PRESSURE METRICS FOR DEFORMATION SPACES OF QUASIFUCHSIAN GROUPS WITH PARABOLICS

LIEN-YUNG “NYIMA” KAO (高連庸)

JOINT WITH **HARRY BRAY** AND **DICK CANARY**

THE GEORGE WASHINGTON UNIVERSITY
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OUTLINE

- Goal:
  - Dynamics:
    - A study of non-compact systems
  - Geometry:
    - Construct a metric on quasifuchsian spaces
    - Entropy rigidity type results
- What have been done
- Setup and our results
- Model Example: Teichmüller spaces of closed surfaces
- From Fuchsian to quasifuchsian
- From compact to non-compact
  - How does the proofs look like in compact cases
  - What failed in non-compact cases and remedies

Let $S = S_{g,n}$ be an orientable surface with g genus and n punctures and negative Euler characteristic.

FUCHSIAN REPRESENTATIONS:

$\rho \in \underline{\mathrm{Hom}(\pi_1(\mathrm{S}), \mathrm{PSL}(2,\mathbb{R}))}$ is discrete & faithful

orientation preserving isometries on $\mathbb{H}$

$X_\rho := \rho(\pi_1(S))/\mathbb{H}$

is a hyperbolic surface

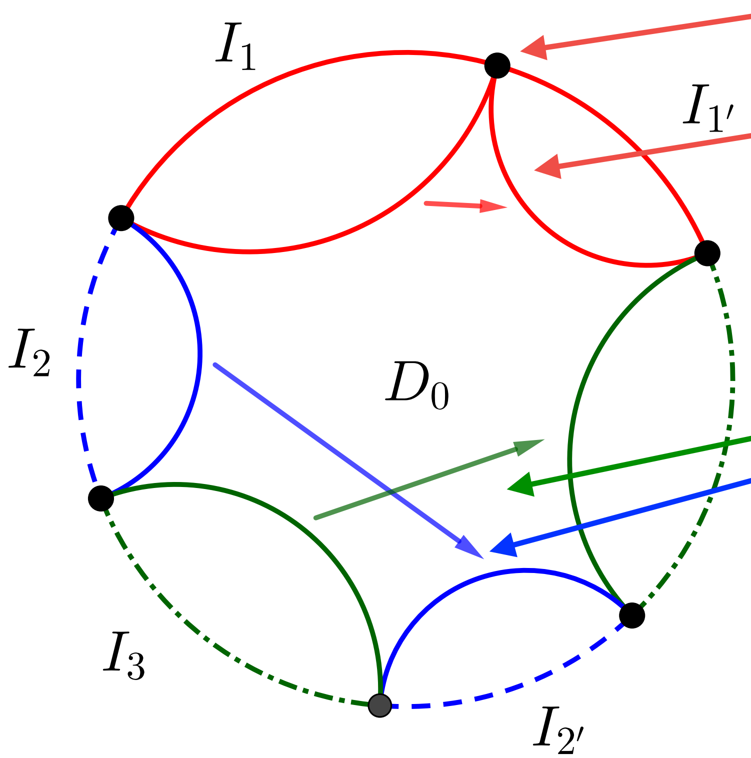
cusp

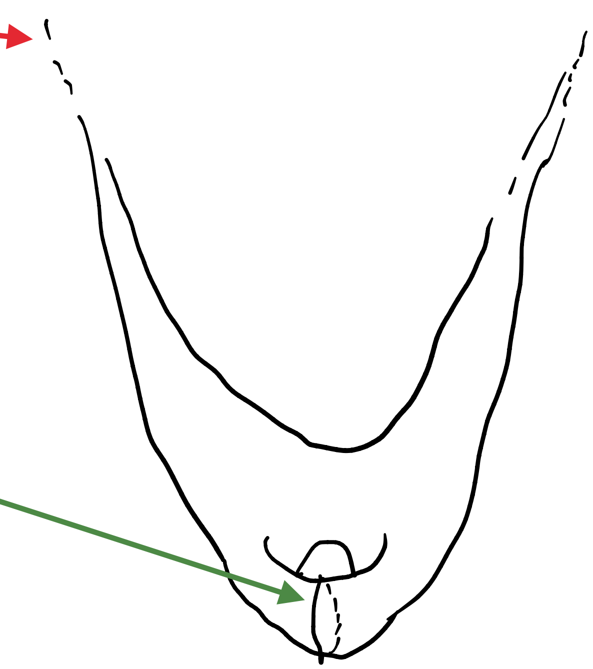
parabolic element

has only one fixed point
which is on the boundary

hyperbolic element

has two fixed points
which are both on the boundary





QUASIFUCHSIAN REPRESENTATIONS:

Let $\pi_1(S) \cong \Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group s.t. $\Gamma\backslash\mathbb{H}^2$ has finite area.

We say $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ *quasifuchsian* if there exists a quaisconformal home-omorphism $\phi : \mathbb{C} \to \mathbb{C}$ such that $\rho(\gamma) = \phi\gamma\phi^{-1}$ for all $\gamma \in \Gamma$.

orientation preserving isometries on $\mathbb{H}^3$

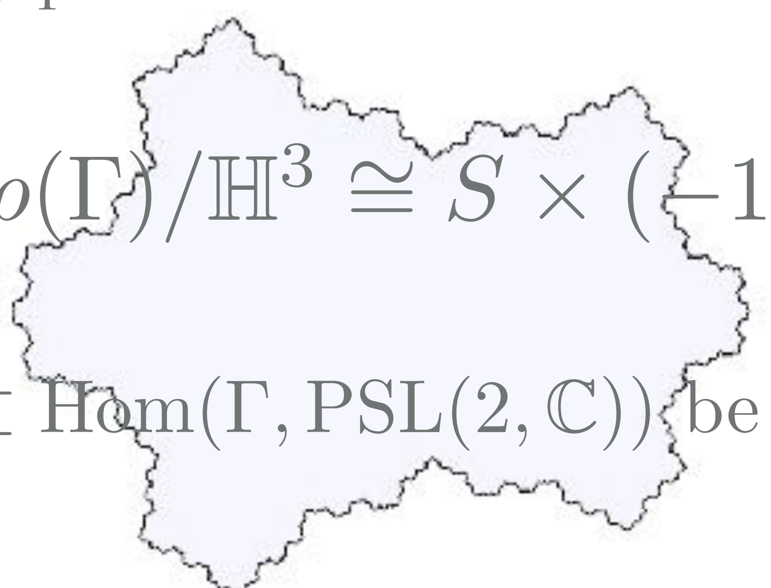
maps small circles to small ellipse with bdd axes ratio,

i.e. $\limsup_{r\to 0} \dfrac{\sup\{(d(\phi(z),\phi(x)) : \ d(x,z) = r\}}{\inf\{(d(\phi(z),\phi(x)) : \ d(x,z) = r\}} \leq K$

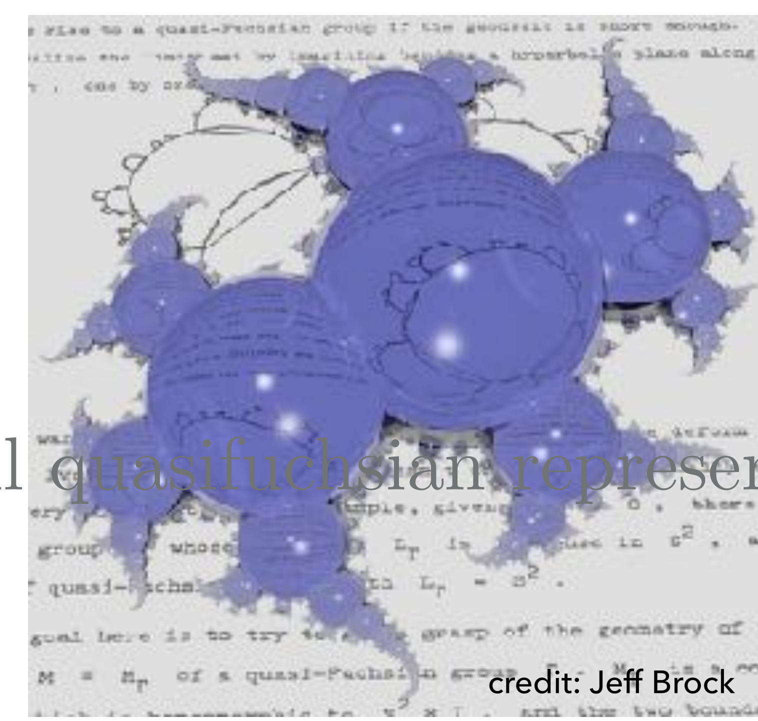
Equivalently, $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ is quasifuchsian if and only if $\rho(\partial S)$ is parabolic and $\rho(\Gamma)$ preserves a Jordan curve in $\hat{\mathbb{C}}$.

$$N_\rho := \rho(\Gamma)/\mathbb{H}^3 \cong S \times (-1,1)$$

$QC(\Gamma) \subset \mathrm{Hom}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))$ be the space of all quasifuchsian representations.



credit: Jeff Brock



credit: Jeff Brock

TEICHMULLER SPACE

$$\mathcal{T}(S) := \mathrm{Hom}_{\mathrm{F}}^{\mathrm{tp}}(\pi_1(\mathrm{S}), \mathrm{PSL}(2,\mathbb{R}))/\sim$$

= conjugacy classes of type-preserving finite area Fuchsian reps.

if there exists an isomorphism $\iota : \rho_1(\pi_1(S)) \to \rho_2(\pi_1(S))$ sending hyperbolic elements to hyperbolic elements and parabolic elements to parabolic elements

$X_\rho := \rho(\pi_1(S))/\mathbb{H}$ has finite area

QUASIFUCHSIAN SPACE

$$QF(S) = QC(\Gamma)/\mathrm{PSL}(2,\mathbb{C}) \subset \mathrm{Hom}_{tp}(\Gamma, \mathrm{PSL}(2,\mathbb{C}))//\mathrm{PSL}(2,\mathbb{C})$$

$$QF(S) \cong \mathcal{T}(S) \times \mathcal{T}(S)$$

Bers simultaneous uniformization theorem

The Fuchsian locus $F(S) \subset QF(S)$ is the set of fuchsian reps

INTERSECTION NUMBER

$$\mathrm{I}(\rho_1, \rho_2) = \lim_{T\to\infty} \frac{1}{|R_{\rho_1}(T)|} \sum_{\gamma \in R_{\rho_1}(T)} \frac{l_2[\gamma]}{\boxed{l_1}[\gamma]}$$

the translation length:

$l_1[\gamma] = \min\{d(x, \rho_1(\gamma)\cdot x) : \ x \in \mathbb{H}^2\}$

where $R_{\rho_1}(T) = \{\gamma : l_1[\gamma] < T\}$

Remark 1: also known as *geodesic stretch*

Remark 2: alternatively, $\mathrm{I}(\rho_1, \rho_2) := \lim_{n\to\infty} \frac{l_2[\gamma_n]}{l_1\boxed{[\gamma_n]}}$

$\gamma_n \in \pi_1(S)$ is equidistributed w.r.t. the Bowen-Margulis measure as $n \to \infty$

RENORMALIZED INTERSECTION NUMBER

$$\mathrm{J}(\rho_1, \rho_2) = \frac{h(\rho_2)}{h(\rho_1)} \lim_{T\to\infty} \frac{1}{|R_{\rho_1}(T)|} \sum_{\gamma \in R_{\rho_1}(T)} \frac{l_2[\gamma]}{l_1[\gamma]}$$

WHAT HAVE BEEN DONE (FOR PRESSURE METRICS)

- When S is a **closed surface**:
  - **Thurston**: the Hessian of $\mathrm{I} : \mathcal{T}(S) \times \mathcal{T}(S)$ is **positively definite**, and thus defines a Riemannian metric on $\mathcal{T}(S)$
  - **Wolpert** (86): Thurston's Riemannian metric is exactly the Weil-Petersson metric

    Thermodynamic Formalism
  - **McMullen** (08): recovered Thurston's Riemannian metric by

    the pressure metric
    - Also, **Pollicott, Sharp** (16)
  - **Bridgeman** (10): builded up the pressure for quasifuchsian space
  - **Bridgeman, Canary, Labourie, Sambarino** (15): higher rank generalizations; also, **Pollicott, Sharp** (16, 18)
- When S is **NOT** compact and has cusps (i.e., not convex cocompact)
  - **K.** (19): construct the pressure metric for $\mathcal{T}(S)$

- When S is a metric graph
  - **Pollicott, Sharp** (13), K. (17): the pressure metric geometry is different from Weil-Petersson metric geometry
- General (dynamical) setting:
  - **Giulietti, Kloeckner, Lopes, Marcon** (18): a Riemannian metric of the cohomologus space of normalized pressure zero Hölder potentials of SFT (a slightly different pressure metric)
  - **Lopes, Ruggiero** (19): this space is non-negatively curved

OUR RESULT

Theorem A (Bray, Canary, K.). *If $\rho, \eta \in QC(\Gamma)$, then* $\mathrm{J}(\rho,\eta) \geq 1$ *and the equality holds iff ρ and η are conjugate in* $\mathrm{Isom}(\mathbb{H}^3)$.

Theorem B (Bray, Canary, K.). *The Hausdorff dimension of the limit set $\Lambda(\rho(\Gamma))$ varies analytically in $QC(\Gamma)$. Both $\mathrm{I}(\rho,\eta)$ and $\mathrm{J}(\rho,\eta)$ vary analytically over $QC(\Gamma) \times QC(\Gamma)$.*

$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$

i.e. h_ρ the topological entropy of the geodesic over N_ρ

Theorem C (Bray, Canary, K.). *The pressure form* $\mathbb{P} := \mathrm{Hess}(\mathrm{J}(\rho,\cdot))$ *on $QF(S)$ induces a* $\mathrm{Mod}(S)$*-invariant path metric, which is an analytic Riemannian metric on the complement of the Fuchsian locus.*
Moreover, if $v \in T_\rho(QF(S))$, then $\mathbb{P}(v,v) = 0$ iff ρ is Fuchsian and v is a pure bending vector.

1. $v = \frac{d}{dt}\rho_t$, $\rho_0 \in F(S)$, and $\rho_{-t} = \overline{\rho_t}$ for all t

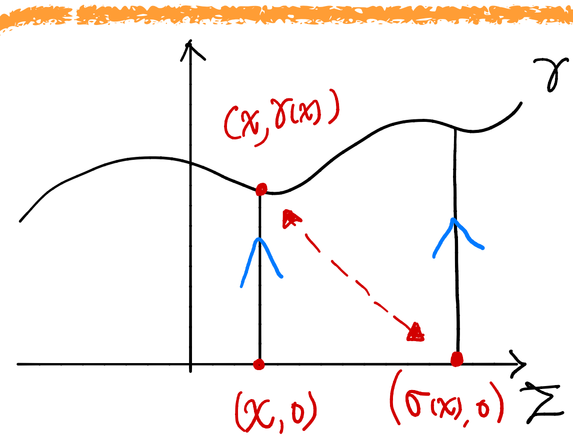
2. $T_\rho QF(S) = T_\rho F(S) \oplus B_\rho$

$d_{\mathbb{P}}(\rho,\eta) = \inf\{\int_0^1 \sqrt{\mathbb{P}(\gamma'(t),\gamma'(t))}\, dt\}$

THE CONSTRUCTION OF PRESSURE METRIC FOR TEICHMÜLLER SPACES OF CLOSED SURFACES

–through symbolic dynamics of the geodesic flow

- Let S be a closed surface and h be a hyperbolic metric on S.
- $g_t : T^1S \to T^1S$ can be understood by the suspension flow:
  - $\sigma_r : \Sigma_r \to \Sigma_r$ where (Σ, σ) is a SFT and the roof function r is constructed by the Bowen-Series coding.



- $\Sigma_r := \{(x,t) : \ x \in \Sigma, \ 0 \le t \le r(x)\}$ with the identification $(x, r(x)) = (0, \sigma(x))$
- the suspension flow σ_t^r : $\Sigma_r \to \Sigma_r$: $\sigma_t^r(x,s) = (x, t+s)$ for $-s \le t \le r(x) - s$

  - More precisely, $r = \log|T'|$ where $T : \Lambda(\Gamma) \to \Lambda(\Gamma)$ is the Bowen-Series map

    the limit set of Γ on $\partial\mathbb{H}^2$

  - r is (bounded) Hölder conti.

- One observe that the symbolic model is stable under perturbation of the hyperbolic metric
  - Structure stability of Anosov flows
- Bowen's formula:

Let (Σ^+, σ, r) be the suspension flow model for geodesic flow over the hyperbolic surface X, then $\boxed{P(-r) = 0}$ where $P(g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{Fix}^n} e^{S_n g(x)}$ is the topological pressure of g

- In general, $P(-h_X \cdot r_X) = 0$ where h_X is the topological entropy of the geodesic flow over X
  - $h_X = \lim_{T\to\infty} \frac{1}{T}\{\lambda : l(\lambda) \le T\}$

m is a eq. state for f
$\iff P(f) = h(m) + \int f \, dm$

- Liouville measure (Bowen-Margulis): $m_L \longleftrightarrow \dfrac{\boxed{m_{r_X}} \times dt}{\int_{\Sigma^+} r_X \, dm_{r_X}}$

- **Dynamics interpretation of the intersection number:** Let $\rho_1,\ \rho_2$ be two cocompact Fuchsian representations, and $\tau_1,\ \tau_2$ be the corresponding roof functions , then

$$\boxed{\mathrm{I}(\rho_1,\rho_2)=\frac{\int_{\Sigma^+}\tau_2\ \mathrm{d}m_{\tau_1}}{\int_{\Sigma^+}\tau_1\ \mathrm{d}m_{\tau_1}}}$$

where is m_{τ_1} the equilibrium state for τ_1 .

Recall: $\mathrm{I}(\rho_1,\rho_2)=\lim_{T\to\infty}\frac{1}{|R_{\rho_1}(T)|}\sum_{\gamma\in R_{\rho_1}(T)}\frac{l_2[\gamma]}{l_1[\gamma]}$

where $R_{\rho_1}(T)=\{\gamma : l_{\rho_1}[\gamma]<T\}$

Idea: the mme μ_{τ_1} of Σ_{τ_1} is equidistributed on closed orbits. On the other hand, μ_{τ_1} is the lift of the eq. state m_{τ_1}.

Intersection Rigidity

Theorem (Thurston). *Let ρ_1, ρ_2 be two fuchsian reps, then $\mathrm{I}(\rho_1, \rho_2) \geq 1$, and the equality holds iff ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}(2, \mathbb{R})$.*

- $0 = P(-\tau_2) = h(m_{-\tau_2}) - \int \tau_2 \, dm_{\tau_2} \geq = h(m_{-\tau_1}) - \int \tau_2 \, dm_{\tau_1}$
- $0 = P(-\tau_1) = h(m_{-\tau_1}) - \int \tau_1 \, dm_{\tau_1}$
- $I(\rho_1, \rho_2) = 1 \iff m_{-\tau_1}$ is an eq. state for $-\tau_2 \iff \underline{\tau_1 \sim \tau_2}$
- $\tau_1 \sim \tau_2 \iff \rho_1 \sim \rho_2$ by marked length spectrum rigidity

cohomology:
$f \sim g \iff f - g = h - h \circ \sigma$

The pressure metric

Theorem (Thurston). *Let $\{\rho_t\} \subset \mathcal{T}(S)$ be an analytic path. Then $\mathrm{I}(\rho_0, \rho_t)$ is real analytic, and $\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{I}(\rho_0, \rho_t)$ is non-degenerate. Hence, the Hessian of I gives a Riemannian metric on $\mathcal{T}(S)$*

- Thermodynamic mapping
  - $\Psi : \mathscr{T}(S) \to \mathrm{P} : \{\text{pressure zero functions}\}/\sim$
  - $d\Psi : T_\rho \mathscr{T}(S) \to T_{-\tau_\rho}\mathrm{P} = \mathrm{Ker} D_{-\tau_\rho}\mathrm{P}$
- Analyticity of pressure
- Pressure metric is given by
  - $$||\dot{\rho}_0||^2_{\mathbb{P}} := ||\dot{\tau}_0||^2_{\mathbb{P}} := \frac{\mathrm{Var}(\dot{\tau}_0, m_{-\tau_0})}{\int \tau_0 \, dm_{-\tau_0}}$$
- Second derivative of pressure

$$0 = \left.\frac{\mathrm{d}^2 P(-\tau_t)}{\mathrm{d}t^2}\right|_{t=0} = (D_{-\tau_0}P)(-\ddot{\tau}_0) + (D^2_{-\tau_0}P)(-\dot{\tau}_0) = -\int \ddot{\tau}_0 \, \mathrm{dm}_{\tau_0} + \mathrm{Var}(-\dot{\tau}_0, \mathrm{m}_{\tau_0})$$

- Relation with I:
  - $$\left.\frac{\mathrm{d}^2 \mathrm{I}(\rho_0, \rho_\mathrm{t})}{\mathrm{d}t^2}\right|_{t=0} = \frac{\int \ddot{\tau}\ \mathrm{dm}_{\tau_0}}{\int \tau_0\ \mathrm{dm}_{\tau_0}} = \frac{\boxed{\mathrm{Var}(\dot{\tau}_0, \mathrm{m}_{\tau_0})}}{\int \tau_0\ \mathrm{dm}_{\tau_0}} = ||\dot{\tau}_0||_\mathrm{P}^2$$
- Degeneracy criterion:

$$\mathrm{Var}(\dot{\tau}_0, \mathrm{m}_{\tau_0}) = 0 \iff \dot{\tau}_0 \sim 0 \iff \dot{\rho}_0 = 0$$

$$\dot{\tau}_0 \in \mathrm{Ker} D_{-\tau_0} P \iff \int \dot{\tau}_0\ dm_{\tau_0} = 0$$

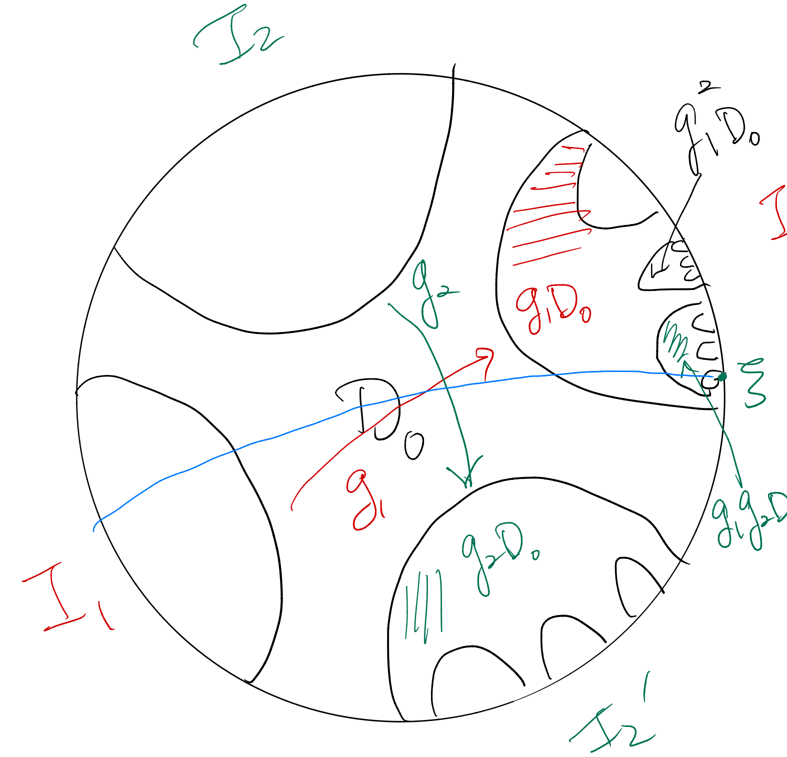
Since $\frac{d\alpha}{dt}$ for any closed geodesic α and $\mathscr{T}(S)$
is determined by finitely many closed geodesics

SUMMARY

- S is a closed surface and $\rho_1,\ \rho_2 \in \mathcal{T}(S)$
- A suspension flow model: $\sigma^{r_i}(x,t) : \Sigma_{r_i} \to \Sigma_{r_i}$
  - (Bowen-Series coding of the geodesics flow)
  - same base space Σ^+ and ``nice" roof functions r_i
  - stable under perturbation
- Bowen's formula: $P(-h(\rho)\cdot r_\rho) = 0$
- Dynamics interpretation of $\mathrm{I}(\rho_1,\rho_2)$
- Good regularity (analyticity) of the pressure P over Σ^+
- Degeneracy criterion: $\mathbb{P}(v,v) = 0 \iff \left.\frac{d}{dt}\right|_{t=0} l_{\rho_t}(\alpha) = 0$ for any closed geodesic α and $\left.\frac{d}{dt}\right|_{t=0} \rho_t = v$

EXAMPLE: CODING FOR SCHOTTKY GROUPS

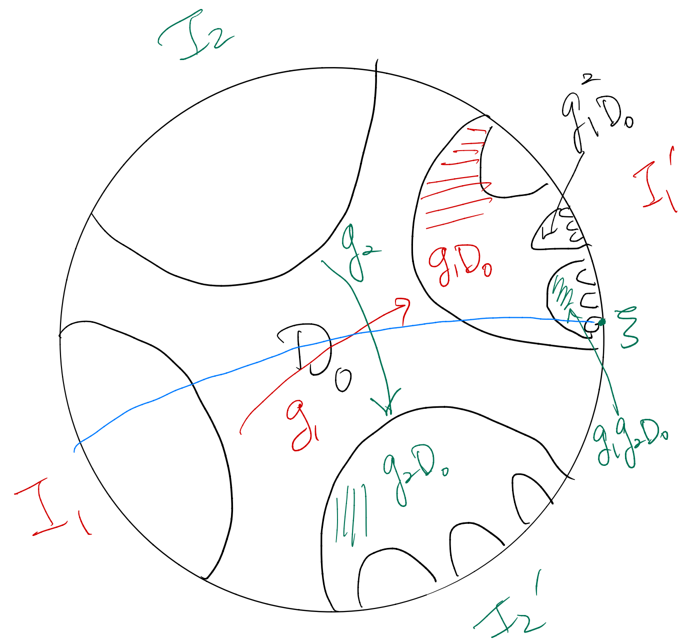
The ideal world: when reps are convex cocompact



- cutting sequence/coding of limit set:
- $\exists! \ \{x_n\} \subset \{g_1, g_2\}^{\mathbb{N}}$ such that $x_1 x_2 \ldots x_n(D_0) \to \xi$ and $\omega : \Sigma^+ \to \Lambda(\Gamma)$

  is a bijection where $\Sigma^+ = \{x = (x_n) \in \{g_1^{\pm}, g_2^{\pm}\}^{\mathbb{N}} : x_n \neq x_{n+1}^{-1}\}$
- What is the associated dynamics on $\Lambda(\Gamma)$
  - Bowen-Series map T: canceling the first element

    $T(\xi) = g_{x_1}^{-1}(\xi)$ where $\omega^{-1}(\xi) = x_1 x_2 \ldots$



Properties:

★ every period orbit $\lambda \in T^1X \leftrightarrow x \in \Sigma^+$ with $\sigma^n(x) = x$

★ let $r(x) := \ln |T'|(w^{-1}(x))$, then $l(\lambda) = r_n(x) := \sum_{i=0}^{n} r(\sigma^i(x))$

$\boxed{B_{\omega(x)}(o, g_{x_1} o)}$ geometric potential

where $B_\xi(x, y) := \lim_{z \to \xi} d(x, z) - d(y, z)$ is the Busemann function

FROM FUCHSIAN TO QUASIFUCHSIAN

Lemma . *If $\rho \in QC(\Gamma)$, then there exists a ρ-equivariant bi-Hölder continuous map*

$$\xi_\rho : \Lambda(\Gamma) \to \Lambda(\rho(\Gamma)).$$

Moreover, if $x \in \Lambda(\Gamma)$, then $\xi_\rho(x)$ varies complex analytically over $QC(\Gamma)$.

★ Suppose $\omega : \Sigma^+ \to \Lambda(\Gamma)$ is a coding for $\Lambda(\Gamma)$, then $\xi_\rho \circ \omega : \Sigma^+ \to \Lambda(\rho(\Gamma))$ for $\Lambda(\rho(\Gamma))$

★ The roof function $\tau_\rho(x) := B_{\xi_\rho(\omega(x))}(o, \rho(g_{x_1})o)$ where $x = x_1 x_2 ...$

★ If $x = \overline{x_1 x_2 ... x_n} \in \mathrm{Fix}^n$, then $l_\rho(g_{x_1} g_{x_2} ... g_{x_n}) = S_n \tau_\rho(x)$

Regularity of τ_ρ ? Behaviors under perturbation?

What is the degenerate criterion ?

FROM COMPACT TO NON-COMPACT

ISSUES

- **cutting sequence is not unique around parabolic limit pts**
- **Bowen-Series map T doesn't expanding at these parabolic pts**
- **regularity of the geometric potentials/first returning map:**
    - one needs at least $|T'|>1$ to have geometric potential well-defined
    - τ_ρ is unbounded
- **regularity of the pressure (phase transition)**
- **reparametrizing the flow might lose thermodynamic data**
    - Cipriano-Iommi
- **Identify where the degeneracy of the variance happens**

REMEDIES

- upgrade **Bowen-Series coding** to **Stadlbauer-Ledrappier-Sarig coding**
  - which code the geodesic flow through a **Countable State Markov Shift**
  - moreover, it is topologically mixing and has BIP
  - nice properties of τ_ρ (locally Hölder and eventually expanding)
- pressure behaves well away from phase transition
  - study the places where phase transitions taking place
  - variance degeneracy criterion
- the reparametrization (time change) function is bounded

SUMMARY

- S is a punctured surface and $\rho, \eta \in QF(S)$
- Stadlbauer-Ledrappier-Sarig coding: $\sigma^{\tau_\rho}(x,t) : \Sigma_{\tau_\rho} \to \Sigma_{\tau_\rho}$
  - same base space Σ^+ and ``nice" roof functions $\tau_\rho(x) := B_{\xi_\rho \circ \omega(x)}(o, \rho(g_{x_1})o)$
  - **Lemma**: stable under perturbation
    - $\tau_\rho(x)$ varies analytically over $QC(\Gamma)$ for all $x \in \Sigma^+$
- Dynamics interpretation of $\mathrm{I}(\rho,\eta)$ remains the same (analyticity)
- Pressure P behaves analytically away from the phase transition
- Degeneracy criterion: $\mathbb{P}(v,v) = 0 \iff \left.\dfrac{d}{dt}\right|_{t=0} h(\rho_t)\cdot l_{\rho_t}(\alpha) = 0$ for any closed geodesic α and $\left.\dfrac{d}{dt}\right|_{t=0} \rho_t = v \iff v$ is a pure bending vector

Thank you!!

感謝!!

kám-siā !! (Taiwanese)