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The Berry Paradox

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Abstract
Berry's Paradox, like Russell's Paradox, is a 'paradox' in name only. It differs from genuine logico-semantic paradoxes such as the Liar Paradox, Grelling's Paradox, the Postcard Paradox, Yablo's Paradox, the Knower Paradox, Prior's Intensional Paradoxes, and their ilk. These latter arise from semantic closure. Their genuine paradoxicality manifests itself as the non-normalizability of the formal proofs or disproofs associated with them. The Russell, the Berry, and the Burali-Forti 'paradoxes', by contrast, simply reveal the *straightforward inconsistency* of their respective existential claims—that the Russell set exists; that the Berry number exists; and that the ordinal of the well-ordering of all ordinals exists. The disproofs of these existential claims are in free logic and are in normal form. They show that certain complex singular terms do not—indeed, *cannot*—denote. All this counsels reconsideration of Ramsey's famous division of paradoxes and contradictions into his Group A and Group B. The proof-theoretic criterion of genuine paradoxicality formally explicates an informal and occasionally confused notion. The criterion should be allowed to reform our intuitions about what makes for genuine paradoxicality, as opposed to straightforward (albeit surprising) inconsistency.

Keywords Logico-semantic paradox · Genuine inconsistency · Free logic · Definite description operator · Berry · Russell · Ramsey

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1 Introduction

This study is in part a somewhat Aristotelian response to [9].[1]

The main work of the present author on the Berry Paradox was done independently, several years ago, and never written up at the time in LaTeX for journal publication. Rosenblatt's study, however, provides the impetus for the present author to set out his own rather different proof-theoretic analysis of the Berry Paradox, which he believes is simpler and more direct.

Rosenblatt and the present author are in agreement on the main metalogical point: the Berry is a straightforwardly refutable existential claim, not a paradox. We reach that shared conclusion by furnishing our respective normal-form refutations (in free logic) of the existential claim that there is such a thing as the 'Berry number'. In this regard our shared treatment of the Berry is like that of the present author's more recent treatment of the Russell Paradox in set theory, in [16], §11.4. Where Rosenblatt and we differ is on the kinds of technical devices used in encoding the reasoning involved, and in our respective assessments of the broader philosophical ramifications of this discovery.

There are two such ramifications. The main one concerns what we regard as a revealed need to re-think Ramsey's famous division of contradictions and paradoxes into his Groups A and B. This will be pursued in §8. Rosenblatt is right when writing in his *Conclusion* (*loc. cit.*, p. 12) that the present author's 'claim that the criterion lends support to Ramsey's distinction is at best unjustified'. The present author urges that the distinction be radically revised, in light of the proof-theoretic explication now available of the notion of logico-semantic paradoxicality.

The other ramification concerns one's overall assessment of the tenability, or fecundity, of our proof-theoretic criterion of logico-semantic paradoxicality. The last sentence of Rosenblatt's abstract reads '…if Tennant's assessment of Russell's paradox holds, few cases may genuinely qualify as paradoxes by his standards.' The assessment of Russell's famous paradox in set theory to which Rosenblatt refers is that it is *not* a genuine paradox at all. Rather, it is a straightforward normal-form disproof of the claim that the so-called 'Russell set' exists. (This would be the set of all and only those sets that are not members of themselves.) The whole point of the present study is to show that the Berry 'paradox', like the Russell 'paradox', can likewise be revealed by sufficiently careful proof-theoretic analysis to be a straightforward normal-form disproof of an existential claim, namely the claim that the 'Berry number' exists. The Russell and the Berry 'paradoxes' are not paradoxes at all.

On this crucial point, Rosenblatt and the present author are in complete agreement. Rosenblatt thinks, however, that such formal discoveries (about the Russell and the

[1] Thanks are owed to Alan Code for directing the author to the folloowing quotes.

…there is this error in deductions: when someone proves through longer steps though it could be done through fewer ones which are actually present in the argument.
Aristotle, *Topics* VIII–11, 162a24–34)
…our account would be adequate, if we achieved a degree of precision appropriate to the underlying material; for precision must not be sought to the same degree in all accounts of things …
(Aristotle, *Nicomachean Ethics* I.3, 1094b13–14)

Berry) threaten the result that ‘few cases may genuinely qualify as paradoxes by [Tennant’s] standards.’ This is a puzzling non-sequitur. It prompts the question: ‘Few cases *of what*, exactly?’. The literature on the proof-theoretic criterion of paradoxicality provides a rich range of genuine logico-semantic paradoxes whose status as such, in conformity with the criterion, has been confirmed by appropriate proof-theoretic analyses. When Rosenblatt writes ‘the reasoning that rules out Russell’s paradox can similarly be applied to some semantic paradoxes’, he is clearly holding on to the classification of the Berry as a semantic *paradox*, rather than re-construing it as a straightforward negative existential in a language-fragment containing a semantic term such as ‘definable’—which is what his own analysis reveals.

2 The Proof-theoretic Criterion of Paradoxicality

Tennant [14] put forward a proof-theoretic criterion of paradoxicality. The intention was to capture *logico-semantic* paradoxicality—the kind of paradoxicality paradigmatically displayed by the Liar Paradox, Grelling’s Paradox, the Knower Paradox, Curry’s Paradox, the Postcard Paradox, the Revenge Paradox, Prior’s Paradox, and Russell’s Paradox of Propositions. Yablo’s Paradox was not known in 1982; but one can add it to this list. These have all been revealed, by close proof-theoretic analysis, to be classifiable as logico-semantic paradoxes according to the proof-theoretic criterion. Their disproofs (or, in the case of the Curry, its proof of an arbitrary sentence) are *not in normal form*. They cannot be brought into normal form by applying reduction procedures. Their reduction sequences fail to terminate. This is because either these sequences loop, or, as with Yablo’s Paradox, they ‘spiral’ *ad infinitum*.

Notice some intentional omissions from the foregoing list, of ‘paradoxes’ that are standardly so-called, but for which the present author would (as we do so now) use scare quotes: the Russell ‘Paradox’ (in set theory); the Berry ‘Paradox’; and the Burali-Forti ‘Paradox’. The scare quotes are warranted because these venerable ‘paradoxes’ are revealed, on appropriately rigorous proof-theoretic analysis, *not* to be of the logico-semantic kind. The present author was motivated by Prawitz’s proof-theoretic analysis of Russell’s paradox[2] to inquire whether we might have discovered, in the non-normalizability of Prawitz’s natural deduction on behalf of Russell, a proof-theoretic criterion of paradoxicality that would apply more generally to the well-known logico-semantic paradoxes.[3] Ironically, the present author’s subsequent development of introduction and elimination rules for the set-abstraction operator revealed Russell’s ‘Paradox’ to be a straightforward negative existential enjoying a proof in normal form.[4]

The Russell, the Berry, and the Burali–Forti ‘Paradoxes’ were, to be sure, surprising at the time of their discoveries; and are so to logical neophytes upon first acquaintance.

2 Prawitz [6], p. 95.

3 As [12] have put it, at p. 569, ‘The Prawitz–Tennant analysis of paradoxes is a way to characterize paradoxes by their proof-theoretic behavior, looking at the derivation of absurdity generated.’

4 See [16] Chapter 11, ‘Core Logic and the Paradoxes’, especially §11.4 therein, ‘Revisiting Russell’s Paradox’.

But they are straightforward results about the *non-existence* of abstract entities of certain definable kinds. The deductive reasoning in these 'paradoxes' *can* be brought into normal form. They are normal-form disproofs of assumptions of the form 'There exists such a thing as *X*'. Taking the Russell for illustration (since it is by far the best-known 'paradox' of this kind), there is a straightforward normal-form *disproof*, in the *free logic* of sets, of the sentence 'There exists the set of all sets that are not members of themselves'. A free logic is one that can properly handle deductive reasoning involving sentences containing non-denoting (and frequently *necessarily* non-denoting) singular terms. Free logic is a *sine qua non* for the fully adequate regimentation of the deductive reasoning in mathematics, for mathematical language has many grammatically well-formed singular terms that fail (*necessarily*) to denote.

In this study, we undertake a similar analysis of the Berry as has been furnished for the Russell. We conclude that we should *categorize the Berry with the Russell*. In §8 we shall draw philosophers' interest, in light of this outcome, to its consequences for how we should now understand or conceive of paradoxicality in the broadest philosophical terms.

The need to take a more detailed proof-theoretic look at the Berry is of long standing. The final note, on p. 285 of [14], was

> Priest maintains that the reasoning in the definability paradoxes, such as Berry's, does not display the pattern I am suggesting as characteristic of paradoxes; the structure of the proofs concerned should therefore be investigated more closely.

The purpose of this study is to do just that, albeit belatedly.

3 What is Needed for an Analysis of the Berry

The Berry Paradox was first published by Russell [11]. Those who were satisfied with the informally rigorous reasoning required to appreciate the problem it poses did not need to imagine the reasoning being formalized by exploiting the strictly later technicalities of Gödel numbering (which is what Rosenblatt opted to do). It would be only many years later that Gödel invented his method of numerical coding of linguistic expressions. It would therefore be anachronistic, today, to resort—let alone insist on resorting—to Gödel's method in any attempt to furnish an accurate account of the nature of Berry's paradox. So we shall eschew such resort while yet giving—or so we shall contend—a rigorous enough formalization of the 'Berry reasoning' to illuminate exactly what *kind* of 'paradox' it is. Our considered judgment will be: it may well strike some as a 'paradox'; but only because it is surprising. It is certainly not a *genuine* paradox like, say, the Liar. This is because the Berry, on close enough analysis, turns out to be a straightforward disproof of an existential statement—a disproof that is in normal form.

To reveal how this is so, we shall need to provide a sufficiently rigorous account of the formal logic best suited for the regimentation of the kind of reasoning that Berry and Russell arguably engaged in. And this requires consideration of definite descriptions. Russell [10] had only shortly beforehand introduced logicians to his account of the truth conditions of simple-looking predications with definite descriptive phrases as their subject terms. 'The F is G' was analyzed as having the truth conditions of 'There is exactly one F and it is G'. The definite-descriptive *operator* ι (to regiment the word 'the') did not make an appearance in 'that paradigm of analytical philosophy', as Ramsey called it—not even in the context of the offered contextual definition $\exists x(\forall y(y = x \leftrightarrow Fy) \wedge Gx)$. The iota was to come only later in Russell's writings,[5]—and then only by means of the just-stated contextual definition, not by its adoption as a primitive variable-binding term-forming operator. Yet it is the latter way of treating ι that is called for in analyzing the Berry.

Let us see what could have been done on Russell's behalf back in 1908, in order to regiment the Berry reasoning. We shall allow ourselves the tolerable anachronism of deploying a Gentzenian natural deduction system. To be sure, such systems were first made available only in [1]. So they would have had to be invented almost thirty years earlier than they were, in order to be of any use to Russell when he introduced his readers to the Berry Paradox. The point, however, is that one should be able to reveal the essential nature of the paradox regardless of one's method of *formatting* one's formal regimentations of informally rigorous deductive reasoning. The Gentzenian proof theorist can furnish natural deductions to regiment *any* informally rigorous mathematical reasoning, no matter its illustrious vintage, or provenance, dating decades—even centuries—before Gentzen gave us the gift of formalized natural deductions.

Moreover, there is no tension between using Gentzenian methods to formalize the logical structure of the informally rigorous reasoning involved in the Berry Paradox while at the same time eschewing Gödel-numbering of formal expressions as overly technical for the expository and diagnostic aims of one's closer study of it. The kind of reference to linguistic expressions that is involved in the Berry can be faithfully handled without resorting to the technicalities of Gödel-numbering. If that were not the case, one would be at a loss to explain how the 'Berry reasoning' could be followed by any reader innocent of Gödel-numbering—which would include all those between 1908 and 1930.

That Russell did not know of Gentzenian natural deductions in 1908 is neither here nor there from the point of view of those who are willing to avail themselves of developments in proof theory since Russell's day. Indeed, Russell's innocence of such deductions and their combinatorial properties helps to explain his own inability to discern the important difference between a paradox like the Liar and the 'paradox' in set theory that came to bear his name. The correct analysis of the Berry requires not only a correct choice of the *deducibility* relation $\vdash$ (of which one might be apprised regardless of the actual system of proof that generates it); it requires also a grasp of the concept of a proof (or disproof) being in *normal form*—hence also a grasp of the process of *normalizing* proofs (or disproofs) that might not be in normal form. What obscured a proper understanding of the Berry in Russell's day was not only that $\vdash$ at that time was not in service of a *free* logic; it was also that it was three decades before Gentzen's bequest to Logic of natural deductions and the crucial concepts of normal form, normalization, and normalizability.

[5] See [11] at p. 253.

4 Natural-Deduction Rules for the Definite Description Operator

In order to show that a given singular term fails to denote, we need to employ a *free* logic. As just intimated, such a logic is so-called because it is free of the dogmatic and incorrect Fregean assumption that every singular term denotes. We adopt the abbreviation $\exists!t$ for $\exists x\, x = t$ ('t exists'). Free logic contains the Rule of Atomic Denotation:

$$\frac{A(\ldots, t, \ldots)}{\exists!t}, \text{ where } A \text{ is a primitive predicate}$$

which, ironically in this context, is a very Russellian idea—for Russell required the existence of all the arguments involved in any true atomic predication.

The rule of introduction in free logic for the variable-binding abstraction operator ι that forms definite descriptive terms from predicates is the following.

$$
\iota\text{-I}\qquad
\begin{array}{c}
\begin{array}{ccc}
\underbrace{{}^{(i)}\overline{\varphi(a)}\ ,\ \overline{\exists!a}^{(i)}} & \qquad & \overline{a=t}^{(i)} \\
\vdots & & \vdots
\end{array}\\
\underline{a=t \qquad \exists!t \qquad \varphi(a)}_{(i)}\\
t=\iota x\varphi(x)
\end{array}
\quad , \quad \text{where } a \text{ is parametric.}
$$
[6]

Note how the canonical conclusion

$$t=\iota x\varphi(x)$$

of ι-I has t on its left-hand side, as a placeholder for *any singular term whatsoever*, including the *parameters* (conventionally a, b, c, ...) that can be used for reasoning involving existentials and universals.[7] On the right-hand side of the identity is a definite descriptive term, formed by means of a *dominant* occurrence of the variable-binding abstraction operator ι. This operator may be applied to a formula φ to form the definite-descriptive term $\iota x\varphi$ if, but only if, the variable x has a free occurrence in φ.

Let us call such a rule for the introduction of a variable-binding term-forming operator a *single-barreled* rule. The rule concerns a *single* occurrence of the operator in 'as dominant a position as possible' within the conclusion, which is an identity statement.

The elimination rules corresponding to the introduction rule stated above for ι are the following three, each one employing the canonical identity statement

$$t=\iota x\varphi(x)$$

[6] Note that since $=$ is an *atomic* binary predicate, the assumption $a=t$ in the rightmost subordinate proof implies $\exists!a$ (by free logic's Rule of Atomic Denotation). So it is not necessary to have $\exists!a$ as a further dischargeable assumption in that subordinate proof.

[7] We are following the notational conventions of [6], a seminal study in proof theory. Prawitz, however, did not treat the definite description operator.

as its major premise (to the left, immediately above the inference stroke). The minor premises (or subproofs) of the first and third rules correspond, respectively, to the first and third immediate subproofs of the introduction rule. This is a convincing sign that the elimination rules are in harmony with the introduction rule that begets them.

ι-E$_1$ $\dfrac{t=\iota x\varphi(x) \quad u=t}{\varphi(u)}$; or, in parallelized form:

$$\frac{t=\iota x\varphi(x)\quad u=t\quad \begin{array}{c}\overline{\varphi(u)}^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (i)}$$

ι-E$_2$ $\dfrac{t=\iota x\varphi(x)}{\exists!t}$

ι-E$_3$ $\dfrac{t=\iota x\varphi(x)\quad \exists!u \quad \varphi(u)}{u=t}$

Note that the rule ι-E$_2$ is an instance of the Rule of Atomic Denotation.

The rules for the identity predicate in free logic involve the addition of just one existential presupposition (in the Rule of Reflexivity of Identity), and no change at all to the usual and familiar Rule of Substitutivity of Identicals:

Reflexivity $\dfrac{\exists!t}{t=t}$ Substitutivity $\dfrac{\varphi \quad t=u}{\psi}$, where $\varphi^t_u = \psi^t_u$

Lemma 1

$$\frac{t=\iota x\varphi(x)\quad \begin{array}{c}\overline{\varphi(t)}^{\,(i)}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (i)}$$

Proof

$$\frac{t=\iota x\varphi(x)\quad \dfrac{\dfrac{t=\iota x\varphi(x)}{\exists!t}}{t=t}\quad \begin{array}{c}\overline{\varphi(t)}^{\,(1)}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (1)\ \iota\text{-E}_1}$$

□

Note that in Lemma 1 the conclusion θ could be $\perp$. Lemma 1 is a purely *logical* result, in the free logic of definite descriptions and the identity predicate, with the iota as a primitive variable-binding term-forming operator.

5 Regimentation of the Reasoning in the Berry Paradox

The foregoing system of free logic will now be used to regiment, as a natural deduction, the informally rigorous deductive reasoning in the Berry Paradox. The deduction will turn out to be a *disproof* of the claim that the 'Berry number' exists. Moreover, this disproof will be in *normal form*. It will follow, by the proof-theoretic criterion of paradoxicality, that the so-called Berry Paradox is a paradox in name only. It is *really* a straightforward 'negative existential'. We proceed to furnish the details.

Choose a quite large natural number k. (How large? …see below.) The Berry Paradox concerns the supposedly least natural number (call it β)[8] *not* definable in fewer than k words. But the definition 'the least natural number not definable in fewer than k words' would define β in fewer than k words. The number k is chosen to be large enough to ensure that this is so.

It seems there is a problem here. So much so that some have called it a paradox. Let us investigate whether it really is one.

To define a natural number is to provide a unary predicate $\varphi(x)$ that it uniquely satisfies. So a true identity

$$n = \iota x \varphi(x)$$

defines the number n. We could also say that the foregoing identity *uniquely specifies* the number n.

Consider now an effectively decidable syntactic condition Φ on such defining terms $\iota x \varphi(x)$. The statement

$$\Phi(\iota x \varphi(x))$$

says that the defining *term* $\iota x \varphi(x)$ satisfies the (effectively decidable syntactic) condition Φ,[9] And if we write

$$n \text{ is } \Phi\text{-definable}$$

this will mean that *for some definite descriptive term* $\iota x \varphi(x)$ *satisfying condition* Φ *we have* $n = \iota x \varphi(x)$. Let us abbreviate this italicized claim about n and Φ to

$$\text{Def}(n, \Phi).$$

and bear in mind that it embeds the existential 'for some definite descriptive term $\iota x \varphi(x)$ satisfying condition Φ'.[10] This embedded existential can be re-parsed as 'for some predicate $\varphi(x)$ for which $\Phi(\iota x \varphi(x))$ holds'. Thus $\text{Def}(n, \Phi)$ abbreviates

for some predicate $\varphi(x)$ for which $\Phi(\iota x \varphi(x))$ holds, we have $n = \iota x \varphi(x)$.

We shall now invoke the standard proof-theoretic method of defining a new, and slightly more complicated, concept in terms of concepts already in hand. The method

[8] We do not know yet whether there *is* such a number as 'it' to be so-called. The symbol β is an abbreviation for a more complex singular term that has yet to be formed. And the 'paradoxical' *discovery* will be made that the latter more complex singular term necessarily fails to denote.

[9] Strictly speaking, one should emphasize with some mnemonic device like corner quotes that the term $\iota x \varphi(x)$ is being *mentioned* not used, in the foregoing statement. A more scrupulous rendering would be

$$\Phi(\ulcorner \iota x \varphi(x) \urcorner).$$

The experienced reader will, we trust, permit us (once this caveat has been noted) to simply omit the corner quotes, both for ease of display and for readability.

[10] From now on we shall suppress repetition of the adjectival phrase 'effectively decidable syntactic' when speaking of the condition Φ. It must, however, be borne in mind.

is to provide Introduction and Elimination Rules for the new concept. We call it a *pasigraph*. In the statement of these rules one will be using concepts already in hand. And these rules must be in harmony, in the same way that Introduction and Elimination Rules for the familiar logical operators are in harmony.

We can give the following Introduction and Elimination rules for our pasigraph $\mathrm{Def}(n, \Phi)$.

$$\text{Def-I}\ \frac{n=\iota x\varphi(x) \quad \Phi(\iota x\varphi(x))}{\mathrm{Def}(n, \Phi)} \qquad \text{Def-E}\ \frac{\mathrm{Def}(n, \Phi) \qquad \begin{array}{c} {}^{(i)}\overline{\underbrace{n=\iota xF(x)\ ,\ \Phi(\iota xF(x))}}^{\,\square\,}{}^{(i)} \\ \vdots \\ \theta \end{array}}{\theta}\,(i)$$

In the rule Def-E, the predicate parameter F (not to be conflated with φ) occurs only where indicated—and hence, not in θ, nor in any side-assumption of the subordinate proof indicated by the descending dots—and the box between the two discharge strokes means that at least one of the indicated assumptions must have been used in the subordinate proof of θ. This latter condition is to ensure that one's reasoning in accordance with these rules remains *relevant*.

As it happens, the rule Def-E will not need to be used in the course of our logical reasoning that demonstrates the non-existence of the Berry number. The elimination rule Def-E is stated alongside its introductory companion Def-I here only for presentational completeness. That we do not need to apply it is neither here nor there. Consider this analogy: in the proof of the Law of Non-Contradiction $\neg(\varphi \wedge \neg\varphi)$, no application needs to be made of the rule $\wedge$-I. There should be no general expectation that, with a pair of harmoniously balanced rules @-I and @-E, and a particular deductive problem involving their operator @, *both* of the rules will need to be applied to provide a solution to the problem.

The reduction procedure for these rules (which establishes their harmony) should be obvious for the reader who is well versed in proof theory. But it will be worth stating the procedure for the record, in the interests of the reader who is not. Here is how to get rid of any occurrence of $\mathrm{Def}(n, \Phi)$ that stands as the conclusion of an Introduction and as the major premise of the corresponding Elimination.

$$\frac{\dfrac{\begin{array}{cc}\Pi & \Sigma \\ n=\iota x\varphi(x) & \Phi(\iota x\varphi(x))\end{array}}{\mathrm{Def}(n, \Phi)}\,\text{Def-I} \qquad \begin{array}{c} \underbrace{\Delta\ ,\ {}^{(i)}\overline{n=\iota xF(x)}\,{}^{\square}\,,\overline{\Phi(\iota xF(x))}{}^{(i)}} \\ \Xi \\ \theta \end{array}}{\theta}\,(i)\ \text{Def-E} \quad \rightsquigarrow \quad \begin{array}{c} \begin{array}{cc} \Pi & \Sigma \end{array} \\ \underbrace{\Delta, n=\iota x\varphi(x)\ ,\Phi(\iota x\varphi(x))} \\ \Xi^F_\varphi \\ \theta \end{array}$$

Note that Ξ^F_φ within the reduct is the proof that results from Ξ by uniformly substituting φ for the predicate parameter F therein. Such substitution, because of the parametric conditions on F, leaves the premises in Δ, and the overall conclusion θ, undisturbed.

Berry's would-be definition or specification of the number β (or should one say 'definition or specification of the would-be—or supposedly existing—number β'?) is the following definite description:

$$\iota n(\neg\text{Def}(n, \Phi) \wedge \forall y(y < n \rightarrow \text{Def}(y, \Phi))).$$

In words: 'the number n such that for no definite descriptive term $\iota x\varphi x$ satisfying condition Φ is it the case that $n = \iota x\varphi x$ but for every number y less than n there is some definite descriptive term $\iota x\varphi x$ satisfying condition Φ such that $y = \iota x\varphi x$'. More pithily: 'the least number not definable by a definite descriptive term satisfying condition Φ'.

It is *this* well-formed singular term *of English* that is shown by the Berry reasoning (suitably 'informally rigorized') not to denote.

Let us abbreviate this term in the first instance as

$$\iota n(\neg\text{Def}(n, \Phi) \wedge \xi(n))$$

—so '$\xi(n)$' is short for '$\forall y(y < n \rightarrow \text{Def}(y, \Phi))$'; and in the second instance as

$$\iota n\psi(n)$$

—so '$\psi(n)$' is short for '$\neg\text{Def}(n, \Phi) \wedge \forall y(y < n \rightarrow \text{Def}(y, \Phi))$'.

The denotation of the term $\iota n\psi(n)$—*should it exist* (and this is by no means a foregone conclusion)—is what we might call 'the Berry number'.

The foregoing abbreviations have been introduced for reasons of conveniently compact proof-display that will emerge below. Using the unabbreviated expressions instead of their abbreviations would produce a proof-display too wide to be contained on a journal page. The following is an easily surveyable summary of the abbreviations in question:

$\iota n(\neg\text{Def}(n, \Phi) \wedge \forall y(y < n \rightarrow \text{Def}(y, \Phi)))$ abbreviates to
$\iota n(\neg\text{Def}(n, \Phi) \wedge \xi(n))$, which in turn abbreviates to
$\iota n\psi(n)$.

It is important to appreciate the following three simple points about the notations and abbreviations that we have chosen.

1. The condition Φ is so chosen that we have $\Phi(\iota n\psi(n))$.

Historical example (Berry's own, *via* Russell): $\Phi(x)$ is 'x contains at most eighteen syllables'; and $\iota n\psi(n)$ is 'the least integer not nameable in fewer than nineteen syllables'. *By inspection* we have, in this case, $\Phi(\iota n\psi(n))$. And note that this example of Φ is an effectively decidable 'syntactic' condition, on the understanding that 'syllable' admits of a suitably rigorous explication. In verifying (1), we simply *counted the syllables* in the English descriptive term that we are abbreviating as $\iota n\psi(n)$, and found that there were at most eighteen of them. The English descriptive term was 'the least integer not nameable in fewer than nineteen syllables'.

2. By way of reminder: Φ is an effectively decidable syntactic property of *syntactic* entities—here, descriptive terms in English. We are not interested in *non*-syntactic properties of these descriptive terms, such as ‘having been uttered within earshot of a philosopher’, or ‘having enjoyed, once upon a time, a denotation, but having lost it since then because of a bloody revolution’.

3. It could turn out, for some effectively decidable syntactic property Φ and some unary formula φ, that $\Phi(\iota x\varphi(x))$ holds **even though the term $\iota x\varphi(x)$ fails to denote**.

This is the crucial point. We have *no guarantee*, from any of the foregoing, that the term $\iota x\varphi(x)$ must denote. Indeed, it will emerge that it *cannot*.

All that we now need to invoke in order to get our analysis of the Berry reasoning underway is the lemma $\Phi(\iota n\psi(n))$. In the Berry example, straightforward reasoning from basic observable (or inspectable) facts yielded the conclusion that $\Phi(\iota n\psi(n))$. More generally—allowing, say, for a change of language from English, or a focus on words rather than syllables—it will always be the case that $\Phi(\iota n\psi(n))$ admits of straightforward *proof in normal form*.

Suppose we establish it, then, by means of some normal-form proof or warrant Π, by choosing Φ appropriately. (The proof Π will have a role to play presently, in our proof of Metatheorem 1.) We would then be able to construct the following formal *dis*proof (a *reductio ad absurdum*), in free logic, of the existential claim $\exists!\iota n\psi(n)$—i.e., of the claim that ‘the Berry number’ β exists.

Metatheorem 1 *There is a normal-form disproof of* $\exists!\iota n\psi(n)$*—that is, of*

$$\exists!\iota n(\neg Def(n,\Phi)\wedge\forall y(y<n\rightarrow Def(y,\Phi))).$$

Proof Here is our witnessing normal-form disproof Ω for this metalogical claim. Note that this normal-form disproof has the aforementioned proof Π (itself in normal form) embedded as a subproof at top right.

$$\Omega:\quad \dfrac{\exists!\iota n\psi(n) \qquad \dfrac{\begin{matrix}{}^{(2)}\overline{}\\ a=\iota n\psi(n),\ \text{i.e.,}\\ a=\iota n(\neg\mathrm{Def}(n,\Phi)\wedge\xi(n))\end{matrix} \qquad \dfrac{\dfrac{\dfrac{{}^{(1)}\overline{}}{\neg\mathrm{Def}(a,\Phi)\wedge\xi(a)}}{\neg\mathrm{Def}(a,\Phi)} \qquad \dfrac{\begin{matrix}{}^{(2)}\overline{}\\ a=\iota n\psi(n)\end{matrix}\quad \begin{matrix}\Pi\\ \Phi(\iota n\psi(n))\end{matrix}}{\mathrm{Def}(a,\Phi)}{\scriptstyle(\mathrm{Def\text{-}I})}}{\bot}{\scriptstyle(\neg\text{-E})}}{\bot}{\scriptstyle(1)\ \mathrm{L1}}}{\bot}{\scriptstyle(2)\ (\exists\text{-E})}$$

□

Note that in this disproof, apart from purely logical rules, only the Introduction Rule for the pasigraph Def finds application; also, no use is made of the internal structure of the second conjunct $\xi(n)$ of $\psi(n)$, which is

$$\forall y(y<n\rightarrow \mathrm{Def}(y,\Phi)).$$

That second conjunct finds its way into the specification $\iota n\psi(n)$ only in order to make plausible the thought that the term $\iota n\psi(n)$ satisfies the condition Φ if we so choose Φ as to ensure that it is satisfied by at most finitely many definite descriptive terms, with the term $\iota n\psi(n)$ itself being one of them. In a nutshell, ψ is so constructed and Φ so chosen as to make available the embedded subproof Π—which, as we have already stressed, will be in normal form. (The proof Π will also have, as its premises, only obvious syntactic truths.) But the rest of the overall disproof (which is also in normal form) proceeds with complete disregard for the internal structure of the second conjunct of ψ.

The disproof is constructive. It reveals a straightforward inconsistency. This realization dissolves the appearance of logico-semantic paradox on the part of the Berry. Like Russell's, Berry's 'paradox' is in name only. It is a completely different kettle of fish from the Liar and its ilk—*which are many and varied*, all of them affording the discovery that the reduction sequences of their associated disproofs (which are *not* in normal form) do not terminate. This is for the deep reason of self-reference in the case of genuine logico-semantic paradoxes. The Russell and the Berry do not partake of this. They simply make us realize that certain complex terms do not—indeed, *cannot*—denote.

6 An Aside on the Least Number Principle

In this section we address the status of the Least Number Principle and its potential to re-instate the Berry as a paradox, in a manner rather different from that of Rosenblatt (*loc. cit.*).

Suppose one is theorizing only about the natural numbers (i.e., what Berry and Russell called integers), *using only the language of arithmetic*. Typically this language would be taken to be that of first-order logic with identity, based on (i) the usual logical operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\exists$, and $\forall$, along with the definite description operator ι, and (ii) the extralogical primitives 0, s, $+$, and $\times$. The presupposed restriction on the permitted language will emerge as important in due course. We need, however, to stress here that 'the language of arithmetic' can be taken to be a rather open-textured metamathematical term. We can countenance extensions of the language of first-order Peano Arithmetic by means of higher-order quantifiers and/or the introduction of grammatically primitive truth- or satisfaction- or provability-predicates, along with devices such as corner-quoting to give these extending expressions free rein to make their contributions to *a priori* science within the extended language. Those contributions, however, must be *a priori*—i.e., 'arithmetical' in a sensibly open-textured sense. There should be no 'empirical' or otherwise contingent content introduced, within legitimate extensions of the arithmetical language, to create empirically tinged substituends in axiom schemata such as Mathematical Induction or the Least Number Principle.[11]

[11] We owe to an anonymous referee awareness of the need to clarify that ι can of course earn its keep in any language of arithmetic, and that one can investigate what happens, for arithmetic as an *a priori* science, if one adds to one's language (say) a primitive truth predicate. Such additions clearly take one 'beyond Peano', yet without overstepping any principled limitation on what 'the language of arithmetic' might contain in the way of interesting new primitive and enriching expressions. Investigations of the kind just described would include [2, 3, 5, 13, 15]. They all remain steadfastly within the domain of the *a priori*.

The axiom schema of *Ordinary* Mathematical Induction is

$$(P0 \wedge \forall x(Px \rightarrow Psx)) \rightarrow \forall z Pz.$$

Constructively equivalent to Ordinary Mathematical Induction is *Strong* Mathematical Induction:

$$\forall x(\forall y(y < x \rightarrow Py) \rightarrow Px) \rightarrow \forall z Pz,$$

where $y < x$ is defined as $\exists w\, x = y + sw$ (so that we are still within the language of arithmetic).

Closely related (indeed: classically equivalent) to Strong Mathematical Induction is the following *Least Number Principle*:

$$\forall x(\neg Px \rightarrow \exists y(\neg Py \wedge \forall z(z < y \rightarrow Pz))).$$

This tells us that if the universal claim $\forall x\, Px$ *has* a counterexample at all, then there is a *least* number that serves as such a counterexample. Note that the uniqueness of such a number is not explicitly claimed (even though it would be unique, should it exist).

The Least Number Principle constructively implies Strong Induction on Decidable Predicates; and conversely. It follows that the constructivist is entitled to the reasoning that follows in the remainder of this section—since the predicates involved are indeed decidable.

We need now to address the possibility[12] that the Berryite (i.e., that character who regards the Berry as a genuine paradox) could try to invoke the Least Number Principle to generate a contradiction if they are presented with a proof of the conclusion (which we have already arrived at) that *there is no such number as the least number not definable by a definite descriptive term satisfying condition* Φ:

$$\neg\exists x\, x = \iota n(\neg\text{Def}(n, P) \wedge \forall y(y < n \rightarrow \text{Def}(y, \Phi))).$$

The Berryite will try to demonstrate—using the Least Number Principle

$$\forall x(\neg Px \rightarrow \exists y(\neg Py \wedge \forall z(z < y \rightarrow Pz)))$$

—the existential conclusion

$$\exists x\, x = \iota n(\neg\text{Def}(n, \Phi) \wedge \forall y(y < n \rightarrow \text{Def}(y, \Phi))).$$

To do this, the Berryite will take, for the schematic predicate Px in the Least Number Principle, the expression $\text{Def}(x, \Phi)$. This will generate the following instance of the Least Number Principle:

$$\forall x(\neg\text{Def}(x, \Phi) \rightarrow \exists y(\neg\text{Def}(y, \Phi) \wedge \forall z(z < y \rightarrow \text{Def}(z, \Phi)))).$$

12 We are indebted to an anonymous referee for raising this possibility.

The latter is constructively equivalent to the conditional

$$\exists x\neg\mathrm{Def}(x,\,\Phi)\;\rightarrow\;\exists y(\neg\mathrm{Def}(y,\,\Phi)\wedge\forall z(z<y\rightarrow\mathrm{Def}(z,\,\Phi))).$$

The Berryite will then urge—when $\Phi(x)$ says that x contains at most (say) eighteen syllables—‘Surely there is *some* number x that is large enough not to be denoted by any descriptive term satisfying the condition Φ—i.e. such that $\neg\mathrm{Def}(x,\,\Phi)$?’. *If* we acquiesce (a big ‘if’!), this gives the Berryite the antecedent $\exists x\neg\mathrm{Def}(x,\,P)$ of the conditional just displayed. They will then detach to conclude the existential

$$\exists y(\neg\mathrm{Def}(y,\,\Phi)\wedge\forall z(z<y\rightarrow\mathrm{Def}(z,\,\Phi))),$$

and then exclaim

> *This* demonstrates the existence of the least number w such that $\neg\mathrm{Def}(w,\,\Phi)$!—that is, the least number not definable by any descriptive term containing at most eighteen syllables.

But all this is to ignore a fatal flaw in the Berryite's reasoning at the very outset. The Principle of Mathematical Induction and the Least Number Principle are intended to hold *only for properties expressible in the language of arithmetic*. These principles are *not* allowed to be invoked when the substituend expressions for their schematic predicate P are themselves *not* (either primitively, or by abbreviatory definitions) in the language of arithmetic. Take a close look at what sorts of expressions find their way into $\mathrm{Def}(w,\,\Phi)$, to contribute to the latter's meaning. The rule Def-I:

$$\frac{n=\iota x\varphi(x)\quad\Phi(\iota x\varphi(x))}{\mathrm{Def}(n,\,\Phi)}$$

shows the potentially *non-arithmetical* expressions φ and Φ as culprit constituents disqualifying the Berryite's invocation of $\mathrm{Def}(x,\,\Phi)$ as a substituend for the schematic predicate Px in the Least Number Principle.

But now: how should we respond to any Berryite offering of a respectable-looking, not immediately dismissible argument for the conclusion that $\mathrm{Def}(x,\,\Phi)$ *is*, after all, a kosher such substituend? We would be confronted then with the following overall logical structure of a *reductio*:

$$\dfrac{\dfrac{\begin{array}{c}\overline{\exists!\iota n\psi(n)}^{\,(1)}\\ \Omega\\ (\text{Metatheorem 1})\\ \bot\end{array}}{\neg\exists!\iota n\psi(n)}{}^{(1)}\qquad\dfrac{\dfrac{\overset{\text{Berryite: ‘surely?’:}}{\exists x\neg\mathrm{Def}(x,\,\Phi)}\quad\overset{\text{Least Number Principle:}}{\exists x\neg\mathrm{Def}(x,\,\Phi)\rightarrow\exists y(\neg\mathrm{Def}(y,\,\Phi)\wedge\forall z(z<y\rightarrow\mathrm{Def}(z,\,\Phi)))}}{\exists y(\neg\mathrm{Def}(y,\,\Phi)\wedge\forall z(z<y\rightarrow\mathrm{Def}(z,\,\Phi))),\ \text{i.e.,}}}{\exists!\iota n\psi(n)}}{\bot}$$

which, upon normalizing by one step of negation reduction, becomes

$$\dfrac{\dfrac{\overset{\text{Berryite: `\textit{surely}?':}}{\exists x \neg \mathrm{Def}(x, \Phi)} \quad \overset{\text{Least Number Principle:}}{\exists x \neg \mathrm{Def}(x, \Phi) \rightarrow \exists y(\neg \mathrm{Def}(y, \Phi) \wedge \forall z(z < y \rightarrow \mathrm{Def}(z, \Phi)))}}{\exists y(\neg \mathrm{Def}(y, \Phi) \wedge \forall z(z < y \rightarrow \mathrm{Def}(z, \Phi))), \text{ i.e.,}}}{\exists ! \iota n \psi(n)}$$

$$\Omega$$

$$\bot$$

To the extent that the Berryite is morally certain about their existential supposition ‘*surely*?’, they should now be convinced that their extension of Peano Arithmetic would be provably inconsistent if its main axiom scheme (Mathematical Induction, or the Least Number Principle—it matters not which) were to be allowed substituends in the Berryite’s proper extension of whatever language of arithmetic, in the charitably open-textured sense explained above, was serving the aims of a *consistent a priori* science of number. The intellectually honest thing for the Berryite to do would be to bite one or other of these two bullets:

(i) give up their assertion $\exists x \neg \mathrm{Def}(x, \Phi)$; and/or

(ii) concede that the expression $\mathrm{Def}(x, \Phi)$ is not a legitimate substituend in the Least Number Principle;

and then: *simply accept* Ω *as showing that there is no such thing as the Berry number*.

7 Paradoxes v. ‘paradoxes’

The proof-theoretic criterion of logico-semantic paradoxicality has confirmed an extensive array of such paradoxes. Moreover, a potential *infinity* of different logico-semantic paradoxes will be revealed as such, according to our conjecture,[13] that the proof-theoretic criterion of logico-semantic paradoxicality would reveal as paradoxical all those paradoxes in the semantic sense analyzed by [4].[14] It should be noted that while the most familiar logico-semantic paradoxes involve single sentences, this need not be the case in general. As Kripke importantly observed, logico-semantic paradox can be generated by *finite sets* of sentences, some of them even claims about contingent empirical matters (such as who said what to whom). The proof-theoretic criterion of paradoxicality extends straightforwardly to deal with such cases: the disproofs of the relevant *sets* of sentences will not be normalizable. Even at this stage one should not underestimate the fecundity of the criterion in confirming instances of what have been regarded, intuitively, as logico-semantic paradoxes. Those instances are many and varied. The ‘structural similarity’ they enjoy to the Liar (their simplest possible representative) is that *the reduction sequences of their associated disproofs*

[13] *Loc. cit.* pp. 282–3.

[14] Lack of space precludes any more thorough investigation of whether the truth of our conjecture is hostage to any particular evaluational schema (weak Kleene; strong Kleene; supervaluational) used for the iterated evaluations employed in Kripke’s account of semantic paradoxicality. We are endebted here to an anonymous referee for raising this possibility. It is certainly worthy of further study.

do not terminate. *This* is what imparts uniformity and generality to the logico-semantic paradoxes. They form, so to speak, a logical 'natural kind'.

By contrast with logico-semantic paradoxicality arising from semantic closure, one can see that the Russell and the Berry are *not* logico-semantic paradoxes. Once we properly diagnose the logical structure of the reasoning involved in them, we realize that certain complex singular terms do not—indeed, *cannot*—denote.

Surprising though that may be (the etymology of 'paradox' is from the Greek for 'contrary to expectation'), the resolution of the air of paradox with the Russell and the Berry lies in a proof of a negative existential. *There is no such thing.* Russell's Paradox shows that there is no such thing as

$$\{x|\neg x \in x\}$$

—though it might be hard at first to see this, for oneself, right away. (It certainly was for Frege.) Berry's Paradox shows that there is no such thing as

$$\iota n(\neg \mathrm{Def}(n, \Phi) \wedge \forall y(y < n \rightarrow \mathrm{Def}(y, \Phi)))$$

—though it might be hard at first to see this, for oneself, right away. (It certainly has been for many a theorist of paradox, even those with the benefit of a proper understanding of the definite descriptive operator iota.) But, in the final analysis, that there are no such things as these two (the Russell set and the Berry number) is no more puzzling than the fact—much easier for anyone to see, for themselves, right away—that there is no such thing as

$$\iota x \neg x = x.$$

The necessary non-existence in each of these three cases is a straightforwardly logical matter, establishable within the framework of the *right kind of logic* for handling abstractive terms in general. That is the framework of *single-barreled* introduction rules for abstraction operators, and their harmoniously balanced elimination rules, within a *free logic*.

8 Re-visiting Ramsey's Groupings

This section can be read as a response (or re-connection) to Rosenblatt's objection that our shared finding on the true nature of the Berry reveals that the present author's proof-theoretic criterion of paradoxicality suffers from a lack of uniformity and generality. We propose that one should embrace our analysis of the Berry and *revise* the view that we expressed in [16], to the effect that our finding that the Russell Paradox in set theory amounted to a genuine inconsistency rather than a genuine paradox brings the proof-theoretic criterion more closely into line with Ramsey's famous groupings. With the Berry now 'in the same boat' as the Russell (a genuine inconsistency rather than a genuine paradox), it is time to fundamentally re-think those groupings.

We need to quote here at some length from Ramsey's famous essay on Foundations of Mathematics ([7]). The following passage will be found at pp. 20–21 in its reprinting in [8].

> It is not sufficiently remarked, and the fact is entirely neglected in *Principia Mathematica*, that these contradictions [i.e., paradoxes] fall into two fundamentally distinct groups, which we will call A and B. The best known ones are divided as follows:–
> A. (1) The class of all classes which are not members of themselves.
> (2) The relation between two relations when one does not have itself to the other.[15]
> (3) Burali Forti's contradiction of the greatest ordinal.
> B. (4) 'I am lying.'
> (5) The least integer not nameable in fewer than nineteen syllables.
> (6) The least indefinable ordinal.
> (7) Richard's Contradiction.
> (8) Weyl's contradiction about 'heterologisch'.[fn]
> The principle according to which I have divided them is of fundamental importance. Group A consists of contradictions which, were no provision made against them, would occur in a logical or mathematical system itself. They involve only logical or mathematical terms such as class and number, and show that there must be something wrong with our logic or mathematics. But the contradictions of Group B are not purely logical, and cannot be stated in logical terms alone; for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms. So they may be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language.

Our proof-theoretic analysis above of the Berry Paradox is a tipping point. In light of it, we should re-visit Ramsey's A–B classification of what in his day, and with the formal methods available, seemed to earn the label 'paradox'. We would urge revision of our earlier claim, in [16] at pp. 305–6, that revealing the Russell *not* to be a paradox

> brings [the] proof-theoretic criterion of paradoxicality more closely into line with Ramsey's famous …distinction between the 'Group A' contradictions—that is, the mathematical paradoxes such as Russell's Paradox—and the 'Group B contradictions'—that is, the logico-semantical paradoxes such as the Liar.

Rather than being 'more closely in line' with Ramsey's distinction, the proof-theoretic criterialist is now *at odds* with Ramsey. For Ramsey put the Liar and the

[15] This is not much discussed by later writers. Presumably the reasoning for paradox proceeds by asking us to consider the relation $\mathcal{R}$ between two relations R_1 and R_2 that is defined as follows:

$$\mathcal{R}(R_1, R_2) \equiv_{df} \neg R_1(R_1, R_2) \ ;$$

and then to inquire whether $\mathcal{R}(\mathcal{R}, \mathcal{R})$. That will generate a tight back-and-forth between $\mathcal{R}(\mathcal{R}, \mathcal{R})$ and its negation. This 'paradox' is of little interest to the inquirer for whom the 'form of proposition' $\mathcal{R}(\mathcal{R}, \mathcal{R})$ simply cannot make sense, on account of its violation of order- or type-considerations. The present author is unaware of anyone revisiting this 'paradox' in the wake of Ramsey's posing of it. Ramsey does not discuss it in his paper; he only mentions it as belonging to Group A.

paradox of heterologicality into his Group B, *along with the Berry*; and he put the Russell into his Group A. The discovery that the Berry and the Russell are 'paradoxes' *of the same kind* means that Ramsey's groupings have to be re-considered.

Let us therefore cease to use Ramsey's distinction between the 'logical/mathematical' paradoxes and the 'linguistic' ones. Let us use instead the distinction provided by the proof-theoretic criterion of paradoxicality, making use of the formal methods afforded by Gentzen's groundbreaking work on natural deduction (and that of Prawitz, following him). We appreciate now the importance of regimenting mathematical proofs by means of formal proofs in *free* first-order logic. Free logic, recall, is free of the false 'background assumption' (unfortunately adopted by Frege) that every singular term denotes.

The pattern, or new classification, that then asserts itself is as follows. There are the logico-semantic paradoxes that arise from the semantic closure of one's language of choice. And then there are the *negative existential theorems* that can be established, using free logic, in various mathematical theories. The languages involved are not semantically closed. These two groupings, henceforward, should be preferred to Ramsey's. Let us call them LS (for 'logico-semantic') and NE (for 'negative existential'). This will be better than trying to reassign items to Ramsey's Group A and Group B. We include also some more recent paradoxes that Ramsey was not in a position to assign to either of his two Groups.

Here, then, are the two new groupings proposed.

LS	**NE**
Liar Paradox	Russell's Paradox
Grelling's Paradox	Burali-Forti Paradox
Knower Paradox	Berry Paradox
Yablo's Paradox	Paradox of the Least Indefinable Ordinal
Curry's Paradox	Richard's Paradox
Postcard Paradox	
Revenge Paradox	
Prior's Paradox	
Russell's Paradox of Propositions	

For demonstrations that Prior's Paradox and Russell's Paradox of Propositions belong in the **LS** grouping, see [17].

In taking issue with Ramsey's earlier and perhaps more intuitive classification, we are proposing that we should allow an explicating theory of paradox—one which employs appropriate formal methods and logical analyses—to re-educate those earlier intuitions behind Ramsey's groupings.

The logico-semantic paradoxes are identified by the sentences that they involve: the Liar is the prime example. They are deemed genuinely paradoxical because they feature in reasoning leading to one or other of two main kinds of untoward results: either a non-normalizable proof of $\perp$ (as with the Liar) or a non-normalizable proof of an arbitrary conclusion ψ (as with the Curry) from the empty set of assumptions (with the logical route proceeding via the sentence concerned). These proofs typically exploit rules of two kinds. First, there are Introduction and Elimination rules governing the main notions embedded in the litmus sentence(s) (for example, the truth predicate in the Liar). Second, there are the '*id est*' rules identifying the problematic sentence (for example, the inferences back and forth between λ and $\neg T(\ulcorner\lambda\urcorner)$ in the Liar reasoning). The upshot is that in the context of the '*id est*' rules, the conceptual apparatus generating the rules governing the main notions is revealed to be incoherently deployed. And finally, the proof-theoretic criterion for paradoxicality says that this revelation takes a special form: a non-normalizable proof (or disproof). The reduction procedures associated with the Introduction and Elimination rules governing the main notions fail to generate a reduction sequence terminating in a proof (or disproof) in normal form.

The foregoing characterization of logico-semantic paradoxes clearly rests on the favored choice of a system of natural deduction (with Introduction and Elimination rules for the logical operators and for what we have called the main notions embedded in the litmus sentence(s)). Normalization of proofs in this system would be by the methods pioneered by Prawitz and his successors, using the reduction procedures associated with the Introduction and Elimination rules.

Hopefully 'equivalently': the same characterization of logico-semantic paradoxes could be made to rest on a favored choice of a sequent calculus (with Right and Left rules for the logical operators and for what we have called the main notions embedded in the litmus sentence). The 'normalization' of proofs in such a system would be by the Cut-elimination methods pioneered by Gentzen, using the reduction procedures associated with the Right and Left rules.

The 'negative existential' paradoxes are paradoxes in name only; and so-called only because they tend to be surprising on first acquaintance. They reveal no incoherence in the conceptual apparatus deployed. Their proofs are genuine proofs, which can be written in normal form, using axioms and rules of inference that are sound for the branch of mathematics in question.

Data Availability Not applicable.

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