\,(6)\ \text{DILEMMA}}{Q\vartheta\to(\exists p(Qp\wedge p)\wedge\exists p(Qp\wedge\neg p))}\,(7)$$

This final proof Π of Prior's Theorem is a proof in free classical quantified propositional logic, and it is in normal form. So it is definitely not a *paradox*.

The proof Π does reveal, however, that Prioreans need to secure, for Prior's Theorem, the as-yet unsecured premise $\exists!\vartheta$. One way to fix this, and to restore theoremhood, would be to supply extra axioms or rules—perhaps along the line of the 'closure principles' suggested by [2], at pp. 22-3—that would ensure the theoremhood of the existential in question.

Perhaps those who give instances of the modifier Q to underscore just how 'counterintuitive' Prior's Theorem is, are happy to grant the existence of the proposition ϑ.

The following question appears to be open:

Can Prior's Theorem—even in the premise-laden form

$$\exists!\vartheta \vdash Q\vartheta \rightarrow (\exists p(Qp \wedge p) \wedge \exists p(Qp \wedge \neg p))$$

—be proved in free *constructive* quantified propositional logic?

3 On $\forall p(Qp \rightarrow \neg p)$ [= ϑ] along with $\Diamond\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q))$

We are still working with the language for quantified propositional logic. We wish to determine whether a paradox usually attributed to Prior is genuinely paradoxical. Our decision will be affirmative. We shall show that the derivation of $\perp$ (absurdity) that is involved cannot be brought into normal form.

Let Q be any attitudinal modifier. Let ϑ once again be the sentence

$$\forall p(Qp \rightarrow \neg p),$$

which expresses the thought that every Q'd proposition is false. We define K^* to be the sentence

$$\Diamond\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q)),$$

which expresses the thought that it is possible that the Q'd propositions be exactly those that are materially equivalent to ϑ.

Closely related to K^* is the sentence $K^{\rightarrow}$, which we define to be

$$\Diamond\forall q(Qq \leftrightarrow (q \rightarrow \vartheta)),$$

and which expresses the thought that it is possible that the Q'd propositions be exactly those that materially imply ϑ.

The question to be investigated is the following:

Is K^* straightforwardly inconsistent, or is K^* paradoxical?

A normal-form proof of $\perp$ (absurdity) from K^* would take the form

$$\dfrac{\Diamond\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q)) \qquad \begin{array}{c} \overline{\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q))}^{(i)} \\ \Omega \\ \perp \end{array}}{\perp}(i)$$

So, in order to show that K^* is genuinely inconsistent, all we would need to do is find a (normal-form) disproof Ω of

$$\forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q)).$$

In order to work strictly within Core Logic, we shall make so bold as to display proofs using *parallelized* eliminations whose major premises stand proud with no non-trivial proof-work above them. This always carries the risk of sideways spread. Accordingly, we shall seek to minimize such spread by providing labels for stretches of proof to which we need subsequently to refer.

It would be a mistake to think that one would need any strictly classical steps in such a disproof Ω. Core Logic $\mathbb{C}$ should suffice—if indeed *there is* such a disproof to be had. I shall, however, cast doubt on this existence claim. There is good reason to believe that there is no Core disproof Ω, even though one might naively think that there must be one, on the grounds (to be furnished below) that there are in fact two Core proofs Π and Σ of the following forms:

$$\begin{array}{c} \forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q)) \\ \Pi \\ \vartheta \end{array} \qquad \begin{array}{c} \underbrace{\vartheta, \forall q(Qq \leftrightarrow (\vartheta \leftrightarrow q))} \\ \Sigma \\ \bot \end{array}$$

The naive reasoner will think that one can join Π and Σ together, 'cutting' on ϑ, so as to produce the sought disproof Ω.

Alas, this is not the case. The would-be normal-form reduct, which we denote as $[\Pi, \Sigma]$,[12] does not exist. Any attempted sequence of reductions will fail to terminate—it will loop. It follows, by the proof-theoretic criterion for paradoxicality,[13] that K* is *paradoxical*, and not a genuinely inconsistent sentence.

Now for the promised proofs. These are core proofs, given in the form in which major premises of all eliminations stand proud, with no non-trivial proof-work above them.[14] Note that Π embeds Σ, so we give Σ first. We use **p** as the parameter for the ∀-Introduction at the final step of Π.

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12 We use our notation $\left[\begin{array}{c} \Delta \\ \Pi \\ \varphi \end{array}, \begin{array}{c} \underbrace{\varphi, \Gamma} \\ \Sigma \\ \psi \end{array}\right]$ (from [22]) as an alternative to the notation

$$\begin{array}{c} \Delta \\ \Pi \\ \underbrace{(\varphi), \Gamma} \\ \Sigma \\ \psi \end{array}$$

that indicates the result of grafting copies of the proof Π onto undischarged assumption occurrences of φ in the proof Σ. Whichever notation one uses, the aim is to continue with whatever reduction steps might be needed in order to arrive at a proof in normal form. The latter proof must draw its assumptions from $\Delta \cup \Gamma$; and its conclusion must be either ψ or $\bot$.

13 See [18], amended in [19] to take care of non-terminating reduction sequences that 'spiral' instead of looping. See also [23] and [25].

14 See [26] for details.

Let $\begin{matrix}\underbrace{Q\vartheta, \forall q(Qq \leftrightarrow (Q\vartheta \leftrightarrow q))}\\ \Sigma \\ \bot\end{matrix}$ be the following proof:

$$\dfrac{\forall q(Qq\leftrightarrow(Q\vartheta\leftrightarrow q)) \quad \dfrac{\begin{matrix}Q\vartheta\text{, i.e.}\\ \forall p(Qp\rightarrow\neg p)\end{matrix} \quad \dfrac{{\scriptstyle(5)}\dfrac{}{Q\vartheta\leftrightarrow(Q\vartheta\leftrightarrow Q\vartheta)} \quad \dfrac{{\scriptstyle(1)}\dfrac{}{Q\vartheta} \quad \dfrac{}{Q\vartheta}{\scriptstyle(1)}}{Q\vartheta\leftrightarrow Q\vartheta}{\scriptstyle(1)} \quad \dfrac{{\scriptstyle(4)}\dfrac{}{Q\vartheta\rightarrow\neg Q\vartheta} \quad {\scriptstyle(3)}\dfrac{}{Q\vartheta} \quad \dfrac{{\scriptstyle(2)}\dfrac{}{\neg Q\vartheta} \quad Q\vartheta}{\bot}}{\bot}{\scriptstyle(2)}}{\bot}{\scriptstyle(3)}}{\bot}{\scriptstyle(4)}}{\bot}{\scriptstyle(5)}$$

Then $\begin{matrix}\forall q(Qq \leftrightarrow (Q\vartheta \leftrightarrow q))\\ \Pi \\ Q\vartheta\end{matrix}$ is the proof

$$\dfrac{\dfrac{\dfrac{\forall q(Qq\leftrightarrow(Q\vartheta\leftrightarrow q)) \quad \dfrac{{\scriptstyle(8)}\dfrac{}{Q\mathbf{p}\leftrightarrow(Q\vartheta\leftrightarrow\mathbf{p})} \quad \dfrac{}{Q\mathbf{p}}{\scriptstyle(10)} \quad \dfrac{{\scriptstyle(7)}\dfrac{}{Q\vartheta\leftrightarrow\mathbf{p}} \quad \dfrac{}{\mathbf{p}}{\scriptstyle(9)} \quad \begin{matrix}{\scriptstyle(6)}\dfrac{}{\underbrace{Q\vartheta,\ \forall q(Qq\leftrightarrow(Q\vartheta\leftrightarrow q))}}\\ \Sigma\\ \bot\end{matrix}}{\bot}{\scriptstyle(6)}}{\bot}{\scriptstyle(7)}}{\bot}{\scriptstyle(8)}}{\dfrac{\neg\mathbf{p}}{Q\mathbf{p}\rightarrow\neg\mathbf{p}}{\scriptstyle(10)}}{\scriptstyle(9)}}{\begin{matrix}\forall p(Qp\rightarrow\neg p)\text{, i.e.}\\ Q\vartheta\end{matrix}}$$

The penultimate step in the proof Σ is an application of the (parallelized) rule of $\forall$-Elimination, with major premise $\forall p(Qp \rightarrow \neg p)$, i.e. $Q\vartheta$. And that same sentence stands as the conclusion of an application of the rule of $\forall$-Introduction, at the final step in the proof Π. So, in determining the reduct[15]

$$[\Pi, \Sigma]$$

the need will arise, after a single distribution conversion, for a $\forall$-reduction. This will involve substituting, for the parameter **p** in the immediate subproof for $\forall$-Introduction, the term $Q\vartheta$. Upon applying obvious shrinking reductions to the result, we shall find that there is a newly created maximal sentence of the form $Q\vartheta \rightarrow \neg Q\vartheta$. Upon applying the $\rightarrow$-reduction that is therefore called for, we shall find ourselves back at our starting point $[\Pi, \Sigma]$.

So the reduction sequences loops. The would-be disproof of

$$\forall q(Qq \leftrightarrow (Q\vartheta \leftrightarrow q))$$

[15] For the definition of this binary reduction function [,] for first-order logic, see [24], [21], and [26].

cannot be brought into normal form. This shows that it—hence also K*:

$$\Diamond\forall q(Qq\leftrightarrow(Q\vartheta\leftrightarrow q))$$

—is, according to the proof-theoretic criterion of paradoxicality mentioned earlier, a *paradox*, and is not genuinely inconsistent.

4 On $\exists p(Qp\wedge\neg p)$ along with $\Diamond\forall q(Qq\leftrightarrow(\eta\leftrightarrow q))$

Let η be the proposition $\exists p(Qp\wedge\neg p)$. Then Prioreans can try to raise a problem for the now familiar possibilitate

$$\Diamond\forall q(Qq\leftrightarrow(\eta\leftrightarrow q))$$

by reducing to absurdity the immediately embedded *unmodalized* proposition

$$\mathbf{K}:\quad \forall q(Qq\leftrightarrow(\eta\leftrightarrow q)).$$

The crucial thing to determine here is whether the Prioreans' *reductio ad absurdum* of **K** is a *genuine disproof*—that is, a disproof *in normal form*. If not, then the *reductio* on offer—whose reduction-sequence will not terminate—will be classifiable as a *paradox*.

To 'fast forward' in this section, what we shall show is the following. There is indeed a '*reductio*' on offer (which I shall construct on the Priorean's behalf). Moreover, it appears to be *constructive*. It will consist of two core proofs Ω and Σ, of the following overall forms:

$$\begin{array}{cc} \mathbf{K} & \underbrace{\neg\eta,\mathbf{K}} \\ \Omega & \Sigma \\ \neg\eta & \bot \end{array}$$

One might think, then, that one could simply find the reduct

$$\left[\begin{array}{ccc} \mathbf{K} & & \underbrace{\neg\eta,\mathbf{K}} \\ \Omega & , & \Sigma \\ \neg\eta & & \bot \end{array}\right]$$

and that it will be a disproof of **K**. *But*—and here is the now familiar kibosh—the reduct does not exist. The would-be *reductio* of **K** cannot be brought into normal form. As we shall see, the reduction sequence in the rule-governed search for the reduct loops. Accordingly, the proof-theoretic criterion for paradoxicality pronounces that **K** (hence also: the possibilitate $\Diamond$**K**) is paradoxical.

[11], at p. 46, offered a Fitch-style *reductio* of **K**, which he noted was constructive. In his footnote 13, Sbardolini wrote

> [The claim in fn. 5 of [1]] that Prior's paradox requires "classical propositional logic" is thus strictly speaking incorrect.

Sbardolini's claim of constructivity is correct.[16] But more detailed formalization of his Fitchian deduction, filling in all the primitive steps involved in the 'longer' steps to which he helped himself in his formal reasoning, reveals that, though constructive at every local step, his overall *reductio* is *not in normal form*. And, as we shall presently see, a strange and discombobulating result emerges when we try to normalize it. Let us set about doing so.

First, here is the proof $\begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}$:

$$\dfrac{\dfrac{\dfrac{}{\eta}{\scriptstyle(2)}\text{, i.e., } \exists p(Qp\wedge\neg p) \qquad \dfrac{\dfrac{\dfrac{}{Q\pi\wedge\neg\pi}{\scriptstyle(1)}}{\neg\pi} \qquad \dfrac{\dfrac{}{\eta}{\scriptstyle(2)} \qquad \dfrac{\dfrac{\dfrac{}{Q\pi\wedge\neg\pi}{\scriptstyle(1)}}{Q\pi} \qquad \dfrac{\mathbf{K}\text{, i.e., } \forall q(Qq\leftrightarrow(\eta\leftrightarrow q))}{Q\pi\leftrightarrow(\eta\leftrightarrow\pi)}}{\eta\leftrightarrow\pi}}{\pi}}{\bot}}{\bot}{\scriptstyle(1)}}{\neg\eta}{\scriptstyle(2)}$$

Call the subproof that ends with $\bot$ and is an immediate subproof for the step labeled (1)

$$\begin{matrix}\underbrace{Q\pi\wedge\neg\pi,\eta,\mathbf{K}}\\ \Xi\\ \bot\end{matrix}\ .$$

Call the immediate subproof of Ω—the one that ends with the step of $\exists$-E labeled (1)—

$$\begin{matrix}\underbrace{\eta,\mathbf{K}}\\ \Omega_0\\ \bot\end{matrix}\ .$$

Second, here is the proof $\begin{matrix}\underbrace{\neg\eta,\mathbf{K}}\\ \Sigma\\ \bot\end{matrix}$:

$$\dfrac{\neg\eta\text{, i.e., } \neg\exists p(Qp\wedge\neg p) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathbf{K}\text{, i.e., } \forall q(Qq\leftrightarrow(\eta\leftrightarrow q)))}{Q\eta\leftrightarrow(\eta\leftrightarrow\eta)} \qquad \dfrac{\dfrac{}{\eta}{\scriptstyle(1)} \quad \dfrac{}{\eta}{\scriptstyle(1)}}{\eta\leftrightarrow\eta}{\scriptstyle(1)}}{Q\eta} \qquad \begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}}{Q\eta\wedge\neg\eta}}{\exists p(Qp\wedge\neg p)}}{\bot}$$

[16] The claim is reprised, but only fleetingly, in footnote 43 on p. 567 of [12].

Call the right-hand immediate subproof above

$$\dfrac{\begin{matrix}\mathbf{K}\\ \Lambda\\ Q\eta\wedge\neg\eta\end{matrix}}{\exists p(Qp\wedge\neg p)},$$

with a bit of exposed detail that will be helpful in due course.

In order to obtain the reduct $[\Omega, \Sigma]$ (if it exists) we need to perform a $\neg$-reduction. That will result in our needing to obtain the reduct

$$\left[\dfrac{\dfrac{\dfrac{\overset{\mathbf{K},\text{ i.e.,}}{\forall q(Qq\leftrightarrow(\eta\leftrightarrow q)))}}{Q\eta\leftrightarrow(\eta\leftrightarrow\eta)}\qquad \dfrac{\overset{\overline{}(1)}{\eta}\quad\overset{\overline{}(1)}{\eta}}{\eta\leftrightarrow\eta}{\scriptstyle(1)}}{Q\eta}\qquad \begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}}{\dfrac{Q\eta\wedge\neg\eta}{\exists p(Qp\wedge\neg p)}}\ ,\ \begin{matrix}\underbrace{\eta,\mathbf{K}}\\ \Omega_0\\ \bot\end{matrix}\right]$$

Remember that η is $\exists p(Qp\wedge\neg p)$. So to obtain this reduct we need to perform $\exists$-reduction. This results in our then having to obtain the reduct

$$\left[\begin{matrix}\mathbf{K}\\ \Lambda\\ Q\eta\wedge\neg\eta\end{matrix}\ ,\ \left(\begin{matrix}\underbrace{Q\pi\wedge\neg\pi,\eta,\mathbf{K}}\\ \Xi\\ \bot\end{matrix}\right)^{\pi}_{\eta}\right]$$

Let us dwell for a moment on the second argument here, the one after the comma. The substitution of the term η for the parameter π produces the construct

$$\dfrac{\dfrac{Q\eta\wedge\neg\eta}{\neg\eta}\qquad \dfrac{\eta\qquad \dfrac{\dfrac{Q\eta\wedge\neg\eta}{Q\eta}\qquad \dfrac{\overset{\mathbf{K},\text{ i.e.,}}{\forall q(Qq\leftrightarrow(\eta\leftrightarrow q))}}{Q\eta\leftrightarrow(\eta\leftrightarrow\eta)}}{\eta\leftrightarrow\eta}}{\eta}}{\bot}$$

whose immediate subproof on the right is of the conclusion η from a set of premises that includes η itself. So it should be replaced by a single occurrence of η. We are now, therefore, seeking to obtain the reduct

$$\left[\begin{matrix}\mathbf{K}\\ \Lambda\\ Q\eta\wedge\neg\eta\end{matrix}\ ,\ \dfrac{\dfrac{Q\eta\wedge\neg\eta}{\neg\eta}\qquad\eta}{\bot}\right]$$

—that is, the reduct

$$\left[\ \dfrac{\dfrac{\dfrac{\begin{matrix}\mathbf{K},\ \text{i.e.,}\\ \forall q(Qq\leftrightarrow(\eta\leftrightarrow q)))\end{matrix}}{Q\eta\leftrightarrow(\eta\leftrightarrow\eta)}\qquad \dfrac{\overline{\eta}^{\,(1)}\quad \overline{\eta}^{\,(1)}}{\eta\leftrightarrow\eta}{\scriptstyle(1)}}{Q\eta}\qquad \begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}}{Q\eta\wedge\neg\eta}\ ,\quad \dfrac{\dfrac{Q\eta\wedge\neg\eta}{\neg\eta}\qquad \eta}{\bot}\ \right],$$

now calling for us to apply an $\wedge$-reduction. Doing so produces

$$\dfrac{\begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}\quad \eta}{\bot}$$

—that is,

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\overline{\eta}^{\,(2)}\ \text{i.e.,}\\ \exists p(Qp\wedge\neg p)\end{matrix}\qquad \dfrac{\dfrac{\overline{Q\pi\wedge\neg\pi}^{\,(1)}}{\neg\pi}\qquad \dfrac{\overline{\eta}^{\,(2)}\qquad \dfrac{\dfrac{\overline{Q\pi\wedge\neg\pi}^{\,(1)}}{Q\pi}\qquad \dfrac{\begin{matrix}\mathbf{K},\ \text{i.e.,}\\ \forall q(Qq\leftrightarrow(\eta\leftrightarrow q))\end{matrix}}{Q\pi\leftrightarrow(\eta\leftrightarrow\pi)}}{\eta\leftrightarrow\pi}}{\pi}}{\bot}}{\bot}{\scriptstyle(1)}}{\neg\eta}{\scriptstyle(2)}\qquad \eta}{\bot}$$

And *this* boils down to

$$\dfrac{\begin{matrix}\eta,\ \text{i.e.,}\\ \exists p(Qp\wedge\neg p)\end{matrix}\qquad \dfrac{\dfrac{\overline{Q\pi\wedge\neg\pi}^{\,(1)}}{\neg\pi}\qquad \dfrac{\eta\qquad \dfrac{\dfrac{\overline{Q\pi\wedge\neg\pi}^{\,(1)}}{Q\pi}\qquad \dfrac{\begin{matrix}\mathbf{K},\ \text{i.e.,}\\ \forall q(Qq\leftrightarrow(\eta\leftrightarrow q))\end{matrix}}{Q\pi\leftrightarrow(\eta\leftrightarrow\pi)}}{\eta\leftrightarrow\pi}}{\pi}}{\bot}}{\bot}{\scriptstyle(1)}$$

So our attempt to obtain the reduct

$$\left[\ \begin{matrix}\mathbf{K}\\ \Omega\\ \neg\eta\end{matrix}\ ,\ \begin{matrix}\overbrace{\neg\eta,\mathbf{K}}\\ \Sigma\\ \bot\end{matrix}\ \right]$$

has simply returned to us the immediate subproof $\begin{matrix}\overbrace{\eta,\mathbf{K}}\\ \Omega_0\\ \bot\end{matrix}$ of Ω!

This is a new phenomenon in attempted normalization of disproofs associated with paradoxes. We have obtained, in Ω_0, a disproof in normal form certainly—but the sought reduct was supposed to have only **K** as a premise. Instead, Ω_0 disproves **K** only *modulo* η that is, Ω_0 derives $\perp$ only from the pair $\{\mathbf{K}, \eta\}$ of premises, not from **K** on its own.

What this suggests is that perhaps there cannot be a cut-admissibility metatheorem for quantified propositional logic.

It would not help for the sympathizer to propose trying, instead, to apply Dilemma as a way to solve the problem. The proposal would be, in light of the two (core) disproofs that we *do* have—namely,

$$\begin{array}{c} \underbrace{\eta\,,\mathbf{K}} \\ \Omega_0 \\ \perp \end{array} \quad \text{and} \quad \begin{array}{c} \underbrace{\neg\eta\,,\mathbf{K}} \\ \Sigma \\ \perp \end{array}$$

—to form the 'strictly classical' disproof

$$\dfrac{\begin{array}{c} \underbrace{\overset{(i)}{\overline{\eta}}\;,\;\mathbf{K}} \\ \Omega_0 \\ \perp \end{array} \qquad \begin{array}{c} \underbrace{\overset{(i)}{\overline{\neg\eta}}\,,\;\mathbf{K}} \\ \Sigma \\ \perp \end{array}}{\perp}\,(i)$$

of just the premise **K**.

But such supposedly classical disproofs can—or should—*always* admit of constructivization, in the obvious and familiar way. That would involve—in the case at hand—applying a step of $\neg$-I at the end of Ω_0, and 'grafting' the resulting proof of $\neg\eta$ from just the premise **K** onto the premise-occurrences of $\neg\eta$ in the proof Σ. That is basically what we set out to do, in beginning with the search for the reduct

$$\left[\begin{array}{c} \mathbf{K} \\ \Omega \\ \neg\eta \end{array} \;,\; \begin{array}{c} \underbrace{\neg\eta,\mathbf{K}} \\ \Sigma \\ \perp \end{array}\right].$$

That search, we now see, has been thwarted. The reduction sequence did not terminate with the desired (dis)proof. It terminated, to be sure! It did not loop, or spiral. Instead, it just came to a useless dead end, serving up an immediate subproof (namely, Ω_0) of one of the two proofs (namely, Ω) that we started with. And Ω_0 is not a proof of the sought result. It has η as an undischarged assumption.

I propose to impose a new and slightly more exigent demand on any normalization process of the kinds undertaken above. The new demand is that whenever one thinks one is able to use two normal proofs[17]

$$\begin{array}{c}\underbrace{\varphi\,,\,\Delta}\\ \Pi\\ \bot\end{array}\quad\text{and}\quad\begin{array}{c}\underbrace{\neg\varphi\,,\,\Gamma}\\ \Sigma\\ \psi\end{array}$$

(thus far discovered; and where, note, Π ends with $\bot$) to form a 'strictly classical'-*looking* proof

$$\dfrac{\begin{array}{c}\underbrace{\overset{(i)\,\_\_}{\varphi}\ ,\ \Delta}\\ \Pi\\ \bot\end{array}\qquad\begin{array}{c}\underbrace{\overset{(i)\,\_\_}{\neg\varphi}\ ,\ \Gamma}\\ \Sigma\\ \psi\end{array}}{\psi}(i)\ ,$$

one should desist. This is because it is always the case that one is (and *should* be) able to 'avoid the classicism' and obtain the sought result *constructively*. In the idiom of 'grafting', the sought result would be [a reduct of]

$$\begin{array}{c}\underbrace{\dfrac{\begin{array}{c}\underbrace{\overset{(i)\,\_\_}{\varphi}\ ,\ \Delta}\\ \Pi\\ \bot\end{array}}{(\neg\varphi)}(j)\quad,\quad\Gamma}\\ \Sigma\\ \psi\end{array}$$

A question now arises for the paradox theorist who says that the criterion of paradoxicality is that the reduction sequence does not terminate in a normal disproof. How should we classify failures so to terminate? Does it count as a failure if the reduction sequence terminates after finitely many steps, but in a disproof other than what we were 'supposed' to get?

I am proposing that we prevent this question from arising in the first place. It arose above because we arrived at the normal proof

$$\begin{array}{c}\underbrace{\eta,\mathbf{K}}\\ \Omega_0\\ \bot\end{array}$$

rather than at a proof of $\bot$ from the single premise $\mathbf{K}$, which is what we were supposed to get. If we heed the demand just formulated—the demand, that is, to avoid the 'clas-

[17] Note that we are generalizing here from the case at hand, in which $\Delta = \Gamma = \{\mathbf{K}\}$ and $\psi = \bot$.

sicizing' stratagem in dogged pursuit of a (now, 'classical-looking') proof of $\bot$ from the single premise **K**—then we shall see that the reduction sequence of *constructivized* proofs does indeed enter a loop.

The implicit supposition has always been that what we were supposed to get was a disproof that would establish the inconsistency in question. The terminal disproof should not help itself to extraneous premises in doing so. But that is what would be happening here, if we were to fail to heed the demand just imposed.

But what if our earnest reducer simply wishes to halt the reduction process at Ω_0, and does not even venture to try the (misguided) classicizing stratagem? Perhaps arriving at two normal disproofs of **K**, one *modulo* η and the other *modulo* $\neg\eta$, *is*, after all, a proof-theoretic marker of *some kind of* paradoxicality on the part of **K** itself. Perhaps we should allow for this sort of 'final outcome' in the case of the so-called *intensional* paradoxes? Only further research into their variety and idiosyncratic behaviors in attempted normalization will produce an answer.

5 On Russell's Paradox of Propositions

The proof-theoretic criterion of paradoxicality deals well with yet another paradox about propositions, due to Russell. Let $\S\varphi$ be the proposition that φ. In [10], at §500, we read the following:

> If m be a class of propositions, the proposition "every m is true" may or may not be itself an m. But there is a one-one relation of this proposition to m: if n be different from m, "every n is true" is not the same proposition as "every m is true".

We propose the following regimentation of this, as a rule of inference in a 'logic of propositions':

$$\frac{\S\forall p(Mp \to p) = \S\forall p(Np \to p)}{\forall p(Mp \leftrightarrow Np)}$$

Note that we are seeking to avoid mention of classes. Thus we write Mp (proposition p has the property M) instead of $p \in M$ (proposition p is in the class M). We employ quantification over propositions, so that 'every proposition with property M is true' can be regimented as '$\forall p(Mp \to p)$'. We also contrapose Russell's conditional, since this simplifies the drawing of inferences. Since Russell himself is talking of *classes* m and n of propositions, their being 'different' from each other is just a matter of their being distinct classes—equivalently, a matter of those classes' respective defining properties M and N not being coextensive. So, if the two propositions "every n is true" and "every m is true" *are* identical, it follows that M and N are coextensive—which is what our rule states.

Russell went on to define a particular class w (or corresponding property W) of propositions, as follows.

> Consider now the whole class of propositions of the form "every m is true", and having the property of not being members of their respective m's. Let this class be w ...

Since we are choosing to regiment with (higher-order) predicates M rather than terms m for classes of propositions, we propose the following characterization of a predicate W (corresponding to Russell's class w) in terms of an introduction rule and a corresponding elimination rule.

$$W\text{-Intro}\quad \frac{q = \S\forall p(\Phi p \to p) \quad \neg\Phi q}{Wq}$$

$$W\text{-Elim}\qquad \frac{Wq \qquad \begin{array}{c} \underbrace{\overline{q = \S\forall p(Fp \to p)}^{\,(i)}\ ,\ \overline{\neg Fq}^{\,(i)}} \\ \vdots \\ \theta \end{array}}{\theta}\,(i) \qquad \text{where } F \text{ is parametric}$$

The W-reduction procedure would be as follows:

$$\frac{\dfrac{\begin{array}{c}\Delta\\ \Pi\\ q = \S\forall p(\Phi \to p)\end{array} \quad \begin{array}{c}\Gamma\\ \Sigma\\ \neg\Phi q\end{array}}{Wq} \qquad \begin{array}{c}\underbrace{\Xi\ ,\ \overline{q = \S\forall p(Fp \to p)}^{\,(i)}\ ,\ \overline{\neg Fq}^{\,(i)}}\\ \Omega\\ \theta\end{array}}{\theta}\,(i) \quad \rightsquigarrow \quad \begin{array}{c}\underbrace{\Xi\ ,\ \begin{array}{c}\Delta\\ \Pi\\ q = \S\forall p(\Phi p \to p)\end{array},\ \begin{array}{c}\Gamma\\ \Sigma\\ \neg\Phi q\end{array}}\\ \Omega^F_\Phi\\ \theta\end{array}$$

The concept W is now available to be a constituent of propositions.

Let π be the proposition $\S\forall q(Wq \to q)$. Note that so 'defining' π presupposes that such a proposition exists. Note further that, if this proposition *does* exist, then it lies within the scope of the quantifier $\forall q$ in the very sentence that expresses it.

Russell's reasoning to a contradiction can now be regimented as follows. We construct two proofs, which we shall call Σ and Π. Each of them has

$$\pi = \S\forall q(Wq \to q)$$

as a premise. *Modulo* this premise, Σ *disproves* $W\pi$; whereas Π *proves* $W\pi$:

$$\begin{array}{c}\underbrace{W\pi\ ,\ \pi = \S\forall q(Wq \to q)}\\ \Sigma\\ \bot\end{array} \qquad \begin{array}{c}\pi = \S\forall q(Wq \to q)\\ \Pi\\ W\pi\end{array}$$

(Details of Σ and Π will follow.) It would therefore appear that the common premise $\pi = \S\forall q(Wq \to q)$ of Σ and of Π is inconsistent. This inconsistency would presumably be brought out by grafting copies of the proof Π of $W\pi$ onto the latter's occurrences in premise position within Σ:

$$\begin{array}{c}\underbrace{\begin{array}{c}\pi = \S\forall q(Wq \to q)\\ \Pi\\ (W\pi)\end{array}\ ,\ \pi = \S\forall q(Wq \to q)}\\ \Sigma\\ \bot\end{array}$$

Let us call this *the accumulation of* Π *on* Σ. It would be a disproof of $\pi = \S\forall q(Wq \to q)$ if *but only if* it can be converted into normal form. It is not, as it stands, in normal form, because (as we shall presently see) one of the graft-occurrences of $W\pi$ is maximal. It stands as the conclusion of an application of W-I (the terminal step in Π), and as the major premise for an application of W-E within Σ.

A single step of existential elimination now reveals the inconsistency of the existential generalization $\exists p(p = \S\forall q(Wq \to q))$. Note that the accumulation of Π on Σ occurs here as the immediate subproof for $\exists$-E, in which the term π plays the role of the *parameter for existential elimination*:

$$\dfrac{\exists p(p = \S\forall q(Wq\to q)) \qquad \begin{matrix}\overline{\pi = \S\forall q(Wq\to q)}^{(1)}\\ \Pi \\ \underbrace{(W\pi)\ ,\ \overline{\pi = \S\forall q(Wq\to q)}^{(1)}}\\ \Sigma \\ \bot\end{matrix}}{\bot}\,(1)$$

Here now are the details of both Σ and Π. Note that Π embeds Σ.

$$\begin{matrix}\underbrace{W\pi\ ,\ \pi = \S\forall q(Wq\to q)}\\ \Sigma\\ \bot\end{matrix} \quad : \quad \dfrac{W\pi \qquad \dfrac{\overline{\neg F\pi}^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\pi = \S\forall p(Fp\to p)}^{(1)} \quad \pi = \S\forall q(Wq\to q)}{\S\forall p(Fp\to p) = \S\forall q(Wq\to q)}}{\forall p(Fp\leftrightarrow Wp)}}{F\pi\leftrightarrow W\pi} \qquad W\pi}{F\pi}}{\bot}}{\bot}\,(1)\ W\text{-Elim}$$

$$\begin{matrix}\pi = \S\forall q(Wq\to q)\\ \Pi\\ W\pi\end{matrix} \quad:\quad \dfrac{\pi = \S\forall q(Wq\to q) \qquad \dfrac{\begin{matrix}\underbrace{\overline{W\pi}^{(2)}\ ,\ \pi = \S\forall q(Wq\to q)}\\ \Sigma\\ \bot\end{matrix}}{\neg W\pi}\,(2)}{W\pi}\,W\text{-Intro}$$

The question we must now address is whether the accumulation of Π on Σ can be brought into normal form. Answer: it cannot be; the reduction sequence loops. We shall see that the accumulation of Π on Σ calls for a W-reduction. The resulting reduct then calls for a shrinking reduction. After that shrinking, $\neg W\pi$ has a maximal occurrence. The needed $\neg$-reduction then takes us back to the accumulation of Π on Σ.

In summary, we conclude that the intensional paradoxes, in their natural-deductive forms presented here, provide confirming support for our criterion of paradoxicality.

It remains to be investigated whether that support might be susceptible to the methods employed in [7], by means of which non-normalizable disproofs associated with paradoxes are related somehow to disproofs of ‘equivalents’ in normal form.

6 A Residual Issue

There is an important issue that we must set aside here for want of space, but which must be subjected to much more detailed future investigation. It would be very helpful, though, to explain its importance here, even if at the cost of a few pages more; and to impart some idea of the current state of play concerning it, in extant investigations. Doing so also provides an opportunity to clarify the much wider methodological setting within which the current study of the intensional paradoxes occupies its own special niche.

The preceding investigation of (non-)normalizability of disproofs associated with, and (so we would maintain) *criterial for*, paradoxicality leaves us with one outstanding agendum. As we have already intimated, it deserves much deeper analysis and/or explication than we have space for here; and, very importantly, it holds out a prospect of potential re-education of intuitions that would require deeper and more extensive study to elucidate. It is the combined matter of *admissible reduction procedures* and how they bear on the question of *identity of proofs*.

We shall refer to this matter as ‘the Ekmanesque predicament’ that *potentially* arises for the proof-theoretic criterion of paradoxicality. Ekman called for an altogether new kind of reduction to remove what he regarded as a ‘redundant’ sentence occurrence within a proof. The sentence occurrence was not of the familiar ‘maximal’ kind (namely, the conclusion of a step of introduction for its dominant operator, and also the major premise of a step of elimination of that operator). Instead, it could even be *atomic*, and therefore have no dominant operator at all. The new kind of reduction procedure that Ekman called for was the following:

$$\dfrac{B \to A \qquad \dfrac{A \to B \quad \begin{matrix}\Pi \\ A\end{matrix}}{B^*}}{A} \quad \rightsquigarrow \quad \begin{matrix}\Pi \\ A\end{matrix}\,.$$

Here it is the lone asterisked occurrence of atom B that is diagnosed as ‘redundant’. The use of this newly available reduction procedure for ‘normalizing’ proofs and disproofs could result in non-terminating reduction sequences starting from intuitively non-paradoxical but inconsistent sets of sentences. An example of such a set is $\{A \to \neg A, \neg A \to A\}$.

The interested reader will find this predicament sourced in [4], and then raised and addressed (from differing standpoints) in [34], [35], [13], [25], [31], [26] (§11.3.2), [16], [17], and [32] (especially Chapter 6 for the topic of current concern, which is co-authored with Peter Schroeder-Heister).

In [17], the authors write as follows:

> Although we are strongly sympathetic to the Prawitz-Tennant analysis, in [[16]] … we suggested that certain results by Ekman … can be naturally seen as showing that the proposed criterion for paradoxicality overgenerates. [That is, in their words from the Abstract, 'there are derivations which are *intuitively non-paradoxical* but which fail to normalize'—Author.] To solve the overgeneration problem, we argued that *the notion of reduction* underlying the criterion must be appropriately qualified, by requiring the reductions to *preserve the identity of the proofs* represented by derivations (or more philosophically, by requiring the reductions to be *meaning-theoretically justified*). … [W]ithout a criterion for the acceptability of reduction procedures, the Prawitz–Tennant analysis overgenerates even when reformulated using general elimination rules. (p. 620) [All emphases added.]

At the very least this quotation reveals that there are several centrally important notions to get clear about: *intuitive* paradoxicality and *intuitive* non-paradoxicality; admissible reduction; identity of proofs; and meaning-theoretic justifiability. All of these need to be explicated. They must then co-function appropriately in any overall solution to the Ekmanesque predicament that can claim reflective equilibrium. It is quite beyond the scope of this particular study to undertake this task. For this study is confined to adding more positive applications (in the form of the hitherto proof-theoretically unexamined intensional paradoxes) to the list of claimed successes for the proof-theoretic criterion of paradoxicality.

That is not to claim, to be sure, that this study's positive conclusions about the intensional paradoxes being genuine paradoxes might eventually have to be re-assessed from the reflectively equilibrated standpoint to which we can aspire after reaching an overall solution to the Ekmanesque predicament. Small and unanticipated technical details could matter enormously.

The reader should bear in mind, though, that the project of formally explicating intuitive or informal concepts in logic and philosophy can lead one to seriously reconsider certain of one's intuitive judgements that one might have held, unreflectively, at the very outset. This has already happened with the present author's own initial intuitive judgments in his investigations of paradoxicality from the proof-theoretic point of view. We have already said some words (see §1.4) about our evolving views involving Russell's Paradox in set theory. But this theme takes on larger significance in the current context, for which a few more words would be appropriate. The present author has arrived at a (for him, stable and convincing) meta-theoretical determination that the logico-semantic paradoxes are a completely different kettle of fish from the so-called 'paradoxes' of Russell, Burali-Forti, Mirimanoff, and others in set theory, *and* the famous Berry Paradox about numerical definability. The proof-theoretic criterion, *applied from within a properly formulated free logic furnished with the right kind of introduction and elimination rules for variable-binding term-forming operators*, reveals that all these latter named 'paradoxes' are not paradoxes at all; they are *straightforward inconsistencies*, established by normal-form disproofs of mistaken existential assumptions. Overall, as already mentioned in §1.4, this calls for a major revision of the Group A and Group B classification of the paradoxes in [9].

One irony is that our formal approach via his criterion for paradoxicality was inspired by the treatment in [8] of Russell's Paradox in set theory (which was the only paradox of any kind that Prawitz studied in his seminal monograph). The irony is that the present author eventually arrived at the carefully considered view that Russell's Paradox in set theory is not a genuine paradox at all. This judgement, however, depends on having sorted out other important details about the right choice of logic (especially for the purpose of formalizing mathematical reasoning).

After our initial study in 1982 that proposed the proof-theoretic criterion of paradoxicality, we undertook our more detailed investigations of Core Logic in pursuit not only of constructivity, but of relevance as well; and realized, through our work on both number-theoretic neo-logicism[18] and set-theoretic neo-logicism[19] that a major *mis*conception on Frege's part was that a so-called 'logically perfect' language had to be one in which every logico-grammatically singular term denoted.

The Fregean tradition had been blind to the need for a *free* logic, which is crucial for any *Begriffsschrift* that is designed to be adequate for the formalization of all the informally rigorous reasoning that is to be found in mathematics—including, for example, the discovery that there can be no such thing as the set of all sets that are not members of themselves. Frege's blindness to free logic brought him the devastation of Russell's Paradox. And that same blindness came to haunt subsequent neo-Fregeans more generally (such as [36]) in their own continuation of Frege's ill-conceived *un*free logic and Frege's ill-advised choice of 'double-barreled' abstraction principles—Basic Law V in the case of Frege, and Hume's Principle in the case of Wright.

We advocate instead the careful free-logical employment of natural-deduction rules for the number- and set-abstraction operators. We call them *single-barreled* rules, because they seek to furnish the precise normative constraints on our deployment of (possibly non-denoting) abstractive terms in canonical identity statements of the form $t = @x\varphi(x)$, where @ is a variable-binding term-forming operator (such as $\#x\varphi(x)$ for natural numbers; and $\{x|\varphi(x)\}$ for sets).

So the irony we are seeking to explain is this: it was Prawitz's discovery of non-normalizability of his (highly *suboptimal* choice of) regimentation of the reasoning behind Russell's Paradox that inspired the criterion for paradoxicality in general, and especially for logico-semantic paradoxicality; yet our own pursuit of a properly designed proof-system for mathematical reasoning, using free core logic[20] and single-barreled natural-deduction rules for abstractive terms, led us to the discovery that Russell's so-called paradox in set theory is not a genuine paradox at all.

Might one venture the suggestion that something similar could well turn out to be the case with (perhaps mistaken?) initial intuitions—pending proper explication—about identity-conditions for proofs as abstract objects; admissible reduction rules; and intuitive paradoxicality (and non-paradoxicality, for that matter). We still need to attain greater clarity about the following issues:

1. Derivations in certain formal systems of logic vs. proofs conceived of as abstract objects that are somehow *represented* by various formal derivations.

[18] See [28] and papers cited therein.

[19] See [29] and papers cited therein.

[20] Core Logic for the constructive case; Classical Core Logic for the classical one.

2. The contribution of (Classical) Core Logic $\mathbb{C}^+$ in clarifying (1), given how in $\mathbb{C}^+$ there is a direct structural isomorphism between natural deductions and their corresponding sequent proofs.
3. The potential contribution of the grandly named 'Global Anti-Dilution Precondition on Rule Applications' in [20] at pp. 351-2, governing the formation of sequent proofs:

   In any application of a sequent rule

   $$\frac{S_0, \ldots, S_n}{T}$$

   we must ensure that no sequent in the subproof of any premis[e] sequent S_i is a sub-sequent of the conclusion sequent T.

4. The bearing on our conception of the identity-conditions of proofs when proofs are viewed as *objects of search*,[21] in the course of which one seeks to preserve and maximize any possible epistemic gains (such as learning, in searching for a proof of a given sequent, that one has stumbled across a proof of a proper subsequent of it).
5. Whether shrinking reductions can be justified on the basis of (4), *without* going so far as to accept any of Ekman's proposed new reduction procedures as justified.
6. Whether the Ekmanesque predicament might be found to afflict our foregoing treatment of the intensional paradoxes, in some unavoidable way.

The present author hopes to return to the pages of this journal with whatever clarity he might be so fortunate as to achieve in addressing these further issues.

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[21] In this connection the reader might be interested in [30].

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