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The Logic for Mathematics without Ex Falso Quodlibet

Neil Tennant*

Department of Philosophy, The Ohio State University, Columbus, Ohio 43210, U.S.A.

ABSTRACT

Informally rigorous mathematical reasoning is relevant. So too should be the premises to the conclusions of formal proofs that regiment it. The rule *Ex Falso Quodlibet* induces *spectacular* irrelevance. We therefore drop it. The resulting systems of Core Logic $\mathbb{C}$ and Classical Core Logic $\mathbb{C}^+$ can formalize all the informally rigorous reasoning in constructive and classical mathematics respectively. We effect a revised match-up between deducibility in Classical Core Logic and a new notion of *relevant* logical consequence. It matches better the deducibility relation of Classical Core Logic than does the Tarskian notion of consequence. It is *implosive*, not explosive.

1. INTRODUCTION

Philosophers and mathematicians inquire after the best choices of formal logical systems within which to regiment the deductive reasoning that is found in constructive and classical mathematics. Regimentation is the *apotheosis* of the Aristotelian ideal of 'perfecting' deductive arguments. It was Euclid who began what is known today as reasoning with 'informal rigor' in mathematics. Formal logical systems enable us to perfect that reasoning. Which formal logical system to use for this purpose is a *methodological* choice. Once the right choice is made, formal logic itself can then provide many insights, and give rise to many controversies — enabling both their formulation and their resolution — within the philosophy of mathematics.

Since [Frege, 1879] (the famous *Begriffsschrift*) the main foundational task of formal logic has been the fully detailed formalization of the informally rigorous deductive reasoning that is now to be found in all branches of mathematics. The great breakthrough of that year was a finally perfected understanding of the logical grammar of multiple quantification. In Frege's hands, fully formal

*Orcid.org/0000-0002-3523-2296. E-mail: tennant.9@osu.edu.

logic was to make the correctness of proof an effectively decidable matter. This was important *epistemologically*: only on the basis of a fully formalized proof could one attain absolute certainty that if its premises were collectively true, then so too would be its conclusion. Formal deduction, *par excellence*, preserves truth. And on Frege's own foundational approach, which was logicist, this would mean that *Arithmetik* could attain the same epistemological status as *Logik* — in Kantian terms, it would be *analytic a priori*.

For the less logicistically inspired methodologist in mathematics (such as, say, Peano or Hilbert), fully formalized deductive reasoning would still serve an importantly analogous purpose. It would enable the mathematical axiomatizer to provide a logical foundation for all the truths that were to be asserted on the basis of proof from their chosen axioms. The axioms themselves needed only to be *a priori*; their status as (possibly) *analytic* was of lesser importance. A fully formal logic would ensure that any mathematical theorem derivable from those axioms by fully formal proof would also be *a priori*.

At that early juncture — the late 1800s — the philosophical and epistemological importance of fully formal logic was firmly established. It matters not that the two Kantian distinctions (analytic / synthetic, and *a priori* / *a posteriori*) were later to be questioned or abandoned. The contribution to epistemology of *logical guarantee of truth transmission* was intact. Nor does it matter that mathematical ontologists could disagree among themselves about whether abstract objects such as numbers 'really' exist (nominalists vs. platonists); or that a methodological divide would emerge between those who insisted on constructivity in all our mathematical reasoning (for example, Brouwer), and those who did not; or that differences of opinion would develop over whether there could be completed infinite totalities (Cantor and Hilbert), or only potentially infinite ones (Kronecker, Weyl, and other predicativists).

To repeat the main theme: the contribution to epistemology of *logical guarantee of truth transmission* remained intact. It would always serve both sides of any of these debates in equal measure. Even when, as in the debate between classicists and intuitionists, there was disagreement over *which rules of inference* were permitted, both sides had a clear understanding of how their respective positions were to be represented, in terms of the formal logical resources of which they could avail themselves. The only change to be made in the main theme was that the contribution to epistemology of *logical guarantee of truth transmission — for the respectively appropriate conception of truth*[1] — remained intact.

From its very inception, then, the development and deployment of the fully formal logic of modern times has been intertwined with various philosophies of mathematics, providing always a stable backbone for honest mathematical conduct. And that backbone could be tended, and improved. One could

[1] For the classicist: truth as determinate and bivalent; for the intuitionist or constructivist: truth as warranted assertability.

make alterations to accommodate the conviction that it would be an error to take every grammatically well-formed singular term as denoting some abstract object. Such alterations lead to a *free* logic, which was surprisingly late in its development in the decades following the *Begriffsschrift*. The ensuing tweaking of the rules governing the quantifiers ∀ and ∃, to accommodate 'existential presuppositions' regarding singular terms (including parameters) were minor indeed. But the result has been to make for a more perfect union of fully formal logic with the informally rigorous reasoning in mathematics that it (the logic) was designed to formalize.

It is remarkable how stable is the resulting logical behavior of the two quantifiers. The disputes over correct choice of logical system are always focused on the logical behavior of the *connectives*, not the quantifiers. With the *quantifiers*' having been the last expressions in informally rigorous mathematical practice (as initiated by Cauchy and Weierstraß) to have been formalized properly (by Frege, with his introduction of bound-variable notation), it is rather ironic that the methodological hubbub since then has been confined to the 'much simpler' *connectives*. To be sure, the classical reasoner has ways of proving existential statements that are not condoned by the constructivist. But this discrepancy has its ultimate roots in the classicist's use of strictly classical rules of *negation*. Both the classicist and the constructivist frame and use the same introduction and elimination rules for the *existential quantifier*.

The axioms and rules of formal logic themselves evolved and changed, both in their number and in their forms. The first formal systems, of Frege and of Hilbert, were not at all 'natural'. In a nutshell: too many axioms (or axiom schemata), and too few genuine rules. Formal systems of deduction gradually improved, to the point where in [Gentzen, 1935], via [Prawitz, 1965], we finally reached a formulation, in terms of *rules of inference*, that was both logically clarifying and philosophically illuminating.

There remains, however, one striking feature of standard informally rigorous deductive reasoning in mathematics that has for far too long escaped serious consideration: the issue of *relevance*. Mathematicians *never* (or: never *need* to) reason or infer irrelevantly. The ultimate premises for their proofs of any mathematical theorem — their so-called mathematical *axioms* — are always *relevant to* the theorems that are deduced from them. Their step-by-step deductive progress from those axioms to their theorems is manifestly relevant at every stage. At no stage in mathematical reasoning is there ever utter thematic discontinuity, or a rupturing lurch from one set of ideas to entirely disconnected ones. There is always a chain of expressive connections. Mathematical reasoning is always tightly knitted so that it can unfold. We provide this purposely metaphorical and intuitive imagery in order to underscore the need for a formal explication of what, in informally rigorous mathematical practice, inspires it.

The questions that animate this study are, first, this rhetorical one:

> *Why hasn't formal logic finally caught up with this striking feature of mathematical reasoning, and done it justice?*

Second, by way of analytic and formal focus:

> *How best does one update or reform or tweak the fully formal logic that the tradition has bequeathed us, so as to capture fully, and endorse only, such deductive reasoning as is genuinely relevant?*

Note that this issue is orthogonal to the question whether one should choose only a constructive formal logical system, or the non-constructive, classicized one that is more widespread in current mathematical practice. Intuitionist and constructive mathematicians reason every bit as relevantly as do their classical *confrères*. It would be an added bonus, for any approach to relevance, if it could be shown that the best way to 'relevantize' is exactly the same for a system of constructive reasoning as it is for a system of non-constructive reasoning.

So: *how does one capture such relevance?* This is simultaneously a question for mathematicians, logicians, *and philosophers of mathematics*. The optimal answer to this question will make use of already well-established metalogical concepts and seek only to 'fine tune' systemic details so as to achieve the goal of relevance. What sort of systems might then be the result? We shall begin our own answer here by displaying our hand(s) right away.

2. THE CORE SYSTEMS

Our discussion is motivated by the need to reconsider the relationship between the semantic relation of logical consequence and the syntactic relation of deducibility, in light of the delineation of the two deductive systems (at first order) of Core Logic $\mathbb{C}$, which is constructive, and its classicized extension Classical Core Logic $\mathbb{C}^+$, which of course is non-constructive.[2] For the reader unacquainted with the two Core systems, some easily accessible explanation of them will be in order at the outset.[3] First, in §2.1, we state the rules for the (unfree) systems, for easy reference, in familiar natural-deduction and sequent-calculus formats. Next, in §2.2, we shall discuss the way in which the two systems capture *relevant* deductions *par excellence*. This is foundationally important, because, as stressed in our introductory remarks, *all mathematical reasoning is relevant*, in the intuitive sense of 'relevant' that we claim can be precisely explicated — see the Appendix — and captured in the Core systems.

2.1. Rules of Natural Deduction and Sequent Calculus

With our natural deduction rules, when we append a *diamond* ($\Diamond$) to the discharge stroke over an assumption, this means that the assumption need not have been used as a premise in the subproof; but, if it has been used, it gets discharged. When we append a *box* ($\Box$) to the discharge stroke over an assumption, we are registering explicitly the requirement that the assumption *must* have been used. (If boxes are placed *between* discharge strokes, the requirement is that at least one of the indicated assumptions must have been used.)

[2] For a full account of these systems, see [Tennant, 2017] and earlier publications cited therein.

[3] At this point, the reader might benefit from a quick look ahead to Figure 1 in §2.4.

We use vertical ellipses to gesture at possible proof-work implicitly there. We shall supply explicit reminders of sets of undischarged side-assumptions (the Δs and Γs below). We shall be meticulous in annotating discharge rules with their boxes or diamonds.[4] Note that dischargeable assumptions are separated by commas from their accompanying sets of side-assumptions. This underscores the fact that they (the dischargeable assumptions) are not to be taken as members of those accompanying sets.

Note the following important features. All the elimination rules are in *parallelized* form, and their major premises *stand proud*, with no non-trivial proof-work above them. Hence, all proofs are in *normal form*. There is no rule of *Ex Falso Quodlibet* (henceforth: EFQ).

With our rules for the Sequent Calculus, note that the only 'structural' rule is that of Reflexivity; proofs, after all, need to get started. There is no rule of THINNING; there is no rule of CUT. Nor do we have any need for Gentzen's structural rules of INTERCHANGE and CONTRACTION, since we are dealing with (single conclusion) *set* sequents (even in the classical case), and not, as Gentzen did, with (possibly multiple-conclusion) *sequence* sequents.

What follows is a list of Introduction (resp., Elimination) rules in Natural Deduction, paired with their corresponding Right (resp., Left) rules in Sequent Calculus.

$$(\neg\text{I})\quad \dfrac{\begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\ ,\ \Delta}}\\ \vdots\\ \bot\end{array}}{\neg\varphi}{\scriptstyle (i)} \qquad (\neg\text{R})\quad \dfrac{\varphi\,,\,\Delta:\bot}{\Delta:\neg\varphi} \qquad (\neg\text{E})\quad \dfrac{\neg\varphi \qquad \begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array}}{\bot} \qquad (\neg\text{L})\quad \dfrac{\Delta:\varphi}{\Delta,\neg\varphi:\bot}$$

$$(\wedge\text{I})\quad \dfrac{\begin{array}{c}\Delta_1\\ \vdots\\ \varphi\end{array} \qquad \begin{array}{c}\Delta_2\\ \vdots\\ \psi\end{array}}{\varphi\wedge\psi} \qquad (\wedge\text{R})\quad \dfrac{\Delta_1:\varphi \quad \Delta_2:\psi}{\Delta,\Delta_2:\varphi\wedge\psi} \qquad (\wedge\text{E})\quad \dfrac{\varphi\wedge\psi \qquad \begin{array}{c}\overset{(i)\text{---}\square\text{---}(i)}{\underbrace{\varphi\,,\psi\ \ ,\ \Delta}}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (i)} \qquad (\wedge\text{L})\ \left\{\begin{array}{c}\dfrac{\Delta,\varphi,\psi:\theta}{\Delta,\varphi\wedge\psi:\theta}\\[2ex] \dfrac{\Delta,\varphi:\theta}{\Delta,\varphi\wedge\psi:\theta}\\[2ex] \dfrac{\Delta,\psi:\theta}{\Delta,\varphi\wedge\psi:\theta}\end{array}\right.$$

$$(\vee\text{I})\quad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array}}{\varphi\vee\psi} \quad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ \psi\end{array}}{\varphi\vee\psi} \qquad (\vee\text{R})\quad \begin{array}{c}\dfrac{\Delta:\varphi}{\Delta:\varphi\vee\psi}\\[2ex] \dfrac{\Delta:\psi}{\Delta:\varphi\vee\psi}\end{array} \qquad (\vee\text{E})\quad \dfrac{\varphi\vee\psi \qquad \begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\varphi\ ,\Delta_1}}\\ \vdots\\ \theta/\bot\end{array} \qquad \begin{array}{c}\overset{\square\text{---}(i)}{\underbrace{\psi\ ,\Delta_2}}\\ \vdots\\ \theta/\bot\end{array}}{\theta/\bot}{\scriptstyle (i)} \qquad (\vee\text{L})\quad \dfrac{\Delta_1,\varphi:\theta/\bot \quad \Delta_2,\psi:\theta/\bot}{\Delta_1,\Delta_2,\varphi\vee\psi:\theta/\bot}$$

The compressed notation '$\theta/\bot$' here indicates that if either of the subordinate conclusions for ($\vee$E) is $\bot$, then one may bring down as the main conclusion the other subordinate conclusion. The sequent rule ($\vee$L) is to be understood in the same way.

[4] This represents an increase in explicitness over the presentation of discharge rules by Gentzen and Prawitz.

(→I) $\dfrac{\begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\varphi\ ,\ \Delta}}\\ \vdots\\ \bot\end{array}}{\varphi\to\psi}{\scriptstyle(i)}\quad \dfrac{\begin{array}{c}\overset{\Diamond\text{—}(i)}{\underbrace{\varphi\ ,\ \Delta}}\\ \vdots\\ \psi\end{array}}{\varphi\to\psi}{\scriptstyle(i)}$ (→R) $\left\{\begin{array}{l}\dfrac{\Delta,\varphi:\bot}{\Delta:\varphi\to\psi}\\[2ex] \dfrac{\Delta:\psi}{\Delta\backslash\{\varphi\}:\varphi\to\psi}\end{array}\right.$ (→E) $\dfrac{\varphi\to\psi\quad \begin{array}{c}\Delta_1\\ \vdots\\ \varphi\end{array}\quad \begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\psi\ ,\Delta_2}}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$ (→L) $\dfrac{\Delta_1:\varphi\quad \Delta_2,\psi:\theta}{\Delta_1,\Delta_2,\varphi\to\psi:\theta}$

(∃I) $\dfrac{\begin{array}{c}\Delta\\ \vdots\\ \varphi^x_t\end{array}}{\exists x\varphi}$ (∃R) $\dfrac{\Delta:\varphi^x_t}{\Delta:\exists x\varphi}$ (∃E) $\dfrac{\exists x\varphi^{@}\quad \begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\varphi^x_a\ ,\ \Delta^{@}}}\\ \vdots\\ \theta^{@}\end{array}}{\theta^{@}}{\scriptstyle(i)}$ (∃L) $\dfrac{\Delta,\varphi^x_a:\theta}{\Delta,\exists x\varphi:\theta}$ where a does not occur in the bottom sequent

(∀I) $\dfrac{\begin{array}{c}\Delta^{@}\\ \vdots\\ \psi(a)\end{array}}{\forall x\psi(x)}$ (∀R) $\dfrac{\Delta:\psi(a)}{\Delta:\forall x\psi(x)}$ where a does not occur in the bottom sequent (∀E) $\dfrac{\forall x\psi\quad \begin{array}{c}\overset{(i)\text{—}\cdots\square\cdots\text{—}(i)}{\underbrace{\psi^x_{t_1},\ldots,\psi^x_{t_n}\ ,\Delta}}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle(i)}$ where $t_1,\ldots,t_n$ are closed terms (∀L) $\dfrac{\Delta,\psi^x_{t_1},\ldots,\psi^x_{t_n}:\theta}{\Delta,\forall x\psi:\theta}$

The foregoing rules form the system $\mathbb{C}$ of *Core Logic*. The system $\mathbb{C}^+$ of *Classical Core Logic* is obtained by adding the following Classical Rule of Dilemma (Natural Deduction form on the left, Sequent Calculus form on the right).

$$\dfrac{\begin{array}{c}{\scriptstyle(i)}\text{—}\square\\ \varphi\\ \vdots\\ \psi\end{array}\qquad \begin{array}{c}\square\text{—}{\scriptstyle(i)}\\ \neg\varphi\\ \vdots\\ \psi\end{array}}{\psi}{\scriptstyle(i)}\qquad \dfrac{\begin{array}{c}{\scriptstyle(i)}\text{—}\square\\ \varphi\\ \vdots\\ \psi\end{array}\qquad \begin{array}{c}\square\text{—}{\scriptstyle(i)}\\ \neg\varphi\\ \vdots\\ \bot\end{array}}{\psi}{\scriptstyle(i)}\qquad\qquad \dfrac{\Delta,\varphi:\psi\qquad \Gamma,\neg\varphi:\psi/\bot}{\Delta,\Gamma:\psi}\ .$$

2.2. The Pursuit of Relevance

A minimal requirement of relevance for any system of formal proof is that there be no proof of either the sequent $A,\neg A:B$ (the 'positive' form of the First Lewis Paradox) or the sequent $A,\neg A:\neg B$ (the 'negative' form thereof).

Every Classical Core proof (hence every Core proof) is in 'normal form' of an exigent kind. In sequent-calculus terms, as already mentioned, the so-called structural rules of CUT and of THINNING:

$$\text{CUT}\quad \dfrac{\Delta:\varphi\quad \varphi,\Gamma:\psi}{\Delta,\Gamma:\psi}\qquad\qquad \text{THINNING}\quad \dfrac{\Delta:}{\Delta:\varphi}$$

are *absent* from the Core systems. The reason for this is that the presence of *either one* of these two structural rules would engender Lewis's First Paradox, which is anathema to *all* relevance logicians. As far as *deducibility* (⊢) is concerned, adopting Gentzen's unrestricted CUT rule would oblige one to surrender

Disjunctive Syllogism ($A \vee B, \neg A : B$), on pain of reinstating the First Lewis Paradox:

$$\frac{A : A \vee B \qquad A \vee B, \neg A : B}{A, \neg A : B}\,\textsc{cut}\,.$$

Disjunctive Syllogism is ubiquitous in mathematical reasoning; so unrestricted CUT must go. And to avoid reinstating the First Lewis Paradox every relevance logician (of whatever 'school') also has to surrender THINNING:

$$\frac{\dfrac{A : A}{A, \neg A :}}{A, \neg A : B}\,\textsc{thinning}\,.$$

2.3. Two Schools of Relevance Logic

Relevance logicians divide roughly into two schools. The most familiar one, given its greater vintage, is the Anderson–Belnap school of relevance logic.[5] This school concentrates on the task of 'relevantizing' *the object-linguistic conditional connective* $\rightarrow$. As far as the metalinguistic relation of *deducibility* ($\vdash$) is concerned, they are committed to the *unrestricted transitivity of deduction* afforded by the CUT rule. Because of that, as explained above, they have to surrender Disjunctive Syllogism ($A \vee B, \neg A : B$), on pain of reinstating the First Lewis Paradox.

By contrast with the Anderson–Belnap school's approach to relevance, the more recent approach of the Core logician is to concentrate on the task of 'relevantizing' the metalinguistic *turnstile* $\vdash$ of deducibility[6] (and, accordingly, whatever 'double' turnstile $\models$ of relevant logical consequence we might be able to match to the relevantized single turnstile $\vdash$). The Core logician does not need to surrender Disjunctive Syllogism. And Lewis's First Paradox remains blocked because the structural rules of CUT and of THINNING, as already observed, are not available in the Core systems.

The following table summarizes the fundamental contrast between the Anderson–Belnap approach to relevance (**A–B**) and that of the Core logician (**Core**). Strong claims in this diagram will receive their justification presently.

	A–B	**Core**
Focus of relevantizing	$\rightarrow$	$\vdash$
What kind of transitivity of deduction results	Unrestricted (CUT)	Slightly restricted, for epistemic gain
Disjunctive Syllogism?	No	Yes
Adequate for mathematics?	No	Yes

[5] The *locus classicus* is [Anderson and Belnap, 1975].
[6] This approach originated in [Tennant, 1979].

The last line in this diagram is significant and important. The Core systems are designed to achieve the most fundamental of Frege's purposes in devising a formal logic: that of providing *effectively checkable proofs* that establish the truth of mathematical theorems with the same degree of certainty that we profess regarding the truth of the mathematical axioms that are employed as premises in those proofs. And we aim to capture *all* of those mathematical theorems. In these fundamental regards, *none* of the Anderson–Belnap systems of relevance logic passes epistemological muster. *None* of them enjoys a metatheorem to the effect that every logical consequence of a satisfiable set of mathematical axioms at first order can be established as a theorem on the basis of those axioms by means of the first-order system of relevance logic in question. In welcome contrast, the Core systems *do* enjoy such metatheorems.

The parlous predicament of the A–B tradition of relevance logic for the formalization of mathematical reasoning is starkly illustrated by Friedman and Meyer [1992]. Friedman exhibited a theorem of classical Peano Arithmetic that cannot be derived from the Peano Axioms using only the logical resources of the Anderson–Belnap system R of relevance logic. This is dispositive. It shows that R is *not* adequate for the formalization of all the reasoning in classical mathematics — all of which is *intuitively* relevant.[7] And this is the case even in the most modest and central branch of mathematics, namely the theory of the natural numbers.

2.4. More on the Core Systems

As already noted, there is no Classical Core proof (hence no Core proof) of either the sequent $A, \neg A : B$ (the 'positive' form of the so-called First Lewis Paradox) or the sequent $A, \neg A : \neg B$ (the 'negative' form thereof).[8,9] Yet there

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[7] In the words of an anonymous referee, '. . . it shows that resources beyond what are provided by R are necessary if we wish to preserve all "classical" mathematical theorems.' Classical Core Logic provides those resources.

[8] Nor is there any $\mathbb{C}^+$-proof of the 'dual' irrelevance $A : B \vee \neg B$. This follows from the main theorem of [Tennant, 2015b] establishing that the relevance relation $\mathcal{R}(\Delta, \varphi)$ holds for any first-order $\mathbb{C}^+$-proof of φ whose premises form the set Δ. If Δ is non-empty and φ is not $\perp$, then $\mathcal{R}(\Delta, \varphi)$ entails that some atom in φ occurs (with the same parity) in some member of Δ. In fact, if Δ is non-empty and φ is not $\perp$, then $\mathcal{R}(\Delta, \varphi)$ entails an even stronger condition on φ and Δ; but the weaker one we have just stated is all that one needs in order to conclude that the sequent $A : B \vee \neg B$ is not provable in $\mathbb{C}^+$.

[9] The well-known system of Minimal Logic, due to Johansson [1936], admits the negative form $A, \neg A : \neg B$ of the First Lewis Paradox. So Johansson's attempted *Reduzierung* of Intuitionistic Logic to a relevant core was a failure. With the system of Core Logic, Johansson's aim is achieved.

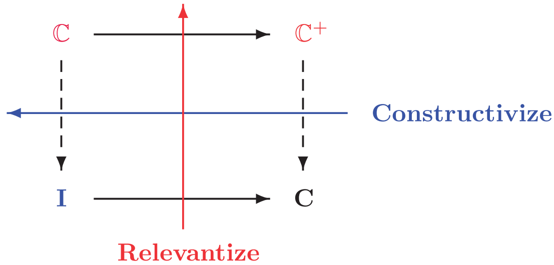

Fig. 1. How $\mathbb{C}$ and $\mathbb{C}^+$ stand to **I** and **C**. Left-to-right is classicizing; top-to-bottom is de-relevantizing. System-inclusions are Left-to-Right and Top-to-Bottom.

is a Core proof of Disjunctive Syllogism:

$$\dfrac{A \vee B \qquad \dfrac{\overline{A}^{(1)} \quad \neg A}{\bot} \qquad \overline{B}^{(1)}}{B}\,(1)\,\vee\mathrm{E}\ .$$

The two Core systems stand to the well-known (and irrelevant) systems **I** of Intuitionistic and **C** of Classical Logic in the way illustrated in Figure 1.

We stress: avoiding the First Lewis Paradox is a *minimal* requirement of relevance. But proofs in $\mathbb{C}^+$ (hence also proofs in $\mathbb{C}$) display a *much more exacting* form of relevance between their premise sets and their conclusions. (Classical) Core Logic does not achieve its status as a relevance logic *simply* by being paraconsistent (*i.e.*, simply by avoiding explosion). Rather, it achieves that status by having rules of inference so carefully crafted that a remarkable 'relation of relevance' $\mathcal{R}(\Delta, \varphi)$ holds between the premise set Δ and the conclusion φ of any (classical) core proof. (See the Appendix for the technical details justifying this claim.) There are no comparable relevance results for any of the systems in the Anderson–Belnap school of relevance logics.

This relation $\mathcal{R}(\Delta, \varphi)$ of relevance goes well beyond anything previously conceived by way of a 'variable-sharing' requirement. 'Variable sharing' is the term of art employed for systems of *propositional* logic for which 'variable' means 'propositional atom'. By contrast the 'variable' sharing described by the relation $\mathcal{R}$ for Classical Core Logic concerns not just propositional *atoms*, but also *extra-logical predicates* in the first-order case. See [Tennant, 2015b] for details. The formal explication $\mathcal{R}(\Delta, \varphi)$ captures the informal notion that primitive linguistic expression occurrences, with suitable parities of occurrences within premises and conclusions of proofs, establish unfailingly that the former are relevant to the latter. That paper ended with a thorough survey of extant variable-sharing results for different propositional systems in the Anderson–Belnap tradition, and showed how the various degrees of 'relevance' they achieved did not (and perhaps cannot) match that of the system $\mathbb{C}^+$ of Classical Core Logic. The challenge was issued at the end of that paper to followers of Anderson and Belnap to furnish a system of relevance logic of their favored kind that would match the deductive power *and the proven achievement of relevance* of the Core systems. It has thus far gone unanswered.

How is this systemic feature of robust relevance achieved by the Core systems? Every Classical Core proof (hence every Core proof), as we have already stressed, is in 'normal form' of an exigent kind. In Core *natural deductions*, as we have already seen, all eliminations are in parallelized form, their major premises standing proud with no non-trivial proof-work above them. In Core *sequent-calculus* proofs, neither cuts nor thinnings are allowed. In addition to these changes to the Gentzenian systems, certain rules governing the logical operators are importantly tweaked. As we saw in §2.1, the rule of →I is tweaked with a new part, and the rule of ∨E is tweaked by being more liberal with respect to the two case-proof conclusions. The natural deduction rules of Gentzen *via* Prawitz have attained canonical status, but that does not mean that their formulations of the various introduction and elimination rules admit of no improvement. The Core logician commends the two tweaks just mentioned, in order to free our logic from the need to have EFQ.

The resulting systems of natural deduction and of sequent calculus then enjoy a beautiful correspondence. Proofs come 'in pairs' — natural deductions and sequent proofs — whose members are structurally isomorphic to each other. So it does not really matter, with the Core systems, whether one deals with natural deductions, or with sequent proofs. They are, as it were, one and the same. This should have struck the reader's eye as soon as it was cast on the rules that were stated in §2.1.

2.5. A Very Important Property of the Core Systems

We are dealing with single-conclusion sequents. These are of the form $\Delta : \{\varphi\}$ or $\Delta : \emptyset$ (the latter often written as $\Delta : \bot$), where Δ is a finite set of sentences. $\Gamma_1 : \Phi_1$ is a *subsequent* of $\Gamma_2 : \Phi_2$ just in case either $\Gamma_1 \subseteq \Gamma_2$ or $\Phi_1 \subseteq \Phi_2$; and it is a *proper* subsequent if either of those containments is proper.

In $\mathbb{C}$ there is of course a proof of the sequent $A, \neg A : \emptyset$, which is a *proper subsequent* of $A, \neg A : B$. This is just a special case of a wholly general property of the proof system $\mathbb{C}$ (and also, incidentally, of $\mathbb{C}^+$). That property can be explained as follows.

Suppose there is a proof (Π, say) of the sequent $\Delta : \varphi$ and a proof (Σ, say) of the sequent $\varphi, \Gamma : \psi$. Suppose further that 'unrestricted' Cut appears to fail, for want of a proof of the notional 'target sequent' $\Delta, \Gamma : \psi$. *This failure of 'unrestricted' Cut does not matter.* This is because a metatheorem guarantees that there *will be* a proof — let us call it $[\Pi, \Sigma]$ — of *some proper subsequent* of the target sequent $\Delta, \Gamma : \psi$.[10]

[10] For this metatheorem see [Tennant, 2012] for the case of $\mathbb{C}$, and [Tennant, 2015a] for the case of $\mathbb{C}^+$.

The Core logician can in effect comfort the traditional, ‘unrestricted’ Cutter by providing a proof of *an even better result.* We can re-frame such ‘failures’ of Cut by saying that they are situations in which Cut *ought* to fail. They are situations in which such failure is more than compensated for by concomitant *epistemic gain*. It ought also to be emphasized that the aforementioned binary function [,] is *computable*. Another way of characterizing this happy property of the Core systems is to say that Cut is in general *admissible, with potential epistemic gain* — despite the fact that Gentzen's ‘unrestricted’ structural rule of CUT is not *in* the systems.[11]

Now, in addition to the welcome provision of *effectiveness* of the method of determination just mentioned (of proof $[\Pi, \Sigma]$ from the proofs Π and Σ), a further welcome feature would be that the effective method would in general furnish, if not a proof of the exact ‘target’ sequent $\Delta, \Sigma : \psi$, then at least a proof of *some proper subsequent* thereof. Why would this be welcome? — because it is *epistemically gainful* to have a proven sequent that is as ‘perfectly valid’ as possible. A valid sequent is *perfectly* valid just in case it has no valid *proper* subsequents. Every sentence appearing in a perfectly valid sequent is necessary for its validity.

It is in this sense that we can show that for the Core systems, despite their not containing the rule of CUT, that rule is nevertheless *admissible with potential epistemic gain*. That is to say, once again:

> Given any proof Π of the sequent $\Delta : \varphi$, and any proof Σ of the sequent $\varphi, \Gamma : \psi$, one can effectively determine a proof $[\Pi, \Sigma]$ of *some subsequent* of $\Delta, \Gamma : \psi$.

It is because the Core systems ‘admit’ of *‘Cut’ with epistemic gain* that they are completely adequate for the formalization of all of mathematics (constructive mathematics in the case of $\mathbb{C}$, classical mathematics in the case of $\mathbb{C}^+$). We shall have more to say about this in due course.

The question now arises: how can we make ‘semantic sense’ of what these Core systems of proof afford us? Put in terms that will be familiar to a contemporary logician: Is there some kind of ‘double turnstile’ (for a relation of ‘semantic’ consequence) that we can match up somehow to the rather different and new ‘single turnstile’ of deducibility in a Core system?

3. THE MODERN NOTION OF LOGICAL CONSEQUENCE

First we shall clarify some notation. We have already been speaking of sentences φ, ψ and sets Δ, Γ of sentences; and now we shall be speaking of models M. The first-order language that contains these sentences will be presupposed in the background.

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[11] We are adverting here to the notion of an admissible rule of inference from premise-sequents to a conclusion-sequent that was introduced by Hiż [1959].

When a model M makes a sentence φ true, we shall write[12]

$$M \Vdash \varphi.$$

When a model makes every sentence in Δ true, we shall say that it *satisfies* Δ, and can write

$$M \Vdash \Delta\,.$$

We shall say that a set Δ of sentences *has* a model just in case there is one that satisfies Δ. We shall often express the same claim by saying 'Δ is satisfiable'. The metalinguistic statement

$$\Delta \models \varphi$$

means that every model satisfying Δ makes φ true. Formally, in the metalanguage:

$$\forall M(M \Vdash \Delta \;\Rightarrow\; M \Vdash \varphi).$$

One can read $\Delta \models \varphi$ also as 'Δ logically implies φ', or as 'φ is a logical consequence of Δ'. Note that this notion of logical consequence is a *semantic* one, defined entirely with reference to models and makings-true.

We have used here the familiar and standard definition of logical consequence that is usually attributed jointly to Bolzano and Tarski. It yields the *classical* conception of consequence when the models are classical ones and the embedded relation '$\Vdash$' is taken to represent the classical relation of 'making true'. It yields the *intuitionistic* conception of consequence when the models are intuitionistic ones (such as Kripke models) and the embedded relation '$\Vdash$' is taken to represent the intuitionistic relation of 'forcing'.[13]

Lemma 1. If Δ is unsatisfiable, then for all φ $\Delta \models \varphi$.

Proof. Suppose that Δ is unsatisfiable. Let φ be an arbitrary sentence. Let Φ be any property. It is trivially true that every model satisfying Δ has property Φ. Formally, in the metalanguage:

$$\forall M(M \Vdash \Delta \;\Rightarrow\; \Phi(M)).$$

[12] This is a carefully considered departure from established conventions in logical terminology. We aim to avoid using the symbol '$\models$' for this particular relation between models and sentences. We thereby reserve that symbol for the relation of logical consequence between sets of sentences and sentences, which will be defined next.

[13] See [Fitting, 1969; Tennant, 1978].

This is because, by supposition, *there are no models satisfying* Δ; whence there are no counterexamples to the universal generalization just displayed. Now take for Φ the property of making φ true:

$$\forall M(M \Vdash \Delta \;\Rightarrow\; M \Vdash \varphi);\text{ i.e., } \Delta \models \varphi.$$

But φ was an arbitrary sentence. Thus for all φ $\Delta \models \varphi$. □

Lemma 1 says that the traditional semantic relation of logical consequence is *explosive*. This holds regardless of whether the models M and the relation $\Vdash$ that they bear to sentences are conceived of, and constituted, intuitionistically or classically. Lemma 1 holds for *intuitionistic* logical consequence as well as for *classical* logical consequence. And this is all for the simple reason that both the intuitionist and the classicist agree that *if there are no Fs, then all Fs are Gs.* The Core logician, too, agrees with this italicized claim. There is a Core proof of the sequent $\neg\exists xFx : \forall x(Fx \rightarrow Gx)$. *This does not mean, however, that the Core logician has to accept that the logical consequence relation is explosive.* The overall aim of this study is to show that the relation of logical consequence should be taken, on the contrary, to be *implosive* — *nothing but absurdity* follows from an unsatisfiable set. Details will emerge in due course.

Along with the standard formal semantics for first-order extensional languages with identity, the tradition provides us also with systems of finitary, effectively checkable, proof. It is by means of such proofs that mathematical reasoning can ideally be 'regimented', and the semantic validity of arguments can come to be *known* with *certainty*. The best and most extensively investigated of the various systems of formal proof that have been invented since Frege wrote his *Begriffsschrift* derive from the work of Gentzen [1935] on systems of natural deduction and on sequent calculi.

Gentzen's systems of formal proof (in the present author's assessment) enjoy a significant advantage over the others. When we said above that mathematical reasoning *can ideally* be 'regimented' as formal proofs, we left the 'can ideally' unexplicated. But when the formal proof systems are the Gentzenian ones, the 'can ideally' becomes a 'can feasibly' — not just a 'can, *in principle*, even if it takes until the heat death of the universe to accomplish'. Of course, it would remain to explicate the notion of feasibility; and in this regard we would be content to say that it means 'within polynomial time', as this is understood in algorithmic complexity theory. This remains a mere conjecture at this stage, but it is one for which the present author has a great deal of anecdotal evidence. Suffice it to say that the *feasible* translatability, into formal proofs, of informally rigorous proofs of mathematical theorems from axioms is a topic calling for a great deal of further, more highly-focused, research. And even if the Gentzenian systems end up not furnishing proofs within feasible computational reach (in producing regimentations of informally rigorous proofs), we are confident that the Gentzenian systems are significantly better, in this regard, than any of their competitors.

With reference to whichever formal system $\mathcal{S}$ of proof one chooses, the metalinguistic statement

$$\Delta \vdash_{\mathcal{S}} \varphi$$

will mean that there is an $\mathcal{S}$-proof, of the conclusion φ, whose premises (*i.e.*, undischarged assumptions) are in Δ. Frequently the subscript is suppressed, when $\mathcal{S}$ is known from the context. The symbol $\vdash$, known as the 'single turnstile', represents the relation of *deducibility* in the system in question.

There is debate between formal semanticists who prioritize models (in terms of which the semantic relation $\models$ of logical consequence is defined), and proof-theoretic semanticists who prioritize rules of inference (in terms of which the syntactic notion $\vdash$ of deducibility is defined), over the question of how best to characterize the meanings of logical operators. This debate, however, is orthogonal to the concerns of this study. We advance no particular preference for either formal semantics or proof theory as the more foundational of the two approaches. The study can be undertaken, and its main arguments and conclusions assessed, from a neutral standpoint from which the main concern is only that there be a (suitably re-conceived) match-up between the two approaches (proof-theoretic and model-theoretic).

The main conclusions of this study may well be regarded by many a logician trained in the current orthodox tradition as too radical. Still, we ask both the philosophically and the technically minded reader to assess the *pro*s and *con*s carefully. We have thus far appealed to the very familiar notions of model and of formal proof; we wish to emphasize, at this stage, that *these* will be taken pretty much as 'given'. At certain points we have, to be sure, caviled at particular aspects of the formulation of rules of inference in the Gentzenian tradition. But the net result has been one of merely 'tweaking' those rules. The major thrust of natural deduction (or the closely related sequent calculus) remains the same.

What we are taking issue with in this study, and focusing on henceforth, is the construction-via-definition of the main resulting metalogical *concepts* (of deducibility and of logical consequence) that the tradition at present affords us. We believe those concepts can be radically amended, without injury or loss to the epistemological role of deductive logic, so as to change logic from being *explosive* to being *implosive*.

4. THE ABSURDITY CONSTANT AND THE RULE EFQ

Natural-deduction systems are standardly formalized with the use of the so-called absurdity constant $\perp$ ('*falsum*') as an official sentence of the formal language. The present author has argued elsewhere that one can forgo having $\perp$ as a sentential constant, and regard it simply as a punctuation mark within proofs, keeping it visible only for convenience, to mark the end of a *dis*proof.[14] That 'punctuation mark' conception of the role of $\perp$ could be

[14] See [Tennant, 1999].

rigorously respected and applied throughout this study; but it would be at some cost in the immediate comprehensibility of the main heterodox proposal being put forward for serious consideration. We have therefore decided, in this exposition, to go along with the standard conception of $\perp$ as a (non-embeddable) sentential constant.

No model makes the absurdity constant $\perp$ of the object language true.[15] This we shall express inferentially by means of the metalogical rule

$$\alpha \ : \quad \frac{\mathbf{M} \Vdash \perp}{\underline{\perp}},$$

where $\underline{\perp}$ is the absurdity constant in the metalanguage. (We shall have occasion to apply the rule (α) in due course.)

In what is curiously like a case of the tail wagging the dog, the explosiveness of logical consequence (the *semantic* relation, be it classical or intuitionistic) is frequently nowadays presented as justification for the presence in one's system of natural deduction of a very problematic *rule of inference* in which the absurdity symbol $\perp$ features. This is the rule EFQ, also known as *Ex Contradictione Quodlibet* or the Absurdity Rule:

$$\frac{\perp}{\varphi}.$$

The rule EFQ along with more prosaically expressed views that lend themselves to formalized expression as EFQ have been part of the tradition in Classical Logic going back at least as far as Peter Abelard (1079–1142).[16] EFQ has survived even into the modern-day Intuitionistic Logic that Brouwer began employing in the early 1900s, and that was belatedly formalized by Heyting [1930] as a strict subsystem of Classical Logic. So EFQ has come to enjoy a firm and unchallenged place in the aforementioned systems of natural deduction due to Gentzen and Prawitz. Note also that commitment to EFQ has been on the *syntactic* side. It is a formal *rule of inference*. EFQ is at home in virtually every orthodox system of formal proof.[17] And these systems pre-dated the development of modern model theory with its clarificatory definition of the explosive semantic relation $\models$ of logical consequence.

[15] Note that on the aforementioned 'punctuation-mark' conception of the role of $\perp$ this would have to be re-expressed as 'No model makes any sentence and its negation true.'

[16] I am indebted here to Peter King. He points out that Abelard maintained that two of the eight principles generally taken to cover propositional logic (*Dialectica* 288.23–34), namely "not-(if not-p then p)" and "not-(if p then not-p)", have to be rejected since if accepted they would entail anything (*Dialectica* 290–292).

[17] As already pointed out, the main system that dropped EFQ was the Minimal Logic of Johansson [1936]. But unfortunately it still proves the negative form $A, \neg A : \neg B$ of the First Lewis Paradox; so Johansson's attempt failed.

5. SOUNDNESS AND COMPLETENESS

We ended §3 by speaking of a 'match-up' between the proof-theoretic and model-theoretic approaches. The well-known soundness and completeness metatheorems for systems of proof with respect to the language's formal semantics have standardly been taken to constitute such a match-up:

$$\text{Soundness:} \qquad \Delta \vdash \varphi \;\;\Rightarrow\;\; \Delta \models \varphi;$$

$$\text{Completeness:} \qquad \Delta \models \varphi \;\;\Rightarrow\;\; \Delta \vdash \varphi.$$

Gödel [1930] established the strong Completeness Theorem for Classical Logic; and the method of proof was much improved by Henkin [1950]. So for nine decades at least, there has been *détente* between proof theory and model theory.

When a significant reform is proposed and effected on one side of the accord, however, pressure can arise for accommodations on the other side, to restore the balance of power. *Rapprochement* is called for. A great deal of interest, therefore, attaches to the question of what metalogical form could be devised for a 'suitably re-conceived' match-up between a newly defined notion of deducibility, such as the one provided by the system $\mathbb{C}^+$ of Classical Core Logic,[18] and a newly defined notion of 'genuine' logical consequence, when EFQ is to be eschewed as a rule of inference. $\mathbb{C}^+$ eschews EFQ, in pursuit of relevance. So the extension of the deducibility relation of $\mathbb{C}^+$ is *properly* included in that of Classical Logic **C**. Let us now explore what adjustments have been, or could be, made in the relationship between deducibility and consequence, in pursuit of this relevantist reform of Classical Logic.

The soundness and completeness metatheorems for $\mathbb{C}^+$ that we have at present take the following forms. The reader is reminded that the semantic relation $\models$ of logical consequence is the conventional, classical, explosive one.

$$\text{Soundness:} \qquad \Delta \vdash_{\mathbb{C}^+} \varphi \;\;\Rightarrow\;\; \Delta \models \varphi;$$

$$\text{Completeness:} \qquad \Delta \models \varphi \;\;\Rightarrow\;\; (\Delta \vdash_{\mathbb{C}^+} \varphi \text{ or } \Delta \vdash_{\mathbb{C}^+} \bot).$$

In these statements, as in the original ones above, Δ is taken to be an arbitrary *set* of sentences; thus Δ could be infinite. And φ is taken to be an arbitrary *sentence* (φ could be $\bot$). Remember that the definition of deducibility requires only that the premises of the witnessing proof be members of Δ, not that they exhaust Δ.

When $\bot$ is the conclusion, both **C** and $\mathbb{C}^+$ achieve completeness in the same way (for arbitrary sets Δ of sentences).

$$\text{Classical:} \quad \Delta \models \bot \;\;\Rightarrow\;\; \Delta \vdash_{\mathbf{C}} \bot.$$

$$\text{Classical Core:} \quad \Delta \models \bot \;\;\Rightarrow\;\; \Delta \vdash_{\mathbb{C}^+} \bot.$$

[18] See [Tennant, 2017].

Likewise, for *satisfiable* sets Δ of sentences, they achieve completeness in the same way.

$$\text{Classical:} \quad \Delta \models \varphi \;\Rightarrow\; \Delta \vdash_{\mathbf{C}} \varphi.$$

$$\text{Classical Core:} \quad \Delta \models \varphi \;\Rightarrow\; \Delta \vdash_{\mathbb{C}^+} \varphi.$$

The crucial difference between the 'completenesses' of **C** and of $\mathbb{C}^+$ is manifested only in the case where Δ is an *unsatisfiable* set of sentences (*i.e.*, $\Delta \models \bot$) and φ is not $\bot$. For in *that* case we have (expanding now to make the formerly implicit universal quantifications explicit)

$$\text{Classical:} \quad \forall\Delta\,\forall\varphi{\neq}\bot(\Delta \models \bot \;\Rightarrow\; \Delta \vdash_{\mathbf{C}} \varphi);\ \text{but}$$

$$\text{Classical Core:} \quad \neg\forall\Delta\,\forall\varphi{\neq}\bot(\Delta \models \bot \;\Rightarrow\; \Delta \vdash_{\mathbb{C}^+} \varphi).$$

The system **C** 'achieves' completeness here in two simple steps. Under the governing supposition $\Delta \models \bot$, we know we already have $\Delta \vdash_{\mathbf{C}} \bot$; so by one step of EFQ it follows that $\Delta \vdash_{\mathbf{C}} \varphi$.

The corresponding metalinguistic conditional for $\mathbb{C}^+$ fails to hold in full generality simply because EFQ is not a rule of the system, and is not derivable in it either. (Note that $A, \neg A \models B$; but, because $\mathbb{C}^+$ ensures *relevance* between the premises and the conclusion of any proof, we have $A, \neg A \nvdash_{\mathbb{C}^+} B$.)

In order to achieve a different *rapprochement* between the deducibility relation of $\mathbb{C}^+$ and some sort of matching semantic consequence relation, we can consider paring down the conventional relation $\models$ in some appropriate way.

To this end we shall introduce a new semantic relation of 'genuine' logical consequence, symbolized by '$\Vdash$'. Thus we shall read '$\Delta \Vdash \varphi$' as 'Δ *genuinely* logically implies φ'. The genuineness of such logical implication consists not only in its coinciding with ordinary logical consequence $\models$ ('double-turnstile') when Δ is satisfiable, but also in $\bot$'s being the only possible consequence of Δ when the latter is *un*satisfiable. Logicians could perhaps refer to the symbol $\Vdash$ as 'double-double-turnstile'. When $\Delta \Vdash \varphi$ holds, we shall say that the argument (or sequent) $\Delta : \varphi$ is *genuinely valid*. We shall re-visit the metalogical forms of the soundness and completeness conditionals, but now with the deducibility relation for $\mathbb{C}^+$ being matched to the new relation $\Vdash$ of genuine logical consequence.

We should say, by way of anticipatory reassurance at this juncture, that we shall be leaving untouched the major feature of proofs, which is that they should be effectively checkable for correctness (which of course entails that they must be finite). Nowhere will it be proposed that rules may be applied only if (for example) the premises for the rule application form a consistent set. This is for the obvious reason that according to Church's undecidability theorem there is no effective method for checking whether such a condition would be met — so any proof calling for consistency checks would not be effectively checkable for correctness.

The main concerns of the remainder of our study will be: Why, in addition to the Introduction and Elimination rules for negation — in which, understandably, $\perp$ has to occur — do Intuitionistic and Classical Logic have EFQ? Can it be justified? Could it be rejected? Could it be replaced by something else? *Should* it be replaced by something else? And finally: Can we answer these questions in a heterodox fashion while still deploying the fundamental notion of a *model* from formal semantics, and the fundamental notion of a formal *proof* from proof theory (even if the latter notion needs some modification should we decide to reject EFQ altogether)?

6. CONDITIONS THAT A PROOF SYSTEM OUGHT TO SATISFY

Consider a logical system $\mathcal{S}$ — for definiteness, a system of natural deduction — satisfying the following conditions:

1. $\mathcal{S}$-proofs are finite. Hence, the set of premises of any $\mathcal{S}$-proof is finite. The relation 'Π is an $\mathcal{S}$-proof of the sentence φ whose premises form the set Δ' is effectively decidable.
2. If there is an $\mathcal{S}$-proof of $\perp$ whose premises form the set Δ, then Δ has no model.
3. If Δ is a set of sentences that has no model, then there is an $\mathcal{S}$-proof of absurdity ($\perp$) from premises in Δ.
4. If Δ has a model and there is an $\mathcal{S}$-proof of φ from premises in Δ, then every model of Δ makes φ true.
5. If Δ has a model and every model of Δ makes φ true, then there is an $\mathcal{S}$-proof of φ from premises in Δ.
6. The premise set Δ and the conclusion φ of any $\mathcal{S}$-proof are mutually relevant.[19]

These six conditions suffice for the remainder of the discussion in this paper.

The present author would like to place on record, however, that there is much to be said in favor of one more condition. In all likelihood it (like Condition 6 above) has not been much considered by formal logicians, even when they address the matter of a logic's adequacy unto the deductive demands of mathematics and science.

7. Given any proof P in a mathematics journal or textbook, written by an expert mathematician, and acknowledged within the pertinent mathematical community as meeting their standards of informal rigor, the following task will be feasible for proponents of system $\mathcal{S}$: extract from the proof P its conclusion φ, along with the set Δ of its premises; then regiment P as a fully formal $\mathcal{S}$-proof of φ from Δ, in such a way as merely to supply

[19] An explication of this relation of relevance is the aforementioned syntactic relation $\mathcal{R}(\Delta, \varphi)$ defined in [Tennant, 2015b]. The reader will recall that Condition 6 has been advanced here as a major aim for a formal system that could fully formalize deductive reasoning in mathematics.

missing detail, while preserving at the macro-level all the various 'lines of argument' that an expert mathematician can discern within P.

This is a much more exigent requirement on the relationship between informally rigorous proofs and their formal regimentations than any that was broached by Hamami [2022]. The reader is now asked to set (7) aside.

Let us focus on Conditions (2)–(5).

(2) can be thought of as "soundness of $\mathcal{S}$ with respect to unsatisfiable sets of sentences".
(3) can be thought of as "completeness of $\mathcal{S}$ with respect to unsatisfiable sets of sentences".
(4) can be thought of as "soundness of $\mathcal{S}$ with respect to satisfiable sets of sentences".
(5) can be thought of as "completeness of $\mathcal{S}$ with respect to satisfiable sets of sentences".

Note that (3) entails that every logically false sentence φ can be refuted in $\mathcal{S}$; that is, there is an $\mathcal{S}$-proof of $\perp$ from (the singleton of) φ.

Observation 1. The empty set (of sentences) is not merely satisfiable, that is, satisfied in *some* model; it is satisfied in *every* model. This is because every model makes every one of the empty set's members true — because there are none.

Note that (4) entails that any $\mathcal{S}$-theorem (*i.e.*, sentence with an $\mathcal{S}$-proof whose premise set is empty) is (by Observation 1) a logical truth, *i.e.*, true in every model.

Note that (5) entails that every logically true sentence φ can be proved outright in $\mathcal{S}$; that is, φ is the conclusion of an $\mathcal{S}$-proof whose premise set is empty.

Lemma 2. (1), (2), (3), (4), and (5) do not collectively entail that if there is an $\mathcal{S}$-proof of $\perp$ whose premises form the set Δ, then there is an $\mathcal{S}$-proof of any sentence whatsoever from Δ.

Proof. The system $\mathbb{C}^+$ of Classical Core Logic meets requirements (1), (2), (3), (4), and (5). There is a $\mathbb{C}^+$-proof of $\perp$ from the set $\{A, \neg A\}$;[20] but there is no $\mathbb{C}^+$-proof of B from $\{A, \neg A\}$. □

Note that Lemma 2 would be trivially true if one were to add condition (6) to the list of five conditions that it mentions.

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[20] Indeed, this is a proof in Core Logic $\mathbb{C}$.

Lemma 2 tells us that (1), (2), (3), (4), and (5) do not force the logic to be explosive. We are morally certain that a strengthening of Lemma 2 can be achieved by incorporating (7); but at this stage this remains a conjecture.

It is worth stressing that we are engaging in an *a prioristic* methodological investigation, insofar as it involves merely reflecting on, and analyzing, the clear and distinct *deductive needs* of both mathematicians and natural scientists. The aim is to formulate (as we have done above) *conditions of adequacy* on any logical system that is to satisfy those needs. Once we have our logic, we can proceed to develop our theories. This is the proper order of foundational study. It is the *opposite* of the recently touted 'abductive methodology' according to which we are presumed to have prior access to theories in general (both mathematical and scientific) and then somehow — by means of criteria both vague and implausible — work backwards from those theories (closed, one might ask, under *what* relation of logical consequence or deducibility, exactly?) to some choice of logical system the preference for which is based on intuitive and unexplicated notions of simplicity, elegance, fit with evidence, and the like. For an account of such an abductive (and *a posterioristic*) methodology, see [Williamson, 2017].

7. THE QUESTION

The question now posed for consideration is this:

> *What extra condition, if any, not already entailed by (1), (2), (3), (4), and (5) can possibly be required of any logical system of proof that is to be adequate for the formal regimentation of the deductive reasoning that is involved in mathematics and science?*

Here, *being adequate* is to be understood as being able to furnish whatever proofs and disproofs might be required as fully formal regimentations of the deductive reasoning that is involved in these areas of intellectual endeavor. We include, of course, proving theorems in mathematics from decidable sets of mathematical axioms; discovering inconsistencies in proposed axiom sets; making scientific predictions from scientific hypotheses combined with statements of initial and boundary conditions for experiments; and discovering any conflicts that could arise between such predictions and the statements of observations and measurements that might result from experimental testing.

Having raised the foregoing question, we shall pause to anticipate and dispose of one possible reply to it. It involves the controversial rule EFQ. The reply in question would be

> *Surely you would wish your logic to be able to derive from a contradictory set of premises any sentence you like? That is, surely you would wish to have EFQ in your proof system? This important virtue of a logical system of proof is, by your own Lemma 2, not entailed by your conditions (1), (2), (3), (4), and (5).*

The answer to this question is an emphatic *No*. Note that this question *cannot even be raised* by one who accepts the relevance condition (6) on $\mathcal{S}$-proofs. For EFQ directly violates relevance. The extra feature that the questioner thinks is a virtue is actually a vice.

8. EFQ AND EXPLOSION

The rule EFQ, despite being so widely adopted as a 'primitive' rule of conventional natural deduction, is not readily acceptable. It can be irksome to experts in logic, a thorn in the logic teacher's side, and a stumbling block for students of logic. Perhaps Abelard was right (and, in retrospect, deservedly so?) when he wrote to Heloise '... odiosum me mundo reddidit logica' ('... logic has made me hated in the world').[21]

The simple Gentzen–Prawitz proof

$$\dfrac{\dfrac{A \quad \neg A}{\bot}\,{\scriptstyle \neg\text{Elim}}}{B}\,{\scriptstyle \text{EFQ}}$$

shows that EFQ leads to the notorious First Lewis Paradox

$$\dfrac{A \quad \neg A}{B} \ .$$

All relevance logicians, and a great majority of beginners in logic, reject this result (as a would-be valid inference). This is because there need not be any meaning connection at all between A and B; and they have the intuition, quite rightly, that there *should* be such a connection between the premises and the conclusion of any genuinely valid argument. More generally, whenever one has a disproof (that is, a collective *reductio ad absurdum*) Π of a set Δ of sentences:

$$\begin{array}{c}\Delta\\ \Pi\\ \bot\end{array} ,$$

the rule EFQ permits one to conclude further that Δ logically implies any sentence ψ whatsoever:

$$\begin{array}{c}\Delta\\ \Pi\\ \dfrac{\bot}{\psi}\,{\scriptstyle \text{EFQ}}\end{array} ,$$

regardless of whether ψ enjoys any connection of meaning with the premises in Δ. This is why EFQ itself is sometimes called 'Explosion'. It blows up any

[21] From a letter to Heloise, in V. Cousin, ed., *Opera Petri Abaelardi*. 2 vol. Paris: A Durand, 1849, 1859, 1.680–81; also in PL178 c375–378; trans. Betty Radice, *The Letters of Abelard and Heloise*. Harmondsworth: Penguin, 1974, pp. 270–271.

inconsistent set of sentences to be the whole language. The sad irony is that the belated adoption of the simple-minded, and therefore explosive, semantic relation $\models$ of logical consequence is now taken to provide some kind of retrospective justification for having EFQ as a rule of inference, in the minds of many orthodox-logic lobbyists.

9. ALLEGED LOSSES AT THE META-LEVEL UPON GIVING UP EFQ

As far as provabilities *within* the system are concerned (at the *object* level), we have already seen that EFQ is dispensable. But the supporter of EFQ is likely to complain that in giving up EFQ other kinds of losses will be incurred, these ones at the *meta*-level. The present author has countered elsewhere the complaint that the so-called Deduction Theorem

$$\Delta \vdash \varphi \rightarrow \psi \quad \Leftrightarrow \quad \Delta, \varphi \vdash \psi$$

fails in the left-to-right direction. We shall not reprise the considerations here.[22] We shall turn instead to the rather more subtle question of valid arguments supposedly having to be 'closed under substitutions'.

9.1. Substitution Instances of Proofs

One of the 'sacrosanct' features of formal logic is widely supposed to be that proofs should be closed under uniform substitutions for their primitive extra-logical expressions. Consequently, and more importantly, the 'substitutionist dogma' maintains

> for any such substitution σ applied to a proof Π of φ from Δ (which proof establishes the argument $\Delta : \varphi$ as valid) the resulting proof $\sigma\Pi$ must establish, as valid also, the resulting argument $\sigma\Delta : \sigma\varphi$.[23]

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[22] See Tennant [2018].

[23] One striking piece of evidence that the substitutionist dogma has a strong hold is to be found in [Williamson, 2017]. Williamson abandons the *a prioricity* of foundational considerations that should determine one's choice of logic. He opts instead for an 'abductive methodology' to determine that choice. Yet he still clings to the substitutionist dogma. He states a supposedly stripped-down set of requirements on the relations of logical consequence that can so much as enter into consideration. At p. 327 he writes

> Logical consequence in the sense of $\models$ obeys the standard structural rules for a consequence relation. That is, the following hold for all sentences α and β of L and all sets Γ and Δ of sentences of L:

Assumption	$\{\alpha\} \models \alpha$
Monotonicity (Thinning)	If $\Gamma \models \alpha$ then $\Gamma \cup \Delta \models \alpha$
Cut	If $\Gamma \models \alpha$ and $\Delta \cup \{\alpha\} \models \beta$ then $\Gamma \cup \Delta \models \beta$

To be sure, even on an exigent reading of 'validity', the dogma just stated is true for all proofs of the extremal forms

$$\begin{matrix} \emptyset \\ \Pi \\ \varphi \end{matrix} \quad \text{and} \quad \begin{matrix} \Delta \\ \Pi \\ \bot \end{matrix}$$

— that is, proofs of logical truths and disproofs of unsatisfiable sets of premises. The dogma is *not* true, however, of many a case 'in between' those extremal cases. Along the spectrum of intermediate cases are ones whose premise sets are satisfiable and whose conclusions are falsifiable.

To an examination of such cases we now turn.

9.1.1. Substitutions can turn satisfiable premise sets into unsatisfiable ones

The opponent of EFQ will concede, for example — or adduce by way of challenging the substitutionist dogma — that one can no longer simply assume that if one has a proof Π of a *genuinely* valid sequent $\Delta : \varphi$ (with Δ satisfiable and φ distinct from $\bot$), then any non-trivial substitution instance $\sigma\Pi$ of the proof Π will automatically establish the genuine validity of the sequent $\sigma\Delta : \sigma\varphi$. The opponent of EFQ will point out that non-trivial substitutions, by increasing logical complexity, are liable on occasion to turn a (non-empty) satisfiable set Δ of premises into an unsatisfiable one. In such a case (where $\sigma\Delta$ is unsatisfiable), according to one kind of opponent of EFQ (namely, the one who recommends adopting a more exigent relation $\Vdash$ of 'genuine' consequence in place of the conventional relation $\models$), the absurdity constant $\bot$ will be the *sole* genuine consequence of $\sigma\Delta$. One would simply be in error to think that one had (after such substitution) a proof to the effect that $\sigma\varphi$ genuinely follows from $\sigma\Delta$.

Let us illustrate this general point with a very simple example to show that the taking of substitution instances can wreak havoc with the results of one's deductive labors. One can produce, say, the perfectly innocent little proof

$$\frac{A \quad B}{A \wedge B}$$

(using propositional atoms A and B) and judge it to establish the validity of the argument (or *sequent*)

$$A, B : A \wedge B.$$

Logical consequence also obeys a rule of closure under uniform substitution. [Emphasis added.]

Note that Williamson's talk here is all about the semantic notion of logical consequence, and not about the syntactic notion of deducibility.

Now, without looking too closely, imagine one decides to create from this little proof the substitution instance

$$\frac{A \quad \neg A}{A \wedge \neg A} \tag{1}$$

Although with such a degenerate example it is highly unlikely that one would fail to see the inconsistency that is now staring one in the face, one might not register this, and one might take oneself to have a proof establishing the validity, now, of the sequent

$$A, \neg A : A \wedge \neg A.$$

But (according to the extreme opponent of EFQ) nothing could be more wrong. For the new set of used premises (after the substitution) is not satisfiable. So, according to this opponent, it does not genuinely logically imply *any* sentence *at all* — not even the apparently correctly drawn conclusion $A \wedge \neg A$.

This little example is a highly degenerate one, to be sure. And it takes some epistemic imagination (or humility) to realize that one *might not* register the above inconsistency that would be staring one in the face. But this worry about the cogency of the line of thought being pursued is easily dispelled by considering the possibility of more recondite cases, where the post-substitution inconsistency of the 'new' set of premises is much more difficult to detect. We are *always* giving hostage to fortune in (dogmatically?) assuming that, upon taking a substitution instance $\sigma\Pi$ of a proof Π, that new proof $\sigma\Pi$ will establish/prove/warrant/ensure/make certain ... the *genuine* logical validity of the argument it purports to establish — namely, the argument whose premises are the substitution instances of the premises of the proof Π, and whose conclusion is the substitution instance of the conclusion of the same.

9.1.2. Substitutions can turn perfectly valid sequents into ones that are not perfectly valid

Recall that a valid sequent is called *perfectly* valid just in case every one of its proper subsequents is invalid. In the case of a perfectly valid sequent, every sentence involved (either as a premise or as the conclusion) is needed for validity. We shall see in what follows that substitutions can destroy perfect validity.

Consider the proof

$$\Pi : \quad \cfrac{\cfrac{\cfrac{\overset{(1)}{\overline{A}} \quad A \to B}{B} \qquad B \to C}{C}}{A \to C}{\scriptstyle (1)}$$

of the perfectly valid sequent

$$A \to B,\ B \to C\ :\ A \to C.$$

The proof Π has the following two technically legitimate substitution instances. But they are pointless or unenlightening. The first one results from substituting $P\wedge\neg P$ for A; the second one results from substituting $Q\rightarrow Q$ for C:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{P\wedge\neg P}} \quad (P\wedge\neg P)\rightarrow B}{B} \quad B\rightarrow C}{C}}{(P\wedge\neg P)\rightarrow C}\,{\scriptstyle(1)} \qquad \dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{A}} \quad A\rightarrow B}{B} \quad B\rightarrow(Q\rightarrow Q)}{Q\rightarrow Q}}{A\rightarrow(Q\rightarrow Q)}\,{\scriptstyle(1)}\ .$$

These substitutions turn the erstwhile conclusion $A\rightarrow C$ of Π into logical truths — $(P\wedge\neg P)\rightarrow C$ and $A\rightarrow(Q\rightarrow Q)$ respectively. Accordingly, each of these can be proved from the empty set of premises:

$$\dfrac{\dfrac{\overset{(1)}{\overline{P\wedge\neg P}} \quad \dfrac{\overset{(2)}{\overline{\neg P}} \quad \overset{(2)}{\overline{P}}}{\bot}}{\bot}\,{\scriptstyle(2)}}{(P\wedge\neg P)\rightarrow C}\,{\scriptstyle(1)} \qquad \dfrac{\dfrac{\overset{(1)}{\overline{Q}}}{Q\rightarrow Q}}{A\rightarrow(Q\rightarrow Q)}\,{\scriptstyle(1)}\ .$$

The original proof Π before the substitutions were made no longer serves any purpose with regard to the argument (*i.e.*, the sequent to be proved) that results from either of the substitutions in question.

Let us introduce a word here to describe substitutions like these, and use the word even before it is fully explicated. We shall call these two substitutions, in this particular proof Π, 'silly'.

Next we shall give an example of an even sillier substitution in Π — that of replacing all occurrences of B in Π with an occurrence of A:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{A}} \quad A\rightarrow A}{A} \quad A\rightarrow C}{C}}{A\rightarrow C}\,{\scriptstyle(1)}\ .$$

The result of this substitution is a 'proof' of $A\rightarrow C$ from a set of premises *that includes* $A\rightarrow C$. The other 'premise' is the *logical truth* $A\rightarrow A$, which is of no use at all as a *premise* in any passage of deductive reasoning. It is well known that logically true premises can always be suppressed.

A similar extreme silliness results if one replaces C in Π with A:

$$\dfrac{\dfrac{\dfrac{\overset{(1)}{\overline{A}} \quad A\rightarrow B}{B} \quad B\rightarrow A}{A}}{A\rightarrow A}\,{\scriptstyle(1)}\ .$$

The conclusion is now a logical truth, and the two contingent premises $A \to B$ and $B \to A$ are not needed to establish it.

Silly substitutions like these are the extremal companions of our earlier substitution of $\neg A$ for B in the one-step proof

$$\frac{A \quad B}{A \wedge B}\ ,$$

which produced the proof

$$\frac{A \quad \neg A}{A \wedge \neg A}\ ,$$

whose premise set has no model — as is shown by the one-step proof

$$\frac{\neg A \quad A}{\bot}\ .$$

We see, then, that substitutions run the risk of overdoing things. They can turn satisfiable sets of premises into unsatisfiable ones; and they can turn falsifiable conclusions into unfalsifiable ones. And even when neither of these extremes (unsatisfiable premises, or unfalsifiable conclusion) is its outcome, a substitution can turn [a proof of] a perfectly valid sequent (*i.e.*, a valid sequent that has no valid proper subsequents) into [a proof of] a valid sequent that is not perfectly valid, and whose premise set is satisfiable and whose conclusion is falsifiable. An example demonstrating this possibility is [any proof of] the perfectly valid sequent

$$A \to B,\ A \to C : A \to (B \wedge C).$$

Upon substituting A for B it becomes

$$A \to A,\ A \to C : A \to (A \wedge C),$$

which is valid but not perfectly valid, since it has the valid proper subsequent

$$A \to C : A \to (A \wedge C).$$

9.1.3. More carefully considered reasons why one would wish certain proofs to be closed under substitutions

Why should one wish to have easy (but frequently misguided) access to substitution instances of proofs that one has discovered? What epistemic concern drives this felt need for ‘preservation of persuasive force’ of a proof upon taking a non-trivial substitution instance of it?

More generally: what reason do we ever have, to find pairs of proofs one of which is a non-trivial substitution instance of the other and each of whose proven arguments *is* genuinely valid? One can give an obvious answer to *this* question without succumbing to the temptation to regard every substitution instance of a proof of a genuinely valid argument as establishing a genuinely valid argument. But to obtain the answer in question, one has to reverse the direction of the all-important operation: we have to inquire, not after non-trivial *substitutions* (which *refine* displayed logical structure) but after non-trivial *coarsenings*.

Quine put forward a famous maxim, now known as the Maxim of Shallow Analysis: do not expose more logical structure in a valid argument than what is needed in order to prove it. In other words, try to get by with the *coarsest possible* exposure of logical structure. This is a maxim worth following because, in following it, one never incurs any increased risk of inconsistency among one's premises. For, consider: one is usually working within a mathematical theory whose set of axioms, one is morally certain, has a model. And one may be attempting to derive a theorem φ (in the language of the theory) from some of the axioms along with certain lemmas already proven from them. Call the entertained premise set Δ. What a discerning, logically sophisticated mathematician would do is seize on any opportunity that might present itself to show that some *coarsening* of premises in Δ will allow one to prove the accompanying *coarsening* of the would-be conclusion φ. Call the coarsened sequent that this mathematician seeks to prove, $\Delta^* : \varphi^*$. Note that Δ^* is at *lower* risk — or at least *no greater* risk — of being unsatisfiable than Δ is. So, when the mathematician succeeds in applying Quine's maxim by discovering a proof Π^* of the argument $\Delta^* : \varphi^*$, s/he knows that it most certainly establishes a genuine validity if its desired *substitution instance* does.

This is the only real methodological source of a logician's interest in being able to take substitution instances of found proofs, and regard them as establishing genuinely valid arguments. The value of such pursuits at a 'higher level' (actually, a coarsened level) of revelation of logical structure is that the resulting proofs (the 'starred' ones) could well turn out to be more widely applicable within mathematics at large, even in theories with different primitive non-logical expressions than the theory in which the mathematician happens to be working.

10. HOW CAN EXTREME OPPOSITION TO EFQ FIND FORMAL EXPRESSION?

It would not be unfair to say that the proponent of EFQ (the Explosionist) thinks that any logical encounter with $\perp$ puts one on a slippery logical slope to *everywhere*. We venture to offer here, by contrast, on behalf of the extreme opponent of EFQ, a radical re-framing of the significance that $\perp$ *could* (and perhaps *should*) hold for us deducers. Here is how the re-framing goes.

What $\perp$ tells us is that we should avoid the slippery logical slope altogether. We need to turn around (metaphorically speaking) to re-examine the assumptions that have led us there (*i.e.*, to $\perp$). *Nothing follows* from absurdity (except absurdity itself); and therefore *nothing other than absurdity follows* from any set of assumptions that has led us to absurdity. $\perp$ stops one dead in one's logical tracks, allowing one to go *nowhere* (else).

We propose the following metalogical principle:

> *If one can prove $\perp$ by means of a proof whose premises form the set Δ, then* only *$\perp$ follows (as a genuine logical consequence) from Δ.*

In other words, the proposal is that we should adopt **Implosion** in place of **Explosion**.[24] It is high time to examine the consequences of doing so. It turns out that they are all epistemically benign.

This is rather extraordinary and unexpected. While the advocate of Explosion regards an inconsistent set as a logical big bang, the advocate of Implosion regards it as a logical black hole. There appears to be no prospect here of being able to take an Aristotelian 'middle position' on the matter of the logical power of an incoherent set of sentences.

What would it take to refute the contention that there really are only these two extreme polar opposites? One might think it possible to allow inconsistent premise sets to imply *some* sentences but *not all* of them. For such a position the devil would lie in the details. The basic problem, for the relevantist, would be difficult to banish — the problem of avoiding licensing a sentence (other than $\perp$) as logically implied by a premise set *simply because of a deeply buried inconsistency.* In cases where the implication can be secured without any reliance on a

[24] To the best of the author's knowledge, there are only two contemporary sources for a suggestion along these lines. Neither of them uses the term 'implosion'.

The first contemporary source is [Wagner, 1991]. Wagner's proposal, however, was confined to the limited logical environment of knowledge-base management, and in particular to what he termed 'conservative' or 'skeptical' systems. He disavowed any intention of insisting that the new principle 'ex contradictione nihil sequitur' be applied to logic in its usual and most important application, namely in formalizing mathematical reasoning. He wrote (p. 538)

> ... the classical principle *ex contradictione sequitur quodlibet* has been considered fundamental by most logicians and philosophers. Clearly, it makes sense for mathematics [fn] where it amounts to the postulate that contradictions in a theory must not be tolerated and have to be removed ...

The Implosionist proposal being investigated in this study, however, is that the principle 'ex contradictione nihil [sive falsum] sequitur' should be applied also in logic's *primary* domain of application: *mathematics.*

The second contemporary source is [Priest, 1999]. Priest characterized the 'null' account of negation (p. 141) as one according to which 'a contradiction has no content. Accordingly, $\alpha \wedge \neg\alpha$ entails nothing.' He went on to say (p. 142) that 'the most simple-minded way' 'to make sense of the idea that a contradiction has no content' is to say that

> $\Sigma \models \alpha$ iff Σ is consistent, and $\Sigma \vdash \alpha$.

His focus on consistency and deducibility ($\vdash$) in the definiens show that his relation $\models$ of entailment is thoroughly *syntactic.* On our implosionist approach, however, we seek to characterize a new *semantic* notion of consequence, to match to an independently furnished account of deducibility ($\vdash$).

deeply buried inconsistency, it should be possible to extract from the premise set some members of it that are jointly consistent and that collectively imply the desired sentential conclusion. But in such a situation the Core logician can furnish a proof to that effect; so there is no loss involved in simply eschewing the inconsistent premise set as the 'global' justification for the conclusion in question. What the Implosionist *can* assert at this stage is that there is no extant treatment of logical implication of certain sentences by inconsistent premise sets that does not result in explosion. Moreover, the Implosionist offers a coherent, well-defined notion of genuine logical consequence according to which $\bot$ is the sole logical consequence of any unsatisfiable set of premises. It is clean, simple, and elegant. The Implosionist commends it for careful consideration, on behalf of the Core logician. It may well be all or nothing.

We shall therefore in due course leave behind the orthodoxy of 'all', and undertake to examine with an open mind the heterodox 'nothing'.

11. TOWARDS A SEMANTICAL CONCEPT OF RELEVANT CONSEQUENCE

Here is how the inferentialist (at the meta-level) can capture the notion $\models$ of *standard* logical consequence. The inferentialist supplies for it an introduction rule and a harmoniously balancing elimination rule. In the rule $\models$I, 'M' is a sortal parameter for models, occurring only where indicated; in the rule $\models$E, '$\mathbf{M}$' is a sortal term for a model. Ω is a set of sentences; θ is a sentence.

$$\models\text{I} \quad \begin{array}{c} \overline{M \Vdash \Omega}^{(i)} \\ \vdots \\ \dfrac{M \Vdash \theta}{\Omega \models \theta}{(i)} \end{array} \qquad\qquad \models\text{E} \quad \dfrac{\Omega \models \theta \quad \mathbf{M} \Vdash \Omega}{\mathbf{M} \Vdash \theta}$$

Given these rules, we can prove the following.

Lemma 3 (Transitivity of Tarskian consequence; or, 'cut for $\models$').

$$\frac{\Delta \models \varphi \quad \varphi, \Gamma \models \psi}{\Delta, \Gamma \models \psi}$$

Proof.

$$\dfrac{\dfrac{\varphi, \Gamma \models \psi \qquad \dfrac{\dfrac{\Delta \models \varphi \qquad \dfrac{\overline{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Delta}}{M \Vdash \varphi}{\models\text{E}} \qquad \dfrac{\overline{M \Vdash \Delta, \Gamma}^{(1)}}{M \Vdash \Gamma}}{M \Vdash \varphi, \Gamma}}{M \Vdash \psi}{\models\text{E}}}{\Delta, \Gamma \models \psi}{(1)\models\text{I}}$$

□

The proof just given (in the metalanguage) is a Core proof. It uses the definitional rules of introduction and elimination for the Tarskian notion of logical consequence; but note that it does *not* use any rule of CUT in the metalanguage. It is worth noting that the transitivity of classical *logical consequence* (in the form of unrestricted CUT for $\models$) is constructively (and relevantly) derivable using the inference rules $\models$I and $\models$E above.

11.1. A New Relation of Genuine Validity of Argument

We embark now on the following task:

> Define a formal notion of 'genuine' logical consequence (let us denote it by $|\!\models$) guided by the idea that it consists in the preservation of truth from premises to conclusions under all possible interpretations of the non-logical vocabulary — of which there is at least one — *that make the premises true.*

Thus if we have $\Delta|\!\models\varphi$ (for $\varphi \neq \bot$), then we shall be able to infer that there is at least one interpretation of the non-logical vocabulary involved (*i.e.*, some model M) that satisfies Δ:

$$\exists M\, M \Vdash \Delta.$$

Apart from this novel requirement, the rest of the conceptual content is as before — that is, every model that satisfies Δ will make φ true:

$$\forall M(M \Vdash \Delta \Rightarrow M \Vdash \varphi).$$

We can ensure this outcome for $|\!\models$ by means of the following introduction and elimination rules for this new pasigraph. The sortal parameter M (in italics) for models in the two introduction rules and in the elimination rule $|\!\models\mathrm{E}_2$ is to be understood as occurring only where indicated. Boldface **M** in the three rules in which it occurs is a term for a model.

$$|\!\models\mathrm{I}\quad \begin{array}{c} \overline{M \Vdash \Delta}^{\,(i)} \\ \vdots \\ \bot \\ \hline \Delta|\!\models\bot \end{array}{}^{(i)} \qquad\qquad |\!\models\mathrm{E}_1\quad \dfrac{\Delta|\!\models\bot \quad \mathbf{M}\Vdash\Delta}{\bot}$$

$$|\!\models\mathrm{I}\quad \begin{array}{cc} & \overline{M \Vdash \Delta}^{\,(i)} \\ & \vdots \\ \mathbf{M}\Vdash\Delta & M\Vdash\varphi \\ \hline \multicolumn{2}{c}{\Delta|\!\models\varphi} \end{array}{}^{(i)} \qquad |\!\models\mathrm{E}_2\quad \begin{array}{cc} & \overline{M \Vdash \Delta}^{\,(i)} \\ & \vdots \\ \Delta|\!\models\varphi & \psi \\ \hline \multicolumn{2}{c}{\psi} \end{array}{}^{(i)} \qquad |\!\models\mathrm{E}_3\quad \dfrac{\Delta|\!\models\varphi \quad \mathbf{M}\Vdash\Delta}{\mathbf{M}\Vdash\varphi}$$

LEMMA 4. EFQ is super-fallacious: it has no valid instance.

Proof. For the application of $\|\!\models \mathrm{E}_2$ in the formal metalinguistic proof below we are taking Δ to be the singleton of falsum:

$$\dfrac{\dfrac{\dfrac{}{\bot \|\!\models \varphi}{\scriptstyle(2)} \qquad \dfrac{\dfrac{}{M \Vdash \bot}{\scriptstyle(1)}}{\bot}{\scriptstyle(\alpha)}}{\bot}{\scriptstyle(1)\ \|\!\models \mathrm{E}_2}}{\bot \not\|\!\models \varphi}{\scriptstyle(2)}$$

□

LEMMA 5. Lewis's First Paradox is fallacious. That is, it is not the case that $A, \neg A \|\!\models B$.

Proof.

$$\dfrac{A, \neg A \|\!\models B \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{M \Vdash \{A, \neg A\}}{\scriptstyle(1)}}{M \Vdash \neg A}}{M \nVdash A} \qquad \dfrac{\dfrac{}{M \Vdash \{A, \neg A\}}{\scriptstyle(1)}}{M \Vdash A}}{\bot}}{\bot}{\scriptstyle(1)\ \|\!\models \mathrm{E}_2}$$

□

The relation $\|\!\models$ is transitive in the sense afforded by the following lemma.

LEMMA 6.

$$\left.\begin{array}{r} \Delta \|\!\models \varphi \\ \varphi, \Gamma \|\!\models \psi \\ \Delta, \Gamma \text{ satisfiable} \end{array}\right\} \Rightarrow \ \Delta, \Gamma \|\!\models \psi.$$

Proof.

$$\dfrac{\exists M\ M \Vdash \Delta, \Gamma \qquad \dfrac{\dfrac{}{\mu \Vdash \Delta, \Gamma}{\scriptstyle(2)} \qquad \dfrac{\varphi, \Gamma \|\!\models \psi \qquad \dfrac{\dfrac{\Delta \|\!\models \varphi \qquad \dfrac{\dfrac{}{M \Vdash \Delta, \Gamma}{\scriptstyle(1)}}{M \Vdash \Delta}}{M \Vdash \varphi}{\scriptstyle\|\!\models \mathrm{E}_3} \qquad \dfrac{\dfrac{}{M \Vdash \Delta, \Gamma}{\scriptstyle(1)}}{M \Vdash \Gamma}}{M \Vdash \varphi, \Gamma}}{M \Vdash \psi}{\scriptstyle\|\!\models \mathrm{E}_3}}{\Delta, \Gamma \|\!\models \psi}{\scriptstyle(1)\ \|\!\models \mathrm{I}}}{\Delta, \Gamma \|\!\models \psi}{\scriptstyle(2)}$$

□

11.2. The Reason Why the Core Systems Are Perfect for the Formalization of Mathematical Reasoning

One frequently encounters criticism of Core Logic (for intuitionistic mathematical reasoning) and of Classical Core Logic (for classical mathematical reasoning) from the following under-informed vantage point.

> Mathematical theories are based on effectively decidable sets of axioms; and mathematicians prove their mathematical theorems by interpolating many lemmas on the way to them, from those axioms. As an interpolant, a lemma might stand as the major premise of an elimination for the proof of the eventual theorem. Consider, for example, a proof Π of lemma λ from a set Δ of axioms, combined with a subsequent proof Σ of the eventual theorem θ from lemma λ in combination with further axioms Γ:
>
> $$\underbrace{\begin{array}{c}\Delta\\ \Pi\\ (\lambda)\end{array} \quad , \quad \Gamma}_{\begin{array}{c}\Sigma\\ \theta\end{array}}$$
>
> Some of those premise occurrences of λ in the proof Σ might well be major premises for eliminations. But, even if they aren't, they all have non-trivial proof-work (namely, Π) above them; so this overall proof that we wish to be able to construct *cannot be a proof in either of the Core systems.* Moreover, if you try to turn it into a proof of the latter kind, there will in general be hyperexponential explosion in the length of proof, as is well known from the proof-theoretic literature on normalization and cut-elimination.

This objection from a conventional proof theorist has to be taken seriously — and it can be, head-on. The objector grants us soundness of classical core proof. We therefore know, on the basis of the proofs Π and Σ, that

$$\Delta \models \lambda \qquad \text{and} \qquad \lambda, \Gamma \models \theta.$$

Moreover, the mathematical reasoners whose reasoning is to be regimented are all morally certain that $\Delta \cup \Gamma$ is satisfiable. We share that belief. So now we know that

$$\Delta \mathrel{|\!\!\models} \lambda \qquad \text{and} \qquad \lambda, \Gamma \mathrel{|\!\!\models} \theta.$$

At this point we can appeal to our metalinguistic Lemma 6 and conclude (at the meta-level) that

$$\Delta, \Gamma \mathrel{|\!\!\models} \theta.$$

That is, we know that θ is true in *any* model of the axioms. We have *rigorously justified certainty* in the truth of the theorem that has been proved. We have *not even* needed to construct an 'overall' Core, or Classical Core, proof of θ from $\Delta \cup \Gamma$. All that we need to construct (as users of the Core systems) are the proofs of the bits of reasoning that lie *strictly between* the interpolants (the lemmas) that have been employed on the deductive journey from axioms to theorem. There is no specter of hyperexponential explosion in length of proof to have us quaking in our regimenting boots. There is no point *at all* in confounding the *hyperlogarithmic reduction* in overall length of proof that is effected by judicious lemma interpolation. And *that* is why the Core systems provide such clean, direct, elegant, and economical regimentations of informally rigorous mathematical reasoning *as we find it*. The Core systems are designed to regiment informally rigorous mathematical reasoning *exactly as it is found on the page*, lemmas and all. This is the methodological and epistemological impact of Conditions (1)–(6) (in §6). It is also the reason why we conjecture that Classical Core Logic meets Condition 7.

11.3. The 'Logical Power' of an Unsatisfiable Set

Logicians (and mathematicians, and thinkers in general) have to be hypervigilant. The threat of unsatisfiability is everywhere. One can be proceeding innocently, making permitted inferential moves in an unsuspecting way, when suddenly an unsatisfiability visits itself upon us. Note that we speak here of unsatisfiability, not inconsistency, even though at first order a set of sentences is unsatisfiable if and only if it is inconsistent. Inconsistency can visit itself upon us by means of an explicit disproof — a proof of $\perp$ from the accumulated premises in question. The danger under discussion here is more insidious than that. It is that our accumulated premises can form an unsatisfiable set without our being aware that this is so. If we *become* aware that this is so, it will be by virtue of a proof of $\perp$ from the premises in question. On closer examination one can then (usually) detect the source of the inconsistency, come to understand how things had gone wrong, and take steps to revise one's premises (one's ultimate starting points) so as to form a satisfiable set.

The Implosionist maintains that one can learn to live with the exigent restriction proposed here on consequences of unsatisfiable sets of sentences.[25] Remember, we have (3) to fall back on:

> If Δ is a set of sentences that has no model, then there is an $\mathcal{S}$-proof of absurdity ($\perp$) from premises in Δ.

A formally correct proof 'of' a sentence φ (not: $\perp$) 'from' a set Δ of used premises establishes that the sequent $\Delta : \varphi$ is genuinely valid (*i.e.*, that $\Delta \Vdash \varphi$) *provided only that* the set Δ is itself satisfiable. And unfortunately, because of Church's Undecidability Theorem for first-order logic, there is no effective method for determining, of any given set of premises of a proof, *whether* it is indeed satisfiable (equivalently: consistent in $\mathbb{C}^+$). One just has to live with the ever-present specter of as yet undetected unsatisfiability of the premise set of any formally correct proof that one has constructed.

That predicament, however, does not afflict the Implosionist alone. It also afflicts the *Explosionist*. Having EFQ in one's system does not in any way

[25] Note that one could replace '*can*' here with '*must*', if one cannot go along with '*cannot*'.

afford protection against the specter of as yet undetected unsatisfiability of the premise set of any formally correct proof that one has constructed.

If Δ is indeed incoherent (unsatisfiable), then we can, in principle, discover this to be the case — Condition (3) above. *And we can do so without making any use of EFQ*. In the meantime, however, we might think we have made some deductive progress on the basis of Δ — when in fact we have made *no progress at all*. (Note that this remark applies equally well to the Explosionist.) No proof of ours 'of' a sentence φ 'from' an *un*satisfiable premise set Δ (actually making use of all of them) can possibly establish the genuine validity of the argument (sequent) $\Delta{:}\varphi$. For *no* proof can.[26]

11.4. What Does a Proof of φ ($\neq \bot$) from Δ Show, if Δ is Unsatisfiable?

A final nagging worry that the orthodox logician might have (whose orthodoxy involves subscribing to EFQ) is the following. What sense can be made of a proof of the form

$$\begin{array}{l}\Delta\\ \Pi\,,\ \text{ where } \varphi \neq \bot\\ \varphi\end{array}$$

in cases where the set Δ of premises is unsatisfiable? We shan't always be able to recognize when such a case obtains, for any particular set Δ of premises that we have used. As already observed above, we have no effective method (because of Church's Theorem) of deciding *whether* any given finite set Δ of sentences has a model. Nor, for the same reason, can we tell whether the conclusion φ, although syntactically distinct from $\bot$, is logically false; or whether it is logically true; or whether it is contingent. All we can know, on the basis of Π, is that *if* Δ has any model at all, *then* φ will be true in it. But, for all we know, Δ *might have no model*. In that case there will be a proof Σ, say, of $\bot$ from some $\Delta' \subseteq \Delta$. Without loss of generality this proof Σ can be taken to be a classical core proof.

It would appear that the subscriber to EFQ will find no epistemic value at all in having such a proof Π as displayed above, unless already morally certain (on the basis of whatever reflective reasons) that its set Δ of premises is satisfiable. And that happens to be the case in mathematics only when Δ is a subset of some set of axioms (such as those for Peano Arithmetic) firmly believed to be consistent (hence satisfiable). The extreme *opponent* of EFQ — the Implosionist — is no worse off, epistemically, for insisting that arguments with conclusions other than $\bot$ can be genuinely valid only if their

[26] We are talking here about *genuine* progress of an *apodeictic* kind — the kind that involves presenting the set Δ of one's axioms-for-use, furnishing a proof of φ from (and using all of) those axioms, and then claiming that one knows, on the basis of one's proof, that φ is true. That kind of 'progress' is illusory if the set Δ is unsatisfiable.

premise sets are satisfiable. Both parties to the debate about the acceptability of EFQ itself find themselves in the same epistemic predicament, with proof Π in hand, when they are *not* morally certain that its premise set Δ is satisfiable.

If this 'tie' in such a situation is to be broken, then it should, arguably, be broken in favor of the Implosionist. For the Explosionist should be worried that a disproof of Δ might be lurking *within* his EFQ-using proof Π of the form displayed above, because of the way that the applications of EFQ therein can render the as yet not explicitly detected inconsistency of the premise set Δ less clear and distinct. The Implosionist at least has some confidence that a Classical Core proof Π of φ from Δ, since it *lacks* any applications of EFQ, will be unlikely to be 'trading on' any hidden inconsistency of its premise set. Any disproof Σ of that premise set is likely to be quite unrelated, in its line of argument, to the proof Π, whose conclusion φ could *not* have been arrived at courtesy of any applications of EFQ.

12. HOW DOES IMPLOSION COHERE WITH THE GÖDEL PHENOMENA?

The quick answer to this section's titular question is: it coheres just fine. At the metalevel, any system of $\mathcal{S}$-proof of the kind under consideration here (indeed: even a constructive one),[27] regardless of whether it permits EFQ, delivers derivations of Gödel's famous Incompleteness Theorems. This is the case even for arithmetical theories that are closed under full Classical Logic. Our epistemic situation with regard to the consistency of arithmetic (or, equivalently: *its having the natural model* $\mathbb{N}$ as a model for its axioms) is unchanged, even for one who eschews EFQ.

That we live with the specter of inconsistency in our *mathematical* theorizing — of all areas — is underscored by Gödel's Second Incompleteness Theorem. It states that

> No consistent, sufficiently strong theory T of arithmetic can prove its own consistency statement Con_T (which is formulated, via coding, in the language of T).

No such theory T, then, can really validate its own theorems. When we have a proof of φ from axioms of T, we are unable to *prove*, from the vantage point of T, that T *has a model* and that φ is therefore true in any such model.

The ensuing discussion will reach its conclusion independently of the question of how best to express the consistency of T in the language of T. We note without proof that the derivability conditions (on the logic of the object language and the provability predicate for T) to which one appeals in the metaproof of the Second Incompleteness Theorem, using Con_T in a form that really does express the consistency of T, will be satisfied by any logic $\mathcal{S}$ that

[27] See [Tennant, 2023].

this study concerns.[28] The reader should revisit the requirements (1), (2), (3), (4), (5), and (6) that were laid down on a logic $\mathcal{S}$ in §6.

Suppose our theory T is, for example, the infinitely axiomatized theory PA of Peano Arithmetic. Suppose we have an $\mathcal{S}$-proof of φ 'in PA', *i.e.* an $\mathcal{S}$-proof of φ whose premises are axioms of PA. Then, if PA is satisfiable at all, we presume that one of the models of PA is the standard model $\mathbb{N}$; whence, by (4), the PA-theorem φ is true in $\mathbb{N}$. That is what underlies our proof-based pursuit of theorems derivable from the axioms of PA. We believe those theorems are true in $\mathbb{N}$, because we believe that $\mathbb{N}$ *is* a model of PA.

But what if PA *has no model*? The only way we could discover this to be the case would be to have an actual $\mathcal{S}$-proof of $\perp$ from a certain finite subset (Γ, say) of the infinite axiom set of PA; call such a proof an $\mathcal{S}$-*disproof*. We would then at the very least have to give up, as erstwhile presumed truths in $\mathbb{N}$, every $\mathcal{S}$-theorem whose every known $\mathcal{S}$-proof has Γ included in its premise set. *Prima facie*, this would still leave in the running all those $\mathcal{S}$-theorems of PA that we have proved whose premise sets do *not* include any premise set (consisting of PA-axioms) of any known $\mathcal{S}$-disproof. Certain subsets of the set of axioms of PA might still have models. But it is hard to predict how the mathematical community would react to a first-ever, confirmed, checked, correct $\mathcal{S}$-proof of $\perp$ from premises that are all axioms of PA. Would there be a scramble to identify such subsets?

Such a discovery would certainly rock the mathematical community's boat. It is hard to know whether they would be 'all hands on deck' to determine a new communal choice of axioms for the natural numbers, or jumping ship to different epistemic islands whose axiom sets had not *yet* been proved to be inconsistent.

We contend, though, that whatever might transpire in this eventuality will have nothing to do with whether mathematicians believe that any set of sentences whose inconsistency, surprisingly, has just been revealed, logically implies any sentence whatsoever. The Implosion vs. Explosion debate is orthogonal to these epistemic concerns. For mathematicians, proving $\perp$ from any of their favorite sets of axioms *will* be 'bad enough'. EFQ will play no role at all in how they react to such a formal discovery. Moreover, EFQ will play no essential role at all in *leading to* such a discovery. If there really is an inconsistency in PA, but thus far undetected, then (as (3) and Lemma 2 together tell us) there is an EFQ-eschewing $\mathcal{S}$-proof of $\perp$ from axioms of PA, 'out there', waiting to be discovered.

And the same holds for any theory in place of PA.

APPENDIX
FORMAL EXPLICATION OF RELEVANCE IN CLASSICAL CORE LOGIC

Core Logic, in both its constructive and its classical forms, is a *relevant* logic, in an interesting and deeper sense than that provided merely by the assurance

[28] See in this connection [Jeroslow, 1973; Raatikainen, 2021].

that the logic does not allow derivation of the Lewis Paradox (in either its positive or its negative form). We confine ourselves here to the propositional system in explaining the formal explication of deductive relevance that is to be had from Core Logic. This task involves spelling out the details of a very exigent form of 'variable-sharing' (or, in our terminology, sharing of *atoms*). To this end we need to supply the following definitions. The reader is assumed to be familiar with the notions of positive and negative subformula occurrences within a formula.

DEFINITION 1. $\pm\varphi \equiv_{\mathrm{df}}$ some atom occurs both positively and negatively in φ. (Note that $\pm$ is a metalinguistic predicate, not a function sign.)

DEFINITION 2. $\varphi \approx \Delta \equiv_{\mathrm{df}}$ some atom has the ***same*** parity (positive or negative, at some occurrence) in φ as it has in some member of Δ.

DEFINITION 3. Suppose $\varphi \neq \psi$. Then $\varphi \bowtie \psi \equiv_{\mathrm{df}}$ some atom has the ***opposite*** parity at some occurrence in φ from the parity it has at some occurrence in ψ.

DEFINITION 4. $\varphi_1, \ldots, \varphi_n$ $(n > 1)$ is a $\bowtie$-*path connecting* φ_1 *to* φ_n *in* $\Delta \equiv_{\mathrm{df}}$ for $1 \leq i \leq n$, φ_i is in Δ, and for $1 \leq i < n$, $\varphi_i \bowtie \varphi_{i+1}$.

DEFINITION 5. A set Δ of formulae is $\bowtie$-*connected* $\equiv_{\mathrm{df}}$ for all φ, ψ in Δ, if $\varphi \neq \psi$, then there is a $\bowtie$-path connecting φ to ψ in Δ.

DEFINITION 6. A *component* of Δ is an inclusion-maximal $\bowtie$-connected subset of Δ (where the $\bowtie$-connections are established via members of Δ).

Relevance Metatheorem about Core Logic. A Classical Core proof of a conclusion φ from a set Δ of undischarged assumptions establishes that Δ is *relevantly connected both within itself and to* φ, in the sense that exactly one of the following three conditions holds:

(1) Δ is non-empty, φ is $\perp$, and:
if Δ is a singleton $\{\delta\}$, then $\pm\delta$; otherwise, Δ is $\bowtie$-*connected.*
(2) Δ is non-empty, φ is not $\perp$, and:
the components $\Delta_1, \ldots, \Delta_m$ $(m \geq 1)$ of Δ are such that for $1 \leq i \leq m$, we have $\varphi \approx \Delta_i$.
(3) Δ is empty, φ is not $\perp$, and $\pm\varphi$.

Cases (1) and (3) cover the two logical extremes. In case (1) we have a proof of the joint inconsistency of the premises in Δ. In that case Δ itself is the only component of Δ. In case (3) we have a proof of a logical theorem φ. In that case φ will contain some atom both positively and negatively.

Case (2) covers the 'middle range', so to speak, and it is this case that reveals the most interesting structure involving both Δ and φ. The set Δ of premises is partitioned into components $\Delta_1, \ldots, \Delta_n$ $(n \geq 1)$, each of which, if not a singleton, is $\bowtie$-connected. Moreover, each component Δ_i bears a special relation to φ, to wit: some atom occurs with the same parity in φ as it does in *some member* of Δ_i.

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