Airy Stress Function

2-D equilibrium without body forces:

\[ \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \]

What if these relations were satisfied automatically?

\[ \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \]

\( \phi \) is some scalar function.

Plugging in:

\[ \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi}{\partial y^2} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial^2 \phi}{\partial x \partial y} \right) = \frac{\partial^3 \phi}{\partial x \partial y^2} + \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0 \]

i.e. equilibrium is always valid, regardless of \( \phi \)

But is it that easy?

The function must still satisfy (a modified version of) compatibility:

\[ \nabla^2 (\sigma_{xx} + \sigma_{yy}) = \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 0 \]

\[ \nabla^4 \phi = \frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \]

biharmonic equation
If the biharmonic equation is satisfied, $\phi$ is the solution to a problem! Which problem? Who knows!

Example: uniaxial tension

\[ \sigma_{xx} = \frac{F}{A} \quad \sigma_{yy} = 0 \quad \tau_{xy} = 0 \]
\[ \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \]

$\phi = Cy^2$  $\sigma_{xx} = 2C$

Pure shear:

\[ \sigma_{xx} = 0 \quad \sigma_{yy} = 0 \quad \tau_{xy} = 3 \]

$\phi = Bxy$  $\sigma_{xx} = B = 3$

$\phi = 3xy$
what about both?

\[ \phi = \text{const} + 3xy \]

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15 governing equations to solve 15 unknowns:

- internal stress: 6 unique \([\sigma]\)
- internal strain: 6 unique \([e]\)
- displacements: 3 unique \([u]\)
Boundary conditions: what do you know from the problem statement.

- These are what make each problem unique!
- Where does each condition apply in the problem?

General test taking tips

- Calm down everything will be alright!
- What is actually important in the problem?
- Skip hard questions or when you get stuck
- Write something down (but not too much)
- Explain in words if you have to
- Set yourself up to get every point possible